Analysis I

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Contents

1 Introduction

This course is a first introduction to real analysis. We'll set the scene with exploring basic properties of real numbers. Then we'll go on to define what it means for a sequence to converge, and for a series to converge, and we'll explore these ideas in detail, allowing us to give careful justifications of some familiar results, as well as proving some that you might not have seen before.

There are several resources that will help you as you study the course:

- the lecture videos
- the slides that go with the lecture videos
- these notes
- the notes by Dr Hilary Priestley
- the problems sheets
- the additional content on Moodle
- each other
- your college tutors.

When I made the lecture videos, I used a mix of slides and handwriting, but everything handwritten also appears on a typed slide. I'm making the slides available alongside the videos, in case you would like a copy of these to view or annotate. These notes (the document you're reading now!) include everything in the slides. You are encouraged to read them alongside the notes by Dr Hilary Priestley.

Acknowledgements

These notes, and the lectures they accompany, are extremely closely based on the notes produced by Dr Hilary Priestley. The same applies to the problems sheets.

I would like these notes to be as useful as possible. If you (whether student or tutor) think that you've noticed a typo, or mistake, or part that is unclear, please check the current, up-to-date, notes on the website, to see whether I've already fixed it. If not, please email me (vicky.neale@maths) or put a note in the Moodle discussion, and I'll do something about it, and (with your permission) thank you here.

Thanks to Kendall De, Orson Hart, Aidan Strong and another (a student who preferred not to be named) for helping to fix glitches in these notes and the accompanying problems sheets.

2 Axioms for arithmetic in $\mathbb{R}$

What are the real numbers?

We are going to work with the real numbers in this course. What is a real number? We are not, in this course, going to define the real numbers — we aren't going to say what they *are*. Instead, we'll focus on how they behave — what they *do*. We'll identify the key properties of the real numbers, and the goal is to deduce everything from these properties. That leaves the task of showing that there is a set with the required properties (that is, constructing the real numbers), but that is not a task for this course.

So what are these key properties? They are all properties that will probably feel 'obvious'. For us, they are *assumptions*. We assume that the real numbers have these properties. You might feel surprised that there aren't more assumptions, but, as we'll see, we can *deduce* the other familiar (and not so familiar) properties from these assumptions. The goal is to avoid assuming more than we have to.

One question to have in the back of your mind through this section is "What other structures also have these properties?" We know that the rational numbers share some properties with the real numbers, and also the complex numbers share some properties with the real numbers. But we also know that in some way the rational numbers are definitely different from the real numbers, and similarly for complex numbers, and we should see this come up. If the rational numbers share all of our key properties of the real numbers, then we haven't assumed enough properties of the real numbers!

The flavour of this section will be about deducing basic properties, one careful step at a time. Some people enjoy this way of working; others, well, not so much. If you're in the latter category, don't worry: most of this course will have quite a different flavour (although it will still be about giving careful, rigorous proofs of results that might feel 'obvious'). But, at its heart, mathematics is built on the sort of axiomatic reasoning that we'll see in this section, so it's good to have a sense of how that works.

Notation. We write $\mathbb{R}$ for the set of real numbers. We write $\mathbb{Q}$ for the set of rational numbers, and $\mathbb{C}$ for the set of complex numbers.

Axioms for arithmetic in $\mathbb{R}$

One really important feature of $\mathbb{R}$ is that we can do 'arithmetic'. We have operations of addition and multiplication. (Note that I didn't mention subtraction or division. Can you think why?) Here is a careful statement of the key properties of addition and multiplication.

- For every a, $b \in \mathbb{R}$ there is a unique real number $a+b$, called their *sum*.
- For every a, $b \in \mathbb{R}$ there is a unique real number $a \cdot b$, called their *product*.
- For $a \in \mathbb{R}$ there is a unique real number $-a$ called its *negative* or its *additive inverse*.
- For $a \in \mathbb{R}$ with $a \neq 0$ there is a unique real number $\frac{1}{a}$ called its *reciprocal* or its *multiplicative inverse*.
- There is a special element $0 \in \mathbb{R}$ called *zero* or the *additive identity*.
- There is a special element $1 \in \mathbb{R}$ called *one* or the *multiplicative identity*.

For all a, b, $c \in \mathbb{R}$, we have

- $a + b = b + a$ (+ is commutative)
- $a + (b + c) = (a + b) + c$ (+ is associative)
- $a + 0 = a$ (additive identity)
- $a + (-a) = 0$ (additive inverses)
- $a \cdot b = b \cdot a$ (· is commutative)
- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (· is associative)
- $a \cdot 1 = a$ (multiplicative identity)
- if $a \neq 0$ then $a \cdot \frac{1}{a} = 1$ (multiplicative inverses)
- $a \cdot (b + c) = a \cdot b + a \cdot c$ (· distributes over +)

- $0 \neq 1$ (to avoid total collapse)

This is a long list of properties, called *axioms*.

I have given a name to each axiom, so that in proofs we can say which property we are using at each stage. Some people prefer to number the axioms, but there is no standard way to do this, and I find it hard to remember the numbers and easier to remember the names.

I think of the axioms as coming in bundles, to help me to remember them. The first four tell us that 'addition behaves nicely'. The next four tell us that 'multiplication behaves nicely'. Then there is an axiom that tells us that 'addition and multiplication interact nicely', and a final technical detail to clarify that $\mathbb{R} \neq \{0\}$.

As you study abstract algebra (such as linear algebra and group theory) this year, you'll find yourself coming across lists of axioms such as this again.

Definition. Let $\mathbb{F}$ be a set with operations $+$ and $\cdot$ that satisfy the axioms above. Then we say that $\mathbb{F}$ is a *field*.

Example. We've just said that $\mathbb{R}$ is a field. The rational numbers $\mathbb{Q}$ form a field. The complex numbers $\mathbb{C}$ form a field. You'll meet other fields too, in other courses. The integers $\mathbb{Z}$ do not form a field.

In the next section, we'll use these axioms to deduce the basic properties of arithmetic in $\mathbb{R}$. This reasoning would apply equally to any field. We'll need to make further assumptions about $\mathbb{R}$, but we'll postpone this till we've studied basic arithmetic.

3 Properties of arithmetic in $\mathbb{R}$

In the last section, we saw the arithmetic axioms for $\mathbb{R}$. Now we'll deduce some properties of arithmetic in $\mathbb{R}$.

Proposition 1. *Let a, b, c, x, y be real numbers.*

(i) *If $a + x = a$ for all a then $x = 0$ (uniqueness of 0).*

(ii) *If $a + x = a + y$ then $x = y$ (cancellation for $+$).*

(iii) $-0 = 0$.

(iv) $-(-a) = a$.

(v) $-(a + b) = (-a) + (-b)$.

(vi) *If $a \cdot x = a$ for all $a \neq 0$ then $x = 1$ (uniqueness of 1).*

(vii) *If $a \neq 0$ and $a \cdot x = a \cdot y$ then $x = y$ (cancellation for $\cdot$).*

(viii) *If $a \neq 0$ then $\frac{1}{\frac{1}{a}} = a$.*

(ix) $(a + b) \cdot c = a \cdot c + b \cdot c$.

(x) $a \cdot 0 = 0$.

(xi) $a \cdot (-b) = -(a \cdot b)$. *In particular,* $(-1) \cdot a = -a$.

(xii) $(-1) \cdot (-1) = 1$.

(xiii) *If $a \cdot b = 0$ then $a = 0$ or $b = 0$. If $a \neq 0$ and $b \neq 0$ then $\frac{1}{a \cdot b} = \frac{1}{a} \cdot \frac{1}{b}$.*

Remark.

- (ii) shows the uniqueness of $-a$, the additive inverse of a.
- (vii) shows the uniqueness of $\frac{1}{a}$, the multiplicative inverse of a (if $a \neq 0$).
- As we'll see shortly, (i)–(v) can be proved using only the four axioms about $+$.
- Similarly, (vi)–(viii) can be proved using only the four axioms about $\cdot$.

- (ix)–(xiii) between them use all the axioms.
- It's worth proving results like this in a sensible order! Once we've proved a property, we can add it to the list of properties we can assume in subsequent parts. You'll see that we prove some later parts using earlier parts.

Proof. (i) Suppose that $a + x = a$ for all a. Then

$$\begin{aligned} x &= x + 0 \quad \text{(additive identity)} \\ &= 0 + x \quad \text{(+ is commutative)} \\ &= 0 \quad \text{(by hypothesis, with } a = 0\text{).} \end{aligned}$$

(ii) Suppose that $a + x = a + y$. Then

$$\begin{aligned} y &= y + 0 \quad \text{(additive identity)} \\ &= y + (a + (-a)) \quad \text{(additive inverses)} \\ &= (y + a) + (-a) \quad \text{(+ is associative)} \\ &= (a + y) + (-a) \quad \text{(+ is commutative)} \\ &= (a + x) + (-a) \quad \text{(hypothesis)} \\ &= (x + a) + (-a) \quad \text{(+ is commutative)} \\ &= x + (a + (-a)) \quad \text{(+ is associative)} \\ &= x + 0 \quad \text{(additive inverses)} \\ &= x \quad \text{(additive identity).} \end{aligned}$$

(iii) We have $0 + 0 = 0$ (additive identity)

and $0 + (-0) = 0$ (additive inverses)

so $0 + 0 = 0 + (-0)$, so $0 = -0$ (cancellation for + (ii)).

(iv) We have

$$\begin{aligned}(-a)+a &= a+(-a) \quad (+\text{ is commutative})\\ &= 0 \quad (\text{additive inverses})\end{aligned}$$

and $(-a)+(-(-a)) = 0$ (additive inverses),

so $(-a)+a = (-a)+(-(-a))$,

so $a = -(-a)$ (cancellation for $+$ (ii)).

(v) Exercise (see Sheet 1).

(vi)–(viii) Exercise — similar to (i), (ii), (iv).

(ix) (This is another form of distributivity, similar to the axiom but different!)

We have

$$\begin{aligned}(a+b)\cdot c &= c\cdot(a+b) \quad (+\text{ is commutative})\\ &= c\cdot a + c\cdot b \quad (\cdot\text{ distributes over }+)\\ &= a\cdot c + b\cdot c \quad (\cdot\text{ is commutative – twice}).\end{aligned}$$

(x) We have $a\cdot(0+0) = a\cdot 0 + a\cdot 0$ ($\cdot$ distributes over $+$),

and also

$$\begin{aligned}a\cdot(0+0) &= a\cdot 0 \quad (\text{additive identity})\\ &= a\cdot 0 + 0 \quad (\text{additive identity})\end{aligned}$$

so $a\cdot 0 = 0$ (cancellation for $+$ (ii))

(xi) We have

$$\begin{aligned} a \cdot b + a \cdot (-b) &= a \cdot (b + (-b)) \quad (\cdot \text{ distributes over } +) \\ &= a \cdot 0 \quad (\text{additive inverses}) \end{aligned}$$

and $a \cdot b + (-(a \cdot b)) = 0$ (additive inverses),

so $a \cdot (-b) = -(a \cdot b)$ (cancellation for $+$ (ii)).

(xii) We have

$$\begin{aligned} (-1) \cdot (-1) &= -((-1) \cdot 1) \quad ((\text{xi}) \text{ with } a = -1,\, b = 1) \\ &= -(-1) \quad (\text{multiplicative identity}) \\ &= 1 \quad ((\text{iv})). \end{aligned}$$

(xiii) Suppose, for a contradiction, that $a \neq 0$, $b \neq 0$ but $a \cdot b = 0$. Then

$$\begin{aligned} 0 &= \left(\frac{1}{a} \cdot \frac{1}{b}\right) \cdot 0 \quad ((\text{x})) \\ &= 0 \cdot \left(\frac{1}{a} \cdot \frac{1}{b}\right) \quad (\cdot \text{ is commutative}) \\ &= (a \cdot b) \cdot \left(\frac{1}{a} \cdot \frac{1}{b}\right) \quad (\text{hypothesis}) \\ &= (b \cdot a) \cdot \left(\frac{1}{a} \cdot \frac{1}{b}\right) \quad (\cdot \text{ is commutative}) \\ &= \left((b \cdot a) \cdot \frac{1}{a}\right) \cdot \frac{1}{b} \quad (\cdot \text{ is associative}) \\ &= (b \cdot \left(a \cdot \frac{1}{a}\right)) \cdot \frac{1}{b} \quad (\cdot \text{ is associative}) \\ &= (b \cdot 1) \cdot \frac{1}{b} \quad (\text{multiplicative inverses}) \\ &= b \cdot \frac{1}{b} \quad (\text{multiplicative identity}) \\ &= 1 \quad (\text{multiplicative inverses}) \end{aligned}$$

and this is a contradiction ($0 \neq 1$).

So if $a \cdot b = 0$ then $a = 0$ or $b = 0$.

Note that on the way we showed that if $a \neq 0$ and $b \neq 0$ then $a \cdot b \neq 0$ and $(a \cdot b) \cdot \left(\frac{1}{a} \cdot \frac{1}{b}\right) = 1$ so $\frac{1}{a \cdot b} = \frac{1}{a} \cdot \frac{1}{b}$ (cancellation for $\cdot$ (vii)).

$\square$

From now on, we can use all of these properties. We shan't give such detailed, one-axiom-at-a-time, derivations in the remainder of the course — but we could, if we needed to!

Remark. (This remark is not part of the course!) You might be concerned that if we don't go back to the axioms every time then we might overlook an unproved step, or might make a mistake. But going back to the axioms every time is not practical: a research paper might take tens of pages to give a proof, referring back to other results, which themselves build on results, which build on results, ... This is where *proof verification* comes in: computers can take care of this detailed checking, leaving humans to focus on things that humans are good at (like having creative ideas for proofs).

One interesting project in this area (there are others, not just this one) is Xena `https://xenaproject.wordpress.com/what-is-the-xena-project/` — this is aimed at undergraduates, which is why I'm mentioning it. Some of you might find it interesting. There's an article about proof verification and Xena in the London Mathematical Society Newsletter, pages 32–36 of `https://www.lms.ac.uk/sites/lms.ac.uk/files/files/NLMS_484-forweb2.pdf`.

Now back to the course ....

Notation. From now on, we use more familiar notation. We write

$$a - b \text{ for } a + (-b)$$
$$ab \text{ for } a \cdot b$$
$$\frac{a}{b} \text{ for } a \cdot \left(\frac{1}{b}\right)$$
$$a^{-1} \text{ sometimes for } \frac{1}{a}.$$

The associativity of addition and multiplication means that we can write expressions like $a + b + c$ and xyz, without needing to write brackets.

Definition. Take $a \in \mathbb{R} \setminus \{0\}$.

Define $a^0 = 1$.

We define positive powers of a inductively: for integers $k \geqslant 0$, we define $a^{k+1} = a^k \cdot a$.

For integers $l \leqslant -1$, we define $a^l = \dfrac{1}{a^{-l}}$.

Remark. Note that with this definition $a^1 = a$ and $a^2 = a \cdot a$ (as we'd want).

Lemma 2. *For $a \in \mathbb{R} \setminus \{0\}$ we have $a^m a^n = a^{m+n}$ for m, $n \in \mathbb{Z}$.*

Proof. Exercise (see Sheet 1). □

This finishes this section on arithmetic in $\mathbb{R}$. In the next section, we'll go on to explore more key properties of $\mathbb{R}$, in addition to it being a field.

4 Ordering the real numbers

When we picture $\mathbb{R}$ in our minds, typically it is not as a scattered collection of numbers. Rather, we often picture them as lying along a number line, usually running from left to right, with 0 in the 'middle'; positive numbers on the right, increasing as we move away from 0; and negative numbers on the left, getting smaller as we move left away from 0. Even in writing that description, I have made assumptions about $\mathbb{R}$: I have assumed that we have notions of 'positive' and 'negative', and that we can compare the sizes of two real numbers. This section is about formalising those assumptions.

Here are our axioms for the usual ordering on $\mathbb{R}$.

There is a subset $\mathbb{P}$ of $\mathbb{R}$ such that for a, $b \in \mathbb{R}$

- if a, $b \in \mathbb{P}$ then $a + b \in \mathbb{P}$ (+ and ordering)
- if a, $b \in \mathbb{P}$ then $a \cdot b \in \mathbb{P}$ ($\cdot$ and ordering)
- exactly one of $a \in \mathbb{P}$, $a = 0$ and $-a \in \mathbb{P}$ holds (positive, negative or 0).

The elements of $\mathbb{P}$ are called the *positive numbers*. The elements of $\mathbb{P}\cup\{0\}$ are called the *non-negative numbers*.

We write $a < b$, or $b > a$, exactly when $b - a \in \mathbb{P}$.

We write $a \leqslant b$, or $b \geqslant a$, exactly when $b - a \in \mathbb{P} \cup \{0\}$.

Now, just as with the axioms for arithmetic, we can infer useful properties from these axioms. First, some important properties of the ordering.

Proposition 3. *Take a, b, $c \in \mathbb{R}$. Then*

(i) $a \leqslant a$; *(reflexivity)*

(ii) if $a \leqslant b$ and $b \leqslant a$ then $a = b$; *(antisymmetry)*

(iii) if $a \leqslant b$ and $b \leqslant c$ then $a \leqslant c$, and similarly with $<$ in place of $\leqslant$; *(transitivity)*

(iv) exactly one of $a < b$, $a = b$ and $a > b$ holds. *(trichotomy)*

Proof. (i) We have $a - a = 0 \in \mathbb{P} \cup \{0\}$ (additive inverses).

(ii) Suppose that $a \leqslant b$ and $b \leqslant a$.

If $a - b = 0$ or $b - a = 0$ then $a = b$ (properties of $+$) and we are done.

If not, then $b - a \in \mathbb{P}$ and $a - b \in \mathbb{P}$.

But $b - a = -(a - b)$ (properties of $+$),

so then $a - b \in \mathbb{P}$ and $-(a - b) \in \mathbb{P}$, contradicting 'positive, negative or 0'.

(iii) Note that $c - a = c + (-a) = c + 0 + (-a) = c + (-b) + b + (-a) = (c - b) + (b - a)$ (properties of $+$)

so if $a < b$ and $b < c$ then $a < c$ ($+$ and ordering).

The cases where $a = b$ and/or $b = c$ are straightforward, and give the result for $\leqslant$.

(iv) This follows from 'positive, negative or 0'.

□

In the next section, we'll explore the interaction between the ordering and basic arithmetic.

5 Inequalities and arithmetic

The next result has some useful results about inequalities and arithmetic, which will save us from having to go back to the axioms every time.

Proposition 4. *Take a, b, $c \in \mathbb{R}$.*

(i) $0 < 1$.

(ii) $a < b$ *if and only if* $-b < -a$. *In particular,* $a > 0$ *if and only if* $-a < 0$.

(iii) *If* $a < b$ *then* $a + c < b + c$.

(iv) *If* $a < b$ *and* $0 < c$ *then* $ac < bc$.

(v) $a^2 \geqslant 0$, *with equality if and only if* $a = 0$.

(vi) $a > 0$ *if and only if* $\frac{1}{a} > 0$.

(vii) *If* $a,\ b > 0$ *and* $a < b$ *then* $\dfrac{1}{b} < \dfrac{1}{a}$.

Furthermore, (ii), (iii) and (iv) hold with $\leqslant$ *in place of* $<$.

Proof. (i) By trichotomy, we have $0 < 1$ or $0 = 1$ or $0 > 1$.

But 'to avoid total collapse' $0 \neq 1$. So it suffices to rule out $0 > 1$.

Suppose, for a contradiction, that $0 > 1$.

Then $-1 \in \mathbb{P}$ (by definition of $>$) so $(-1) \cdot (-1) \in \mathbb{P}$ ($\cdot$ and ordering).

But $(-1) \cdot (-1) = 1$ (Proposition 1 (xii)),

so $0 < 1$ — but this contradicts trichotomy.

So $0 < 1$.

(ii) Using properties of addition, we have

$$\begin{aligned} a < b &\Leftrightarrow b - a \in \mathbb{P} \\ &\Leftrightarrow (-a) - (-b) \in \mathbb{P} \\ &\Leftrightarrow -a > -b. \end{aligned}$$

(iii) Assume that $a < b$.

Then $(b + c) - (a + c) = b - a > 0$ so $a + c < b + c$.

(iv) Assume that $a < b$ and $0 < c$.

Then $bc - ac = (b - a)c > 0$ ($\cdot$ and ordering).

(v) Certainly $a^2 = 0$ if and only if $a = 0$ (Proposition 1 (x) and (xiii)).

If $a \neq 0$, then exactly one of a and $-a$ is positive, and either way $a^2 = a \cdot a = (-a) \cdot (-a) > 0$ ($\cdot$ and ordering).

(vi) Suppose, for a contradiction, that $a > 0$ and $\frac{1}{a} < 0$, so $a > 0$ and $-\frac{1}{a} > 0$.

Then $-1 = -\left(a \cdot \dfrac{1}{a}\right) = a \cdot \left(-\dfrac{1}{a}\right) > 0$. But this contradicts (i).

Similarly if $a < 0$ and $\frac{1}{a} > 0$.

(vii) Suppose that a, $b > 0$ and $a < b$.

Then $\frac{1}{a}$, $\frac{1}{b} > 0$ by (vi),

so $a \cdot \dfrac{1}{a} \cdot \dfrac{1}{b} < b \cdot \dfrac{1}{a} \cdot \dfrac{1}{b}$ by (iv),

so $\frac{1}{b} < \frac{1}{a}$.

$\square$

Now we can prove a useful inequality (you'll have an opportunity to apply it on Sheet 1).

Theorem 5 (Bernoulli's Inequality)**.** *Let x be a real number with $x > -1$. Let n be a positive integer. Then $(1 + x)^n \geqslant 1 + nx$.*

Proof. By induction on n. Fix $x > -1$.

$n = 1$: clear.

induction step: suppose the result holds for some $n \geqslant 1$, that is, $(1 + x)^n \geqslant 1 + nx$.

Note that $1 + x > 0$, and $nx^2 \geqslant 0$ (since $n > 0$ and $x^2 \geqslant 0$ by Proposition 4 (v)).

Then

$$
\begin{aligned}
(1+x)^{n+1} &= (1+x)(1+x)^n \quad \text{(by definition)} \\
&\geqslant (1+x)(1+nx) \quad \text{(induction hypothesis and Prop 4 (iv))} \\
&= 1+(n+1)x+nx^2 \quad \text{(properties of arithmetic)} \\
&\geqslant 1+(n+1)x \quad (\text{since } nx^2 \geqslant 0).
\end{aligned}
$$

So, by induction, the result holds. □

In the next section, we'll move on to consider the modulus of a real number.

6 The modulus of a real number

Definition. Let $a \in \mathbb{R}$. The *modulus* $|a|$ of a is defined to be

$$
|a| := \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0. \end{cases}
$$

(It is also sometimes called the *absolute value* of a.)

Remark. The modulus is well defined (that is, this is a legitimate definition) thanks to the 'positive, negative or 0' property (essentially trichotomy).

Here are some basic properties of the modulus.

Proposition 6. *Take a, b, $c \in \mathbb{R}$. Then*

(i) $|-a| = |a|$;

(ii) $|a| \geqslant 0$;

(iii) $|a|^2 = a^2$*;*

(iv) $|ab| = |a||b|$*;*

(v) $-|a| \leqslant a \leqslant |a|$*;*

(vi) if $c \geqslant 0$*, then* $|a| \leqslant c$ *if and only if* $-c \leqslant a \leqslant c$*; and similarly with weak inequalities* ($\leqslant$, $\geqslant$) *replaced by strict* ($<$, $>$)*.*

Proof. (i), (ii) Immediate from the definition, since $a > 0$ if and only if $-a < 0$.

(iii) Check using the definition and trichotomy – go through the cases and also use $(-a)(-a) = a^2$.

(iv) Check the cases using the definition and trichotomy.

(v) If $a \geqslant 0$, then $-|a| \leqslant 0 \leqslant a = |a|$.

If $a < 0$, then $-|a| = a < 0 \leqslant |a|$.

(vi) Assume that $c \geqslant 0$.

($\Rightarrow$) Suppose that $|a| \leqslant c$. Then, by (v), $-c \leqslant -|a| \leqslant a \leqslant |a| \leqslant c$, and we're done by transitivity (Proposition 3).

($\Leftarrow$) Suppose that $-c \leqslant a \leqslant c$. Then $-a \leqslant c$ and $a \leqslant c$. But $|a|$ is a or $-a$, so $|a| \leqslant c$.

Similarly for the version with strict inequalities.

□

Theorem 7 (Triangle Inequality)**.** *Take* a*,* $b \in \mathbb{R}$*. Then*

(i) $|a + b| \leqslant |a| + |b|$*;*

(ii) $|a + b| \geqslant ||a| - |b||$.

Remark. (ii) is called the *Reverse Triangle Inequality.*

Proof. (i) We have $-|a| \leqslant a \leqslant |a|$ and $-|b| \leqslant b \leqslant |b|$, by Proposition 6.

We can add these (see Sheet 1 Q2); using properties of addition, we get $-(|a|+|b|) \leqslant a+b \leqslant |a|+|b|$.

By Proposition 6 (vi) (with $c = |a|+|b| \geqslant 0$), this gives $|a+b| \leqslant |a|+|b|$.

(ii) By (i), we have $|a| = |a+b+(-b)| \leqslant |a+b| + |-b| = |a+b| + |b|$,

so $|a+b| \geqslant |a| - |b|$.

Similarly (swap a and b), $|a+b| \geqslant |b| - |a|$.

Now $||a| - |b||$ is $|a| - |b|$ or $|b| - |a|$, so $|a+b| \geqslant ||a| - |b||$.

□

7 The complex numbers

You met (or renewed your acquaintance with) the set $\mathbb{C}$ of complex numbers in the course *Introduction to Complex Numbers* ???, so we shan't repeat much of that material here. We'll just focus on revisiting the complex numbers with the new perspective of having considered axioms for $\mathbb{R}$.

In particular, as we said earlier, $\mathbb{C}$ is a field (under the usual addition and multiplication). If you are feeling enthusiastic, then check the axioms!

But $\mathbb{C}$ is fundamentally different from $\mathbb{R}$ because there is no ordering on $\mathbb{C}$ that satisfies the ordering axioms.

Exercise. Prove this!

As you saw in the Complex Numbers course, the Triangle Inequality holds in $\mathbb{C}$, and so does the Reverse Triangle Inequality.

So far, we know that $\mathbb{R}$ is an ordered field. But we still haven't captured everything that's important about $\mathbb{R}$. The rational numbers $\mathbb{Q}$ also form an ordered field, but there are fundamental differences between $\mathbb{Q}$ and $\mathbb{R}$. Intuitively, in $\mathbb{Q}$ there are 'gaps' in the number line (such as at $\sqrt{2}$), whereas this is not the case in $\mathbb{R}$. In the next chunk of the course, we'll explore this in detail, to try to identify what additional property or properties we need to assume hold in $\mathbb{R}$.

8 Upper and lower bounds

In the next few sections of the course, we going to explore the difference between $\mathbb{Q}$ and $\mathbb{R}$, as discussed in the last paragraph of the previous section.

One helpful example to have in mind is the square root of two. We know that $\sqrt{2}$ is not in $\mathbb{Q}$ (you have probably seen a proof of this elsewhere). But there *is* a positive real number that squares to give 2. One goal of these sections of the course is to prove the existence of this positive real number (which we call $\sqrt{2}$).

The key property, which $\mathbb{R}$ has and $\mathbb{Q}$ does not, is called *completeness*. But before we get there, we need a few preliminary definitions.

Definition. Let $S \subseteq \mathbb{R}$. Take $b \in \mathbb{R}$. We say that

- b is an *upper bound* of S if $s \leqslant b$ for all $s \in S$;
- b is a *lower bound* of S if $s \geqslant b$ for all $s \in S$;
- S is *bounded above* if S has an upper bound;
- S is *bounded below* if S has a lower bound;
- S is *bounded* if S is bounded above and below.

Example. I could have put various examples and non-examples here, but instead I've put them in a short Moodle quiz for you to try. You'll get immediate feedback on your answers, to help you explore and check your understanding of the definitions. Please go to the Moodle course page for Analysis I, and try quiz 8.1, before you read on to the next section.

9 The (nearly) empty section

Sorry, this is here just to resolve section/video numbering issues!

10 Supremum, infimum and completeness

Some upper bounds are more interesting than others. The set $[0, 1]$ has upper bounds including 15, 1, 1.7 and infinitely many more. Of these, 1 feels special. This is the focus of our next definition.

Definition. Let $S \subseteq \mathbb{R}$. We say that $\alpha \in \mathbb{R}$ is the *supremum* of S, written $\sup S$, if

(i) $s \leqslant \alpha$ for all $s \in S$; (α is an upper bound of S)

(ii) if $s \leqslant b$ for all $s \in S$ then $\alpha \leqslant b$ (α is the least upper bound of S).

Remark. If S has a supremum, then $\sup S$ is unique. (Check you can show this!)

Now that we have defined the supremum, we can state our final key property of $\mathbb{R}$ (in addition to the properties that make it an ordered field).

Completeness axiom for the real numbers Let S be a non-empty subset of $\mathbb{R}$ that is bounded above. Then S has a supremum.

Remark. There are two conditions on S here: non-empty, and bounded above. They are both crucial!

It is easy to forget the non-empty condition, but it has to be there: the empty set does not have a supremum, because every real number is an upper bound for the empty set — there is no *least* upper bound.

The condition that S is bounded above is also necessary: a set with no upper bound certainly has no supremum.

Example.

- Let $S = [1, 2)$. Then 2 is an upper bound, and is the least upper bound: if $b < 2$ then b is not an upper bound because $\max(1, 1 + \frac{b}{2}) \in S$ and $\max(1, 1 + \frac{b}{2}) > b$. Note that in this case $\sup S \notin S$.

- Let $S = (1, 2]$. Then we again have $\sup S = 2$, and this time $\sup S \in S$.

The supremum is the least upper bound of a set. There's an analogous definition for lower bounds.

Definition. Let $S \subseteq \mathbb{R}$. We say that $\alpha \in \mathbb{R}$ is the *infimum* of S, written $\inf S$, if

- $s \geqslant \alpha$ for all $s \in S$; (α is a lower bound of S)
- if $s \geqslant b$ for all $s \in S$ then $\alpha \geqslant b$ (α is the greatest lower bound of S).

Let's explore some useful properties of sup and inf.

Proposition 8.

(i) Let S, T be non-empty subsets of $\mathbb{R}$, with $S \subseteq T$ and with T bounded above. Then S is bounded above, and $\sup S \leqslant \sup T$.

(ii) Let $T \subseteq \mathbb{R}$ be non-empty and bounded below. Let $S = \{-t : t \in T\}$. Then S is non-empty and bounded above. Furthermore, $\inf T$ exists, and $\inf T = -\sup S$.

Remark. (ii) and a similar result with sup and inf swapped essentially tell us that we can pass between sups and infs. Any result we prove about sup will have an analogue for inf. Also, we could have phrased the Completeness Axiom in terms of inf instead of sup. Proposition 8(ii) tells us that we don't need separate axioms for sup and inf.

Proof. (i) Since T is bounded above, it has an upper bound, say b.

Then $t \leqslant b$ for all $t \in T$, so certainly $t \leqslant b$ for all $t \in S$, so b is an upper bound for S.

Now S, T are non-empty and bounded above, so by completeness each has a supremum.

Note that $\sup T$ is an upper bound for T and hence also for S, so $\sup T \geqslant \sup S$ (since $\sup S$ is the *least* upper bound for S).

(ii) Since T is non-empty, so is S.

Let b be a lower bound for T, so $t \geqslant b$ for all $t \in T$.

Then $-t \leqslant -b$ for all $t \in T$, so $s \leqslant -b$ for all $s \in S$, so $-b$ is an upper bound for S.

Now S is non-empty and bounded above, so by completeness it has a supremum.

Then $s \leqslant \sup S$ for all $s \in S$, so $t \geqslant -\sup S$ for all $t \in T$, so $-\sup S$ is a lower bound for T.

Also, we saw before that if b is a lower bound for T then $-b$ is an upper bound for S.

Then $-b \geqslant \sup S$ (since $\sup S$ is the *least* upper bound),

so $b \leqslant -\sup S$.

So $-\sup S$ is the greatest lower bound.

So $\inf T$ exists and $\inf T = -\sup S$.

$\square$

You might be wondering how all this relates to familiar notions of maximum and minimum so let's explore that.

Definition. Let $S \subseteq \mathbb{R}$ be non-empty. Take $M \in \mathbb{R}$. We say that M is the *maximum* of S if

(i) $M \in S$; (M is an element of S)

(ii) $s \leqslant M$ for all $s \in S$ (M is an upper bound for S).

Remark.
- If S is empty or S is not bounded above then S does not have a maximum. (Check this!)
- Let $S \subseteq \mathbb{R}$ be non-empty and bounded above, so (by completeness) $\sup S$ exists.

  Then S has a maximum if and only if $\sup S \in S$.

  Also, if S has a maximum then $\max S = \sup S$.

  (Check this!)

Definition. Let $S \subseteq \mathbb{R}$ be non-empty. Take $m \in \mathbb{R}$. We say that m is the *minimum* of S if

(i) $m \in S$; (m is an element of S)

(ii) $s \geqslant m$ for all $s \in S$ (m is a lower bound for S).

Here is a key result about the supremum, which we'll use *a lot*. It is a quick consequence of the definition, but it will be useful to have formulated it in this way.

Proposition 9 (Approximation Property). *Let $S \subseteq \mathbb{R}$ be non-empty and bounded above. For any $\varepsilon > 0$, there is $s_\varepsilon \in S$ such that $\sup S - \varepsilon < s_\varepsilon \leqslant \sup S$.*

Proof. Take $\varepsilon > 0$.

Note that by definition of the supremum we have $s \leqslant \sup S$ for all $s \in S$.

Suppose, for a contradiction, that $\sup S - \varepsilon \geqslant s$ for all $s \in S$.

Then $\sup S - \varepsilon$ is an upper bound for S, but $\sup S - \varepsilon < \sup S$. Contradiction.

So there is $s_\varepsilon \in S$ with $\sup S - \varepsilon < s_\varepsilon$. $\square$

11 Existence of roots

Now that we have identified the completeness property of $\mathbb{R}$, we are ready to prove that $\mathbb{R}$ contains a square root of 2.

Theorem 10. *There exists a unique positive real number α such that $\alpha^2 = 2$.*

Proof. **Existence** Let $S = \{s \in \mathbb{R} : s > 0,\ s^2 < 2\}$.

Idea: argue that S has a supremum, and show that $\sup S$ has the required properties.

Note that S is non-empty (eg $1 \in S$)

and S is bounded above, because if $x > 2$ then $x^2 > 4$ (properties of ordering) so $x \notin S$, so 2 is an upper bound for S.

So, by completeness, S has a supremum. Let $\alpha = \sup S$.

Note that certainly $\alpha > 0$ (since $1 \in S$ so $\alpha \geqslant 1$).

By trichotomy, we have $\alpha^2 < 2$ or $\alpha^2 = 2$ or $\alpha^2 > 2$.

Idea: show that if $\alpha^2 < 2$ or $\alpha^2 > 2$ then we get a contradiction.

Case 1 Suppose, for a contradiction, that $\alpha^2 < 2$.

Then $\alpha^2 = 2 - \varepsilon$ for some $\varepsilon > 0$.

Idea: consider $\alpha + h$ for a small $h > 0$. Later on, we'll choose h small enough that $(\alpha+h)^2 < 2$, and that will be a contradiction because $\alpha+h \in S$ and $\alpha + h > \sup S$.

Note that $\alpha \leqslant 2$ (we said earlier that 2 is an upper bound for S).

For $h \in (0,1)$ we have

$$\begin{aligned}(\alpha + h)^2 &= \alpha^2 + 2\alpha h + h^2 \\ &= 2 - \varepsilon + 2\alpha h + h^2 \\ &\leqslant 2 - \varepsilon + 4h + h \\ &\leqslant 2 - \varepsilon + 5h\end{aligned}$$

so let $h = \min(\frac{\varepsilon}{10}, \frac{1}{2})$ and then $(\alpha + h)^2 < 2$.

Now $\alpha + h \in S$ and $\alpha + h > \sup S$. This is a contradiction.

So it is not the case that $\alpha^2 < 2$.

Case 2 Suppose, for a contradiction, that $\alpha^2 > 2$.

Then $\alpha^2 = 2 + \varepsilon$ for some $\varepsilon > 0$.

Idea: consider $\alpha - h$ for a small $h > 0$. Later on, we'll choose h small enough that $(\alpha - h)^2 > 2$, and that will lead to a contradiction because $\alpha - h < \sup S$.

For $h \in (0,1)$ we have

$$\begin{aligned}(\alpha - h)^2 &= \alpha^2 - 2\alpha h + h^2 \\ &= 2 + \varepsilon - 2\alpha h + h^2 \\ &\geqslant 2 + \varepsilon - 4h\end{aligned}$$

so choose $h = \min(\frac{\varepsilon}{8}, \frac{1}{2}, \frac{\alpha}{2})$ and then $(\alpha - h)^2 > 2$ (and also $\alpha - h > 0$).

Now $\alpha - h < \sup S$, so by the Approximation property there is $s \in S$ with $\alpha - h < s$.

But then $2 < (\alpha - h)^2 < s^2 < 2$, which is a contradiction.

So it is not the case that $\alpha^2 > 2$.

Hence, by trichotomy, $\alpha^2 = 2$.

Uniqueness Suppose that β is also a positive real number such that $\beta^2 = 2$.

Aim: $\alpha = \beta$.

Then $0 = \alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta)$

and $\alpha + \beta > 0$, so $\alpha = \beta$. $\square$

Proposition 11. *$\mathbb{Q}$ is not complete (with the ordering inherited from $\mathbb{R}$).*

Proof. If $\mathbb{Q}$ were complete, then the proof of Theorem 10 would work just as well in $\mathbb{Q}$. But we know that there is not an element of $\mathbb{Q}$ that squares to 2. So $\mathbb{Q}$ is not complete. $\square$

Theorem 12. *Let n be an integer with $n \geqslant 2$, and take a positive real number r. Then r has a real n^{th} root.*

Proof. Exercise. (See Sheet 2 for the case of the cube root of 2.) $\square$

12 More consequences of completeness

In this course, we write $\mathbb{N}$ for the set of positive integers, so $\mathbb{N} = \mathbb{Z}^{>0}$.

Theorem 13 (Archimedean property of $\mathbb{N}$)**.** *$\mathbb{N}$ is not bounded above.*

Proof. Idea: if there's an upper bound then we can find a natural number just less than it, and add 1.

Suppose, for a contradiction, that $\mathbb{N}$ is bounded above.

Then $\mathbb{N}$ is non-empty and bounded above, so by completeness (of $\mathbb{R}$) $\mathbb{N}$ has a supremum.

By the Approximation property with $\varepsilon = \frac{1}{2}$, there is a natural number $n \in \mathbb{N}$ such that $\sup \mathbb{N} - \frac{1}{2} < n \leqslant \sup \mathbb{N}$.

Now $n + 1 \in \mathbb{N}$ and $n + 1 > \sup \mathbb{N}$. This is a contradiction. $\square$

Corollary 14. *Let $\varepsilon > 0$. Then there is $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \varepsilon$.*

Proof. If not, then $\frac{1}{\varepsilon}$ would be an upper bound for $\mathbb{N}$. This would contradict Theorem 13. $\square$

Theorem 15. *Let S be a non-empty subset of $\mathbb{Z}$.*

(i) If S is bounded below, then S has a minimum.

(ii) If S is bounded above, then S has a maximum.

Proof. (i) Assume that S is bounded below.

Then, by completeness (applied to $\{-s : s \in S\}$), S has an infimum.

Secret aim: $\inf S \in S$.

By the Approximation property (with $\varepsilon = 1$), there is $n \in S$ such that $\inf S \leqslant n < \inf S + 1$. Aim: $\inf S = n$.

Suppose, for a contradiction, that $\inf S < n$.

Write $n = \inf S + \delta$, where $0 < \delta < 1$.

By the Approximation property (with $\varepsilon = \delta$), there is $m \in S$ such that $\inf S \leqslant m < \inf S + \varepsilon = n$.

Now $m < n$ so $n - m > 0$

but $n - m$ is an integer, so $n - m \geqslant 1$.

Now $n \geqslant m + 1 \geqslant \inf S + 1$. This is a contradiction.

So $n = \inf S \in S$ so $\inf S = \min S$.

(ii) Similar.

□

Proposition 16. *Take $a, b \in \mathbb{R}$ with $a < b$. Then*

(i) there is $x \in \mathbb{Q}$ such that $a < x < b$ (the rationals are dense in the reals); and

(ii) there is $y \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < y < b$ (the irrationals are dense in the reals).

Proof. Exercise (see Sheet 2). □

Summary of our work so far

$\mathbb{R}$ is a complete ordered field.

This sums up the key properties we have identified as our assumptions about $\mathbb{R}$. From this, we shall develop the theory of real analysis.

13 Countability

The concept of countability gives us a way to distinguish between sets by comparing their 'sizes'. This will be a quick introduction. You can find more information in the supplementary notes by Dr Hilary Priestley, on the Moodle Analysis I course.

Our main tool for comparing the 'sizes' of two sets is to ask whether there is a bijection between them. In this section and the next, we'll often be using the notions of bijection, injection and surjection. If you don't feel confident with the definitions of these, then I recommend you remind yourself of the definitions before you continue with this section.

Definition. Let A be a set. We say that A is *finite* if $A = \emptyset$ or there exists $n \in \mathbb{N}$ such that there is a bijection $f : A \to \{1, 2, \ldots, n\}$. We say that A is *infinite* if it is not finite

Remark.

- A subset of a finite set is finite.
- A non-empty finite subset of $\mathbb{R}$ is bounded above (in fact, has a maximum) and so a subset of $\mathbb{R}$ that is not bounded above is infinite.
- $\mathbb{N}$ is not bounded above (by the Archimedean property) so is infinite.

Definition. Let A be a set. We say that A is

- *countably infinite* if there is a bijection $f : A \to \mathbb{N}$;
- *countable* if there is an injection $f : A \to \mathbb{N}$;
- *uncountable* if A is not countable.

Remark. There are variations on the details of these definitions, so it's worth checking carefully if you're looking at a book or other source. For example, some people say 'countable' where we are using 'countably infinite'.

Here are a couple of useful properties.

Proposition 17. *Let A be a set.*

(i) A is countable if and only if A is countably infinite or finite.

(ii) If there is an injection $f : A \to B$ and an injection $g : B \to A$, then there is a bijection $h : A \to B$.

Proof. Not in this course. See Priestley's supplementary notes on countability. $\square$

Proposition 18. *Each of the following sets is countably infinite.*

(i) $\mathbb{N}$

(ii) $\mathbb{N} \cup \{0\}$

(iii) $\{2k-1 : k \in \mathbb{N}\}$

(iv) $\mathbb{Z}$

(v) $\mathbb{N} \times \mathbb{N}$.

Remark. It might feel surprising that the set of odd natural numbers 'has the same size as' the set of all natural numbers!

Proof. (i) Clear.

(ii) Define $f : \mathbb{N} \cup \{0\} \to \mathbb{N}$ by $f(n) = n + 1$. This is a bijection.

(iii) Define $f : \mathbb{N} \to \{2k-1 : k \in \mathbb{N}\}$ by $f(n) = 2n - 1$.

(iv) Idea: line up the integers as 0, 1, −1, 2, −2, 3, −3, ...

Define $f : \mathbb{Z} \to \mathbb{N}$ by

$$f(k) = \begin{cases} 2k & \text{if } k \geqslant 1 \\ 1 - 2k & \text{if } k \leqslant 0. \end{cases}$$

This is a bijection.

(v) Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $f((m, n)) = 2^{m-1}(2n - 1)$.

Claim f is a bijection.

Proof of claim

injective: If $f((m_1, n_1)) = f((m_2, n_2))$ then

$$2^{m_1-1}(2n_1 - 1) = 2^{m_2-1}(2n_2 - 1),$$

so, by uniqueness of prime factorisation in $\mathbb{N}$, $2^{m_1-1} = 2^{m_2-1}$ and $2n_1 - 1 = 2n_2 - 1$,

so $m_1 = m_2$ and $n_1 = n_2$.

surjective: Take $k \in \mathbb{N}$.

Then $k = 2^r(2s+1)$ for some r, $s \geqslant 0$

(consider the set $T = \{t \in \mathbb{Z}^{\geqslant 0} : \frac{k}{2^t} \in \mathbb{N}\}$ — this is non-empty and bounded above so has a maximum).

Then $k = f(r+1, s+1)$.

□

14 More on countability

We can build new countable sets from old. This is a very helpful way to prove that a set is countable!

Proposition 19. *Let A, B be countable sets.*

(i) If A and B are disjoint, then $A \cup B$ is countable.

(ii) $A \times B$ is countable.

Remark. In (i), we don't need the condition that A and B are disjoint, but it makes life easier for our proof.

Proof. Since A and B are countable, there are injections $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$.

(i) Idea: to list elements of $A \cup B$, alternate taking elements from A, B.

Define $h : A \cup B \to \mathbb{N}$ by

$$h(x) = \begin{cases} 2f(x) - 1 & \text{if } x \in A \\ 2g(x) & \text{if } x \in B. \end{cases}$$

This is an injection (because f and g are).

(ii) Define $h : A \times B \to \mathbb{N}$ by $h((a, b)) = 2^{f(a)}3^{g(b)}$.

By the uniqueness of prime factorisation in $\mathbb{N}$, this is an injection.

$\square$

Theorem 20. $\mathbb{Q}^{>0}$ *is countable.*

Proof. Define $f : \mathbb{Q}^{>0} \to \mathbb{N}$ by $f(\frac{p}{q}) = 2^p 3^q$ where p, $q \in \mathbb{Z}^{>0}$ and $\mathrm{hcf}(p, q) = 1$.

This is an injection (by uniqueness of prime factorisation in $\mathbb{N}$). $\square$

Corollary 21. $\mathbb{Q}$ *is countable.*

Proof. We can write $\mathbb{Q} = \mathbb{Q}^{>0} \cup \{0\} \cup \mathbb{Q}^{<0}$. This is a disjoint union.

We have just seen that $\mathbb{Q}^{>0}$ is countable, and similarly so is $\mathbb{Q}^{<0}$, and $\{0\}$ is finite and hence countable.

Hence, by Proposition 19, $\mathbb{Q}$ is countable. $\square$

Our next task will be to show that $\mathbb{R}$ is uncountable. We're going to do this using decimal expansions. That means that we need to know that decimal expansions exist. Ideally we'd study sequences and series for a bit, then look at decimal expansions, then prove that $\mathbb{R}$ is uncountable. But I'd like to wrap up this section on countability now. So we're going to assume a fact about decimal expansions, and deduce that $\mathbb{R}$ is uncountable. I encourage you to revisit the fact later in the course — I'll let you know when we've covered the relevant theory. You could also (now) look at Dr Hilary Priestley's supplementary notes on decimal expansions and the uncountability of $\mathbb{R}$ (on the Moodle Analysis I page).

Fact Every real number has a decimal expansion, and if we require that we choose a non-terminating expansion (such as $0.24999\ldots$ for $\frac{1}{4}$) rather than a terminating one (such as 0.25 for $\frac{1}{4}$) where there is a choice, then this decimal expansion is unique.

Theorem 22. $\mathbb{R}$ *is uncountable.*

Remark. The proof strategy we are going to use is called *Cantor's diagonal argument.*

Proof. It suffices to show that $(0, 1]$ is uncountable.

Note that certainly $(0, 1]$ is not finite (by Corollary 14 of the Archimedean property).

Suppose, for a contradiction, that $(0, 1]$ is countably infinite. List the elements as x_1, x_2, x_3, $\ldots$. (If you like, we have a bijection $\mathbb{N} \to (0, 1]$, and $x_1 = f(1)$, $x_2 = f(2)$, $x_3 = f(3)$, $\ldots$.)

Each has a non-terminating decimal expansion (where relevant choosing the non-terminating option):

$$\begin{aligned}
x_1 &= 0.a_{11}a_{12}a_{13}a_{14}\ldots \\
x_2 &= 0.a_{21}a_{22}a_{23}a_{24}\ldots \\
x_3 &= 0.a_{31}a_{32}a_{33}a_{34}\ldots \\
&\vdots \\
x_k &= 0.a_{k1}a_{k2}a_{k3}a_{k4}\ldots \\
&\vdots
\end{aligned}$$

Construct a real number $x \in (0, 1]$ with decimal expansion $0.b_1b_2b_3\ldots$

where

$$b_k = \begin{cases} 5 & \text{if } a_{kk} = 6 \\ 6 & \text{if } a_{kk} \neq 6. \end{cases}$$

Then $x \neq x_k$ for all k, because x differs from x_k in the k^{th} decimal place, so x is not on our list, which supposedly contained all elements of $(0, 1]$. This is a contradiction. $\square$

Remark. The only significance of the choice of 5 and 6 as the key digits when defining x was that we didn't involve 0 or 9, to avoid issues with non-unique decimal expansions.

There are many further interesting things to say about countability, but not in this course. We need to move on to consider sequences. Before you continue to the next section, try to come up with a collection of examples of sequences that you can use as your personal repertoire for testing the definitions and results we'll look at. Try to find some 'typical' and some 'extreme' examples of sequences that you think converge, and of sequences that you think don't converge.

15 Introduction to sequences

Remark. You are already familiar with the sine and cosine, exponential and logarithm functions, and with raising a number to a non-integer power. At the moment, though, we haven't formally defined these. We'll do so later in the course (using infinite series—that's why it needs to wait till later), and this term and later in other Analysis courses you'll explore and prove the familiar properties of these function.

All this means that at the moment our collection of functions we've defined is rather small, and doesn't give the richness we'd like when exploring examples of sequences. So for now we'll work with these familiar functions (trig, exponential, log)—we'll assume that they exist and have the properties we expect. You can do this on the problems sheets too. When we come to defining them later on, you can watch out to see that we don't have any circular arguments!

Notation. When we use logarithms, these will all be to the base e. We write $\log x$ for $\log_{\mathrm{e}}(x)$. We don't write $\ln x$.

For $a > 0$ and $x \in \mathbb{R}$, we define $a^x = \mathrm{e}^{x \log a}$. (Of course this relies on definitions of the exponential and logarithm functions, which will come later.)

Remark. Examples can be really useful. I don't mean worked examples (although these can also be really useful), I mean examples of objects that do or don't have certain properties. I'll include some examples in these notes and the accompanying videos. You'll find additional examples in Dr Hilary Priestley's lecture notes, on the Moodle page for Analysis I, and I encourage you to work through those too. I also encourage you to try your own examples (and non-examples), to help you to deepen your experience and understanding of the definitions and results we'll meet.

Example. Here are some informal examples of sequences.

- $\dfrac{3}{10}, \dfrac{33}{100}, \dfrac{333}{1000}, \dfrac{3333}{10000}, \dots$ are approximations to $\frac{1}{3}$, each better than the previous.
- $\dfrac{14}{10}, \dfrac{141}{100}, \dfrac{1414}{1000}, \dfrac{14142}{10000}, \dots$ are approximations to $\sqrt{2}$, each better than the previous.

- Take $\varepsilon > 0$. Then, by the Archimedean property, there is $N \geqslant 1$ such that $0 < \frac{1}{N} < \varepsilon$. Now for all $n \geqslant N$ we have $0 < \frac{1}{n} \leqslant \frac{1}{N} < \varepsilon$. We see that apart from finitely many terms at the start, the terms of the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ all lie within distance ε of 0. This is the case for any positive real number ε.

- $1, -1, 2, -2, 3, -4, 4, -4, \ldots$ is another sequence, and intuitively it feels as though it does not tend to a limit.

- $7, 1.2, -5, 2, 324, -9235.32, \ldots$ is another sequence—there is no clear pattern to the terms (I just typed them wherever my fingers landed), but it is still a sequence.

What exactly is a sequence?

Definition. A *real sequence*, or *sequence* of real numbers, is a function $\alpha : \mathbb{N} \to \mathbb{R}$. We call $\alpha(n)$ the n^{th} term of the sequence.

We usually write a_n for $\alpha(n)$, and say that α defines the sequence (a_n) with terms $a_1, a_2, a_3, a_4, \ldots$. We might also write this as $(a_n)_{n \geqslant 1}$ or $(a_n)_{n=1}^{\infty}$.

Similarly, a *complex sequence* is formally a function $\alpha : \mathbb{N} \to \mathbb{C}$, and we write it as (a_n), where now $a_n \in \mathbb{C}$ for $n \geqslant 1$.

Remark.

- The order of the terms in a sequence matters!

- We write (a_n) for the sequence, and a_n for a term of the sequence.

- Much of the theory relating to sequences applies to both real and complex sequences. Sometimes, though, we'll need to focus only on real sequences—for example if we're using inequalities. In this case we'll carefully specify that we're working with real sequences. If we don't specify, and just say 'sequences', then it applies equally to real and complex sequences. We'll also have a section (and corresponding video) at the end of this block concentrating on complex sequences.

Example.

- Let $a_n = (-1)^n$. Then the first few terms of the sequence are $-1, 1, -1, 1, -1, 1, \ldots$.

- Let $a_n = \frac{\sin n}{2n+1}$. Then the first few terms of the sequence are
$$\frac{1}{3}\sin 1, \frac{1}{5}\sin 2, \frac{1}{7}\sin 3, \ldots .$$

- Let
$$a_n = \begin{cases} 0 & \text{if } n \text{ is prime} \\ 1 + \frac{1}{n} & \text{otherwise.} \end{cases}$$
Then the first few terms of the sequence are $2, 0, 0, \frac{5}{4}, 0, \frac{7}{6}, 0, \frac{9}{8}, \ldots$.

- Let $a_n = n$. Then the first few terms of the sequence are $1, 2, 3, 4, 5, \ldots$.

Definition. We can make new sequences from old. Let (a_n), (b_n) be sequences and let c be a constant. Then we can define new sequences 'termwise': $(a_n + b_n)$, $(-a_n)$, (a_nb_n), (ca_n), $(|a_n|)$. If $b_n \neq 0$ for all n, then we can also define a sequence $(\frac{a_n}{b_n})$.

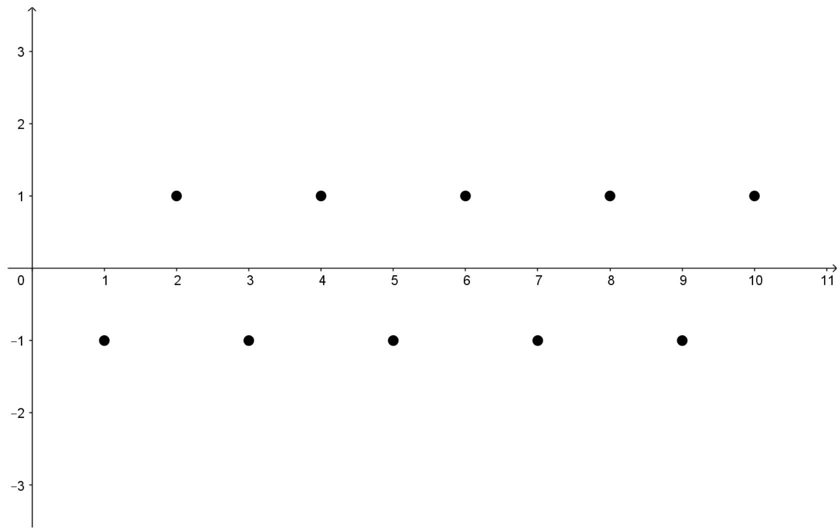
Example. Let $a_n = (-1)^n$ and $b_n = 1$ for $n \geqslant 1$.

Then the first few terms of $(a_n + b_n)$ are $0, 2, 0, 2, 0, 2, \ldots$; and $(-a_n) = ((-1)^{n+1})$; and $(|a_n|) = (b_n)$.

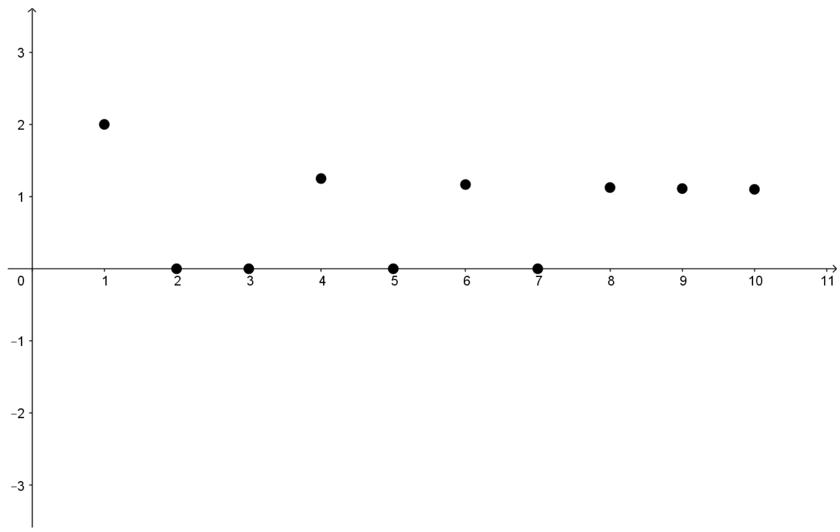
16 Convergence of a sequence

Before we see a formal definition of convergence, let's consider some examples informally. Here's one way I like to visualise a sequence. These are examples from the previous section. Each graph plots the points (n, a_n) for $1 \leqslant n \leqslant 10$.

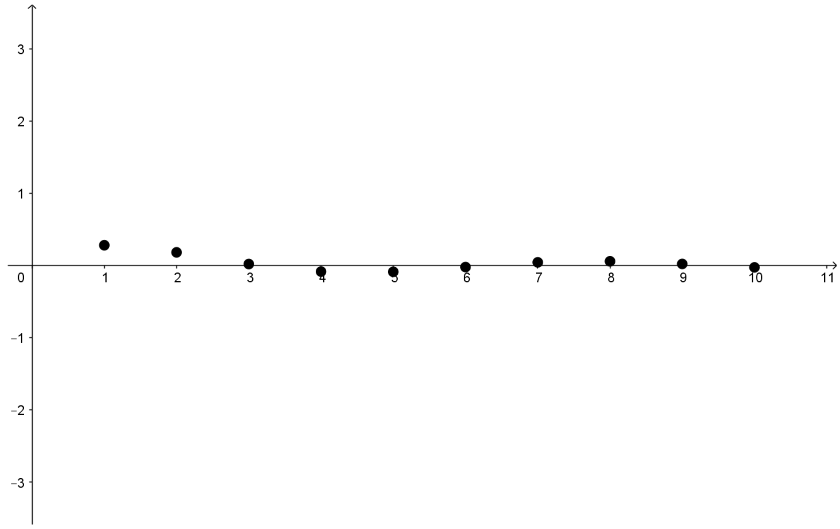
- $a_n = (-1)^n$



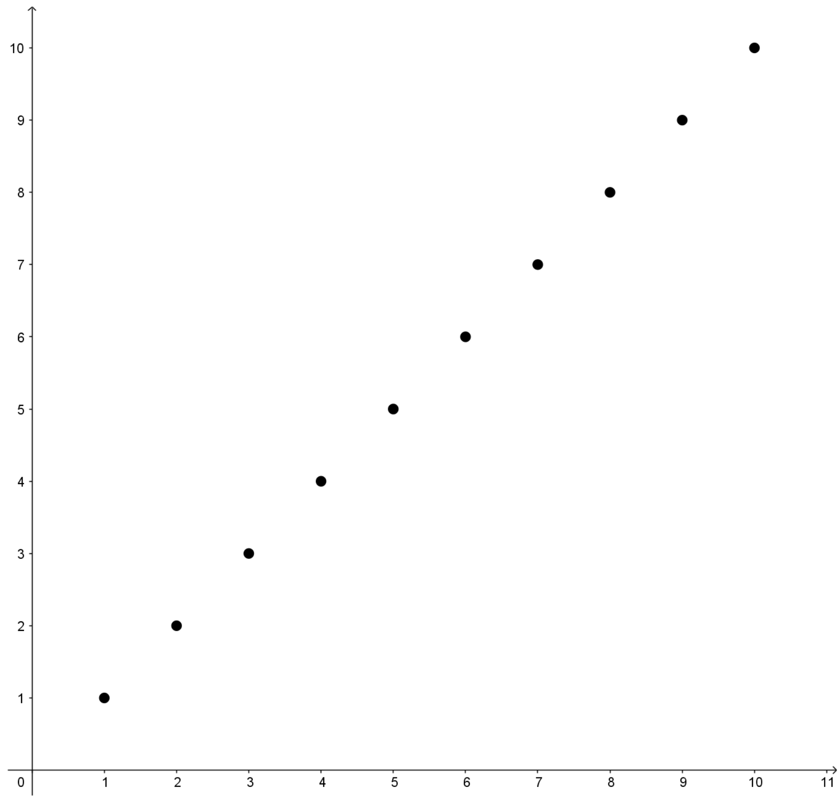
- $a_n = \begin{cases} 0 & \text{if } n \text{ is prime} \\ 1 + \frac{1}{n} & \text{otherwise} \end{cases}$



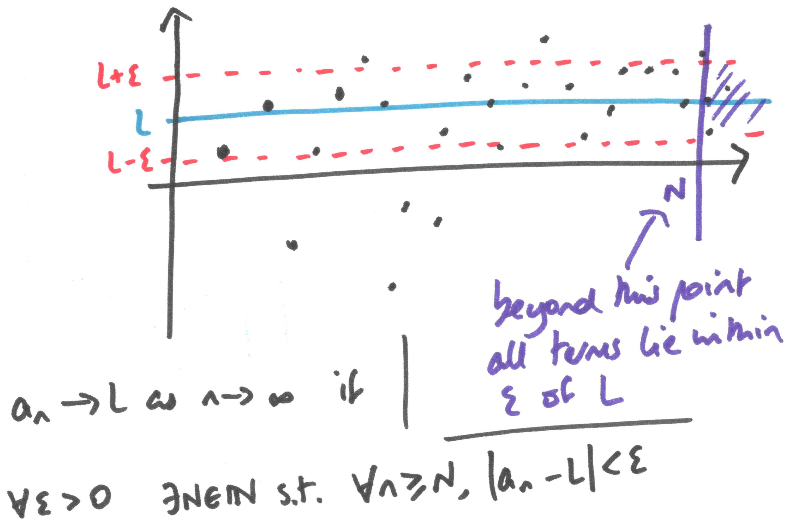
- $a_n = \frac{\sin n}{2n+1}$



- $a_n = n$



Here is an unofficial picture of the definition of convergence.



Definition. Let (a_n) be a real sequence, let $L \in \mathbb{R}$. We say that (a_n) *converges* to L as $n \to \infty$ if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n \geqslant N, \ |a_n - L| < \varepsilon.$$

In this case we write $a_n \to L$ as $n \to \infty$, and we say that L is the *limit* of (a_n).

Remark.
- We might also say that (a_n) *tends to* L as $n \to \infty$, and we might also write that $\lim_{n\to\infty} a_n = L$.
- N can depend on ε, and almost always will.
- The 'order of the quantifiers' matters. We wrote "$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ldots$". This order allows N to depend on ε. If we wrote "$\exists N \in \mathbb{N}$ such that $\forall \varepsilon > 0 \ldots$" that would be something quite different.

  We could replace $n \geqslant N$ in the definition by $n > N$, and $|a_n - L| < \varepsilon$ by $|a_n - L| \leqslant \varepsilon$, without changing the definition. (Check this!) But it's crucial that we have $\varepsilon > 0$ not $\varepsilon \geqslant 0$. (Check this!)

- I put 'the' limit in the definition. We'll see later that if it exists then it's unique.

Definition. Let (a_n) be a real sequence. We say that (a_n) *converges*, or *is convergent*, if there is $L \in \mathbb{R}$ such that $a_n \to L$ as $n \to \infty$. If (a_n) does not converge, then we say that it *diverges*, or *is divergent*.

Intuitively, the first thousand or million terms of a sequence shouldn't affect whether it converges. We'll prove a result that makes this precise, but first we need a quick definition.

Definition. Let (a_n) be a sequence. A *tail* of (a_n) is a sequence (b_n), where for some natural number k we have $b_n = a_{n+k}$ for $n \geqslant 1$. That is, (b_n) is the sequence obtained by deleting the first k terms of (a_n).

Lemma 23 (Tails Lemma). *Let (a_n) be a sequence.*

(i) If (a_n) converges to a limit L, then every tail of (a_n) also converges, and to this same limit L.

(ii) If a tail $(b_n) = (a_{n+k})$ of (a_n) converges, then (a_n) converges.

Proof. (i) Take a tail of (a_n): take $k \geqslant 1$ and let $b_n = a_{n+k}$ for $n \geqslant 1$.

Assume that (a_n) converges to a limit L.

Take $\varepsilon > 0$.

Then there is N such that if $n \geqslant N$ then $|a_n - L| < \varepsilon$.

Now if $n \geqslant N$ then $n + k \geqslant N$ so $|a_{n+k} - L| < \varepsilon$, that is, $|b_n - L| < \varepsilon$.

So (b_n) converges and $b_n \to L$ as $n \to \infty$.

(ii) Assume that $(b_n) = (a_{n+k})$ converges.

Then there is $L \in \mathbb{R}$ such that $b_n \to L$ as $n \to \infty$.

Take $\varepsilon > 0$.

Then there is N such that if $m \geqslant N$ then $|b_m - L| < \varepsilon$, that is, $|a_{m+k} - L| < \varepsilon$.

Now if $n \geqslant N + k$ then $n = m + k$ where $m \geqslant N$, and so $|a_n - L| < \varepsilon$.

So (a_n) converges and $a_n \to L$ as $n \to \infty$.

□

Example. We'll see later (soon!) that there are other ways to prove convergence, not only directly from the definition. But for now we've only got the definition (and the Tails Lemma), so let's get some practice using what we've got so far.

- **Claim.** $\frac{1}{n} \to 0$ as $n \to \infty$.

  Proof. Take $\varepsilon > 0$.

  Then there is $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$ (by the Archimedean property).

  For $n \geqslant N$ we have $|\frac{1}{n} - 0| = \frac{1}{n} \leqslant \frac{1}{N} < \varepsilon$.

  So $\frac{1}{n} \to 0$ as $n \to \infty$. □

- **Claim.** Let $a_n = 1 + (-1)^n \frac{1}{\sqrt{n}}$ for $n \geqslant 1$. Then $a_n \to 1$ as $n \to \infty$.

  Proof. Take $\varepsilon > 0$.

  Aim: want N such that if $n \geqslant N$ then $|a_n - 1| < \varepsilon$

  that is, $|(1 + (-1)^n \frac{1}{\sqrt{n}}) - 1| < \varepsilon$

  that is, $\frac{1}{\sqrt{n}} < \varepsilon$, that is, $\frac{1}{\varepsilon} < \sqrt{n}$.

Take $N = \lceil \frac{1}{\varepsilon^2} \rceil + 1$.

Here $\lceil x \rceil$ *denotes the* ceiling function*: it is defined to be the smallest integer greater than or equal to* x*. Informally, if* x *is an integer then take that value; otherwise, round up to the next integer.*

If $n \geqslant N$, then

$$\begin{aligned} & n > \frac{1}{\varepsilon^2} \\ & \text{so } \sqrt{n} > \frac{1}{\varepsilon} \\ & \text{so } \frac{1}{\sqrt{n}} < \varepsilon \\ & \text{so } |a_n - 1| < \varepsilon. \end{aligned}$$

So $a_n \to 1$ as $n \to \infty$. □

•

Claim. Let $a_n = \dfrac{n\cos(n^3+1)}{5n^2+1}$ for $n \geqslant 1$. Then $a_n \to 0$ as $n \to \infty$.

Proof. Take $\varepsilon > 0$.

Aim: want N such that if $n \geqslant N$ then $|a_n - 0| < \varepsilon$,
that is, $\left|\dfrac{n\cos(n^3+1)}{5n^2+1}\right| < \varepsilon$

but $|\cos(n^3+1)| \leqslant 1$ so it's enough to ensure that $\left|\dfrac{n}{5n^2+1}\right| < \varepsilon$

and $5n^2+1 \geqslant 5n^2$ so it's enough to ensure that $\left|\frac{n}{5n^2}\right| < \varepsilon$

that is, $\frac{1}{5n} < \varepsilon$, that is, $n > \frac{1}{5\varepsilon}$.

Take $N = \lceil \frac{1}{\varepsilon} \rceil + 1$.

If $n \geqslant N$, then $n > \frac{1}{5\varepsilon}$ so

$$|a_n| = \left| \frac{n\cos(n^3+1)}{5n^2+1} \right| \leqslant \frac{1}{5n} < \varepsilon.$$

So $a_n \to 0$ as $n \to \infty$. □

Remark. Here are some top tips!

- We don't need the smallest possible N. It's (almost always) not even interesting to know what it is. So make your life easier! If an inequality (in the right direction) helps, then go for it.

- Be careful to make sure that the logic flows in the right direction, and that you've set out the logic explicitly. Hopefully the examples we've just seen help you to have ideas of how to do this.

- The definition officially says $N \in \mathbb{N}$, but we don't really care whether N is a natural number. If we have a value that works, then we can always choose a natural number larger than it.

- We think of ε as a small positive real number, but we are obliged to prove it for *all* $\varepsilon > 0$. But if we can prove it for say $0 < \varepsilon < 1$ then that's enough—if N works for a certain ε then it works for all larger values too. So you can work with a smaller range of ε, such as $0 < \varepsilon < 1$, if that is most convenient (but it would be a good idea to mention briefly why this is sufficient).

- It's really worth becoming comfortable with inequalities and modulus. In the examples, it was nicer to use the absolute values to write things like $|a_n - L| < \varepsilon$, rather than $-\varepsilon < a_n - L < \varepsilon$. If you prefer the second at the moment, then I recommend practising to get used to the first!

Working directly from the definition is often painful or impractical. Our next goal is to prove a result that will give a more convenient strategy for proving convergence in some circumstances.

17 Limits: first key results

The next result is extremely useful in practice! We'll see a more general version later, but even this version is strong enough to be useful.

Proposition 24 (Sandwiching, first version). *Let (a_n) and (b_n) be real sequences with $0 \leqslant a_n \leqslant b_n$ for all $n \geqslant 1$. If $b_n \to 0$ as $n \to \infty$, then $a_n \to 0$ as $n \to \infty$.*

Remark. You might like to draw yourself a diagram to develop your intuition for what this result says.

Proof. Idea: if N works for b_n then it works for a_n too.

Assume that $0 \leqslant a_n \leqslant b_n$ for all n, and that $b_n \to 0$ as $n \to \infty$.

Take $\varepsilon > 0$.

Since $b_n \to 0$, there exists N such that if $n \geqslant N$ then $|b_n| < \varepsilon$.

Now if $n \geqslant N$ then $0 \leqslant a_n \leqslant b_n < \varepsilon$, so $|a_n| < \varepsilon$.

So $a_n \to 0$ as $n \to \infty$. □

Example.

- **Claim.** $\frac{1}{2^n} \to 0$ as $n \to \infty$.

  Proof. We have $2^n \geqslant n$ for $n \geqslant 1$ (can prove this by induction),

  so $0 \leqslant \frac{1}{2^n} \leqslant \frac{1}{n}$ for $n \geqslant 1$, and $\frac{1}{n} \to 0$,

  so by Sandwiching $\frac{1}{2^n} \to 0$ as $n \to \infty$. □

- **Claim.** Let $a_n = \dfrac{n\cos(n^3+1)}{5n^2+1}$ for $n \geqslant 1$ (we saw this example earlier). Then $a_n \to 0$ as $n \to \infty$.

  Proof. Idea: apply Sandwiching to $(|a_n|)$.

  We have

  $$0 \leqslant \left|\frac{n\cos(n^3+1)}{5n^2+1}\right| \leqslant \frac{1}{5n} \leqslant \frac{1}{n}$$

  for $n \geqslant 1$,

  and $\frac{1}{n} \to 0$ as $n \to \infty$,

  so, by Sandwiching, $|a_n| \to 0$ as $n \to \infty$.

  But (looking back at the definition) we see that $|a_n| \to 0$ if and only if $a_n \to 0$. □

Here are two key sequences; it will be useful later to have studied them. (You can also think of them as further worked examples.)

Lemma 25. *(i) Take $c \in \mathbb{R}$ with $|c| < 1$. Then $c^n \to 0$ as $n \to \infty$.*

(ii) Let $a_n = \frac{n}{2^n}$ for $n \geqslant 1$. Then $a_n \to 0$ as $n \to \infty$.

Proof. (i) Write $|c| = \frac{1}{1+y}$ where $y > 0$.

Take $\varepsilon > 0$.

Let $N = \lceil \frac{1}{y\varepsilon} \rceil + 1$. (When writing this proof, we might leave this line blank and fill it in at the end!)

Take $n \geqslant N$.

By Bernoulli's inequality (since $y > 0$ and $n \geqslant 1$) we have $(1+y)^n \geqslant 1 + ny$, so

$$|c^n| = \frac{1}{(1+y)^n} \leqslant \frac{1}{1+ny} \leqslant \frac{1}{Ny} < \varepsilon.$$

So $c^n \to 0$ as $n \to \infty$.

(ii) Note that if $n \geqslant 2$ then $2^n = (1+1)^n \geqslant \binom{n}{2}$ (by the binomial theorem).

Take $\varepsilon > 0$.

Let $N = \lceil 2 + \frac{2}{\varepsilon} \rceil$.

For $n \geqslant N$, we have

$$|a_n - 0| = \frac{n}{2^n} \leqslant \frac{n}{\binom{n}{2}} = \frac{2}{n-1} \leqslant \frac{2}{N-1} < \varepsilon.$$

So $a_n \to 0$ as $n \to \infty$.

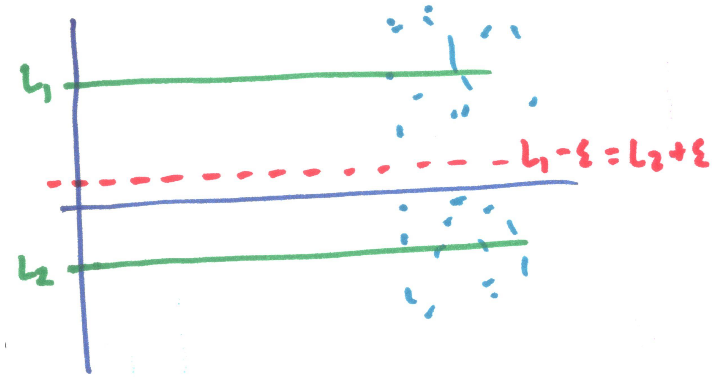
$\square$

As promised earlier, let's show that if a sequence converges, then its limit is unique.

Theorem 26 (Uniqueness of limits)**.** *Let (a_n) be a convergent sequence. Then the limit is unique.*

Proof. Assume that $a_n \to L_1$ and $a_n \to L_2$ as $n \to \infty$. Aim: $L_1 = L_2$.

Idea: contradiction. If $L_1 \neq L_2$, then eventually all the terms are really close to L_1, and also to L_2, and that's not possible.



Suppose, for a contradiction, that $L_1 \neq L_2$.

Let $\varepsilon = \dfrac{|L_1 - L_2|}{2} > 0$.

Since $a_n \to L_1$ as $n \to \infty$, there is N_1 such that if $n \geqslant N_1$ then $|a_n - L_1| < \varepsilon$.

Also, since $a_n \to L_2$ as $n \to \infty$, there is N_2 such that if $n \geqslant N_2$ then $|a_n - L_2| < \varepsilon$.

For $n \geqslant \max\{N_1, N_2\}$ we have $|a_n - L_1| < \varepsilon$ and $|a_n - L_2| < \varepsilon$, so

$$\begin{aligned} |L_1 - L_2| &= |(L_1 - a_n) + (a_n - L_2)| \\ &\leqslant |L_1 - a_n| + |a_n - L_2| \text{ by the triangle inequality} \\ &< 2\varepsilon = |L_1 - L_2|. \end{aligned}$$

This is a contradiction.

So $L_1 = L_2$. □

18 Limits: modulus and inequalities

Proposition 27. *Let (a_n) be a convergent sequence. Then $(|a_n|)$ also converges. Moreover, if $a_n \to L$ as $n \to \infty$ then $|a_n| \to |L|$ as $n \to \infty$.*

Proof. Say $a_n \to L$ as $n \to \infty$.

Take $\varepsilon > 0$.

Then there is N such that if $n \geqslant N$ then $|a_n - L| < \varepsilon$.

Now if $n \geqslant N$ then, by the Reverse Triangle Inequality, we have

$$||a_n| - |L|| \leqslant |a_n - L| < \varepsilon.$$

So $(|a_n|)$ converges, and $|a_n| \to |L|$ as $n \to \infty$. $\square$

Remark. We could instead have proved Proposition 27 using the Sandwiching Lemma, since $a_n \to L$ as $n \to \infty$ if and only if $|a_n - L| \to 0$ as $n \to \infty$ (check this using the definition of convergence).

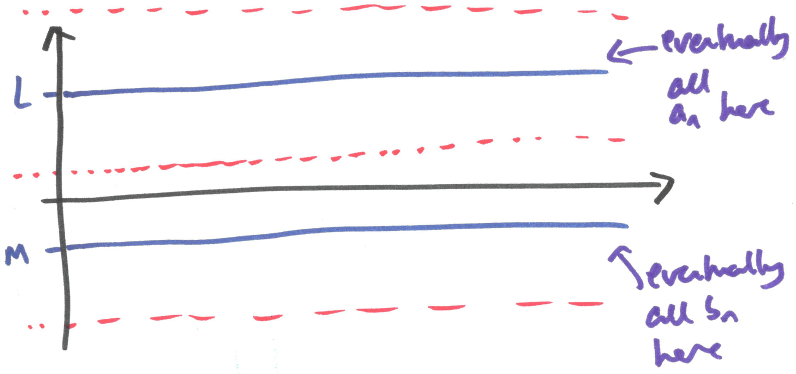
Now let's think about inequalities. If (a_n) is a convergent sequence and $a_n > 0$ for all n, then what can we say about the limit? It's not the case that the limit must be positive. For example, if $a_n = \frac{1}{n}$ then $a_n > 0$ for all n but $a_n \to 0$. But it's hard to see how a sequence of positive terms could have a negative limit.

Proposition 28 (Limits preserve weak inequalities)**.** *Let (a_n) and (b_n) be real sequences, and assume that $a_n \to L$ and $b_n \to M$ as $n \to \infty$, and that $a_n \leqslant b_n$ for all n. Then $L \leqslant M$.*

Remark.

- This includes the special case where $a_n = 0$ for all n: Proposition 28 says that if $b_n \geqslant 0$ for all n, and $b_n \to M$ as $n \to \infty$, then $M \geqslant 0$. (This is because the constant sequence $0, 0, 0, \ldots$ certainly converges to 0.)
- A common mistake is to use the non-result that limits preserve strict inequalities. As we've seen, this is not true. Please try not to do this!

Proof. Suppose, for a contradiction, that it is not the case that $L \leqslant M$, so (by trichotomy) $L > M$.



Let $\varepsilon = \frac{1}{2}(L - M) > 0$.

Since $a_n \to L$ as $n \to \infty$, there is N_1 such that if $n \geqslant N_1$ then $|a_n - L| < \varepsilon$.

Since $b_n \to M$ as $n \to \infty$, there is N_2 such that if $n \geqslant N_2$ then $|b_n - M| < \varepsilon$.

Now for $n \geqslant \max\{N_1, N_2\}$ we have $a_n > L - \varepsilon$ and $b_n < M + \varepsilon$,

so $L - \varepsilon < a_n \leqslant b_n < M + \varepsilon$,

so $L - M < 2\varepsilon = L - M$. This is a contradiction. □

We saw a sandwiching result earlier. Here is a generalisation.

Proposition 29 (Sandwiching)**.** *Let (a_n), (b_n) and (c_n) be real sequences with $a_n \leqslant b_n \leqslant c_n$ for all $n \geqslant 1$. If $a_n \to L$ and $c_n \to L$ as $n \to \infty$, then $b_n \to L$ as $n \to \infty$.*

Proof. Take $\varepsilon > 0$.

Since $a_n \to L$ as $n \to \infty$, there is N_1 such that if $n \geqslant N_1$ then $|a_n - L| < \varepsilon$.

Since $c_n \to L$ as $n \to \infty$, there is N_2 such that if $n \geqslant N_2$ then $|c_n - L| < \varepsilon$.

Then for $n \geqslant \max\{N_1, N_2\}$ we have $L - \varepsilon \leqslant a_n \leqslant b_n \leqslant c_n \leqslant L + \varepsilon$,

so $|b_n - L| < \varepsilon$.

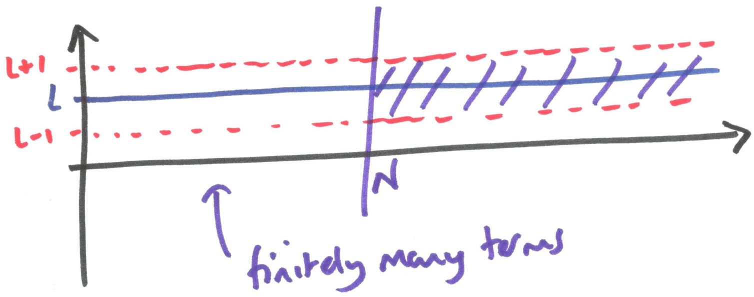
So $b_n \to L$ as $n \to \infty$. □

19 Bounded and unbounded sequences

Definition. Let (a_n) be a sequence. We say that (a_n) is *bounded* if the set $\{a_n : n \geqslant 1\}$ is bounded, that is, there is M such that $|a_n| \leqslant M$ for all $n \geqslant 1$. If (a_n) is not bounded then we say that it is *unbounded*.

Proposition 30 (A convergent sequence is bounded)**.** *Let* (a_n) *be a convergent sequence. Then* (a_n) *is bounded.*

Remark. Proposition 30 tells us that if (a_n) is unbounded then (a_n) diverges.



Proof.

Assume that $a_n \to L$ as $n \to \infty$.

Then (taking $\varepsilon = 1$) there is N such that if $n \geqslant N$ then $|a_n - L| < 1$ so

$$|a_n| = |(a_n - L) + L| \leqslant |a_n - L| + |L| < 1 + |L|.$$

Let $M = \max\{|a_1|, |a_2|, \dots, |a_N|, |L| + 1\}$.

Then $|a_n| \leqslant M$ for all $n \geqslant 1$. □

Remark.

- As remarked earlier, if (a_n) is unbounded then (a_n) diverges. So, for example, (2^n) diverges.
- Unboundedness is not the same as divergence. The converse of Proposition 30 is not true. A bounded sequence can diverge. For example, let $a_n = (-1)^n$. Then $|a_n| \leqslant 1$ for all $n \geqslant 1$, so (a_n) is bounded.

  Claim. $((-1)^n)$ does not converge.

Proof. Suppose, for a contradiction, that $(-1)^n \to L$ as $n \to \infty$.

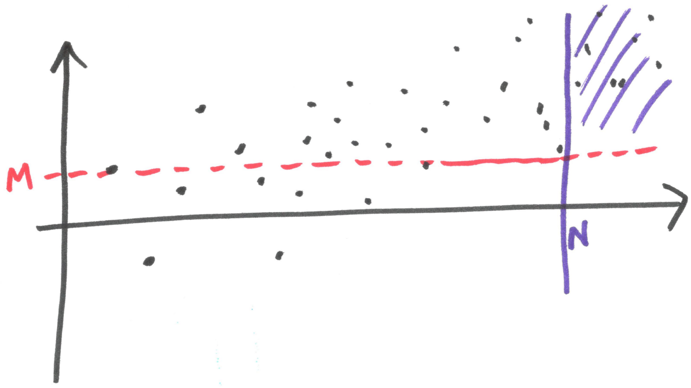
Then (taking $\varepsilon = 1$) there is N such that if $n \geqslant N$ then $|(-1)^n - L| < 1$.

In particular ($n = 2N$) we have $|L - 1| < 1$ so $L > 0$,

and ($n = 2N + 1$) we have $|L + 1| < 1$ so $L < 0$.

This is a contradiction. $\square$

What would it mean to say that a sequence tends to infinity?



$\forall M \in \mathbb{R}\ \exists N \in \mathbb{N}$ s.t. $\forall n \geqslant N \quad a_n > M$

Definition. Let (a_n) be a real sequence. We say that (a_n) tends to infinity as $n \to \infty$ if

$$\forall M \in \mathbb{R}\ \exists N \in \mathbb{N} \text{ such that } \forall n \geqslant N,\ a_n > M.$$

In this case we write $a_n \to \infty$ as $n \to \infty$.

Similarly, we say that (a_n) tends to negative infinity as $n \to \infty$ if

$$\forall M \in \mathbb{R}\ \exists N \in \mathbb{N} \text{ such that } \forall n \geqslant N,\ a_n < M.$$

In this case we write $a_n \to -\infty$ as $n \to \infty$.

Remark. This is a separate definition from our earlier definition of convergence, and ∞ is definitely not a real number. Results about convergence to a real number L cannot just be applied by 'taking $L = \infty$'—this would be highly illegal!

Example.

- Let $a_n = n^2 - 6n$ for $n \geqslant 1$.

  Claim. $a_n \to \infty$ as $n \to \infty$.

  Proof. Fix $M > 0$. (It suffices to prove the result for $M > 0$.)

  We want N such that if $n \geqslant N$ then $n^2 - 6n \geqslant M$

  but $n^2 - 6n = (n-3)^2 - 9$

  so we are done if $(n-3)^2 \geqslant M + 9$

  that is, we are done if $n - 3 \geqslant \sqrt{M+9}$

  Let $N = \lceil 4 + \sqrt{M+9} \rceil$.

  If $n \geqslant N$, then $n - 3 \geqslant \sqrt{M+9} > 0$,

  so $(n-3)^2 \geqslant M + 9$,

  so $n^2 - 6n \geqslant M$.

  So $a_n \to \infty$ as $n \to \infty$. □

- Let $a_n = \begin{cases} 0 & \text{if } n \text{ prime} \\ n & \text{otherwise.} \end{cases}$

  Then (a_n) does not tend to infinity, because there are infinitely many primes: for any $N \in \mathbb{N}$, there is a prime n with $n > N$, and then $a_n = 0$.

Lemma 31. *(i) If $\alpha < 0$, then $n^\alpha \to 0$ as $n \to \infty$.*

(ii) If $\alpha > 0$, then $n^\alpha \to \infty$ as $n \to \infty$.

Proof. (i) Take $\varepsilon \in (0,1)$. We have

$$\begin{aligned} & n^\alpha < \varepsilon \\ \Leftrightarrow\ & \mathrm{e}^{\alpha \log n} < \varepsilon \\ \Leftrightarrow\ & \alpha \log n < \log \varepsilon \\ \Leftrightarrow\ & \log n > \frac{1}{\alpha} \log \varepsilon \text{ (note } \alpha < 0) \\ \Leftrightarrow\ & n > \mathrm{e}^{\frac{1}{\alpha} \log \varepsilon} \end{aligned}$$

so we can take $N = 1 + \lceil \mathrm{e}^{\frac{1}{\alpha} \log \varepsilon} \rceil$.

(ii) Take $M > 0$. We have

$$\begin{aligned} & n^\alpha > M \\ \Leftrightarrow\ & \mathrm{e}^{\alpha \log n} > M \\ \Leftrightarrow\ & \alpha \log n > \log M \\ \Leftrightarrow\ & \log n > \frac{1}{\alpha} \log M \text{ (note } \alpha > 0) \\ \Leftrightarrow\ & n > \mathrm{e}^{\frac{1}{\alpha} \log M} \end{aligned}$$

so we can take $N = 1 + \lceil \mathrm{e}^{\frac{1}{\alpha} \log M} \rceil$.

□

Lemma 32. *Let $c \in \mathbb{R}^{>0}$.*

(i) If $c < 1$, then $c^n \to 0$ as $n \to \infty$.

(ii) If $c = 1$, then $c^n \to 1$ as $n \to \infty$.

(iii) If $c > 1$, then $c^n \to \infty$ as $n \to \infty$.

Proof. (i) This was Lemma 25.

(ii) This is clear from the definition of convergence.

(iii) Exercise. (You could adapt the argument from (i), or use logarithms.)

□

20 Complex sequences

A lot of the theory we have just seen applies equally to complex sequences, but there are some differences. Let's spell out the definition of convergence explicitly.

Definition. Let (z_n) be a complex sequence, let $L \in \mathbb{C}$. We say that (z_n) *converges* to L as $n \to \infty$ if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \text{ such that } \forall n \geqslant N, \; |z_n - L| < \varepsilon.$$

Remark.
- If (z_n) tends to a limit, then this limit is unique, exactly as in Theorem 26.
- We can have a sort of sandwiching for complex sequences, if we use the modulus. If (z_n) and (w_n) are complex sequences, and $|w_n| \leqslant |z_n|$ for all $n \geqslant 1$, and $z_n \to 0$ as $n \to \infty$, then $w_n \to 0$ as $n \to \infty$.

Given a complex sequence (z_n), there are two associated real sequences $(\mathrm{Re}(z_n))$ and $(\mathrm{Im}(z_n))$. The next result relates convergence of (z_n) to convergence of $(\mathrm{Re}(z_n))$ and $(\mathrm{Im}(z_n))$.

Theorem 33 (Convergence of complex sequences)**.** *Let (z_n) be a complex sequence. Write $z_n = x_n + \mathrm{i}y_n$ with x_n, $y_n \in \mathbb{R}$, so that (x_n) and (y_n) are real sequences. Then (z_n) converges if and only if both (x_n) and (y_n) converge. Moreover, in the case where (z_n) converges, we have $\lim\limits_{n\to\infty} z_n = \lim\limits_{n\to\infty} x_n + \mathrm{i} \lim\limits_{n\to\infty} y_n$.*

Proof. Exercise. □

Example.

- Let $z_n = \frac{\mathrm{i}^n}{n}$. Then $|z_n| = \frac{1}{n} \to 0$ as $n \to \infty$ so $z_n \to 0$ as $n \to \infty$.

- Let $z_n = (1+\mathrm{i})^n$. The sequence is

$$1+\mathrm{i}, 2\mathrm{i}, -2+2\mathrm{i}, -4, -4-4\mathrm{i}, -8\mathrm{i}, 8-8\mathrm{i}, 16, \dots .$$

  The real parts are $1, 0, -2, -4, -4, 0, 8, 16, \dots$—this sequence doesn't converge, and hence neither does (z_n).

21 Subsequences

We can make a good informal guess as to what we mean by a subsequence.

Let $(a_n)_{n\geqslant 1}$ be a sequence. Then a subsequence is a sequence $(b_r)_{r\geqslant 1}$, where each b_r is in (a_n), and the terms are in the right order.

Example. Let $a_n = n$ for $n \geqslant 1$. The following are subsequences of (a_n).

- $2, 4, 6, 8, \dots$ — the subsequence (a_{2n})
- $2, 4, 8, 16, \dots$ — the subsequence (a_{2^n})

The following are not subsequences of (a_n).

- $6, 4, 8, \dots$ — the terms are not in the right order
- $2, 4, 0, \dots$ — not all the terms are in (a_n)
- $1, 2, 3, \dots, 2020$ — finite so not a sequence.

Now let's give a formal definition of a subsequence.

Definition. Let $(a_n)_{n\geqslant 1}$ be a sequence. A *subsequence* $(b_r)_{r\geqslant 1}$ of $(a_n)_{n\geqslant 1}$ is defined by a function $f : \mathbb{N} \to \mathbb{N}$ such that f is strictly increasing (if $p < q$ then $f(p) < f(q)$), and $b_r = a_{f(r)}$ for $r \geqslant 1$.

We often write $f(r)$ as n_r. Then $n_1 < n_2 < n_3 < \cdots$ is a strictly increasing sequence of natural numbers, and $b_r = a_{n_r}$ so the sequence (b_r) has terms $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$.

Remark.

- Formally, (a_n) corresponds to a function $\alpha : \mathbb{N} \to \mathbb{R}$ or $\alpha : \mathbb{N} \to \mathbb{C}$. Then a subsequence of (a_n) corresponds to a function $\alpha \circ f$, where $f : \mathbb{N} \to \mathbb{N}$ is strictly increasing.

- Subscripts are 'dummy variables'. We can write (a_n) as (a_r) or (a_m) or (a_α) or (a_x). It is conventional to use a letter close to n in the alphabet, to help us remember that it is a natural number. We can use any letter for the subscripts in the subsequence (b_r), except that if we write our original sequence as (a_n) then we should avoid using n for the subsequence too.

- It's sometimes useful to know that $n_r \geqslant r$ for $r \geqslant 1$. (Exercise: prove this inequality, using induction.)

Proposition 34 (Subsequences of a convergent sequence)**.** *Let (a_n) be a sequence. If (a_n) converges, then every subsequence (a_{n_r}) of (a_n) converges. Moreover, if $a_n \to L$ as $n \to \infty$ then every subsequence also converges to L.*

Remark. So if (a_n) is a sequence, and it has two subsequences that tend to different limits, then (a_n) does not converge. This follows from Proposition 34, and can be a useful strategy for showing that a sequence does not converge.

Proof. Assume that (a_n) converges to L.

Let (a_{n_r}) be a subsequence of (a_n).

Take $\varepsilon > 0$.

Since $a_n \to L$, there is N such that if $n \geqslant N$ then $|a_n - L| < \varepsilon$.

If $r \geqslant N$, then $n_r \geqslant r \geqslant N$ (see remark before this result),

so $|a_{n_r} - L| < \varepsilon$.

So $a_{n_r} \to L$ as $r \to \infty$. $\square$

Example. Let $a_n = \begin{cases} 0 & \text{if } n \text{ is prime} \\ 1 + \frac{1}{n} & \text{otherwise} \end{cases}$.

Claim. (a_n) does not converge.

Proof. Idea: the subsequence of terms with prime subscripts tends to 0, and the subsequence of terms with non-prime subscripts tends to 1, so (a_n) doesn't converge.

Let the primes be $p_1 < p_2 < p_3 < \cdots$. Let $P = \{p_1, p_2, p_3, \dots\}$.

Note that there are infinitely many primes, so $(a_{p_r})_{r \geqslant 1}$ is a subsequence.

We have $a_{p_r} = 0$ for all $r \geqslant 1$, so $a_{p_r} \to 0$ as $r \to \infty$.

Let the elements of $\mathbb{N} \setminus P$ be $n_1 < n_2 < n_3 < \cdots$.

Note that there are infinitely many non-primes, so $(a_{n_r})_{r \geqslant 1}$ is a subsequence.

We have $a_{n_r} = 1 + \frac{1}{n_r}$ for $r \geqslant 1$, and so we see that $a_{n_r} \to 1$ as $r \to \infty$.

So (a_n) has subsequences that converge to different limits, so, by Proposition 34, (a_n) does not converge. $\square$

22 Algebra of Limits — part one

Example. This is an unofficial example. We'll return to it once we've proved some results.

Let $a_n = \dfrac{7n^5 - n\sin(n^2+5n)+3}{4n^5-3n^2+n+2}$.
What can we say about (a_n)?

Intuitively...

- the numerator grows like $7n^5$ — the other terms are much smaller for large n, which is all we care about;

- the denominator grows like $4n^5$

so we might conjecture that $a_n \to \frac{7}{4}$ as $n \to \infty$.

To prove this (and lots more!), we'll prove a bunch of results that are extremely useful in practice. Collectively, these are known as the 'Algebra of Limits', and we'll quote "by AOL" in arguments.

Theorem 35 (Algebra of Limits, part 1)**.** *Let (a_n) and (b_n) be sequences with $a_n \to L$ and $b_n \to M$ as $n \to \infty$. Let c be a constant.*

(i) (constant) If $a_n = c$, so (a_n) is a constant sequence, then $a_n \to c$ as $n \to \infty$.

(ii) (scalar multiplication) The sequence (ca_n) converges, and $ca_n \to cL$ as $n \to \infty$.

(iii) (addition) The sequence $(a_n + b_n)$ converges, and $a_n + b_n \to L + M$ as $n \to \infty$.

(iv) (subtraction) The sequence $(a_n - b_n)$ converges, and $a_n - b_n \to L - M$ as $n \to \infty$.

(v) (modulus) The sequence $(|a_n|)$ converges, and $|a_n| \to |L|$ as $n \to \infty$.

Proof. (i) This is immediate from the definition.

(ii) If $c = 0$, then we're done by (i). So assume that $c \neq 0$.

Take $\varepsilon > 0$.

Since $a_n \to L$, there is N such that if $n \geqslant N$ then $|a_n - L| < \varepsilon$.

Now if $n \geqslant N$ then $|ca_n - cL| = |c||a_n - L| < |c|\varepsilon$.

So (ca_n) converges to cL.

OR...

Take $\varepsilon > 0$.

Since $a_n \to L$, there is N such that if $n \geqslant N$ then $|a_n - L| < \frac{\varepsilon}{|c|}$.

Now if $n \geqslant N$ then $|ca_n - cL| = |c||a_n - L| < \varepsilon$.

So (ca_n) converges to cL.

(iii) Take $\varepsilon > 0$.

Since $a_n \to L$ as $n \to \infty$ there is N_1 such that if $n \geqslant N_1$ then $|a_n - L| < \varepsilon$.

Since $b_n \to M$ as $n \to \infty$ there is N_2 such that if $n \geqslant N_2$ then $|b_n - M| < \varepsilon$.

Let $N = \max\{N_1, N_2\}$. If $n \geqslant N$, then $|a_n - L| < \varepsilon$ and $|b_n - M| < \varepsilon$, so

$$\begin{aligned} |(a_n + b_n) - (L + M)| &\leqslant |a_n - L| + |b_n - M| \text{ (by triangle inequality)} \\ &< 2\varepsilon. \end{aligned}$$

So $(a_n + b_n)$ converges to $L + M$.

(iv) This follows from (ii) and (iii).

(v) This was Proposition 27.

$\square$

Remark. In (iii), I ended up showing that we can make $|(a_n+b_n)-(L+M)|$ less than 2ε by going far enough along the sequence. But the definition says ε, not 2ε, so isn't this a problem?

Well, no, it's not a problem. We need to show that we can make $|(a_n + b_n) - (L + M)|$ less than any positive real number — and that's what we've done. The important thing is that 2 was a (positive) constant: it didn't depend on n.

We could instead have chosen N_1 and N_2 corresponding to $\frac{\varepsilon}{2}$ (so if $n \geqslant N_1$ then $|a_n - L| < \frac{\varepsilon}{2}$ and similarly for b_n), and then we'd have got ε at the end. But if I'd done that then it might have seemed more mysterious: you might have wondered "how would I have known to choose $\frac{\varepsilon}{2}$?"

In practice, sometimes I doodle on scrap paper and consequently know what to choose at the start, and sometimes I just work through and see what happens, and if I get 2ε or 1000ε at the end then it doesn't matter. I illustrated these two alternative approaches in (ii) — but really they're the same, and both are fine.

Example.

Claim. Let $a_n = \dfrac{1}{2^n} + \left(1 + (-1)^n \dfrac{1}{\sqrt{n}}\right) + \dfrac{n\cos(n^3+1)}{5n^2+1}$. Then $a_n \to 1$ as $n \to \infty$.

Proof. We showed earlier that $\dfrac{1}{2^n} \to 0$ and $1 + (-1)^n \dfrac{1}{\sqrt{n}} \to 1$ and also $\dfrac{n\cos(n^3+1)}{5n^2+1} \to 0$ as $n \to \infty$ (see Section 16).

So, by AOL, (a_n) converges, and $a_n \to 0 + 1 + 0 = 1$ as $n \to \infty$. □

Example.

Claim. Let $a_n = (-1)^n + \frac{n}{2^n}$ for $n \geqslant 1$. Then (a_n) does not converge.

Proof. Suppose, for a contradiction, that (a_n) converges.

Note that $\left(\frac{n}{2^n}\right)$ converges (this was an earlier example).

So, by AOL, the sequence with n^{th} term $(-1)^n = a_n - \frac{n}{2^n}$ converges.

But we showed earlier that $((-1)^n)$ does not converge (or we could now note that it has subsequences tending to different limits 1 and -1). This is a contradiction. □

23 Algebra of Limits — part two

Theorem 36 (Algebra of Limits, part 2)**.** *Let (a_n) and (b_n) be sequences with $a_n \to L$ and $b_n \to M$ as $n \to \infty$.*

(vi) (product) The sequence $(a_n b_n)$ converges, and $a_n b_n \to LM$ as $n \to \infty$.

(vii) (reciprocal) If $M \neq 0$, then the sequence $\left(\frac{1}{b_n}\right)$ converges, and $\frac{1}{b_n} \to \frac{1}{M}$ as $n \to \infty$.

(viii) (quotient) If $M \neq 0$, then the sequence $\left(\frac{a_n}{b_n}\right)$ converges, and $\frac{a_n}{b_n} \to \frac{L}{M}$ as $n \to \infty$.

Remark. You might wonder whether the sequences $\left(\frac{1}{b_n}\right)$ and $\left(\frac{a_n}{b_n}\right)$ in (vii) and (viii) are defined. This is a good question. The answer is that — as we'll show in the proof — if $M \neq 0$ then a tail of (b_n) has all its terms nonzero, and hence there's a tail of $\left(\frac{1}{b_n}\right)$ that exists, and similarly for $\left(\frac{a_n}{b_n}\right)$. When we talk about convergence of these sequences, it's enough to consider a tail.

Proof. (vi) We're going to want to study

$$\begin{aligned}|a_n b_n - LM| &= |a_n(b_n - M) + M(a_n - L)| \\ &\leqslant |a_n||b_n - M| + |M||a_n - L|\end{aligned}$$

— this use of the triangle inequality can help us to see how to proceed.

Take $\varepsilon > 0$. We may assume that $\varepsilon < 1$.

Since $a_n \to L$, there is N_1 such that if $n \geqslant N_1$ then $|a_n - L| < \varepsilon$.

Since $b_n \to M$, there is N_2 such that if $n \geqslant N_2$ then $|b_n - M| < \varepsilon$.

Let $N = \max\{N_1, N_2\}$.

If $n \geqslant N$, then $|a_n - L| < \varepsilon$ and $|b_n - M| < \varepsilon$ and $|a_n| < |L| + \varepsilon$, so

$$\begin{aligned}
|a_n b_n - LM| &= |a_n(b_n - M) + M(a_n - L)| \\
&\leqslant |a_n||b_n - M| + |M||a_n - L| \\
&< (|L| + \varepsilon) \cdot \varepsilon + |M| \cdot \varepsilon \\
&< \varepsilon(1 + |L| + |M|).
\end{aligned}$$

Since $1 + |L| + |M|$ is constant, this is enough to show that $(a_n b_n)$ converges, and the limit is LM.

(vii) Assume that $M \neq 0$.

Idea: (1) eventually b_n is close to M, so can't be 0. (2) $\left|\dfrac{1}{b_n} - \dfrac{1}{M}\right| = \dfrac{|b_n - M|}{|M||b_n|}$ – eventually the numerator is small, and $|b_n|$ is close to $|M|$.

Take $\varepsilon > 0$.

Since $b_n \to M$ and $|M| > 0$, there is N_1 such that if $n \geqslant N_1$ then $|b_n - M| < \frac{|M|}{2}$, so (by the Reverse Triangle Inequality)

$$|b_n| \geqslant ||b_n + (M - b_n)| - |M - b_n|| > \frac{|M|}{2} > 0.$$

So the tail $(b_n)_{n \geqslant N_1}$ has all terms nonzero, so we can consider the sequence $\left(\frac{1}{b_n}\right)_{n \geqslant N_1}$.

Also, there is N_2 such that if $n \geqslant N_2$ then $|b_n - M| < \varepsilon$.

Let $N = \max\{N_1, N_2\}$. If $n \geqslant N$, then

$$\left|\frac{1}{b_n} - \frac{1}{M}\right| = \frac{|M - b_n|}{|M||b_n|} < \frac{\varepsilon}{|M|} \cdot \frac{2}{|M|}.$$

Since $\frac{2}{|M|^2}$ is a positive constant, this shows that $\left(\frac{1}{b_n}\right)_{n \geqslant N_1}$ converges, and the limit is $\frac{1}{M}$.

(viii) This follows from (vi) and (vii).

□

Example. Let $a_n = \dfrac{7n^5 - n\sin(n^2 + 5n) + 3}{4n^5 - 3n^2 + n + 2}$ (we saw this example at the start of Section 22).

Claim. $a_n \to \frac{7}{4}$ as $n \to \infty$.

Proof. Idea: the important terms (for large n) are $7n^5$ and $4n^5$.

We have

$$a_n = \frac{7 - \frac{1}{n^4}\sin(n^2 + 5n) + \frac{3}{n^5}}{4 - \frac{3}{n^3} + \frac{1}{n^4} + \frac{2}{n^5}}.$$

Now

$$0 \leqslant \left|\frac{1}{n^4}\sin(n^2 + 5n)\right| \leqslant \frac{1}{n^4} \leqslant \frac{1}{n}$$

and $\frac{1}{n} \to 0$, so by Sandwiching $\frac{1}{n^4}\sin(n^2 + 5n) \to 0$, and several other terms also tend to 0 (eg by Sandwiching),

so, by AOL, (a_n) converges, and

$$a_n \to \frac{7 - 0 + 0}{4 - 0 + 0 + 0} = \frac{7}{4}$$

as $n \to \infty$. □

Proposition 37 (Reciprocals and infinite/zero limits)**.** *Let (a_n) be a sequence of positive real numbers. Then $a_n \to \infty$ as $n \to \infty$ if and only if $\frac{1}{a_n} \to 0$ as $n \to \infty$.*

Proof. Exercise (using the definitions). □

24 Orders of magnitude

When we're studying a sequence (a_n), it can be really useful to develop some intuition about the behaviour of a_n for large n, in order to make a conjecture about the convergence (or otherwise) of the sequence, and to select a proof strategy. (This is what we did in the example at the end of the last section, for example.)

Example.

- Let $a_n = \dfrac{8n^2 + 1000000n + 1000000}{14n^6 + n^3 + n}$.

  Intuitively, the key term in the numerator is $8n^2$, and the key term in the denominator is $14n^6$. Even with the amusingly large coefficients in the numerator, when n is large these terms will be much smaller than $8n^2$.

  So it feels like the sequence grows roughly like $\frac{8}{14n^4}$, so should tend to 0.

  We can formalise this using AOL. Dividing through top and bottom by n^6 (since this is the key term), we get

  $$a_n = \frac{\frac{8}{n^4} + \frac{1000000}{n^5} + \frac{1000000}{n^6}}{14 + \frac{1}{n^3} + \frac{1}{n^5}} \to \frac{0+0+0}{14+0+0} = 0$$

  as $n \to \infty$.

- We showed in Lemma 25 that $\frac{n}{2^n} \to 0$ as $n \to \infty$.

  This is an example of the idea that 'exponentials beat polynomials'. But while 'exponentials beat polynomials' is a useful slogan for intuition, it is not suitable for rigorous proofs!

- We've seen a couple of examples where we used that $|\cos x| \leqslant 1$ and $|\sin x| \leqslant 1$ for all x — this can be useful.

- We'll show in the next section that $\frac{\log n}{n} \to 0$ as $n \to \infty$. Intuitively, polynomials grow faster than logarithms.

Definition. Let (a_n) and (b_n) be sequences. We write $a_n = O(b_n)$ as $n \to \infty$ if there is a constant $C \in \mathbb{R}^{>0}$ and there is N such that if $n \geqslant N$ then $|a_n| \leqslant C|b_n|$. This is 'big O' notation.

If $b_n \neq 0$ for all n (or all sufficiently large n), then we write $a_n = o(b_n)$ as $n \to \infty$ if $\frac{a_n}{b_n} \to 0$ as $n \to \infty$. This is 'little o' notation.

Remark.

- Sandwiching tells us that if $a_n = O(b_n)$ and $b_n \to 0$ as $n \to \infty$ then $a_n \to 0$ as $n \to \infty$.
- Big O and little o notation give us precise ways to make precise statements about comparative rates of growth of sequences. Please use them precisely!

Example. This example is in a Moodle quiz. Before you read on to the next section, please go to the Moodle course page for Analysis I, and try the quiz for section 24 (it's a short multiple choice quiz).

25 Monotonic sequences

Definition. Let (a_n) be a real sequence.

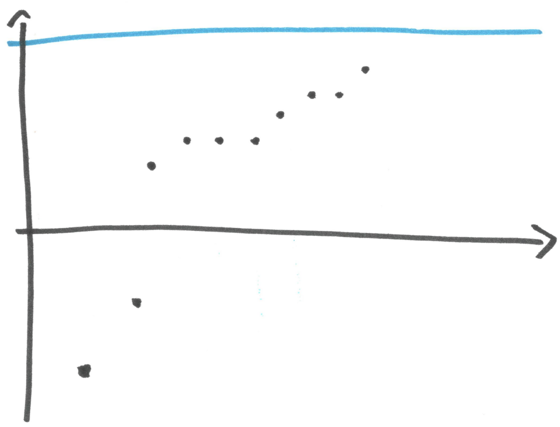
- We say that (a_n) is *monotonic increasing*, or *monotone increasing*, or *increasing*, if $a_n \leqslant a_{n+1}$ for all n.
- We say that (a_n) is *strictly increasing* if $a_n < a_{n+1}$ for all n.
- We say that (a_n) is *monotonic decreasing*, or *monotone decreasing*, or *decreasing*, if $a_n \geqslant a_{n+1}$ for all n.

- We say that (a_n) is *strictly decreasing* if $a_n > a_{n+1}$ for all n.

- We say that (a_n) is *monotonic*, or *monotone*, if it is increasing or decreasing.

Example. Notice that a constant sequence is both increasing and decreasing. This might seem counterintuitive!

We know that a convergent sequence is bounded. What can we say about a bounded monotone sequence?



Theorem 38 (Monotone Sequences Theorem). *Let (a_n) be a real sequence.*

(i) If (a_n) is increasing and bounded above, then (a_n) converges.

(ii) If (a_n) is decreasing and bounded below, then (a_n) converges.

Remark.

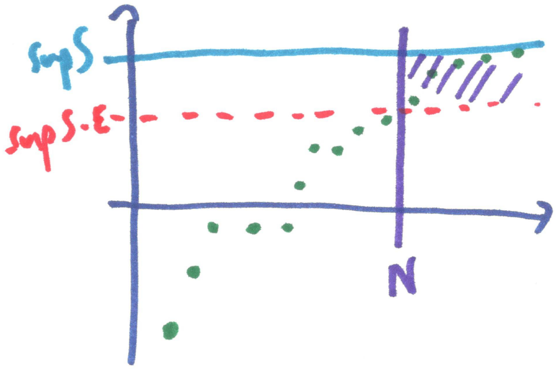
- So 'a bounded monotone sequence converges'.

- The result applies to tails of sequences too: if (a_n) has a tail that is monotone and bounded, then it converges.

Proof. (i) Assume that (a_n) is increasing and bounded above.

Idea: $\{a_n : n \geqslant 1\}$ has a supremum, and (a_n) converges to this.

The set $S = \{a_n : n \geqslant 1\}$ is non-empty and bounded above, so, by Completeness, it has a supremum.



Take $\varepsilon > 0$.

By the Approximation Property, there is N such that $\sup S - \varepsilon < a_N \leqslant \sup S$.

If $n \geqslant N$, then $\sup S - \varepsilon < a_N \leqslant a_n \leqslant \sup S$,

so $|a_n - \sup S| < \varepsilon$.

So (a_n) converges, and $a_n \to \sup S$ as $n \to \infty$.

(ii) If (a_n) is decreasing and bounded below, then $(-a_n)$ is increasing and bounded above, so (ii) follows from (i).

$\square$

Lemma 39. *Let (a_n) be a real sequence that is increasing and not bounded above. Then $a_n \to \infty$ as $n \to \infty$.*

Proof. Take $M \in \mathbb{R}$.

Since (a_n) is not bounded above, there is N such that $a_N > M$.

Then, since (a_n) is increasing, if $n \geqslant N$ then $a_n \geqslant a_N > M$. $\square$

Example. Let $a_n = \left(1 + \frac{1}{n}\right)^n$.

On Sheet 1, you proved that (a_n) is increasing and that (a_n) is bounded above (by 3). So, by the Monotone Sequences Theorem, (a_n) converges. Say $a_n \to L$ as $n \to \infty$. Then, since limits preserve weak inequalities, we see that $2 \leqslant L \leqslant 3$.

(Secretly, we know more about L, but that's strictly unofficial for now.)

Example. Let $c \geqslant 0$. In this example, we'll show that $\sqrt{c}$ exists. (This generalises earlier work on $\sqrt{2}$, and uses a different strategy.)

Define (a_n) by $a_1 = 1$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right)$ for $n \geqslant 1$.

This is a legitimate definition, since (by induction) $a_n \neq 0$ for $n \geqslant 1$.

Claim. (a_n) converges, and if $a_n \to L$ then $L^2 = c$.

Proof.

- (a_n) bounded below:

  by a straightforward induction argument, we have $a_n > 0$ for all n.

- study $a_n^2 - c$:

  for $n \geqslant 1$, we have

$$\begin{aligned}
a_{n+1}^2 - c &= \frac{1}{4}\left(a_n + \frac{c}{a_n}\right)^2 - c \\
&= \frac{1}{4}\left(a_n^2 + 2c + \frac{c^2}{a_n^2}\right) - c \\
&= \frac{1}{4}\left(a_n^2 - 2c + \frac{c^2}{a_n^2}\right) \\
&= \frac{1}{4}\left(a_n - \frac{c}{a_n}\right)^2 \\
&\geqslant 0,
\end{aligned}$$

  so $a_{n+1}^2 \geqslant c$ for $n \geqslant 1$.

- $(a_n)_{n\geqslant 2}$ decreasing:

  for $n \geqslant 2$, we have

  $$a_{n+1} - a_n = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right) - a_n = \frac{1}{2}\left(\frac{c}{a_n} - a_n\right) = \frac{1}{2a_n}(c - a_n^2) \leqslant 0,$$

  so $a_{n+1} \leqslant a_n$ for $n \geqslant 2$.

So, by the Monotone Sequences Theorem, (a_n) converges.

Say $a_n \to L$ as $n \to \infty$.

Then also $a_{n+1} \to L$ as $n \to \infty$ (it's a tail of the sequence).

But if $L \neq 0$ then

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right) \to \frac{1}{2}\left(L + \frac{c}{L}\right)$$

by AOL.

Since limits are unique, we have $L = \frac{1}{2}\left(L + \frac{c}{L}\right)$,

so, rearranging, $L^2 = c$.

Also, we have $a_n > 0$ for all n, and limits preserve weak inequalities, so $L \geqslant 0$.

So $\sqrt{c}$ exists ($L = \sqrt{c}$).

In the case that $L = 0$, since limits preserve weak inequalities and $a_n^2 \geqslant c$ for $n \geqslant 2$ we have $c \leqslant 0$, so $c = 0$ and $L^2 = c$. □

Lemma 40. *We have* $\frac{\log n}{n} \to 0$ *as* $n \to \infty$.

Proof. Let $a_n = \frac{\log n}{n}$.

Then $a_n \geqslant 0$ for all n, so (a_n) is bounded below.

Also, by properties of log we see that $(a_n)_{n\geqslant 100}$ is decreasing.

So, by the Monotone Sequences Theorem, (a_n) converges. Say $\frac{\log n}{n} \to L$ as $n \to \infty$.

Since limits preserve weak inequalities, we have $L \geqslant 0$.

Now

$$a_{2n} = \frac{\log(2n)}{2n} = \frac{\log 2 + \log n}{2n} \to 0 + \frac{L}{2}$$

by AOL,

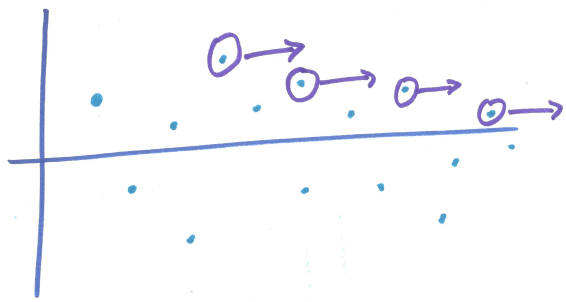
but also (a_{2n}) is a subsequence of (a_n) so $a_{2n} \to L$ as $n \to \infty$.

So, by uniqueness of limits, $\frac{L}{2} = L$, so $L = 0$. □

26 Convergent subsequences

Theorem 41 (Scenic Viewpoints Theorem). *Let (a_n) be a real sequence. Then (a_n) has a monotone subsequence.*

Proof. Idea: consider the 'peaks' of the sequence.



Let $V = \{k \in \mathbb{N} : \text{ if } m > k \text{ then } a_m < a_k\}$. (The elements of V are 'peaks' or 'scenic viewpoints': if $k \in V$ then a_k is higher than all subsequent terms.)

Case 1: V is infinite.

Say the elements of V are $k_1 < k_2 < \cdots$.

Then $(a_{k_r})_r$ is a subsequence of (a_n)

and it is monotone decreasing (if $r < s$ then $k_r < k_s$ so $a_{k_r} > a_{k_s}$).

Case 2: V is finite.

Then there is N such that if $k \in V$ then $k < N$.

Let $m_1 = N$. Then $m_1 \notin V$ so there is $m_2 > m_1$ with $a_{m_2} \geqslant a_{m_1}$.

Also, $m_2 \notin V$ so there is $m_3 > m_2$ with $a_{m_3} \geqslant a_{m_2}$.

Continuing inductively, we construct $m_1 < m_2 < m_3 < \cdots$ such that $a_{m_1} \leqslant a_{m_2} \leqslant a_{m_3} \leqslant \cdots$.

Then $(a_{m_r})_r$ is an increasing subsequence of (a_n). $\square$

Theorem 42 (Bolzano-Weierstrass Theorem)**.** *Let (a_n) be a bounded real sequence. Then (a_n) has a convergent subsequence.*

Proof. By the Scenic Viewpoints Theorem, (a_n) has a monotone subsequence.

This monotone subsequence is bounded (because the whole sequence is), so by the Monotone Sequences Theorem (Theorem 38) it converges. $\square$

Remark.

- This proof of the Bolzano-Weierstrass Theorem was very short, because we did all the work in the Monotone Sequences Theorem and Scenic Viewpoints Theorem! I have another favourite proof of Bolzano-Weierstrass. I've turned it into a quiz 'proof sorter' activity on Moodle.

- The Monotone Sequences Theorem and Scenic Viewpoints Theorem don't make sense for complex sequences. But Bolzano-Weierstrass potentially could ...

Corollary 43 (Bolzano-Weierstrass Theorem for complex sequences)**.** *Let (z_n) be a bounded complex sequence. Then (z_n) has a convergent subsequence.*

Proof. Study real and imaginary parts, and repeatedly pass to subsequences

Write $z_n = x_n + \mathrm{i}y_n$ where $x_n, y_n \in \mathbb{R}$.

Say (z_n) is bounded by M, so $|z_n| \leqslant M$ for all n.

Then (x_n) and (y_n) are also bounded by M, and they are real sequences.

By Bolzano-Weierstrass, (x_n) has a convergent subsequence, say $(x_{n_r})_r$.

Now $(y_{n_r})_r$ is a bounded real sequence, so by Bolzano-Weierstrass it has a convergent subsequence, say $(y_{n_{r_s}})_s$.

Note that $(x_{n_{r_s}})_s$ is a subsequence of the convergent sequence $(x_{n_r})_r$ and hence converges.

So, by Theorem 33, $(z_{n_{r_s}})_s$ converges (since its real and imaginary parts converge). $\square$

27 Cauchy sequences

Example. Let (a_n) be a convergent sequence.

Then $a_{n+1} - a_n \to 0$ as $n \to \infty$.

We can prove this directly from the definition (with the triangle inequality), or using tails and the Algebra of Limits.

But it is not the case that if $a_{n+1} - a_n \to 0$ as $n \to \infty$ then (a_n) converges.

For example, consider $a_n = \sqrt{n}$. Certainly (a_n) does not converge. But

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0$$

as $n \to \infty$.

Nonetheless, intuitively it seems that if eventually all the terms of a sequence are bunched up close together then the sequence might converge.

Definition. Let (a_n) be a sequence. We say that (a_n) is a *Cauchy sequence* if

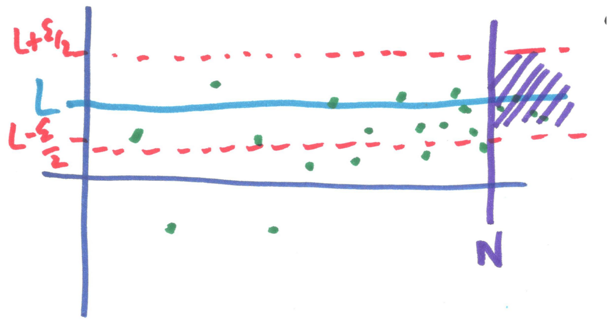
$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall m, n \geqslant N \ |a_n - a_m| < \varepsilon.$$

Remark. Note that this definition makes sense for complex sequences as well as for real sequences.

Proposition 44. *Let (a_n) be a convergent sequence. Then (a_n) is Cauchy.*

Proof. Say $a_n \to L$ as $n \to \infty$.

Take $\varepsilon > 0$.



Since $a_n \to L$, there is N such that if $n \geqslant N$ then $|a_n - L| < \frac{\varepsilon}{2}$.

Take $m, n \geqslant N$. Then $|a_m - L| < \frac{\varepsilon}{2}$ and $|a_n - L| < \frac{\varepsilon}{2}$,

so, by the triangle inequality,

$$\begin{aligned} |a_m - a_n| &= |(a_m - L) + (L - a_n)| \\ &\leqslant |a_m - L| + |a_n - L| < \varepsilon. \end{aligned}$$

So (a_n) is Cauchy. $\square$

Proposition 45. *Let (a_n) be a Cauchy sequence. Then (a_n) is bounded.*

Proof. Idea: use a similar strategy to Proposition 30, where we showed that a convergent sequence is bounded.

Since (a_n) is Cauchy, there is (applying the definition with $\varepsilon = 1$) N such that if $m, n \geqslant N$ then $|a_m - a_n| < 1$.

Now for $n \geqslant N$ we have $|a_n - a_N| < 1$,

so $|a_n| = |(a_n - a_N) + a_N| \leqslant 1 + |a_N|$.

Let $K = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |a_N|\}$.

Then $|a_n| \leqslant K$ for all $n \geqslant 1$.

So (a_n) is bounded. $\square$

Proposition 46. *Let* (a_n) *be a Cauchy sequence. Suppose that the subsequence* $(a_{n_r})_r$ *converges. Then* (a_n) *converges.*

Proof. Idea: eventually all the terms of (a_{n_r}) are really close to the limit L, and eventually all the terms of (a_n) are really close to terms in the subsequence and hence also really close to L.

Say that $a_{n_r} \to L$ as $r \to \infty$.

Take $\varepsilon > 0$.

Then there is N_1 such that if $r \geqslant N_1$ then $|a_{n_r} - L| < \frac{\varepsilon}{2}$.

Also, since (a_n) is Cauchy there is N_2 such that if $m, n \geqslant N_2$ then $|a_m - a_n| < \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$.

Let $r = N$. Then $n_r \geqslant r \geqslant N_1$ so $|a_{n_r} - L| < \frac{\varepsilon}{2}$

and if $n \geqslant N$ then n, $n_r \geqslant N_2$ so $|a_{n_r} - a_n| < \frac{\varepsilon}{2}$,

so

$$\begin{aligned}|a_n - L| &= |(a_n - a_{n_r}) + (a_{n_r} - L)| \\ &\leqslant |a_n - a_{n_r}| + |a_{n_r} - L| < \varepsilon.\end{aligned}$$

So $a_n \to L$ as $n \to \infty$. □

The following result is really useful! We'll use it in later sections.

Theorem 47 (Cauchy Convergence Criterion)**.** *Let* (a_n) *be a sequence. Then* (a_n) *converges if and only if* (a_n) *is Cauchy.*

Proof. ($\Rightarrow$) This was Proposition 44.

($\Leftarrow$) Assume that (a_n) is Cauchy.

Then (a_n) is bounded, by Proposition 45,

so by the Bolzano-Weierstrass Theorem (Theorem 42), (a_n) has a convergent subsequence, say (a_{n_r}).

Then, by Proposition 46, (a_n) converges. $\square$

Remark. One reason this is so useful is that it gives us a way to show that a sequence converges without needing to know in advance what the limit is.

28 Convergence for series

Example. Here are some informal examples of series to set the scene.

- For suitable r, we can consider the geometric series $\displaystyle\sum_{n=0}^{\infty} r^n$ (you might already have some ideas about this series).
- Decimal expansions. When we write $\frac{1}{9} = 0.111\ldots$ or $\frac{1}{9} = 0.\dot{1}$, we mean $\displaystyle\sum_{n=1}^{\infty} \frac{1}{10^n}$.
- We'll define $\mathrm{e} = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$.
- We'll define $\mathrm{e}^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

We'll revisit these examples once we've explored some theory.

Definition. Let (a_k) be a sequence. For $n \geqslant 1$, let

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

This is called a *partial sum* of the *series* $\displaystyle\sum_{k=1}^{\infty} a_k$.

We say that the series $\displaystyle\sum_{k=1}^{\infty} a_k$ *converges* if the sequence (s_n) of partial sums converges. If $s_n \to s$ as $n \to \infty$, then we write $\displaystyle\sum_{k=1}^{\infty} a_k = s$.

If (s_n) does not converge, then we say that $\displaystyle\sum_{k=1}^{\infty} a_k$ *diverges*.

Remark.
- So convergence of series is really a special case of convergence of sequences, rather than a new concept.
- A series is a limit.
- We might sometimes write $\sum\limits_{k\geqslant 1} a_k$ or even $\sum a_k$ instead of $\sum\limits_{k=1}^{\infty} a_k$.
- It would be highly illegal to write something like $\sum\limits_{n=1}^{n} a_n$ — we need to use different letters for quantities that can be different. That's why I've put k as the dummy variable in the sums, because it isn't n (and is still a good letter for a natural number).
- It's sometimes helpful to note that (with the notation above) $a_k = s_k - s_{k-1}$ for $k \geqslant 2$.

Example. Geometric series. Take $z \in \mathbb{C}$. Let $a_k = z^k$ for $k \geqslant 0$, and let $s_n = \sum\limits_{k=0}^{n} z^k$. Then for $n \geqslant 0$ we have

$$s_n = \begin{cases} \frac{1-z^{n+1}}{1-z} & \text{if } z \neq 1 \\ n+1 & \text{if } z = 1. \end{cases}$$

If $|z| < 1$, then $s_n \to \dfrac{1}{1-z}$ as $n \to \infty$, so $\sum\limits_{n=0}^{\infty} z^n$ exists and equals $\dfrac{1}{1-z}$.

If $|z| \geqslant 1$, then (s_n) does not converge and so the series diverges. (One way to see that (s_n) does not converge is to note that if $|z| \geqslant 1$ then $s_n - s_{n-1} = a_n = z^n$ does not tend to 0 as $n \to \infty$.)

Remark. Notice how we worked with partial sums, and determined that the limit exists before writing down $\sum z^n$.

Example. A telescoping series. Let $a_k = \dfrac{1}{k(k+1)}$ for $k \geqslant 1$.

Let $s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$.

Then

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \to 1 \text{ as } n \to \infty \end{aligned}$$

so $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ exists and equals 1.

Remark. Notice how we worked with partial sums, and determined that the limit exists before writing down $\sum \frac{1}{k(k+1)}$.

Example. Let $a_k = (-1)^k$, let $s_n = \sum_{k=1}^{n} (-1)^k$.

Then

$$s_n = \begin{cases} -1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

So (s_n) does not converge, that is, $\sum_{k=1}^{\infty} (-1)^k$ diverges.

Remark. Notice how we worked with partial sums, not the series, and in fact the limit doesn't exist. We definitely didn't write anything dodgy like

$$\sum_{k=1}^{\infty} (-1)^k = (-1+1) + (-1+1) + \cdots = 0,$$

because this would be wrong.

29 Series: first results and a first test for convergence

Proposition 48. *Consider the series* $\sum_{k=1}^{\infty} a_k$. *If* $\sum_{k=1}^{\infty} a_k$ *converges, then* $a_k \to 0$ *as* $k \to \infty$.

Remark. So one way to show that a series diverges is to show that $a_k \not\to 0$. This is disproportionately useful!

Proof. Let $s_n = \sum_{k=1}^{n} a_k$. Then (s_n) converges by assumption. Say $s_n \to s$ as $n \to \infty$.

Then also $s_{n-1} \to s$ as $n \to \infty$,

so by AOL $a_n = s_n - s_{n-1} \to s - s = 0$ as $n \to \infty$. □

Remark. Proposition 48 does not say that if $a_k \to 0$ as $k \to \infty$ then $\sum a_k$ converges. That's because this is false. For example ...

Example. For $n \geqslant 1$, let $s_n = \sum_{k=1}^{n} \frac{1}{k}$. The series $\sum_{k=1}^{\infty} \frac{1}{k}$ is called the *harmonic series*.

Claim. The harmonic series diverges.

Proof. Idea

$$1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{\geqslant \frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{\geqslant \frac{1}{2}}+\cdots+\frac{1}{n}$$

Show that (s_n) is not Cauchy.

Consider $|s_{2^{n+1}} - s_{2^n}|$. We have

$$|s_{2^{n+1}} - s_{2^n}| = \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \geqslant 2^n \cdot \frac{1}{2^{n+1}} = \frac{1}{2}.$$

So (s_n) is not Cauchy, so (s_n) does not converge. □

Remark. It is interesting to study the partial sums of the harmonic series. We'll do this in more detail in a future section.

Proposition 49. *Let (a_k) be a sequence of non-negative real numbers, and let $s_n = \sum_{k=1}^{n} a_k$. Suppose that (s_n) is bounded. Then the series $\sum_{k=1}^{\infty} a_k$ converges.*

Proof. Since $a_k \geqslant 0$ for all k, we see that (s_n) is increasing.

So (s_n) is monotone and bounded, so by the Monotone Sequences Theorem (Theorem 38) it converges, that is, $\sum_{k=1}^{\infty} a_k$ converges. □

Remark. Proposition 49 is a result that can be useful in practice for showing that a series converges. One particularly frequent way to apply it is to show that the partial sums are bounded by comparing with another series that we already know converges. We'll record that as a separate result, but really it's just a special case of Proposition 49, which is in turn just a special case of the Monotone Sequences Theorem.

Theorem 50 (Comparison Test)**.** *Let (a_k) and (b_k) be real sequences. Assume that $0 \leqslant a_k \leqslant b_k$ for all $k \geqslant 1$, and that $\sum_{k=1}^{\infty} b_k$ converges. Then $\sum_{k=1}^{\infty} a_k$ converges.*

Proof. Let $s_n = \sum_{k=1}^{n} a_k$.

Then (s_n) is increasing, since $a_k \geqslant 0$ for all $k \geqslant 1$.

Also,

$$s_n = \sum_{k=1}^{n} a_k \leqslant \sum_{k=1}^{n} b_k \leqslant \sum_{k=1}^{\infty} b_k$$

(since this last series converges),

so (s_n) is bounded.

Hence, by the Monotone Sequences Theorem (or Proposition 49), $\sum_{k=1}^{\infty} a_k$ converges. $\square$

Remark.

- More generally, if there is a positive constant C such that $0 \leqslant a_k \leqslant C b_k$ for $k \geqslant 1$, and if $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges, by a small generalisation of the argument.

- The Comparison Test can also be used to show that a series diverges. If $0 \leqslant a_k \leqslant b_k$ for all k and $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

- We don't need to know the value of $\sum b_k$ to use the Comparison Test, just that it exists.

- Please check the conditions of the Comparison Test very carefully before applying it. Please do not do this by writing things like $\sum_{k=1}^{\infty} a_k \leqslant \sum_{k=1}^{\infty} b_k$. We can't write down $\sum a_k$ (which is, remember, a limit) until we know that the limit exists. So either check the precise conditions of the Comparison Test, or work with partial sums as in Proposition 49.

- The Comparison Test is great!

Example.

Claim. $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

Proof. For $k \geqslant 2$, we have

$$0 \leqslant \frac{1}{k^2} \leqslant \frac{1}{k(k-1)},$$

and $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(k-1)}$ converges (we saw this previously),

so by the Comparison Test we have that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. $\square$

Remark. Note that this tells us nothing about the value of $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$! That is an interesting, but more challenging, problem for another time (not in this course). But we can still use $\sum \frac{1}{k^2}$ in future applications of the Comparison Test, even without knowing the value.

Example. The series $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}$ converges. (As usual, we define $0! = 1$.) This is an exercise on Sheet 5.

We can then define $\mathrm{e} = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}$.

Example. Decimal expansions. I'm not going to go through this example, but now is a good time to revisit it. You'll find the details in Hilary Priestley's supplementary notes on the uncountability of the reals, on Moodle.

30 Series: more results and another test for convergence

In Section 27, we met the Cauchy criterion for convergence (Theorem 47): a sequence (a_n) converges if and only if it is Cauchy. That immediately translates to a useful result for series.

Theorem 51 (Cauchy Convergence Criterion for series)**.** *Let* (a_k) *be a sequence, and write* $s_n = \sum_{k=1}^{n} a_k$*. Then* $\sum_{k=1}^{\infty} a_k$ *converges if and only if*

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n > m \geqslant N \ |s_n - s_m| = \left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon.$$

Proof. Immediate from the Cauchy Convergence Criterion (Theorem 47). □

So far, we have mostly considered series where all the terms are non-negative real numbers. We are interested in other series too, though. When a series can have negative terms (or even non-real), it might be that they enable enough cancellation that a series converges, or it might be that the convergence is so robust that even $\sum |a_k|$ converges. That's what the next definition and result are about.

Definition. Let (a_k) be a sequence. We say that $\sum_{k=1}^{\infty} a_k$ *converges absolutely* if $\sum_{k=1}^{\infty} |a_k|$ converges.

Remark.

- This makes sense for real and complex series.
- The series $\sum |a_k|$ is a series where all the terms are (real and) non-negative. Such series are particularly nice!

Theorem 52 (Absolute convergence implies convergence)**.** *Let* (a_k) *be a sequence. If* $\sum_{k=1}^{\infty} |a_k|$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *converges.*

Proof. Idea: use partial sums and the Cauchy criterion.

Let

$$s_n = \sum_{k=1}^{n} a_k \quad \text{and} \quad S_n = \sum_{k=1}^{n} |a_k|.$$

So we are assuming that (S_n) converges, and want to deduce that (s_n) converges.

For $n > m$, we have

$$|s_n - s_m| = \left|\sum_{k=m+1}^{n} a_k\right| \leqslant \sum_{k=m+1}^{n} |a_k| = |S_n - S_m|.$$

Now $\sum |a_k|$ converges by assumption,

so (S_n) is Cauchy by the Cauchy Convergence Criterion,

so (s_n) is Cauchy by the inequality above,

so $\sum a_k$ converges by the Cauchy Convergence Criterion. $\square$

Example. Let $a_k = (-1)^{3k} \dfrac{\sin^3(k^2)}{k^2+1}$.

Then $0 \leqslant |a_k| \leqslant \frac{1}{k^2}$ for $k \geqslant 1$,

and $\sum \frac{1}{k^2}$ converges,

so by the Comparison Test $\sum |a_k|$ converges,

so $\sum a_k$ converges since absolute convergence implies convergence.

Lemma 53. *Take $p \in \mathbb{R}$. Then $\displaystyle\sum_{k=1}^{\infty} k^{-p}$ diverges for $p \leqslant 1$, and converges if $p > 1$.*

Proof. **Case 1** $p \leqslant 0$. Then $k^{-p} \not\to 0$ as $k \to \infty$, so the series does not converge (by Proposition 48).

Case 2 $p = 1$. This is the harmonic series (see an example in Section 29).

Case 3 $0 < p < 1$. Note that then $k^{-p} > k^{-1} > 0$, and we know that $\sum k^{-1}$ diverges, so by the Comparison Test $\sum k^{-p}$ diverges.

Case 4 $p \geqslant 2$. We already know that $\sum \frac{1}{k^2}$ converges (this was an example near the end of Section 29), and $0 \leqslant k^{-p} \leqslant k^{-2}$, so, by the Comparison Test, $\sum k^{-p}$ converges.

Case 5 $1 < p < 2$. We'll do this later, once we've developed some more theory. $\square$

Example. We know that $\sum \frac{1}{n}$ diverges, and so $\sum \frac{(-1)^n}{n}$ does not converge absolutely. But does it converge? The next result will give us a way to show that it does.

Theorem 54 (Alternating Series Test)**.** *Let (u_k) be a real sequence, and consider the series $\sum_{k=1}^{\infty}(-1)^{k-1}u_k$. If*

- *$u_k \geqslant 0$ for $k \geqslant 1$; and*
- *(u_k) is decreasing, that is, $u_{k+1} \leqslant u_k$ for $k \geqslant 1$; and*
- *$u_k \to 0$ as $k \to \infty$,*

then $\sum_{k=1}^{\infty}(-1)^{k-1}u_k$ converges.

Proof. Idea: consider partial sums, get subsequences that are monotone and bounded.

Let $s_n = \sum_{k=1}^{n}(-1)^{k-1}u_k$.

- (s_{2n}) bounded above: We have

  $$s_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2n-2} - u_{2n-1}) - u_{2n} \leqslant u_1,$$

  so u_1 is an upper bound for (s_{2n}).

- (s_{2n}) is increasing: We have

  $$s_{2n+2} - s_{2n} = u_{2n+1} - u_{2n+2} \geqslant 0.$$

So, by the Monotone Sequences Theorem, (s_{2n}) converges. Say $s_{2n} \to s$ as $n \to \infty$.

Now $s_{2n+1} = s_{2n} + u_{2n+1} \to s + 0 = s$ as $n \to \infty$, by AOL.

So (s_{2n+1}) also converges to s.

Then (by Sheet 4 Q2) (s_n) converges. $\square$

Example.

Claim. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Proof. We have $\frac{1}{n} \geqslant 0$ for all n,

and $\left(\frac{1}{n}\right)_n$ is decreasing,

and $\frac{1}{n} \to 0$ as $n \to \infty$.

Hence, by the Alternating Series Test, $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges,

and so (by AOL) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. □

Example.

Claim. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges.

Proof. Exercise. □

Remark. This remark is not part of the course. A series such as $\sum \frac{(-1)^n}{n}$ that converges but does not converge absolutely is said to *converge conditionally*. Such series are delicate, when compared to more robust series that converge absolutely!

31 More on the Comparison Test

Example. Let $a_k = \dfrac{k^2 + k + 1}{4k^4 - k^2 - 1}$ and consider $\displaystyle\sum_k \frac{k^2 + k + 1}{4k^4 - k^2 - 1}$.

For large enough k, the denominator is positive, so a_k exists and $a_k \geqslant 0$.

Can we apply the Comparison Test?

Idea: a_k grows roughly like $\frac{1}{4k^2}$, so try comparing with that.

For sufficiently large k, we have

$$\begin{aligned}\frac{a_k}{\frac{1}{4k^2}} &= \frac{4k^2(k^2+k+1)}{4k^4-k^2-1}\\ &= \frac{4k^4+4k^3+4k^2}{4k^4-k^2-1}\\ &= \frac{1+\frac{1}{k}+\frac{1}{k^2}}{1-\frac{1}{4k^2}-\frac{1}{4k^4}} \to 1 \text{ as } k\to\infty,\end{aligned}$$

so there is K such that if $k \geqslant K$ then

$$0 \leqslant \frac{a_k}{\frac{1}{4k^2}} \leqslant \frac{3}{2}$$

so

$$0 \leqslant a_k \leqslant \frac{3}{2}\cdot\frac{1}{4k^2}.$$

Now $\displaystyle\sum_k \frac{3}{8k^2}$ converges,

so, by the Comparison Test, $\sum_k a_k$ converges.

(It doesn't matter that we have the inequalities only for large enough k — the first finitely many terms don't affect convergence.)

Remark. Some people like to summarise this strategy in the following result.

Theorem 55 (Limit form of Comparison Test)**.** *Let (a_k), (b_k) be real sequences of positive terms, and assume that there is $L > 0$ such that $\frac{a_k}{b_k} \to L$ as $k \to \infty$. Then $\sum a_k$ converges if and only if $\sum b_k$ converges.*

Proof. Since $\frac{a_k}{b_k} \to L$ as $k \to \infty$ and $\frac{L}{2} > 0$, there is K such that if $k \geqslant K$ then $\left|\dfrac{a_k}{b_k} - L\right| < \dfrac{L}{2}$, and so $\dfrac{L}{2} < \dfrac{a_k}{b_k} < \dfrac{3L}{2}$.

($\Leftarrow$) Then for $k \geqslant K$ we have $0 < a_k < \frac{3L}{2}b_k$, so if $\sum b_k$ converges then so does $\sum \frac{3L}{2}b_k$ and hence, by the Comparison Test, $\sum a_k$ converges.

($\Rightarrow$) Also, for $k \geqslant K$ we have $0 < b_k < \frac{2}{L}a_k$ (noting that $L \neq 0$), so if $\sum a_k$ converges then so does $\sum \frac{2}{L}a_k$ and hence, by the Comparison Test, $\sum b_k$ converges. □

Remark. It was important that, at least for sufficiently large k, the terms a_k and b_k are positive, and it was important that $\frac{a_k}{b_k}$ converges to a positive real number.

Example. Let $a_k = \dfrac{k^2+k+1}{4k^4-k^2-1}$.

Then (as before)

$$\frac{a_k}{\frac{1}{4k^2}} \to 1 \text{ as } k \to \infty,$$

and $a_k > 0$ for sufficiently large k

and $\frac{1}{4k^2} > 0$ for $k \geqslant 1$

and $\sum \frac{1}{4k^2}$ converges

so, by the limit form of the Comparison Test, $\sum a_k$ converges.

32 Ratio Test

Our next test for convergence is really useful. It sometimes gives a self-contained way to decide whether a series converges, rather than having to have an idea already (as is necessary for the Comparison Test, for example). The idea of the next test is essentially comparison with a geometric series. Let's do an example before we state the general result.

Example. Let $a_k = \dfrac{k}{2^k}$ and consider $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$.

We can't directly compare with $\sum \frac{1}{2^k}$.

Idea: the terms a_k decrease nearly like $\frac{1}{2^k}$.

More precisely,

$$\begin{aligned}\frac{a_{k+1}}{a_k} &= \frac{k+1}{2^{k+1}} \Big/ \frac{k}{2^k} \\ &= \frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k} \\ &= \frac{k+1}{k} \cdot \frac{1}{2} \\ &= \left(1+\frac{1}{k}\right) \cdot \frac{1}{2} \to \frac{1}{2} \text{ as } k \to \infty,\end{aligned}$$

where we used AOL at the end.

So there is K such that if $k \geqslant K$ then $\left|\dfrac{a_{k+1}}{a_k} - \dfrac{1}{2}\right| < \dfrac{1}{4}$, so $\dfrac{a_{k+1}}{a_k} < \dfrac{3}{4}$.

Then for $k \geqslant K$ we have $0 < a_k \leqslant \left(\frac{3}{4}\right)^{k-K} a_K$

and $\displaystyle\sum_{k=K}^{\infty} \left(\frac{3}{4}\right)^{k-K} a_K$ converges (geometric series with common ratio $\frac{3}{4}$, and $\left|\frac{3}{4}\right| < 1$)

so, by the Comparison Test, $\sum a_k$ converges.

Theorem 56 (Ratio Test)**.** *Let (a_k) be a real sequence of positive terms. Assume that $\dfrac{a_{k+1}}{a_k}$ converges as $k \to \infty$, say to limit L.*

(i) If $0 \leqslant L < 1$, then $\sum a_k$ converges.

(ii) If $L > 1$, then $\sum a_k$ diverges.

Remark.

- Here, exceptionally, we allow $L = \infty$, and this is covered by the $L > 1$ case.
- If $L = 1$, then the Ratio Test tells us nothing.
- If $\frac{a_{k+1}}{a_k}$ does not tend to a limit as $k \to \infty$, then the Ratio Test tells us nothing.

Proof. (i) Assume that $0 \leqslant L < 1$.

Let $\alpha = \frac{1+L}{2}$, so that $L < \alpha < 1$. Let $\varepsilon = \alpha - L > 0$.

Since $\frac{a_{k+1}}{a_k} \to L$, there is N such that if $k \geqslant N$ then $\left|\frac{a_{k+1}}{a_k} - L\right| < \varepsilon$, so $\frac{a_{k+1}}{a_k} < L + \varepsilon = \alpha$.

Now for $k \geqslant N$ we have $0 < a_k \leqslant \alpha^{k-N} a_N$.

But $\sum\limits_{k \geqslant N} \alpha^{k-N} a_N$ converges (constant times a geometric series with common ratio α, where $|\alpha| < 1$).

So, by the Comparison Test, $\sum a_k$ converges (the first N terms do not affect convergence).

(ii) Assume that $L > 1$.

Case 1 $L \in \mathbb{R}$.

Let $\alpha = \frac{1+L}{2}$, so $1 < \alpha < L$. Let $\varepsilon = L - \alpha > 0$.

Since $\dfrac{a_{k+1}}{a_k} \to L$, there is N such that if $k \geqslant N$ then $\left|\frac{a_{k+1}}{a_k} - L\right| < \varepsilon$, so $\frac{a_{k+1}}{a_k} > L - \varepsilon = \alpha$.

Now for $k \geqslant N$ we have $a_k \geqslant \alpha^{k-N} a_N > 0$,

and so $a_k \not\to 0$ as $k \to \infty$, so $\sum a_k$ diverges.

Case 2 $L = \infty$.

Let $\alpha = 2$.

Since $\dfrac{a_{k+1}}{a_k} \to \infty$, there is N such that if $k \geqslant N$ then $\frac{a_{k+1}}{a_n} > \alpha$.

Then finish as in Case 1.

$\square$

Example. Let $a_k = \dfrac{k}{2^k}$ (we did this before!).

Then $a_k > 0$ for all k, and

$$\frac{a_{k+1}}{a_k} = \frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k} = \left(1 + \frac{1}{k}\right) \cdot \frac{1}{2} \to \frac{1}{2} < 1 \text{ as } k \to \infty,$$

by AOL.

So, by the Ratio Test, $\sum a_k$ converges.

Example. Let $a_k = \frac{1}{k}$.

Then $a_k > 0$ for all k, and

$$\frac{a_{k+1}}{a_k} = \frac{k}{k+1} \to 1 \text{ as } k \to \infty,$$

so the Ratio Test tells us nothing.

Notice how we really had to consider the limit. We have $\frac{a_{k+1}}{a_k} < 1$ for all k, but that's not enough to determine convergence — remember that we already know that this series diverges.

Example. Let

$$a_k = \begin{cases} \frac{1}{2^k} & \text{if } k = 2^m \text{ for some } m \geqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

As it stands, we can't apply the Ratio Test, because the terms aren't all positive.

But we can omit the zero terms (which do not affect the convergence of the series): let $b_m = \dfrac{1}{2^{2^m}}$ for $m \geqslant 1$, and consider $\displaystyle\sum_m b_m$.

Now $b_m > 0$ for all m, and

$$\frac{b_{m+1}}{b_m} = \frac{2^{2^m}}{2^{2^{m+1}}} = \frac{1}{2^{2^m}} \to 0 < 1 \text{ as } m \to \infty,$$

so by the Ratio Test $\sum b_m$ converges and hence $\sum a_k$ converges.

Remark.

- The Ratio Test is brilliant, but please make sure you apply it carefully. Check the conditions!
- It's not always the case that $\frac{a_{k+1}}{a_k}$ converges, so that's why we stated it as a condition in the Ratio Test. Try to avoid assuming that the limit exists.

- We proved the Ratio Test by comparing with a geometric series. So we shouldn't use the Ratio Test to decide whether a geometric series converges!

We can adapt the Ratio Test to study absolute convergence of series even when the terms are not all real and positive.

Corollary 57. *Let (a_k) be a sequence of non-zero (real or complex) numbers. Assume that $\left|\frac{a_{k+1}}{a_k}\right|$ converges as $k \to \infty$, say to limit L.*

(i) If $0 \leqslant L < 1$, then $\sum a_k$ converges absolutely and hence converges.

(ii) If $L > 1$, then $\sum a_k$ diverges.

Remark.
- As before, we allow $L = \infty$ and include this in the case $L > 1$.
- If $L = 1$ then the Ratio Test tells us nothing.

Proof. (i) Apply the Ratio Test to $(|a_k|)$.

(ii) If $L > 1$, then the proof of the Ratio Test as applied to $(|a_k|)$ shows that $|a_k| \not\to 0$, so $a_k \not\to 0$, and so $\sum a_k$ diverges.

□

Remark. We'll see later in the course that the Ratio Test (especially in this form) is extremely helpful for studying power series.

33 Integral Test

In this section, we'll study certain series by considering corresponding integrals. This is a bit surprising, since we currently don't know what integration is. But it's nice to see the link to convergence of series now, so we'll pretend that we know what integration is, and that we know some basic facts about integration. In Analysis III, you'll fill in the details of this — you might like to revisit this section/video after studying Analysis III.

Some (for now unofficial) facts we'll assume:

- Suitably nice functions are integrable (in this section we'll consider only suitably nice functions).
- We can integrate constants: if $c \in \mathbb{R}$ then $\int_k^{k+1} c \, \mathrm{d}x = c$.
- Integration preserves weak inequalities: if $f, g : [a, b] \to \mathbb{R}$ are suitably nice, and $f(x) \leqslant g(x)$ for all $x \in [a, b]$, then $\int_a^b f \leqslant \int_a^b g$.
- If $a < b < c$ and $f : [a, c] \to \mathbb{R}$ is suitably nice, then $\int_a^c f = \int_a^b f + \int_b^c f$.

Theorem 58 (Integral Test)**.** *Let $f : [1, \infty) \to \mathbb{R}$ be a function. Assume that*

- *f is non-negative ($f(x) \geqslant 0$ for all $x \in [1, \infty)$);*
- *f is decreasing (if $x < y$ then $f(x) \geqslant f(y)$);*
- *$\int_k^{k+1} f(x)\mathrm{d}x$ exists for each $k \geqslant 1$.*

Let $s_n = \sum_{k=1}^n f(k)$ and $I_n = \int_1^n f(x)\mathrm{d}x$.

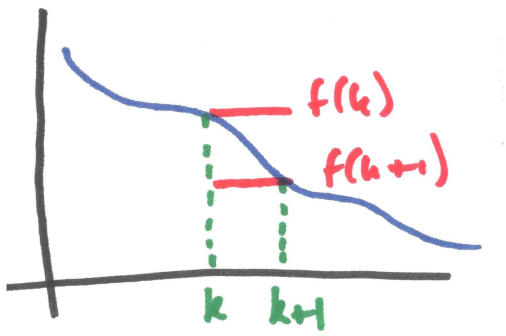
(i) Let $\sigma_n = s_n - I_n$. Then (σ_n) converges, and if we let σ be the limit of (σ_n), then $0 \leqslant \sigma \leqslant f(1)$.

(ii) $\sum f(k)$ converges if and only if (I_n) converges.

Remark.
- The main part of the Integral Test is (ii), and (i) is mostly interesting for helping us to prove (ii), but (as we'll see) (i) is also useful in its own right.

- If f is continuous then $\int_k^{k+1} f(x)\mathrm{d}x$ exists for each $k \geqslant 1$.

Proof. (i) Idea: show that (σ_n) is bounded below and decreasing.



Since f is decreasing, for $x \in [k, k+1]$, we have

$$f(k+1) \leqslant f(x) \leqslant f(k),$$

and so

$$f(k+1) = \int_k^{k+1} f(k+1)\mathrm{d}x \leqslant \int_k^{k+1} f(x)\mathrm{d}x \leqslant \int_k^{k+1} f(k)\mathrm{d}x = f(k).$$

Now

$$\begin{aligned} f(2) &\leqslant \int_1^2 f(x)\mathrm{d}x \leqslant f(1) \\ \text{and } f(3) &\leqslant \int_2^3 f(x)\mathrm{d}x \leqslant f(2) \\ \text{and } &\vdots \\ \text{and } f(n) &\leqslant \int_{n-1}^n f(x)\mathrm{d}x \leqslant f(n-1). \end{aligned}$$

Adding these (finitely many) inequalities gives

$$s_n - f(1) \leqslant I_n \leqslant s_n - f(n)$$

so

$$0 \leqslant f(n) \leqslant s_n - I_n \leqslant f(1)$$

so

$$0 \leqslant \sigma_n \leqslant f(1) \text{ for all } n \geqslant 1.$$

Also,

$$\begin{aligned}\sigma_{n+1} - \sigma_n &= s_{n+1} - I_{n+1} - s_n + I_n \\ &= f(n+1) - \int_n^{n+1} f(x)\mathrm{d}x \leqslant 0\end{aligned}$$

as above.

So (σ_n) is decreasing and bounded below,

so, by the Monotone Sequences Theorem, it converges.

Say $\sigma_n \to \sigma$ as $n \to \infty$.

Then, since limits preserve weak inequalities, and $0 \leqslant \sigma_n \leqslant f(1)$ for all $n \geqslant 1$, we have $0 \leqslant \sigma \leqslant f(1)$.

(ii) If (s_n) converges, then by AOL so does (I_n), since $I_n = s_n - \sigma_n$.

And if (I_n) converges, then by AOL so does (s_n), since $s_n = I_n + \sigma_n$.

□

Example. This is Lemma 53 revisited.

Claim. If $0 < p \leqslant 1$, then $\sum k^{-p}$ diverges, and if $p > 1$ then $\sum k^{-p}$ converges.

Proof. Fix $p > 0$. Define $f : [1, \infty) \to \mathbb{R}$ by $f(x) = x^{-p}$.

Then f is non-negative, and decreasing on $[1, \infty)$, and continuous.

Now for $p \neq 1$ we have

$$\begin{aligned} I_n &= \int_1^n x^{-p} \mathrm{d}x \\ &= \left[\frac{1}{1-p} x^{1-p} \right]_1^n \\ &= \frac{1}{1-p}(n^{1-p} - 1), \end{aligned}$$

so for $p < 1$ we see that (I_n) does not converge,

and for $p > 1$ we see that (I_n) does converge.

Also, for $p = 1$ we have

$$I_n = \int_1^n x^{-p} \mathrm{d}x = [\log x]_1^n = \log n,$$

so (I_n) does not converge.

Hence, by the Integral Test, $\sum k^{-p}$ converges for $p > 1$ and diverges for $0 < p \leqslant 1$. □

Remark. The Integral Test handles $p \geqslant 0$, but not $p < 0$ because in this case the function is not decreasing. Fortunately we can handle $p < 0$ directly, because in this case $k^{-p} \not\to 0$ and so $\sum k^{-p}$ diverges.

Example.

Claim. $\displaystyle\sum_{k \geqslant 2} \frac{1}{k \log k}$ diverges.

Proof. Exercise — use the Integral Test. □

Remark. This series can be useful for counterexamples, because it feels like it 'only just' diverges.

34 Euler's constant and rearranging series

Example. We know that the harmonic series $\sum_k \frac{1}{k}$ diverges. But the Integral Test can give us additional information.

Let $\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$.

Define $f : [1, \infty) \to \mathbb{R}$ by $f(x) = \frac{1}{x}$.

Then f is non-negative, decreasing and continuous, and

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x}\, \mathrm{d}x,$$

so (i) of Theorem 58 tells us that (γ_n) converges as $n \to \infty$, and the limit is in $[0, 1]$.

Let γ be this limit (this is standard notation), so

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \to \gamma \text{ as } n \to \infty,$$

and $0 \leqslant \gamma \leqslant 1$.

So, roughly speaking, the partial sums of the harmonic series grow like $\log n$, and hence tend to infinity rather slowly.

The number γ is known as *Euler's constant*.

It is not known whether γ is rational or irrational.

Example. Let $s_n = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k}$.

Then

$$\begin{aligned}
s_{2n} &= 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) \\
&= (\gamma_{2n} + \log(2n)) - (\gamma_n + \log n) \\
&= \log 2 + \gamma_{2n} - \gamma_n \\
&\to \log 2 \text{ as } n \to \infty,
\end{aligned}$$

and $s_{2n+1} = s_{2n} + \frac{1}{2n+1} \to \log 2$ as $n \to \infty$,

so (by a result on a problems sheet) (s_n) converges to $\log 2$, that is,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \log 2.$$

Remark. The order in which we sum the terms in this series really matters.

It turns out that if we regroup to have the same terms but in another order, with three positive terms followed by one negative, so

$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{4} + \cdots$$

then we instead get $\log 2 + \frac{1}{2}\log 3$ (exercise: show this!). There's yet another version of the series, with yet another value, on Sheet 7.

Definition. Let $g : \mathbb{N} \to \mathbb{N}$ be a bijection (a *permutation* of $\mathbb{N}$). Given a series $\sum a_k$, write $b_k = a_{g(k)}$. Then $\sum b_k$ is a *rearrangement* of $\sum a_k$.

Remark.

- It turns out (no proof in this course!) that if $\sum a_k$ is absolutely convergent, then any rearrangement of $\sum a_k$ also converges, to the same limit. In this sense absolutely convergent series are 'robust'.
- As we have seen, a series that converges but not absolutely (that is, a series that converges conditionally) is less robust. A rearrangement might give a series that converges to a different value, or even that does not converge at all.

Remark. In this course we've seen several tests for convergence of a series:

- the Comparison Test;
- the Alternating Series Test;
- the Ratio Test;

- the Integral Test.

We also saw that absolute convergence implies convergence.

These are the main tools for studying convergence of a series, but they are not the only ones: not every series is susceptible to one of these tests, and there are other convergence tests that can be useful — but they are beyond the scope of the course.

35 Power series

Definition. A *real power series* is a series of the form $\sum_{k=0}^{\infty} c_k x^k$, where $c_k \in \mathbb{R}$ for all $k \geqslant 0$ and x is a real variable.

A *complex power series* is a series of the form $\sum_{k=0}^{\infty} c_k z^k$, where $c_k \in \mathbb{C}$ for all $k \geqslant 0$ and z is a complex variable.

Remark.

- Much of the theory applies equally to real and complex power series, and of course every real power series is also a complex power series. Our focus in this course is mostly on real power series, but sometimes it is at least as convenient, or even more convenient, to work in the more general complex setting and then specialise later.

- We typically want to define a function using a power series. This is why we think of x or z as a variable.

- By convention, when we consider the series $\sum_{k=0}^{\infty} c_k z^k$ at $z = 0$, we mean just c_0. There are no issues about what 0^0 might mean! Every power series converges at $z = 0$, so we do not need to consider this case when studying convergence.

Example. Consider $\sum_{k=0}^{\infty} \frac{z^k}{k!}$. We use the Ratio Test: for $z \neq 0$, we have

$$\left| \frac{z^{k+1}}{(k+1)!} \Big/ \frac{z^k}{k!} \right| = \frac{k!}{(k+1)!}|z| = \frac{|z|}{k+1} \to 0 \text{ as } k \to \infty$$

and $0 < 1$, so by the Ratio Test the series converges absolutely, and hence converges, for all $z \in \mathbb{C}$.

Definition. We define the *exponential function* $\exp : \mathbb{C} \to \mathbb{C}$ by $\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$. We also write e^z for $\exp(z)$.

Example. Consider

$$\sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} \text{ and } \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \text{ and } \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} \text{ and } \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}.$$

Each of these converges for all $z \in \mathbb{C}$. (Exercise: use the Ratio Test to prove this for $z \neq 0$.)

Definition. We define the *sine function* $\sin : \mathbb{C} \to \mathbb{C}$ by

$$\sin(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!},$$

and the *cosine function* $\cos : \mathbb{C} \to \mathbb{C}$ by

$$\cos(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}.$$

Definition. We define the *hyperbolic sine function* $\sinh : \mathbb{C} \to \mathbb{C}$ by

$$\sinh(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$$

and the *hyperbolic cosine function* $\cosh : \mathbb{C} \to \mathbb{C}$ by

$$\cosh(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}.$$

Remark.

- We can go on to define other trig functions such as tan, cosec, sec and cot using these, on suitable domains. We wouldn't expect these further functions to have power series that converge on the whole of $\mathbb{C}$.
- We have defined sin and cos by power series, not by right-angled triangles.
- We need to go on to deduce the usual properties of exp, sin and cos, working from the power series definitions. We'll make a start on that in this course, and you will continue in Analysis II next term.

Remark. We previously proved (as part of AOL) that if (s_n) and (t_n) are convergent sequences, with $s_n \to L$ and $t_n \to M$, then (s_n+t_n) also converges, and $s_n + t_n \to L + M$.

We can apply this to sequences of partial sums, which gives us a way to consider the sum of two series.

To put that more explicitly, let $\sum a_k$ and $\sum b_k$ be convergent series, and write

$$s_n = \sum_{k=1}^{n} a_k \quad \text{and} \quad t_n = \sum_{k=1}^{n} b_k.$$

Then (s_n) and (t_n) converge. Say $s_n \to s$ and $t_n \to t$ (that is, $\displaystyle\sum_{k=1}^{\infty} a_k = s$ and $\displaystyle\sum_{k=1}^{\infty} b_k = t$). Then

$$s_n + t_n = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} (a_k + b_k),$$

so by AOL $(s_n + t_n)$ converges, and $\displaystyle\sum_{k=1}^{\infty} (a_k + b_k) = s + t$.

So we can sum two convergent series, and we can also use AOL to show that we can multiply a series by a (real or complex) number.

Remark. The above remark gives a useful way to show that a series diverges.

If $\sum a_k$ converges and $\sum b_k$ diverges, then $\sum(a_k+b_k)$ diverges too. That's because if $\sum(a_k + b_k)$ converges, then also $\sum((a_k + b_k) - a_k)$ converges, by the remark above.

Exercise: show, through suitable examples, that if $\sum a_k$ and $\sum b_k$ both diverge, then it might be that $\sum(a_k + b_k)$ converges and it might be that it diverges.

Example. From the power series definitions earlier, and this remark about AOL applied to series, we can see that for $z \in \mathbb{C}$ we have

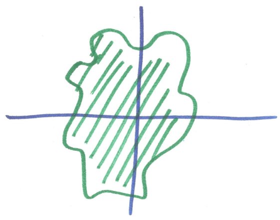
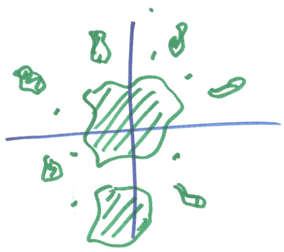
$$\begin{aligned} \cos z &= \frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}) \\ \text{and } \sin z &= \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}) \\ \text{and } \cosh z &= \frac{1}{2}(\mathrm{e}^{z} + \mathrm{e}^{-z}) \\ \text{and } \sinh z &= \frac{1}{2}(\mathrm{e}^{z} - \mathrm{e}^{-z}) \\ \text{and } \mathrm{e}^{\mathrm{i}z} &= \cos z + \mathrm{i}\sin z. \end{aligned}$$

We can also see from the power series definitions that for $z \in \mathbb{C}$ we have $\cos(\mathrm{i}z) = \cosh z$, and other similar relationships between cos and cosh, and between sin and sinh. (Exercise: think about all of these!)

36 Radius of convergence

In this section, it will be more natural to study power series in $\mathbb{C}$. The main goal will be to determine the subset of $\mathbb{C}$ on which a given power series converges. As we'll see, this subset must have a rather specific form. You might already have ideas about this, having tackled Sheet 6 Q4.

We expect that a power series will be more likely to converge for small $|z|$ than for large $|z|$. In principle, the subset of $\mathbb{C}$ on which a given power series $\sum c_k z^k$ converges might be a blob of convergence, or an ink splat of convergence, or an even more complicated region.

It turns out, though, that in fact the relevant shape is not a blob or an ink splat, but a disc. There are some details to be addressed, but that is the secret reason for the following definition — we'll go into the details after the definition.

Definition. Let $\sum c_k z^k$ be a power series. We define its *radius of convergence* to be

$$R := \begin{cases} \sup\{|z| \in \mathbb{R} : \sum |c_k z^k| \text{ converges}\} & \text{if the sup exists} \\ \infty & \text{otherwise.} \end{cases}$$

Remark.

- We certainly have $0 \in \{|z| \in \mathbb{R} : \sum |c_k z^k| \text{ converges}\}$, so the set is non-empty. So this subset of $\mathbb{R}$ has a sup if and only if it is bounded.
- There are other equivalent ways to define the radius of convergence, so if you look at another source then you might see a slightly different definition.

This definition is interesting, but without the following proposition it doesn't get us far. The proposition is more important than it might at first sight appear!

Proposition 59 (Radius of convergence)**.** *Let $\sum c_k z^k$ be a power series with radius of convergence R.*

(i) If $R > 0$ and $|z| < R$, then $\sum c_k z^k$ converges absolutely and hence converges.

(ii) If $|z| > R$, then $\sum c_k z^k$ diverges.

Remark. This proposition says nothing about what happens if $|z| = R$. This is deliberate!

Proof. (i) **Case 1:** $R \in \mathbb{R}$.

Assume that $R > 0$, and take $z \in \mathbb{C}$ with $|z| < R$.

Then there is S with $|z| < S < R$. Let $\varepsilon = R - S > 0$.

Since $R = \sup\{|w| \in \mathbb{R} : \sum |c_k w^k| \text{ converges}\}$, by the Approximation Property there is ρ such that $S = R - \varepsilon < \rho \leqslant R$ and $\sum |c_k \rho^k|$ converges.

Then $0 \leqslant |z| < \rho$ and $\sum |c_k \rho^k|$ converges, so by the Comparison Test $\sum |c_k z^k|$ converges.

Since absolute convergence implies convergence, this shows that $\sum c_k z^k$ converges.

Case 2: $R = \infty$.

Very similar to Case 1.

(ii) Take $z \in \mathbb{C}$ with $|z| > R$

Then we know that $\sum |c_k z^k|$ diverges, by definition of R, but we don't know about $\sum c_k z^k$.

Suppose, for a contradiction, that $\sum c_k z^k$ converges.

Find ρ with $\rho > R$ such that $\sum |c_k \rho^k|$ converges.

Then $c_k z^k \to 0$ as $k \to \infty$, so $(c_k z^k)$ is bounded, so there is M such that $|c_k z^k| \leqslant M$ for all k.

Take ρ with $R < \rho < |z|$.

Then

$$0 \leqslant |c_k \rho^k| \leqslant |c_k z^k| \left|\frac{\rho}{z}\right|^k \leqslant M \left|\frac{\rho}{z}\right|^k$$

and $\sum \left|\frac{\rho}{z}\right|^k$ converges (geometric series with common ratio $\left|\frac{\rho}{z}\right|$, and $\left|\frac{\rho}{z}\right| < 1$),

so, by the Comparison Test, $\sum |c_k \rho^k|$ converges,

contradicting the definition of R.

□

Remark.

- We call $\{z \in \mathbb{C} : |z| < R\}$ the *disc of convergence* for the power series. Proposition 59 shows that this is a useful concept. For a real power series, the corresponding concept is an *interval of convergence.*
- Anything at all can happen on the circle $\{z \in \mathbb{C} : |z| = R\}$! The series might converge everywhere on the circle, or diverge everywhere on the circle, or converge at some points and diverge at others.
- You might like to revisit Sheet 6 Q4 briefly having seen the theory, to see the connections.

Example.

- We have already seen that the exponential, sine and cosine, hyperbolic sine and hyperbolic cosine series have radius of convergence ∞ (using the Ratio Test).

- The geometric series $\sum z^k$ has $R = 1$ (from an example in Section 28).

Example. Consider $\displaystyle\sum_{k=0}^{\infty} \frac{k!}{k^k} x^k$.

For $x \neq 0$, we have

$$\begin{aligned}\left|\frac{(k+1)!}{(k+1)^{k+1}} x^{k+1} \cdot \frac{k^k}{k! x^k}\right| &= \frac{k!}{(k+1)^k} \frac{k^k}{k!} |x| = \left(\frac{k}{k+1}\right)^k |x| \\ &= \left(1 + \frac{1}{k}\right)^{-k} |x| \to \frac{1}{\mathrm{e}} |x| \text{ as } k \to \infty,\end{aligned}$$

so by the Ratio Test the series $\sum \left|\frac{k!}{k^k} x^k\right|$ converges for $|x| < \mathrm{e}$ (so $R \geqslant \mathrm{e}$)

and diverges for $|x| > \mathrm{e}$ (so $R \leqslant \mathrm{e}$).

So $R = e$.

Remark. Note that it was not enough to use the Ratio Test to show that the series converges (absolutely) for $|x| < \mathrm{e}$ — this shows that $R \geqslant \mathrm{e}$, not that $R = \mathrm{e}$.

Example. Consider $\sum c_k x^k$ where $c_k = \begin{cases} 1 & \text{if } k \text{ prime} \\ 0 & \text{otherwise.} \end{cases}$

So the Ratio Test won't work!

For $x = 1$, we see that $c_k x^k \not\to 0$ as $k \to \infty$ (because there are infinitely many primes), so $R \leqslant 1$.

If $|x| < 1$, then $0 \leqslant |c_k x^k| \leqslant |x^k|$,

and $\sum |x^k|$ is a convergent geometric series,

so, by the Comparison Test, $\sum c_k x^k$ converges absolutely and hence converges. So $R \geqslant 1$.

So $R = 1$.

Remark. The Ratio Test is often useful for finding the radius of convergence of a power series, but does not always work. There are more sophisticated strategies that work in other situations, but it is easy to apply them incorrectly, and they are not needed for Prelims.

37 Differentiation Theorem

We have seen that we can define a function using a power series. For example, we defined the exponential, sine and cosine functions in this way. In this section, we're going to explore the derivative of a function defined by a power series, as this will enable us to study some key properties of familiar functions. This will not be an in-depth exploration, since you haven't studied differentiability yet. We'll state a theorem, and then see how extremely useful it can be in practice. We shan't prove it in this course; you'll see a proof in a future analysis course.

Theorem 60 (Differentiation Theorem for real power series)**.** *Let $\sum c_k x^k$ be a real power series with radius of convergence R. Assume that $0 < R \leqslant \infty$. For $|x| < R$, define $f(x) = \displaystyle\sum_{k=0}^{\infty} c_k x^k$.*

Then $f(x)$ is well defined whenever $|x| < R$. Moreover, if $|x| < R$ then the derivative $f'(x)$ exists, and

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{k=0}^{\infty} c_k x^k\right) = \sum_{k=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}(c_k x^k) = \sum_{k=1}^{\infty} k c_k x^{k-1}.$$

Remark.

- The slogan is that "on the disc of convergence, we can differentiate term-by-term".
- The theorem is definitely not obvious! It involves exchanging the order of limiting processes, and that is a delicate business.

Example. We saw that the power series defining the exponential, sine, cosine, sinh and cosh functions have $R = \infty$, so the series converge on $\mathbb{R}$ (and on $\mathbb{C}$), and by the Differentiation Theorem they are differentiable on all of $\mathbb{R}$. Moreover, by the Differentiation Theorem we can differentiate term by term on $\mathbb{R}$.

For example, for $x \in \mathbb{R}$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{k=0}^{\infty}\frac{x^k}{k!}\right) \\
&= \sum_{k=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^k}{k!}\right) \quad \text{by the Differentiation Theorem} \\
&= \sum_{k=1}^{\infty}\frac{kx^{k-1}}{k!} \\
&= \sum_{k=1}^{\infty}\frac{x^{k-1}}{(k-1)!} \\
&= \sum_{k=0}^{\infty}\frac{x^k}{k!} = \mathrm{e}^x.
\end{aligned}$$

To summarise, for all $x \in \mathbb{R}$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x &= \mathrm{e}^x \\
\frac{\mathrm{d}}{\mathrm{d}x}\sin x &= \cos x \\
\frac{\mathrm{d}}{\mathrm{d}x}\cos x &= -\sin x \\
\frac{\mathrm{d}}{\mathrm{d}x}\sinh x &= \cosh x \\
\frac{\mathrm{d}}{\mathrm{d}x}\cosh x &= \sinh x.
\end{aligned}$$

Example.

Claim. $\sin^2 x + \cos^2 x = 1$ for all $x \in \mathbb{R}$.

Proof. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = \sin^2 x + \cos^2 x$.

Then (using properties of differentiability that you'll study in Analysis II next term) h is differentiable on $\mathbb{R}$, and

$$h'(x) = 2\cos x \sin x - 2\sin x \cos x = 0 \text{ for all } x \in \mathbb{R}.$$

This means (using a result you'll see in Analysis II) that h is constant.

But we know from the power series that $\sin 0 = 0$ and $\cos 0 = 1$, so $h(0) = 1$.

So $h(x) = 1$ for all $x \in \mathbb{R}$. $\square$

Remark. It would not be a good plan to try to do this by squaring power series and manipulating terms — this would need a lot of justification.

Example.

Claim. $\mathrm{e}^{a+b} = \mathrm{e}^a \mathrm{e}^b$ for all a, $b \in \mathbb{R}$.

Proof. Fix $c \in \mathbb{R}$, and define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \mathrm{e}^x \mathrm{e}^{c-x}$.

Then (Analysis II) g is differentiable on $\mathbb{R}$, and

$$g'(x) = \mathrm{e}^x \mathrm{e}^{c-x} - \mathrm{e}^x \mathrm{e}^{c-x} = 0 \text{ for all } x \in \mathbb{R}.$$

This means (Analysis II) that g is constant.

But we know from the power series that $\mathrm{e}^0 = 1$, so $g(0) = \mathrm{e}^c$.

So $g(x) = \mathrm{e}^c$ for all $x \in \mathbb{R}$.

This argument works for all $c \in \mathbb{R}$. Take $a, b \in \mathbb{R}$, and apply it with $x = a$, $c = a + b$ to get $\mathrm{e}^{a+b} = \mathrm{e}^a \mathrm{e}^b$. $\square$

Remark. This shows that for all $x \in \mathbb{R}$ we have $\mathrm{e}^x \mathrm{e}^{-x} = \mathrm{e}^0 = 1$. From the power series, we see that $\mathrm{e}^x > 0$ for $x \geqslant 0$, and hence in fact $\mathrm{e}^x > 0$ for all $x \in \mathbb{R}$.

Remark. These examples illustrate a really useful strategy, which can also be used to prove results like trig identities. Watch out for more on this in Analysis II next term!

What is π? We have defined sine and cosine using power series, without mentioning right-angled triangles. We can then define π to be the smallest positive x such that $\sin x = 0$, or $\frac{\pi}{2}$ as the smallest positive x such that $\cos x = 0$. It is not obvious that smallest such values exist; you'll look at this in more detail in Analysis II. You'll then be able to go on and prove that sine and cosine are 2π-periodic, for example.

Example. We see that if $x, y \in \mathbb{R}$ then

$$\mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^{x}(\cos y + \mathrm{i}\sin y).$$

We can then use properties of π to see that $\mathrm{e}^{2\pi\mathrm{i}} = 1$.

You'll study differentiability in $\mathbb{C}$ as part of the Part A Complex Analysis course, when you'll go on to explore many interesting (and surprising) properties of complex functions.

There are some further examples at the end of Hilary Priestley's notes on Moodle, which you'll also see in Analysis II/III.

This brings us to the end of Analysis I, but definitely not the end of analysis.

Building on your knowledge of analysis so far, you might like to consider the following questions, as a warm up for Analysis II.

- Given a function $f : \mathbb{R} \to \mathbb{R}$ and a, $L \in \mathbb{R}$, what does it mean to say that $f(x) \to L$ as $x \to a$?

- What does it mean to say that $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point $x \in \mathbb{R}$?

- Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$ At which points (if any) is f continuous?

- What does it mean to say that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a point $x \in \mathbb{R}$?

Well done and thank you for making it to the end of Analysis I!

To be continued… (in Analysis II)