ANALYSIS I

Countability: Informal Supplementary Notes

The treatment of countability required for Analysis I leaves many natural questions unanswered. This handout goes a little further and outlines some non-examinable material.

Some suggestions for reading (mathematically rather than philosophically oriented)

1. T.M. Apostol, *Mathematical Analysis*, 2nd edition.
2. P.J. Cameron, *Sets, Logic and Categories*, parts of Ch. 1.
3. D. Goldrei, *Classic Set Theory*, parts of Ch. 6.
4. A.G. Hamilton, *Numbers, Sets and Axioms*, Ch.2.

Apostol gives an account geared to applications in analysis. The other books include not-too-formal accounts of set theory. The one by Cameron is particularly suitable for contextual reading and contains a lot of interestng material in an accessible form.

C.1 The set of natural numbers ω, *alias* $\mathbb{N}$, and the Well-Ordering Principle.

INFORMALLY:

The 'standard' Zermelo–Fraenkel Axioms postulated for set theory include an axiom (the Axiom of Infinity) which guarantees the existence of a 'smallest' infinite set, denoted ω, whose members are defined to be the natural numbers.

The set ω comes ready-equipped with a strict total order $<$. It also supports

- a **Principle of Induction**, allowing traditional proofs by induction (this principle is hardwired into the axiom which supplies the set ω);
- the **Primitive Recursion Theorem**, which allows one to define functions on the natural numbers without having to say 'and so on'—a computer science favourite example of definition by recursion is the factorial function $n \mapsto n!$.

Arithmetic operations are then defined on ω by recursion, and their properties established by induction. Moreover ω satisfies the Well Ordering Principle:

(WO) Every non-empty subset of ω has a least element.

Aside. Here we are thinking of the natural numbers as a freestanding mathematical entity rather than as embedded into $\mathbb{R}$. We do not in Analysis I consider how a structure with the properties we have assumed about $\mathbb{R}$ might be constructed. Such a construction would normally start with set theory and the natural numbers derived from axioms of set theory. Then one would proceed successively to build $\mathbb{Z}$, $\mathbb{Q}$ and finally (the hard stage) $\mathbb{R}$. Strategies for the last stage are either (1) order-theoretic ('complete' $\mathbb{Q}$ by adding missing sups and infs or (2) topological (fill gaps by constructing limits for Cauchy sequences in $\mathbb{Q}$). To complete the construction one proves that the axioms for $\mathbb{R}$ which we are assuming for the purposes of this course are indeed satisfied.

Assuming $\mathbb{R}$ as 'given' and the natural numbers sitting inside it via $2 := 1 + 1$, $3 := 2 + 1$, and so on, the (WO) property looks obvious. However it does not follow from our axioms for $\mathbb{R}$ without some extra information about the natural numbers, such as the principle of induction. It makes more mathematical sense to start with the Zermelo-Fraenkel Axioms for set theory, use them to construct ω, show that ω satisfies (WO) and then go on to construct $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. However we do not have time to do this in the Prelims courses, so instead we make some basic assumptions about set theory, including the principle of induction, together with our axioms for $\mathbb{R}$.

C.2 Recap on definitions from Section 5.

A set A is

- **countable** if there exists an injection $f\colon A \to \mathbb{N}$, written $A \preccurlyeq \mathbb{N}$.
- **countably infinite** if there exists a bijection $f\colon A \to \mathbb{N}$, written $A \approx \mathbb{N}$.

Note that the relation $\approx$ ('same size as') is symmetric and transitive so that to show that we have $A \approx \mathbb{N}$ for some given set A it suffices to show $A \approx B$ for some B for which we know that $B \approx \mathbb{N}$. For example we might take $B = \mathbb{N} \times \mathbb{N}$.

C.3 Characterising countably infinite sets.

Let A be an infinite set. It is highly plausible that the following assertions are equivalent:

(1) there is a bijection $f\colon A \to \mathbb{N}$ (that is, $A \approx \mathbb{N}$);
(2) there is an injection $g\colon A \to \mathbb{N}$ (that is, $A \preccurlyeq \mathbb{N}$);
(3) there is a surjection $h\colon \mathbb{N} \to A$.

Let's see what is involved in establishing that they are equivalent.

(1) $\implies$ **(3)**: Trivial. (Similarly for **(1)** $\implies$ **(2)**).

(3) $\implies$ **(2)**: Assume we have a surjective map $h\colon \mathbb{N} \to A$. Then for each $a \in A$ the set

$$h^{-1}(a) = \{\, n \in A \mid h(n) = a \,\}$$

is non-empty. By (WO), there is a least $n_a \in \mathbb{N}$ such that $h(n_a) = a$. Then $g\colon a \mapsto n_a$ sets up an injection from A into $\mathbb{N}$. [(WO) for $\mathbb{N}$ gives us a way to specify a single element from each member of the family $\{h^{-1}(a)\}_{a\in A}$ of sets.]

(2) $\implies$ **(1)**: Assuming (2), there exists an injective map $g\colon A \to \mathbb{N}$, so that the image $g(A)$ is a subset of $\mathbb{N}$ and is infinite. To prove (1) it will suffice to construct a bijection k from $\mathbb{N}$ to $g(A)$. We proceed as follows. Define $k(0)$ to be the first (= least) element in $g(A)$. Then, since A is infinite, $g(A) \neq \{k(0)\}$ so there exists an element of $g(A)$ which is $> k(0)$ and by (WO) there is a least such element; call it $k(1)$. We continue in this way: assuming $k(0), k(1), \ldots, k(m-1)$ have been chosen from $g(A)$ to be the first m elements of $g(A)$, with respect to the order of $\mathbb{N}$. We have not exhausted $g(A)$ because it is infinite, so we can pick $k(m)$ to be the least element of $g(A) \setminus \{k(0), \ldots, k(m-1)\}$. The proof relies heavily on (WO): we need it at each step to pick the next-in-line element from $g(A)$. Doing this ensures that in the end $k\colon N \to g(A)$ is a bijection.

[But this is an 'and so on' construction. To make it rigorous one need a form of the Recursion Theorem.]

We have "proved"

A set A is countably infinite if and only if it is countable but not finite.

CONCLUSION: (1), (2), (3) do give equivalent ways of capturing the notion that an infinite set A is countably infinite, but a considerable amount of set-theoretic machinery is needed to put on a fully mathematically sound footing arguments which seem intuitively convincing.

Note: To get the equivalence of (1)–(3) we needed to exclude the case that A is finite. But the finite case is easy to encompass, so

Given a countable set$A$, there exists a surjective map from $\mathbb{N}$ onto A.

C.4 The union of a countable collection of countable sets A_n ($n \in \mathbb{N}$) is countable.

This is discussed in the cited books, at various levels of formality. The neatest proof makes use of the final claim in C.3, either overtly citing or covertly needing, the so-called Axiom of Choice (AC) to pick one surjection (out of possibly infinitely many) h_n from A_n onto $\mathbb{N}$. Then we can set up a surjection from $\mathbb{N} \times \mathbb{N} \to \bigcup A_n$ by $(m, n) \mapsto h_n(m)$. Since $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$ (recall 5.3(e)), this is sufficient to yield the required result.

For a brief, informal introduction to the Axiom of Choice see for example [2].

C.5 The Schröder–Bernstein Theorem.

Let A and B be sets and assume there exist injective maps $g\colon A \to B$ and $h\colon B \to A$. Then $A \approx B$.

Proof. This does not need any fancy set-theoretic machinery. The most elegant proof makes use of a result known as the Tarski Fixed Point Theorem. This proof is given for example in [2], pp. 18–19. □

SBT is particularly useful because it is often much harder explicitly to construct a bijection than to construct an injection.