

# Prelims Probability

## Sheet 7 — MT23

- Sketch the cumulative distribution function of the following distributions:
  - the (discrete) uniform distribution on  $\{1, 2, \dots, n\}$ ;
  - the (continuous) uniform distribution on  $[a, b]$ ;
  - the exponential distribution with parameter 1;
  - the normal distribution with mean 0 and variance 1.
- For each case below, does there exist a constant  $c$  such that the given function is a probability density function? If so, find  $c$  and find the cumulative distribution function. (In each case, the given function is zero outside the interval  $(0, 1)$ .)
  - $f_1(x) = cx$  for  $0 < x < 1$ .
  - $f_2(x) = cx^{-1}$  for  $0 < x < 1$ .
  - $f_3(x) = cx^{-1/2}$  for  $0 < x < 1$ .
  - $f_4(x) = c(4x^3 - x)$  for  $0 < x < 1$ .
- Let  $U$  be a uniformly distributed random variable on  $[0, 1]$ . Find
  - $\mathbb{E}[U]$  and  $\text{var}(U)$
  - $\mathbb{P}(U < a | U < b)$  for  $0 < a < b < 1$ .
- Let  $X$  be exponentially distributed with parameter  $\lambda$ .
  - Find  $\mathbb{P}(X > x)$
  - Find  $\mathbb{P}(a \leq X \leq b)$  for  $0 < a < b$
  - Show that  $\mathbb{P}(X > a + x | X > a) = \mathbb{P}(X > x)$  for  $a, x > 0$ . [This is the *memoryless property* of the exponential distribution (compare to Question 3 on Sheet 3).]
  - Find  $\mathbb{P}(\sin X > \frac{1}{2})$
  - Let  $c > 0$ . What is the distribution of the random variable  $cX$ ? [Try using part (a).]
  - For  $x \in \mathbb{R}$ , let  $\lceil x \rceil$  denote the *ceiling* of  $x$ ; that is, the smallest integer greater than or equal to  $x$ . Show that the discrete random variable  $\lceil X \rceil$  has a geometric distribution, and find its parameter. [Hint: write the event  $\{\lceil X \rceil = k\}$  as  $\{X \in I\}$  for some interval  $I$ .]

5. Blood plasma nicotine levels in smokers can be modelled by a normal random variable  $X$  with mean 315 and variance  $131^2$ , the units being nanograms per millilitre.
- What is the probability that a randomly chosen smoker has nicotine levels lower than 300?
  - What is the probability that a randomly chosen smoker has nicotine levels between 300 and 500?
  - If 20 smokers are to be tested what is the probability that at most one has a nicotine level higher than 500?

[If  $\Phi(x)$  is the cumulative distribution function of the standard normal distribution then  $\Phi(-0.115) = 0.454$ ,  $\Phi(1.412) = 0.921$ .]

6. The radius of a circle is uniformly distributed on  $[0, b]$ . Find the cumulative distribution function, the probability density function, the expectation and the variance of the random variable representing the area of the circle.
7. Let  $X$  be a continuous random variable taking values in  $[a, b]$  with c.d.f.  $F_X$  which is strictly increasing on  $[a, b]$ .
- Show that the random variable  $F_X(X)$  has a uniform distribution on  $[0, 1]$ .
  - Let  $U$  be a uniform random variable on  $[0, 1]$ . What is the distribution of the random variable  $F_X^{-1}(U)$ , where  $F_X^{-1}$  is the inverse of  $F_X$ ?
  - Suppose that  $U_1, U_2, \dots, U_n$  are a set of computer-generated pseudo-random numbers (assumed to be drawn from a uniform distribution on  $[0, 1]$ ). How would you use them to simulate a random sample  $X_1, X_2, \dots, X_n$  from the distribution with density

$$f(x) = \mu e^{-\mu x}, \quad x \geq 0?$$