# Introduction to Complex Numbers

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**SYLLABUS:**

Complex numbers and their arithmetic. The Argand diagram (complex plane). Modulus and argument of a complex number. Simple transformations of the complex plane. De Moivre's Theorem; roots of unity. Euler's theorem; polar form $re^{i\theta}$ of a complex number. Polynomials and a statement of the Fundamental Theorem of Algebra.

**READING LIST:**

- Richard Earl – *Complex Numbers* on https://www.maths.ox.ac.uk/study-here/undergraduate-study/bridging-gap
- Richard Earl – *Towards Higher Mathematics,* Cambridge, 2017, Chapter 1.

**FURTHER EADING:**

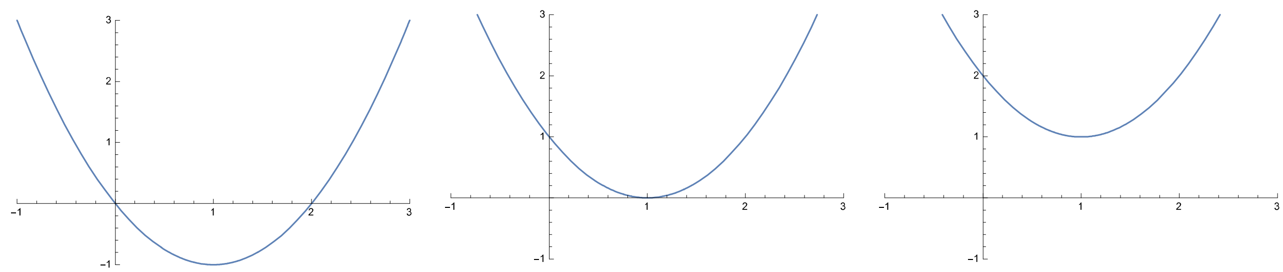
- Roger Fenn – *Geometry,* Springer, Chapter 4
- Liang-shin Hahn – *Complex Numbers and Geometry*, Mathematical Association of America, 1994, Chapter 1.

## 1 The Need for Complex Numbers

It is well known that the two roots of the quadratic equation $ax^2 + bx + c = 0$ ($a, b, c$ real and $a \neq 0$) are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1}$$

and mathematicians have been solving quadratic equations since the time of the Babylonians. When the *discriminant* $b^2 - 4ac$ is positive then these two roots are real and distinct; graphically they are where the curve $y = ax^2 + bx + c$ cuts the $x$-axis. When $b^2 - 4ac = 0$ then we have one real root and the curve just touches the $x$-axis here. But what happens when $b^2 - 4ac < 0$? Then there are no real solutions to the equation as no real number squares to give the negative $b^2 - 4ac$. From the graphical point of view the curve $y = ax^2 + bx + c$ lies entirely above or below the $x$-axis.



distinct real roots — repeated real root — complex roots

It is only comparatively recently that mathematicians have been comfortable with the roots in (1) when $b^2 - 4ac < 0$. During the Renaissance the quadratic would have been considered unsolvable or its roots would have been called *imaginary.* But if we imagine $\sqrt{-1}$ to exist, and that it behaves (adds and multiplies) much the same as other numbers, then the two roots in (1) can be written in the form

$$x = A \pm B\sqrt{-1} \tag{2}$$

where $A = -b/(2a)$ and $B = \sqrt{4ac - b^2}/(2a)$ are real numbers. But what meaning can such roots have? It was this philosophical point which pre-occupied mathematicians until the start of the 19th century when these 'imaginary' numbers started proving so useful (especially in the work of Cauchy) that the philosophical concerns ultimately became side-issues.

**Notation 1** *We shall, from now on, write*

$$i = \sqrt{-1}$$

*though many books, particularly those written for engineers and physicists, use $j$ instead. The notation $i$ was first introduced by the Swiss mathematician Leonhard Euler in 1777. (See p.8 for a brief biography.)*

**Definition 2** *A **complex number** is a number of the form $a + bi$ where $a$ and $b$ are real numbers. We will usually denote a complex number with a single letter like $z$ or $w$. If $z = a + bi$, where $a$ and $b$ are real, then $a$ is known as the **real part** of $z$ and $b$ as the **imaginary part**. We write*

$$a = \operatorname{Re} z \quad \textit{and} \quad b = \operatorname{Im} z.$$

*When we write 'let $z = a + bi$' we will implicitly assume that $a$ and $b$ are real so that $a = \operatorname{Re} z$ and $b = \operatorname{Im} z$.*

- Note that real numbers are complex numbers; a real is just a complex number with zero imaginary part.

**Notation 3** *We write $\mathbb{C}$ for the set of all complex numbers.*

One of the first major results concerning complex numbers, and which conclusively demonstrated their usefulness, was proved rigorously by Argand in 1806. From the quadratic formula (1) we know that all quadratic equations can be solved using complex numbers – what Gauss was the first to prove was the much more general result:

**Theorem 4** ***(Fundamental Theorem of Algebra)*** *Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree $n \geqslant 1$ with real (or complex) coefficients $a_k$. Then the roots of the equation $p(z) = 0$ are complex. That is, there are $n$ (not necessarily distinct) complex numbers $\gamma_1, \ldots, \gamma_n$ such that*

$$a_0 + a_1 z + \cdots + a_n z^n = a_n(z - \gamma_1)(z - \gamma_2) \cdots (z - \gamma_n).$$

*In particular the theorem shows that a degree $n$ polynomial has, counting repetitions, $n$ roots in $\mathbb{C}$.*

The proof of this theorem is far beyond the scope of this text. Note that the theorem only guarantees the *existence* of the roots of a polynomial somewhere in $\mathbb{C}$ unlike the quadratic formula which determines exactly the roots. The theorem gives no hints as to where in $\mathbb{C}$ these roots are to be found.

## 2 Their Algebra

We add, subtract, multiply and divide complex numbers in obvious sensible ways. To **add** or **subtract** complex numbers, we just add or subtract their real and imaginary parts:

$$(a + bi) + (c + di) = (a + c) + (b + d)i; \qquad (a + bi) - (c + di) = (a - c) + (b - d)i. \tag{3}$$

Note that these equations are entirely comparable with adding or subtracting two vectors in the $xy$-plane. Unlike with vectors, we can also **multiply** complex numbers by expanding the brackets in the usual fashion and remembering that $i^2 = -1$:

$$(a + bi)(c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (ad + bc)i. \tag{4}$$

To **divide** complex numbers, we firstly note that $(c + di)(c - di) = c^2 + d^2$ is real. So

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \times \frac{c - di}{c - di} = \left(\frac{ac + bd}{c^2 + d^2}\right) + \left(\frac{bc - ad}{c^2 + d^2}\right) i. \tag{5}$$

**Example 5** *Calculate, in the form $a + bi$, the following complex numbers.*

$$(1 + 3i) + (2 - 6i), \qquad (1 + 3i) - (2 - 6i), \qquad (1 + 3i)(2 - 6i), \qquad (1 + 3i)/(2 - 6i).$$

**Solution**

$$\begin{aligned}
(1 + 3i) + (2 - 6i) &= (1 + 2) + (3 + (-6))i = 3 - 3i; \\
(1 + 3i) - (2 - 6i) &= (1 - 2) + (3 - (-6))i = -1 + 9i; \\
(1 + 3i)(2 - 6i) &= 2 + 6i - 6i - 18i^2 = 2 + 18 = 20.
\end{aligned}$$

Division takes a little more care, and we need to remember to multiply both numerator and denominator by $2 + 6i$.

$$\frac{1 + 3i}{2 - 6i} = \frac{(1 + 3i)(2 + 6i)}{(2 - 6i)(2 + 6i)} = \frac{2 + 6i + 6i + 18i^2}{2^2 + 6^2} = \frac{-16 + 12i}{40} = -\frac{2}{5} + \frac{3}{10} i.$$

■

**Remark 6** *Division of complex numbers is very similar to the method of rationalizing a surd. Recall that to write a quotient such as $(2 + 3\sqrt{2})/(1 + 2\sqrt{2})$ in the form $q_1 + q_2\sqrt{2}$ where $q_1$ and $q_2$ are rational numbers, then we multiply the numerator and denominator by $1 - 2\sqrt{2}$ to get*

$$\frac{2 + 3\sqrt{2}}{1 + 2\sqrt{2}} = \frac{2 + 3\sqrt{2}}{1 + 2\sqrt{2}} \times \frac{1 - 2\sqrt{2}}{1 - 2\sqrt{2}} = \frac{2 + 3\sqrt{2} - 4\sqrt{2} - 12}{1 - 8} = \frac{10}{7} + \frac{1}{7}\sqrt{2}. \qquad ■$$

The number $c - di$,used in (5), as relating to $c + di$, has a special name and some useful properties (Proposition 18).

**Definition 7** *Let $z = a + bi$. The* ***conjugate*** *of $z$ is the number $a - bi$ and is denoted as $\bar{z}$ (or sometimes as $z^*$).*

- Note $z$ is real if and only if $z = \bar{z}$. Also

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}; \qquad \operatorname{Im} z = \frac{z - \bar{z}}{2i}. \tag{6}$$

- Note that two complex numbers are equal if and only if their real and imaginary parts correspond. That is

$$z = w \quad \text{if and only if} \quad \operatorname{Re} z = \operatorname{Re} w \quad \text{and} \quad \operatorname{Im} z = \operatorname{Im} w.$$

  This is called **comparing real and imaginary parts.**

- Note from equation (2) that when the *real* quadratic equation $ax^2+bx+c = 0$ has complex roots then these roots are conjugates of one another. More generally, if $z_0$ is a root of the polynomial $a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0$, where the coefficients $a_k$ are real, then its conjugate $\overline{z_0}$ is also a root (Corollary 19).

We present the following problem because it highlights a potential early misconception involving complex numbers – if we need a new number $i$ as the square root of $-1$, then shouldn't we need another one for the square root of $i$? However $z^2 = i$ is just another polynomial equation, with complex coefficients, and two roots in $\mathbb{C}$ are guaranteed by the fundamental theorem of algebra. They are also quite straightforward to calculate.

**Example 8** *Find all those complex numbers $z$ that satisfy $z^2 = i$.*

**Solution** Suppose that $z^2 = i$ and $z = a + bi$. Then $i = (a + bi)^2 = (a^2 - b^2) + 2abi$. Comparing real and imaginary parts in the above we obtain two simultaneous equations

$$a^2 - b^2 = 0 \text{ and } 2ab = 1.$$

So $b = \pm a$ from the first equation. Substituting $b = a$ into the second equation gives $a = b = 1/\sqrt{2}$ or $a = b = -1/\sqrt{2}$. Substituting $b = -a$ into the second equation gives $-2a^2 = 1$ which has no real solution $a$. So the two $z$ which satisfy $z^2 = i$, i.e. the two square roots of $i$, are $(1 + i)/\sqrt{2}$ and $(-1 - i)/\sqrt{2}$. ■

**Remark 9** *Notice, as with the square roots of real numbers, that the two square roots of $i$ are negative one another, and this is generally the case. However, it's typically the case that neither of the two square roots is more preferential than the other. So we reserve the notation $\sqrt{z}$, denoting a preferred choice of square root of $z$, for a positive number $z$ and define $\sqrt{z}$ to be the positive root of $z$ in such a case. Further the following rules*

$$(xy)^r = x^r y^r, \qquad x^r x^s = x^{r+s}$$

*hold true for positive $x, y$ and real $r, s$. It's unclear what these rules might mean for complex numbers, let alone whether they are true. Below is a famous fallacy of mathematics which shows care needs to be taken with powers of complex numbers.*

**Example 10** *Clearly*

$$\frac{-1}{1} = \frac{1}{-1} \qquad \text{so that} \qquad \sqrt{\frac{-1}{1}} = \sqrt{\frac{1}{-1}}$$

*and as* $\sqrt{a/b} = \sqrt{a}/\sqrt{b}$ *then*

$$\frac{i}{1} = \frac{\sqrt{-1}}{\sqrt{1}} = \frac{\sqrt{1}}{\sqrt{-1}} = \frac{1}{i}.$$

*This rearranges to give* $i^2 = 1$, *which is plainly false as* $i^2 = -1$.

**Example 11** *Use the quadratic formula to find the two solutions of* $z^2 - (3+i)z + (2+i) = 0$.

**Solution** We have $a = 1$, $b = -3 - i$ and $c = 2 + i$ (in the notation of the quadratic formula). So

$$b^2 - 4ac = (-3-i)^2 - 4 \times 1 \times (2+i) = 9 - 1 + 6i - 8 - 4i = 2i.$$

Knowing that $\sqrt{i} = \pm(1+i)/\sqrt{2}$ from Example 8, we see that the two square roots of $2i$ are $\pm(1+i)$. So

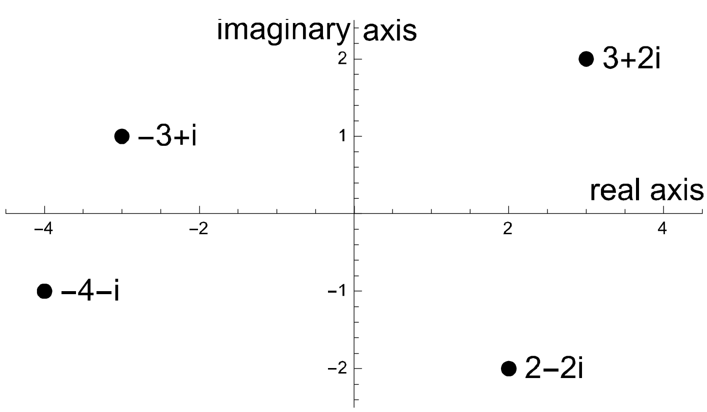
$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{(3+i) \pm \sqrt{2i}}{2} = \frac{(3+i) \pm (1+i)}{2} = \frac{4+2i}{2} \quad \text{or} \quad \frac{2}{2} = 2+i \quad \text{or} \quad 1.$$

Note that the two roots are not conjugates of one another – this need not be the case when the coefficients $a, b, c$ are not all real. ■

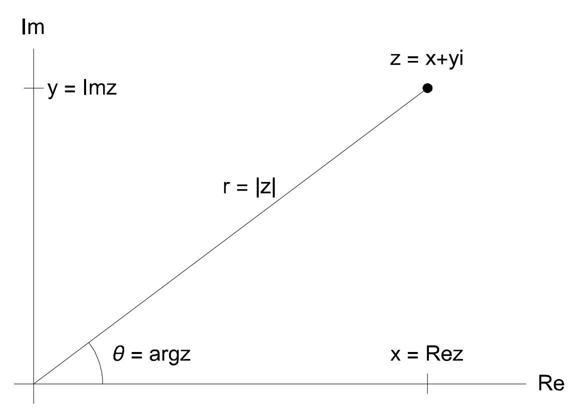
# 3 The Argand Diagram

The real numbers are often represented on the **real line**, each point of which corresponds to a unique real number. This number increases as we move from left to right along the line. The complex numbers, having two components, their real and imaginary parts, can be represented on a plane; indeed $\mathbb{C}$ is sometimes referred to as the **complex plane**, but more commonly, when we represent $\mathbb{C}$ in this manner, we call it the **Argand diagram**[1].

In the Argand diagram the point $(a, b)$ represents the complex number $a + bi$ so that the $x$-axis contains all the real numbers, and so is termed the **real axis**, and the $y$-axis contains all those complex numbers which are purely imaginary (i.e. have no real part) and so is referred to as the **imaginary axis**.



the Argand diagram



cartesian and polar co-ordinates

[1]After the Swiss mathematician Jean-Robert Argand (1768-1822), although the Norwegian mathematician Caspar Wessel (1745-1818) had previously had the same idea for representing complex numbers, but his work went unnoticed.

**Remark 12** *We can think of* $z_0 = a + bi$ *as a point in an Argand diagram, but it is often useful to think of it as a vector as well. Adding* $z_0$ *to another complex number translates that number by the vector* $(a, b)$*. That is the map* $z \mapsto z + z_0$ *represents a translation* $a$ *units to the right and* $b$ *units up in the complex plane. Note also that the conjugate* $\bar{z}$ *of a point* $z$ *is its mirror image in the real axis. So the map* $z \mapsto \bar{z}$ *represents reflection in the real axis. We shall discuss in more detail the geometry of the Argand diagram in §5.* ■

A complex number $z$ in the complex plane can be represented by Cartesian co-ordinates, its real and imaginary parts, $x$ and $y$, but equally useful is the representation of $z$ by **polar co-ordinates**, $r$ and $\theta$. If we let $r$ be the distance of $z$ from the origin and, if $z \neq 0$, we let $\theta$ be the angle that the line connecting $z$ to the origin makes with the positive real axis (see the right diagram above), then we can write

$$z = x + yi = r\cos\theta + (r\sin\theta)i. \tag{7}$$

The relations between $z$'s Cartesian and polar co-ordinates are simple:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta; \qquad\qquad r = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan\theta = y/x.$$

**Definition 13** *The number* $r$ *is called the* ***modulus*** *of* $z$ *and is written* $|z|$*. If* $z = x + yi$ *then* $|z| = \sqrt{x^2 + y^2}$.

- Note that $|z| \geqslant 0$ for all $z$ and that if $|z| = 0$ then $z = 0$. Note also

$$|\operatorname{Re} z| \leqslant |z| \quad \text{and} \quad |\operatorname{Im} z| \leqslant |z|. \tag{8}$$

**Definition 14** *The number* $\theta$ *is called the* ***argument*** *of* $z$ *and is written* $\arg z$*. If* $z = x + yi$ *then*

$$\sin\arg z = \frac{y}{\sqrt{x^2 + y^2}}; \qquad \cos\arg z = \frac{x}{\sqrt{x^2 + y^2}}; \qquad \tan\arg z = \frac{y}{x}.$$

*Note that the argument of* $0$ *is undefined. Note also that* $\arg z$ *is defined only up to multiples of* $2\pi$*. For example, the argument of* $1 + i$ *could be* $\pi/4$ *or* $9\pi/4$ *or* $-7\pi/4$ *etc.. Here* $\pi/4$ *would be the preferred choice as for definiteness we shall take the* ***principal values for argument*** *to be in the range* $0 \leqslant \theta < 2\pi$.

**Notation 15** *For ease of notation we shall write* $\operatorname{cis}\theta$ *for* $\cos\theta + i\sin\theta$*, so that complex numbers in polar form as in (7) will now be written* $z = r\operatorname{cis}\theta$.

**Remark 16** *In due course we will prove Euler's famous result,* $e^{i\theta} = \cos\theta + i\sin\theta$*, and so write the polar form as* $re^{i\theta}$ *after that.* ■

**Proposition 17** *Let* $\alpha$ *and* $\beta$ *be real numbers. Then*

$$\operatorname{cis}(\alpha + \beta) = \operatorname{cis}(\alpha)\operatorname{cis}(\beta); \qquad \overline{\operatorname{cis}\alpha} = \operatorname{cis}(-\alpha) = (\operatorname{cis}\alpha)^{-1}.$$

**Proof** Recalling the formulae for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$ we have

$$\begin{aligned}
\operatorname{cis}(\alpha)\operatorname{cis}(\beta) &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\
&= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \\
&= \cos(\alpha + \beta) + i\sin(\alpha + \beta) = \operatorname{cis}(\alpha + \beta).
\end{aligned}$$

From this we then have $\text{cis}(-\alpha)\text{cis}\,\alpha = \text{cis}\,(-\alpha+\alpha) = \text{cis}(0) = 1$, and finally

$$\overline{\text{cis}\,\alpha} = \overline{\cos\alpha + i\sin\alpha} = \cos\alpha - i\sin\alpha = \cos(-\alpha) + i\sin(-\alpha) = \text{cis}(-\alpha).$$

■

We now prove some important formulae about properties of the modulus, argument and conjugation.

**Proposition 18** *The following identities and inequalities hold for complex numbers $z, w$.*

$$|zw| = |z||w| \tag{9}$$
$$|z/w| = |z|/|w| \ \textit{if}\ w \neq 0 \tag{10}$$
$$\arg(zw) = \arg z + \arg w \ \textit{if}\ z, w \neq 0 \tag{11}$$
$$\arg(z/w) = \arg z - \arg w \ \textit{if}\ z, w \neq 0 \tag{12}$$
$$\arg \bar{z} = -\arg z \ \textit{if}\ z \neq 0 \tag{13}$$
$$z\bar{z} = |z|^2 \tag{14}$$
$$\overline{z \pm w} = \bar{z} \pm \bar{w} \tag{15}$$
$$\overline{zw} = \bar{z}\,\bar{w} \tag{16}$$
$$\overline{z/w} = \bar{z}/\bar{w} \ \textit{if}\ w \neq 0 \tag{17}$$
$$|\bar{z}| = |z| \tag{18}$$
$$|z+w| \leqslant |z| + |w| \tag{19}$$
$$||z| - |w|| \leqslant |z - w| \tag{20}$$

*(19) is known as the* ***triangle inequality***.

**Proof** We prove here a selection of these identities. The remainder are left as exercises.

- Identity (9): $|zw| = |z|\,|w|$. Let $z = a + bi$ and $w = c + di$. Then $zw = (ac - bd) + (bc + ad)i$ so that

$$\begin{aligned} |zw| &= \sqrt{(ac-bd)^2 + (bc+ad)^2} = \sqrt{a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2} \\ &= \sqrt{(a^2+b^2)(c^2+d^2)} = \sqrt{a^2+b^2}\,\sqrt{c^2+d^2} = |z|\,|w|\,. \end{aligned}$$

- Identity (11): $\arg(zw) = \arg z + \arg w$. Let $z = r\,\text{cis}\,\theta$ and $w = R\,\text{cis}\,\Theta$. Then

$$zw = (r\,\text{cis}\,\theta)(R\,\text{cis}\,\Theta) = rR\,\text{cis}\,\theta\,\text{cis}\,\Theta = rR\,\text{cis}\,(\theta + \Theta),$$

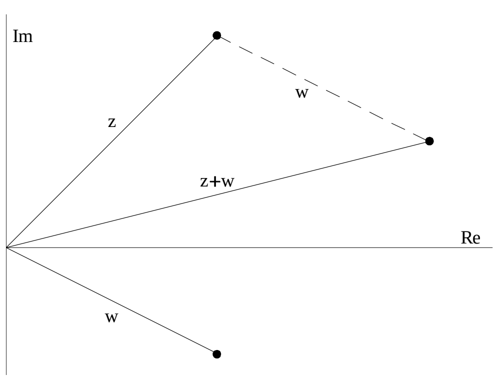
by Proposition 17. We can read off that $|zw| = rR = |z|\,|w|$, which is a second proof of (9), and also that

$$\arg(zw) = \theta + \Theta = \arg z + \arg w \qquad \text{up to multiples of } 2\pi.$$

- Identity (16): $\overline{zw} = \overline{z}\;\overline{w}$. With $z = a + bi$, $w = c + di$ then

$$\overline{zw} = \overline{(ac-bd) + (bc+ad)i} = (ac - bd) - (bc + ad)i = (a - bi)(c - di) = \overline{zw}.$$

- Identity (19): the triangle inequality $|z + w| \leqslant |z| + |w|$.




A geometric proof of this is simple, explains the inequality's name, and is represented diagrammatically to the left.

Note that the shortest distance between 0 and $z + w$ is $|z+w|$. This is the length of one side of the triangle with vertices 0, $z$, $z + w$ and so is shorter in length than the path which goes straight from 0 to $z$ then straight on to $z + w$. The total length of this second path is $|z| + |w|$.

For an algebraic proof, we note for complex numbers $z, w$ using (6), (8), (9) and (18) that

$$\frac{z\overline{w}+\overline{z}w}{2}=\operatorname{Re}(z\overline{w})\leqslant|z\overline{w}|=|z|\,|\overline{w}|=|z|\,|w|\,. \tag{21}$$

Hence, using (14), (15), (21) we have

$$\begin{aligned}
|z+w|^2 &= (z+w)\overline{(z+w)} \\
&= (z+w)(\overline{z}+\overline{w}) \\
&= z\overline{z}+z\overline{w}+\overline{z}w+w\overline{w} \\
&\leqslant |z|^2+2\,|z|\,|w|+|w|^2=(|z|+|w|)^2,
\end{aligned}$$

to give the required result. There is equality only if $z\overline{w}\geqslant 0$ so that $\operatorname{Re}(z\overline{w})=|z\overline{w}|$; if $z$ and $w$ are non-zero this is equivalent to requiring $z=\mu w$ for some $\mu>0$, meaning that $0,z,w$ are on the same half-line from 0.

■


**Leonhard Euler** (1707-1783) (pronounced "oil-er") was a prolific Swiss mathematician (over 800 papers bear his name) and indisputably the greatest mathematician of the 18th century. He made major contributions in many areas of mathematics, but especially in the study of infinite series and in particular he determined the sum of the series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$ to be $\frac{\pi^2}{6}$ (the so-called *Basel problem*). Euler's Identity $e^{i\theta}=\cos\theta+i\sin\theta$, and in particular $e^{i\pi}=-1$, can be proved using infinite series for the exponential and trigonometric functions.

But Euler's name appears widely throughout pure and applied mathematics: Euler's equation is fundamental in the Calculus of Variations (which involves problems such as showing the shortest curve between two points is a straight line); Euler's Method is the simplest numerical method for computing approximate solutions of differential equations; Euler's Theorem in number theory is a generalization of Fermat's Little Theorem; three important centres of a triangle lie on the Euler line. Euler also produced some of the first *topological* results: he famously showed that it was impossible to traverse the seven bridges in Königsberg without repetition, in fact determining more generally which networks (or *graphs*) can be traversed. He also showed for a (convex) polyhedron that $V-E+F=2$ where $V,E,F$ respectively denote the number of vertices (corners), edges, faces that the polyhedron has. Both these results depend on shape (e.g. how the points are connected) rather than geometry (e.g. the lengths of the edges).

**Corollary 19** ***(Conjugate Pairs)*** *Suppose that $z_0$ is a root of the degree $n$ polynomial $p(z)=a_nz^n+\cdots+a_1z+a_0$ where the coefficients $a_k$ are all real. Then the conjugate $\overline{z_0}$ is also a root of $p(z)$. Consequently*

$$p(z)=a_n(z-\alpha_1)\cdots(z-\alpha_r)q_1(z)\cdots q_s(z)$$

*where $\alpha_1,\ldots\alpha_r$ are the real roots of $p(z)$ and $q_1(z),\ldots,q_s(z)$ are real quadratic polynomials with conjugate complex roots. In particular, an odd degree real polynomial has at least one real root.*

**Proof** Note

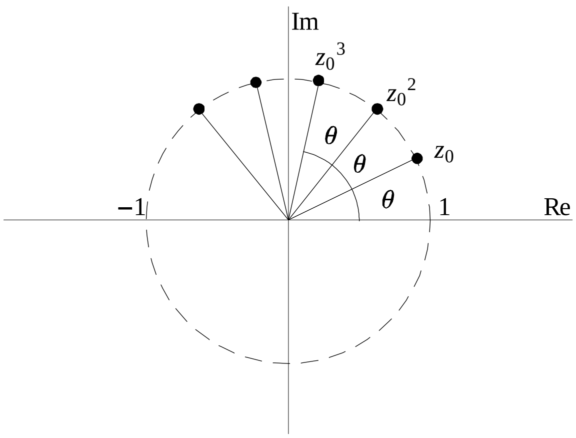
$$\begin{aligned} p(\overline{z_0}) &= a_n(\overline{z_0})^n + a_{n-1}(\overline{z_0})^{n-1} + \cdots + a_1\overline{z_0} + a_0 \\ &= \overline{a_n}(\overline{z_0})^n + \overline{a_{n-1}}(\overline{z_0})^{n-1} + \cdots + \overline{a_1 z_0} + \overline{a_0} \qquad \text{[as the coefficients } a_k \text{ are real]} \\ &= \overline{a_n z_0^n + a_{n-1}z_0^{n-1} + \cdots + a_1 z_0 + a_0} \qquad \text{[using (15) and (16)]} \\ &= \overline{p(z_0)} = \bar{0} = 0. \end{aligned}$$

The fundamental theorem of algebra tells us a polynomial's roots can be found amongst the complex numbers. So the roots of $p(z)$ are either real, call these $\alpha_1, \ldots, \alpha_r$ or come in conjugate complex pairs $\beta_1, \overline{\beta_1}, \ldots, \beta_s, \overline{\beta_s}$. Now

$$(z-\beta_k)(z-\overline{\beta_k}) = z^2 - (\beta_k + \overline{\beta_k})z + \beta_k\overline{\beta_k} = z^2 - (2\operatorname{Re}\beta_k)z + |\beta_k|^2.$$

If we denote this real quadratic as $q_k(z)$ then $p(z) = a_n(z-\alpha_1)\cdots(z-\alpha_r)q_1(z)\cdots q_s(z)$ where $n = r+2s$ from equating the degrees of the polynomials. Note that if $n$ is odd then $r \geqslant 1$ and so $p(z)$ has at least one real root. ■

# 4 Roots of Unity



powers of $z_0$

Consider the complex number $z_0 = \operatorname{cis}\theta$ where $\theta$ is some real number in the range $0 \leqslant \theta < 2\pi$. The modulus of $z_0$ is then 1 and the argument of $z_0$ is $\theta$. In Proposition 18 we proved that $|zw| = |z|\,|w|$ and for $z, w \neq 0$ that $\arg(zw) = \arg z + \arg w$. Hence

$$|z^n| = |z|^n \text{ and } \arg(z^n) = n \arg z$$

for any integer $n$ and $z \neq 0$. So the modulus of $(z_0)^n$ is $1^n = 1$, and the argument of $(z_0)^n$ is $n\theta$ up to multiples of $2\pi$. Putting this another way, we have the following famous theorem due to De Moivre.

**Theorem 20** ***(De Moivre's theorem[2])*** *For a real number $\theta$ and integer $n$ we have that*

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n,$$

*or more succinctly,* $\operatorname{cis}(n\theta) = (\operatorname{cis}\,\theta)^n$.

**Example 21** *Find expressions for* $\cos 4\theta$ *and* $\sin 4\theta$.

[2] Abraham De Moivre (1667-1754), a French protestant who fled religious persecution in France to move to England, is best remembered for this formula but he also made important contributions in probability which appeared in his *The Doctrine Of Chances* (1718).

**Solution** Writing $c$ for $\cos\theta$ and $s$ for $\sin\theta$ we have

$$\cos 4\theta + i\sin 4\theta = (c+is)^4 = c^4 + 4ic^3s - 6c^2s^2 - 4ics^3 + s^4.$$

Comparing real and imaginary parts we have

$$\cos 4\theta = c^4 - 6c^2(1-c^2) + (1-c^2)^2 = 8c^4 - 8c^2 + 1; \qquad \sin 4\theta = 4c^3s - 4c(1-c^2)s = 8c^3s - 4cs.$$

■

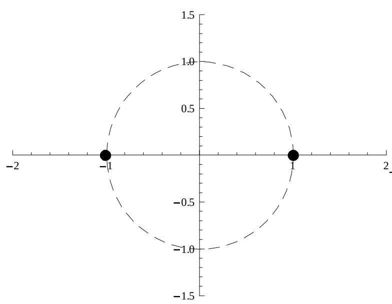
We now apply the ideas from the proof of De Moivre's theorem to the following problem, that of

finding the roots of $z^n = 1$ where $n \geqslant 1$ is an integer.

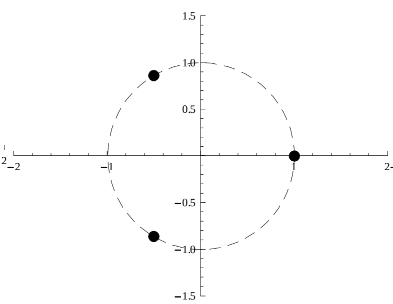
We know from the fundamental theorem of algebra that there are (counting multiplicities) $n$ solutions – these are known as the $n$**th roots of unity**. Let's first solve directly $z^n = 1$ for $n = 2, 3, 4$.

- $n = 2$: we have $0 = z^2 - 1 = (z-1)(z+1)$ and so the square roots of 1 are $\pm 1$.
- $n = 3$: $0 = z^3 - 1 = (z-1)(z^2+z+1)$. The cube roots of unity are then 1 and $(-1 \pm i\sqrt{3})/2$.
- $n = 4$: $0 = z^4 - 1 = (z^2-1)(z^2+1)$, so that the fourth roots of 1 are $\pm 1$ and $\pm i$.

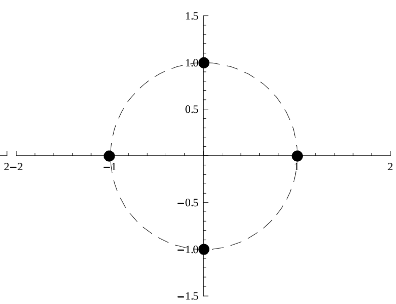
Plotting these roots on Argand diagrams we can see a pattern developing




square roots




cube roots




fourth roots

Returning to the general case, suppose that

$$z = r\,\mathrm{cis}\,\theta \qquad \text{and satisfies} \qquad z^n = 1.$$

Then $z^n$ has modulus $r^n$ and has argument $n\theta$, whilst 1 has modulus 1 and argument 0. Comparing their moduli we see $r^n = 1$ and hence $r = 1$ (as $r > 0$). Comparing arguments we see $n\theta = 0$ up to multiples of $2\pi$. That is, $n\theta = 2k\pi$ for some integer $k$ giving $\theta = 2k\pi/n$. So if $z^n = 1$ then $z$ has the form $z = \mathrm{cis}(2k\pi/n)$ where $k$ is an integer. At first glance there seems to be an infinite number of roots but, as cosine and sine have period $2\pi$, then $\mathrm{cis}(2k\pi/n)$ repeats with period $n$. Hence we have shown:

**Proposition 22** ***(Roots of Unity)*** *The nth roots of unity, that is the solutions of the equation* $z^n = 1$, *are*

$$z = \mathrm{cis}(2k\pi/n) \qquad \textit{where } k = 0, 1, 2, \ldots, n-1.$$

- When plotted these $n$th roots of unity form a regular $n$-gon inscribed within the unit circle with a vertex at 1. More generally, for $c \neq 0$, the $n$ solutions of $z^n = c$ make a regular $n$-gon inscribed in the circle $|z| = |c|^{1/n}$.

**Example 23** *Find all the solutions of the cubic $z^3 = -2 + 2i$.*

**Solution** If we write $-2+2i$ in its polar form we have $-2+2i = \sqrt{8}\,\mathrm{cis}(3\pi/4)$. So if $z^3 = -2+2i$ and $z$ has modulus $r$ and argument $\theta$ then $r^3 = \sqrt{8}$ and $3\theta = 3\pi/4$ up to multiples of $2\pi$,which gives

$$r = \sqrt{2} \quad \text{and} \quad \theta = \pi/4 + 2k\pi/3 \quad \text{for some integer } k.$$

As before we need only consider $k = 0, 1, 2$ (as other values of $k$ lead to repeats) and so the three cube roots are

$$\sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{4}\right) = 1 + i, \qquad \sqrt{2}\,\mathrm{cis}\left(\frac{11\pi}{12}\right), \qquad \sqrt{2}\,\mathrm{cis}\left(\frac{19\pi}{12}\right).$$

■

- Note that the above method generalizes naturally to solving any equation of the form $z^n = c$ where $c$ is a complex number and $n \geqslant 1$ is an integer.

# 5 Their Geometry

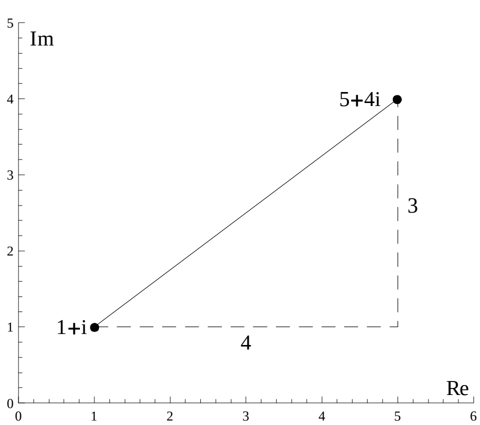
Using the modulus and argument functions, we can measure distances and angles in the Argand diagram.

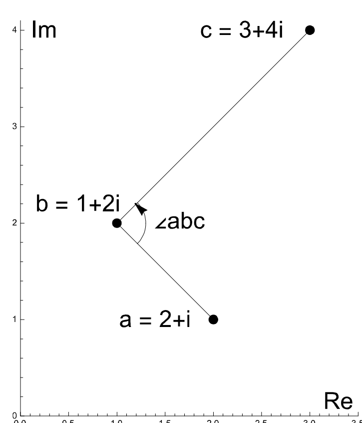
**Definition 24** *Given two complex numbers $z = z_1 + z_2 i$ and $w = w_1 + w_2 i$ then the **distance** between them is $|z - w|$. This follows from Pythagoras' Theorem as*

$$\sqrt{(z_1 - w_1)^2 + (z_2 - w_2)^2} = |(z_1 - w_1) + (z_2 - w_2)i| = |z - w|.$$

**Definition 25** *Given three complex numbers $a, b, c$ then the **angle** $\measuredangle abc$ equals*

$$\arg(c - b) - \arg(a - b) = \arg\left(\frac{c - b}{a - b}\right).$$




On the left we see the distance from $1 + i$ and $5 + 4i$ is

$$\sqrt{(5-1)^2 + (4-1)^2} = 5.$$

In the right diagram $a = 2+i$, $b = 1+2i$ and $c = 3+4i$. The angle at $b$ is

$$\arg\left(\frac{(3+4i) - (1+2i)}{(2+i) - (1+2i)}\right) = \arg(2i) = \pi/2.$$

- Notice that Definition 25 gives a signed angle in the sense that it is measured in an anti-clockwise fashion from the segment $ba$ round to the segment $bc$, and the above formula would give negative this result if the roles of $a$ and $c$ were swapped. Note also that this signed angle is defined only up to multiples of $2\pi$.

**Example 26** *Find the smaller angle $\measuredangle cab$ where $a = 1 + i$, $b = 3 + 2i$ and $c = 4 - 3i$.*

**Solution** The angle $\measuredangle cab$ is given by

$$\arg\left(\frac{b-a}{c-a}\right) = \arg\left(\frac{2+i}{3-4i}\right) = \arg\left(\frac{2+11i}{25}\right).$$

So $\tan\measuredangle cab = 11/2$ with the number $(2 + 11i)/25$ clearly in the first quadrant. (There is a solution to $\tan\theta = 11/2$ in the opposite quadrant.) Hence $\measuredangle cab = \tan^{-1}(11/2) \approx 1.39$ radians. ∎

We can use these definitions of distance and angle to describe various regions of the plane.

**Example 27** *Describe the following regions of the Argand diagram.*

- $|z - a| = r$, where $a$ is a complex number and $r > 0$.

This is clearly the locus of points at distance $r$ from $a$, i.e. the circle with centre $a$ and radius $r$. The regions $|z - a| < r$ and $|z - a| > r$ are then the interior and exterior of the circle respectively.

- $|z - a| = |z - b|$, where $a, b$ are distinct complex numbers.

This is the set of points equidistant from $a$ and $b$ and so is the perpendicular bisector of the line segment connecting $a$ and $b$. The regions $|z - a| < |z - b|$ and $|z - a| > |z - b|$ are then the half-planes either side of this line and which respectively contain $a$ and $b$.

- $\arg(z - a) = \theta$, where $a$ is a complex number and $0 \leqslant \theta < 2\pi$.

This is a half-line emanating from the point $a$ and making an angle $\theta$ with the positive real axis. Note that it doesn't include the point $a$ itself.

**Example 28** *Describe the set of points $z$ which satisfy the equation*

$$\arg\left(\frac{z+1}{z-1}\right) = \frac{\pi}{2}. \tag{22}$$

**Solution** *Method One: parametrization.* We can rewrite (22) as

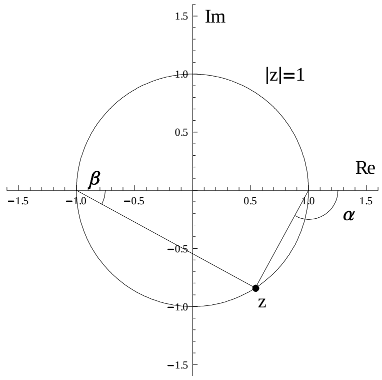
$$1 + \frac{2}{z-1} = \frac{z+1}{z-1} = it \quad \text{where } t > 0,$$

as these points $it$ are precisely those with argument $\pi/2$. Solving for $z$ we have

$$z = \frac{1+ti}{-1+ti} = \left(\frac{t^2-1}{t^2+1}\right) + i\left(\frac{-2t}{1+t^2}\right). \tag{23}$$

It is quite easy to spot, from the first expression for $z$ in (23), that the denominator and numerator have the same modulus and so $|z| = 1$. Those with knowledge of the half-angle tangent formulae may also have spotted them in (23).

So $z$ must lie on the unit circle $|z| = 1$. We see that the real part $x(t)$ in (23) varies across the range $-1 < x(t) < 1$ as $t$ varies across the positive numbers, but the imaginary part $y(t)$ is always negative. Thus (22) is the equation of the lower semicircle from $|z| = 1$, not including the points $-1$ and 1. ■




*Method Two: a geometric approach.* In the diagram on the left $\arg(z-1) = -\alpha$ and $\arg(z+1) = -\beta$ where $\alpha, \beta$ represent the actual magnitudes of the angles irrespective of their sense. By (12) we have

$$\begin{aligned}\arg((z+1)/(z-1)) = \pi/2 \quad &\iff \quad \arg(z+1) - \arg(z-1) = \pi/2 \\ &\iff \quad -\beta - (-\alpha) = \pi/2 \\ &\iff \quad \beta + (\pi - \alpha) = \pi/2\end{aligned}$$

i.e. when the angle at $z$ is a right angle, which holds when $-1$, 1 is the diameter of the circle through $-1$, 1, $z$. This is Thales' Theorem. It can be similarly shown that $\arg((z+1)/(z-1)) = -\pi/2$ on the upper semicircle.

An important class of maps of the complex plane is that of the *isometries*, i.e. the distance preserving maps.

**Definition 29** *We say a map $f$ from $\mathbb{C}$ to $\mathbb{C}$ is an* ***isometry*** *if it preserves distance, that is*

$$|f(z) - f(w)| = |z - w| \qquad \textit{for any complex } z, w.$$

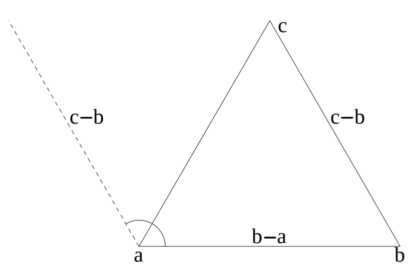
**Example 30** *Of particular note are the following three isometries.*

- $z \mapsto (\operatorname{cis}\theta)\, z$ *which is rotation anticlockwise by $\theta$ about* 0;

  *in particular $z \mapsto iz$ is rotation by a right angle anticlockwise about* 0.
- $z \mapsto z + k$ *which is translation by* $\operatorname{Re} k$ *to the right, and* $\operatorname{Im} k$ *up.*
- $z \mapsto \bar{z}$ *which is reflection in the real axis.*

**Example 31** *The triangle abc in $\mathbb{C}$ (with the vertices taken in anticlockwise order) is equilateral if and only if $a + \omega b + \omega^2 c = 0$ where $\omega = \operatorname{cis}(2\pi/3)$ is a cube root of unity.*

**Solution** Note first that $1 + \omega + \omega^2 = 0$. The triangle $abc$ is equilateral if and only if $c - b$ is the side $b - a$ rotated through $2\pi/3$ anticlockwise – i.e. if and only if




$$\begin{aligned}&c - b = \omega(b - a) \\ \iff\ &\omega a - (1+\omega)b + c = 0 \\ \iff\ &\omega a + \omega^2 b + c = 0 \\ \iff\ &a + \omega b + \omega^2 c = 0.\end{aligned}$$

■

**Proposition 32** *(**Isometries of** $\mathbb{C}$) Let $f$ be an isometry of $\mathbb{C}$.*
*If $f$ is **orientation-preserving** then $f$ has the form $f(z) = az + b$ for some complex $a, b$ with $|a| = 1$. These maps are all **rotations**. If $f$ **reverses orientation** then $f$ has the form $f(z) = a\bar{z} + b$ for some complex $a, b$ with $|a| = 1$. These include, but are not limited to, reflections.*

**Proof** This result (and more generally all the isometries of Euclidean space) are classified the *Geometry* course this term. ■

**Example 33** *Express in the form $f(z) = a\bar{z} + b$ reflection in the line $x + y = 1$.*

**Solution** Knowing from Proposition 32 that the reflection has the form $f(z) = a\bar{z} + b$ we can find $a$ and $b$ by considering where two points are mapped. As 1 and $i$ both lie on the line of reflection then they are both fixed. So

$$1 = a\bar{1} + b = a1 + b; \qquad i = a\bar{i} + b = -ai + b.$$

Substituting $b = 1 - a$ into the second equation we find $a = -i$ and $b = 1 + i$. Hence $f(z) = -i\bar{z} + 1 + i$.

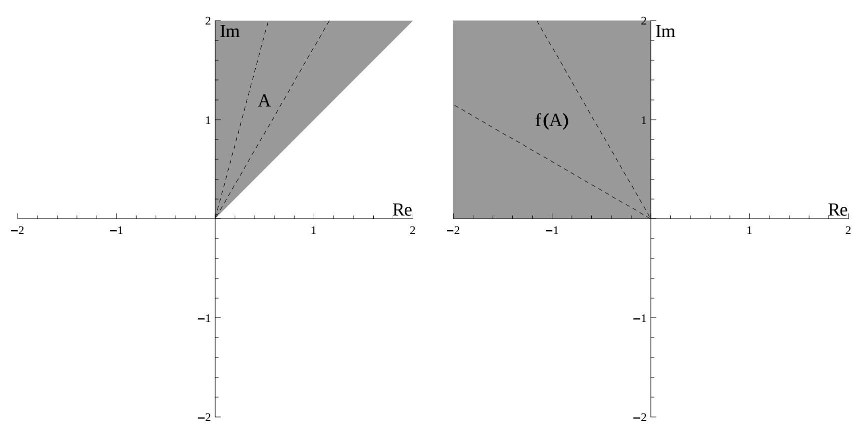
■

There are other important maps of the complex plane which aren't isometries. Here are some examples which involve determining the images of certain regions. The methods used involve either relying on the fact that the map in question has an inverse or parametrizing the region with Cartesian or polar co-ordinates.

**Example 34** *In the two cases below, find the image in $\mathbb{C}$ of the given subset under the given map.*

- *$A$ is the region $\operatorname{Im} z > \operatorname{Re} z > 0$ and $f(z) = z^2$.*

  *A general point in $A$ can be written in the form $z = r\text{cis}\theta$ where $\pi/4 < \theta < \pi/2$ and $r > 0$.*
  *De Moivre's Theorem gives us that $z^2 = r^2\text{cis}2\theta$ which has an argument of $2\theta$ in the range $\pi/2 < 2\theta < \pi$ and independently a modulus of $r^2$ in the range $r^2 > 0$.*
  *So $f(A)$ is the second quadrant.*

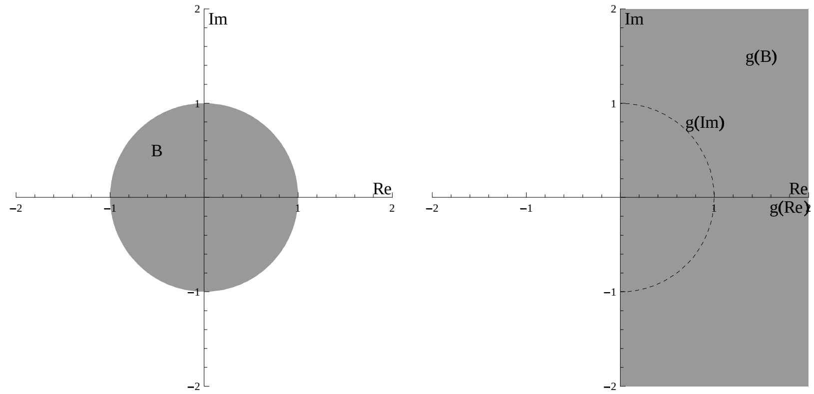
  
  

- *$B$ is the unit disc $|z| < 1$ and $g(z) = \frac{1+z}{1-z}$.*

  *The map $g$ maps from the set $\{z \in \mathbb{C} \colon z \neq 1\}$ to $\{z \in \mathbb{C} \colon z \neq -1\}$, with an inverse $g^{-1}(z) = (z-1)/(z+1)$. To see this we note*

  $$w = (1+z)/(1-z) \quad \Leftrightarrow \quad w - wz = 1 + z \quad \Leftrightarrow \quad w - 1 = z(1+w) \quad \Leftrightarrow \quad z = (w-1)/(w+1).$$

*Note $z$ is in $g(B)$*
*$\Leftrightarrow g^{-1}(z)$ is in $B$*
*$\Leftrightarrow |(z-1)/(z+1)| < 1$*
*$\Leftrightarrow |z-1| < |z+1|$*
*which is the half-plane with points closer to 1 than $-1$, (Example 27) or equivalently the half-plane $Rez > 0$.*




We now prove a selection of basic geometric facts. Here is a reminder of some identities which will prove useful.

$$\operatorname{Re} z = \frac{z+\bar{z}}{2}; \qquad z\bar{z} = |z|^2; \qquad \cos\arg z = \frac{\operatorname{Re} z}{|z|}; \qquad \sin\arg z = \frac{\operatorname{Im} z}{|z|}. \tag{24}$$

**Theorem 35** ***(Cosine Rule)****. Let $ABC$ be a triangle with angles $\hat{A}, \hat{B}, \hat{C}$. Then*

$$|BC|^2 = |AB|^2 + |AC|^2 - 2\,|AB|\,|AC|\cos\hat{A}. \tag{25}$$

**Proof** We can introduce co-ordinates in the plane so that $A$ is at the origin and $B$ is at 1. Let $C$ be at the point $z$. So in terms of our co-ordinates: $|AB| = 1$, $|BC| = |z-1|$, $|AC| = |z|$, $\hat{A} = \arg z$. Hence, by (24),

$$\begin{array}{rclcl} \text{RHS of (25)} & = & 1 + |z|^2 - 2\,|z|\cos\arg z & = & 1 + z\bar{z} - 2\,|z| \times \frac{\operatorname{Re} z}{|z|} \\ & = & 1 + z\bar{z} - 2 \times (z+\bar{z})/2 & = & 1 + z\bar{z} - z - \bar{z} \\ & = & (z-1)(\bar{z}-1) & = & \text{LHS of (25).} \end{array}$$

■

Whilst a simple enough theorem to prove in a geometric manner, Thales' Theorem is arrived at nicely with the use of complex numbers, and the theorem's converse also comes naturally using algebra.

**Theorem 36** ***(Thales[3])*** *Let $A, B$ be distinct points in the plane and let $C$ be the circle with $AB$ as a diameter. Then the point $P$ lies on $C$ if and only if the angle $\measuredangle APB$ is a right angle.*

**Proof** Without loss of generality we may assume that $A, B, P$ have complex co-ordinates $-1, 1, z$ respectively so that the circle $C$ is the circle $|z| = 1$. By the cosine rule, $\measuredangle APB$ is a right angle if and only if

$$2^2 = |z-1|^2 + |z+1|^2 \iff 4 = 2z\bar{z} + 2 \iff |z| = 1.$$

■

[3]After the Greek mathematician and philosopher Thales (c.624BC–c.547BC). Thales might reasonably be considered the first Western philosopher – in that he sought to explain phenomena without reference to mythology – though it is unclear to what extent he was able to prove the various theorems of geometry that are attributed to him. At that time it is more likely that these results were appreciated as useful rules of thumb employed by engineers.

## 6 Euler's Identity

The exponential function $\exp z$ and trigonometric functions, $\sin z$ and $\cos z$, can be defined for complex numbers by the power series

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \qquad \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.$$

A rigorous treatment of these functions appears in the *Analysis I* course this term, but we will somewhat informally discuss properties of these functions.

- These power series converge for all complex numbers $z$.
- Each function is differentiable and

$$\exp' z = \exp z, \qquad \cos' z = -\sin z, \qquad \sin' z = \cos z.$$

- Cosine is an even function and sine is an odd function. Both have period $2\pi$.
- The identity

$$\exp(z+w) = \exp z \times \exp w$$

  holds for all complex numbers $z, w$.
- The identity

$$\cos^2 z + \sin^2 z = 1$$

  holds for all complex numbers $z$. *This does not imply that* $\sin z$ and $\cos z$ are bounded functions. In fact they are unbounded functions.
- **Euler's Identity**

$$\exp z = \cos z + i \sin z$$

  holds for all complex numbers $z$. Consequently $\exp z$ is periodic with period $2\pi i$.

**Proof** Note that the sequence $i^k$, where $k \geqslant 0$, equals $1, i, -1, -i.1.i, -1, -i, \ldots$ repeating with period 4. Hence

$$\begin{aligned}
\exp(iz) &= \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \\
&= \sum_{k=0}^{\infty} \frac{(iz)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(iz)^{2k+1}}{(2k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} \\
&= \cos z + i \sin z.
\end{aligned}$$

■

- The identities

$$\cos z = \frac{\exp(iz) + \exp(-iz)}{2}, \qquad \sin z = \frac{\exp(iz) - \exp(-iz)}{2i}$$

  hold for general complex $z$.

- When $\theta$ is real then $\cos\theta = \mathrm{Re}(\exp i\theta)$ and $\sin\theta = \mathrm{Im}(\exp i\theta)$. This is not true for general complex $z$.

- It can be shown that $\exp\colon \mathbb{R} \to (0,\infty)$ defines a bijection with differentiable inverse $\log\colon (0,\infty) \to \mathbb{R}$, called the *natural logarithm* (and also denoted ln).

- Given positive $a > 0$ and real $x$ we can define

$$a^x = \exp\left(x \log a\right).$$

  Note, with this definition, $e^x = \exp x$ for real $x$.

- For non-zero complex $z$ the equation $\exp w = z$ has a solution $w$. In fact, as exp is periodic, $w = \log z$ can take infinitely many values.

- So for complex numbers $a, b$, the definition

$$a^b = \exp(b \log a)$$

  might take infinitely many values, though only finitely many values if $b$ is rational and just one value if $b$ is an integer.

# 7 Further topics – off syllabus

Below are discussed some further topics relating to complex numbers. **Note that the topics are all beyond the exam syllabus of this short course**, but may be of interest, in particular because they include some details on how to solve cubic equations – the problem which led to the initial use of complex numbers.

## 7.1 Solving Cubic Equations

A cubic equation is one of the form $Az^3 + Bz^2 + Cz + D = 0$ where $A \neq 0$. By dividing through by $A$ if necessary we can assume $A = 1$. Further we can make a substitution to simplify a cubic equation.

**Proposition 37** *The substitution $z = Z - a/3$ turns the equation $z^3 + az^2 + bz + c = 0$ into one of the form $Z^3 + mZ + n = 0$.*

**Proof** The proof is a matter of elementary algebra. ■

This leaves us in a position to introduce Cardano's method for solving such cubics.

**Algorithm 38** ***(Cardano's Method for Solving Cubics)*** *Consider the cubic equation*

$$z^3 + mz + n = 0, \tag{26}$$

*where $m$ and $n$ are real numbers. Let $D$ be such that $D^2 = m^3/27 + n^2/4$. (As $D^2$ is real then $D$ is a real or purely imaginary number.) We then define $t$ and $u$ by*

$$t = -n/2 + D \qquad \text{and} \qquad u = n/2 + D,$$

*and let $T$ and $U$ respectively be cube roots of $t$ and $u$. It follows that $tu$ is real, and if $T$ and $U$ are chosen appropriately, so that $TU = m/3$, then $z = T - U$ is a solution of the original cubic equation.*

**Remark 39** *The* ***discriminant*** *of a cubic is* $\Delta = (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$ *where* $\alpha, \beta, \gamma$ *are the roots of the cubic. It can be shown that* $\Delta = -4m^3 - 27n^2$ *for (26). So* $D^2 = -\Delta/108$*. For* $m \leqslant 0$*, it is the case that (26) has three, two or one real root when* $\Delta < 0$, $\Delta = 0$ *or* $\Delta > 0$. ■

**Proof** Putting $z = T - U$ into the LHS of (26) we get

$$\begin{aligned}
\text{LHS} &= (T-U)^3 + m(T-U) + n \\
&= T^3 - 3UT^2 + 3U^2T - U^3 + mT - mU + n \\
&= t - u + (m - 3UT)(T-U) + n \\
&= (-n/2 + D) - (n/2 + D) + (m - 3UT)(T-U) + n \\
&= (m - 3TU)(T-U).
\end{aligned}$$

Now

$$(TU)^3 = tu = (-n/2 + D)(n/2 + D) = -n^2/4 + D^2 = m^3/27 = (m/3)^3.$$

The cube roots of $tu = (m/3)^3$ are $m/3$, $\omega m/3$ and $\omega^2 m/3$, where $\omega \neq 1$ is a cube root of unity. If $T$ is a cube root of $t$ and $U_0$ is a cube root of $u$, then $TU_0$ is a cube root of $tu$. The other cubes roots of $u$ are $\omega U_0$ and $\omega^2 U_0$; if we choose $U$ appropriately from $U_0$, $\omega U_0$, $\omega^2 U_0$ we have $TU = m/3$. Thus, with these $T$ and $U$, we've shown $z = T - U$ satisfies $z^3 + mz + n = (m - 3TU)(T - U) = 0$. Note that if $T$ and $U$ are chosen carefully (so that $TU = m/3$) then the other two roots are

$$\omega T - \omega^2 U, \qquad \omega^2 T - \omega U$$

as $\omega T, \omega^2 T$ and $\omega U, \omega^2 U$ are respectively cube roots of $t$ and $u$ and $(\omega T)(\omega^2 U) = (\omega^2 T)(\omega U) = TU = m/3$. ■

**Solving The Cubic** The story of how the cubic equation came to be solved is a colourful one, but is also a window on a time when the habits of the mathematical community were very different from today. The three roots of the cubic equation $z^3 + mz + n = 0$ appear in (27) below, where $k$ equals 0, 1 or 2 and $\omega$ is a cube root of unity other than 1. The formula is considerably more complicated than the quadratic formula. A further complication, for the time, came with the unease 16th century mathematicians had with negative numbers, let alone complex ones. So a cubic equation of the form $z^3 + mz = n$ would only have been considered meaningful if $m$ and $n$ were positive and only positive roots would have been of interest; an equation of the form $z^3 = mz+n$ (with $m$ and $n$ again positive) would have been considered an entirely different type of cubic equation. The first person to solve equations of the form $z^3 + mz = n$ was Scipione Del Ferro (1456-1526) who was a mathematician at Bologna University. Before he died, Del Ferro shared, in secret, his solution with a student, Fiore.
Armed with this knowledge, which he thought himself to be the sole possessor of, Fiore rather fancied that he could make a reputation for himself as a teacher of mathematics and challenged the Venetian mathematician, Nicolo Tartaglia (1499-1557), to a competition to solve cubic equations which they would set one another. Unfortunately for Fiore, Tartaglia was more than up to the task as he proved both able to solve equations of the form $z^3+mz = n$ but could also pose and solve equations of the form $z^3 = mz+n$ which were beyond Fiore's ken. Girolamo Cardano (1501-1576, pictured left), heard of Tartaglia's victory and – after some considerable effort on his part – eventually convinced Tartaglia to share his methods, but only after Cardano had made an oath to also keep them secret which Cardano largely did. But the situation became yet more confused when Cardano shared the solution of the cubic with his student and secretary Lodovico Ferrari (1522-1565) who subsequently used it to solve quartic (degree four) equations: both the solutions to the cubic and quartic were now held in secrecy by Cardano's oath to Tartaglia. The situation eventually resolved itself when Cardano heard that Fiore had gained his solution from Del Ferro. Cardano then travelled to Bologna to see Del Ferro's papers and was able to see that his original method had been the same as Tartaglia's. Feeling unburdened of the responsibilities of his oath, Cardano published the solutions to the cubic and quartic in his *Ars Magna* (1545) citing priority with Del Ferro and citing Tartaglia as an independent later solver of the cubic. Despite Cardano acknowledging the two as the original solvers, the method of solution is usually referred to as Cardano's method.

$$z_k = \omega^k \sqrt[3]{-n/2 + \sqrt{n^2/4 + m^3/27}} + \omega^{2k} \sqrt[3]{-n/2 - \sqrt{n^2/4 + m^3/27}} \tag{27}$$

**Example 40** *Find the three roots of* $z^3 - 12z + 8 = 0$.

**Solution** We have $m = -12$ and $n = 8$. As $D^2 = m^3/27 + n^2/4 = -48$ then we can take $D = 4\sqrt{3}i$. We set

$$t = -n/2 + D = -4 + 4\sqrt{3}i; \qquad u = n/2 + D = 4 + 4\sqrt{3}i.$$

Now $T$ is a cube root of $t$. As $|t| = 8$ and $\arg t = 2\pi/3$, then $t$'s three cube roots are

$$T_1 = 2\,\mathrm{cis}\,(2\pi/9); \qquad T_2 = 2\,\mathrm{cis}\,(8\pi/9); \qquad T_3 = 2\,\mathrm{cis}\,(14\pi/9).$$

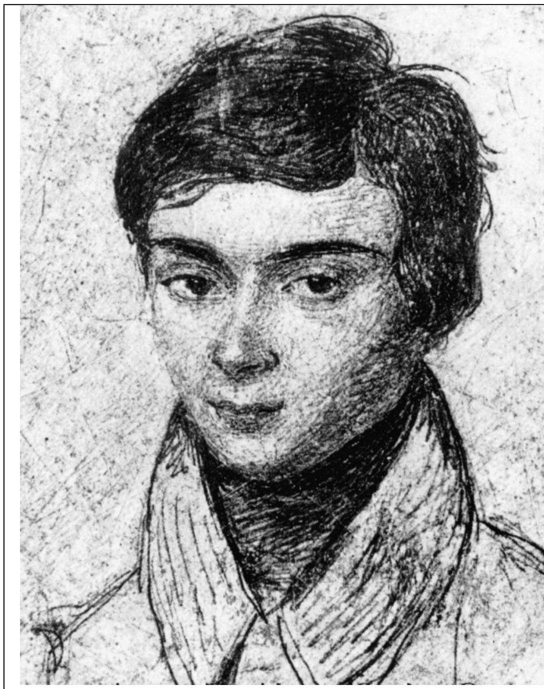
Similarly $|u| = 8$ and $\arg u = \pi/3$ giving

$$U_1 = 2\,\mathrm{cis}\,(7\pi/9); \qquad U_2 = 2\,\mathrm{cis}\,(\pi/9); \qquad U_3 = 2\,\mathrm{cis}\,(13\pi/9)$$

respectively, where the $U_i$ are chosen so that $T_i U_i = m/3 = -4$. Hence the three roots of $z^3 - 12z + 8 = 0$ are

$$T_1 - U_1 = 2(\operatorname{cis}(2\pi/9) - \operatorname{cis}(7\pi/9)) = 4\cos(2\pi/9);$$
$$T_2 - U_2 = 2(\operatorname{cis}(8\pi/9) - \operatorname{cis}(\pi/9)) = 4\cos(8\pi/9);$$
$$T_3 - U_3 = 2(\operatorname{cis}(14\pi/9) - \operatorname{cis}(13\pi/9)) = 4\cos(14\pi/9).$$

Numerically these roots are 3.06418, −3.75877, and 0.69459 to 5 decimal places. Note that the three roots are all real even though none of $t, u, T_i, U_i$ is real. ■



**The Quintic** Despite the successes in the 16th century in solving cubics and quartics, more than two centuries passed before progress was made with the quintic. During that time, perhaps the most insightful view of the mathematics to unfold came in 1770 when Joseph-Louis Lagrange (1736-1813) rederived the solutions for quadratic, cubic and quartic equations through an analysis of permutations of their roots; at that time though it seemed that the mathematical community largely expected quintic equations to be likewise solvable and Paolo Ruffini (1765-1822), Niels Henrik Abel (1802-1829) and Evariste Galois (1811-1832) encountered various difficulties convincing contemporaries of the insolvability of the general quintic. Insolvability, here, means that there is no formula for the solutions of a general quintic polynomial that involves only addition, subtraction, multiplication, division and taking roots, as is the case for polynomials of degree 4 or less; that is to say the general quintic is not *solvable by radicals*. Ruffini gave an incomplete proof of this fact in 1799 but, despite writing to Lagrange several times, his work failed to interest other mathematicians even to the extent of the gap in the proof being pointed out. Ruffini's work contains many important ideas about *permutation groups* and it may have been the novelty of his work that led to it going unheeded. It was to be Abel, instead, who provided the first complete proof in 1824. But it was Galois (pictured left), around 1830, who saw deepest into the problem. Abel had shown that certain quintics are insolvable by radicals, though some (e.g. $x^5 = 0$) clearly are. Galois' genius was to give a criterion showing precisely which polynomials (of any degree) are solvable by radicals. He associated with a polynomial an algebraic structure now known as its *Galois group*. If the roots of the polynomial could be extracted through a succession of everyday algebra and taking roots, then its Galois group could correspondingly be "built up" in a certain technical sense – the group would be what is now known as *solvable*. And the Galois group of the general quintic is not solvable. Galois' legacy in mathematics is enormous and many important ideas relating to *groups* and *fields* – two fundamental structures in modern algebra – can be traced back to his work; in fact finite fields are commonly called *Galois fields* after him. However Galois, too, found it difficult to share his ideas during his lifetime despite three times submitting his work to the Paris Academy. It was only thanks to Joseph Liouville (1809-1882) that Galois' work posthumously reached a wider audience, though this wasn't until 1846.

**Example 41** *By making the substitution $z = k\cos\theta$, and recalling that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, solve the equation $z^3 - 2z - 4 = 0$.*

**Solution** We need that $k^3 : 2k = 4 : 3$ so that $k = \sqrt{8/3}$. The subsitution then gives $\cos 3\theta = 3\sqrt{3/2}$. Clearly this does not have real solutions $\theta$, but using the identity $\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$, we see that

$$\cos x \cosh y = 3\sqrt{3/2}, \qquad \sin x \sinh y = 0.$$

As $y \neq 0$ then we have that $x = n\pi$ where $n = 0, 2, 4$ (with other $n$ giving repetitions or contradictions) and $y = \cosh^{-1}\left(3\sqrt{3/2}\right)$. We then have the three roots

$$\begin{aligned}
z_1 &= \sqrt{\frac{8}{3}} \cos\left(\frac{iy}{3}\right) = \sqrt{\frac{8}{3}} \cosh\left(\frac{y}{3}\right) = 2; \\
z_2 &= \sqrt{\frac{8}{3}} \cos\left(\frac{2\pi}{3} + \frac{iy}{3}\right) = \sqrt{\frac{8}{3}} \left[\frac{-1}{2} \cosh\left(\frac{y}{3}\right) - i\frac{\sqrt{3}}{2} \sinh\left(\frac{y}{3}\right)\right] = -1 - i; \\
z_3 &= \sqrt{\frac{8}{3}} \cos\left(\frac{4\pi}{3} + \frac{iy}{3}\right) = \sqrt{\frac{8}{3}} \left[\frac{-1}{2} \cosh\left(\frac{y}{3}\right) + i\frac{\sqrt{3}}{2} \sinh\left(\frac{y}{3}\right)\right] = -1 + i.
\end{aligned}$$

■

## 7.2 The Extended Complex Plane

There is a useful general equation which encompasses both circles and lines. This fact is less surprising with an appreciation of the extended complex plane, circlines and the Riemann sphere.

**Proposition 42** ***(Circles and Lines in** $\mathbb{C}$**)*** *Let $A$ and $C$ be real and $B$ complex, with $A, B$ not both zero. Then*

$$Az\bar{z} + B\bar{z} + \bar{B}z + C = 0, \tag{28}$$

*represents: (a) a line in direction $iB$ when $A = 0$;*
*(b) a circle, if $A \neq 0$ and $|B|^2 \geqslant AC$, with centre $-B/A$ and radius $|A|^{-1}\sqrt{|B|^2 - AC}$;*
*and otherwise has no solutions. Moreover every circle and line can be represented in the form of (28).*

**Proof** If $A \neq 0$ then we can rearrange (28) as

$$\begin{aligned}
z\bar{z} + (B/A)\bar{z} + (\bar{B}/A)z + C/A &= 0, \\
(z + B/A)\overline{(z + B/A)} &= B\bar{B}/A^2 - C/A, \qquad \text{[as } A \text{ is real and using (15)]} \\
|z + B/A|^2 &= (|B|^2 - AC)/A^2, \qquad \text{[using (14)].}
\end{aligned}$$

If $|B|^2 \geqslant AC$ then this is a circle with centre $-B/A$ and radius $|A|^{-1}\sqrt{|B|^2 - AC}$ and otherwise there are no solutions to (28). Conversely, note that the equation of a general circle is $|z - a| = r$ where $a$ is a complex number and $r \geqslant 0$. This can be rearranged using (14) as

$$z\bar{z} - a\bar{z} - \bar{a}z + (|a|^2 - r^2) = 0$$

which is in the form of (28) with $A = 1$, $B = -a$ and $C = |a|^2 - r^2$.

If $A = 0$ then we have the equation $B\bar{z} + \bar{B}z + C = 0$. If we write $B = u + iv$ and $z = x + yi$ then

$$(u + iv)(x - yi) + (u - vi)(x + yi) + C = 0 \quad \text{which rearranges to} \quad 2ux + 2vy + C = 0, \tag{29}$$

which is the equation of a line. Moreover we see that every line appears in this form by choosing $u, v, C$ appropriately. The line (29) is parallel to the vector $(v, -u)$ or equivalently $v - ui = i(u + iv) = iB$. ■

The fact that lines and circles can both be expressed in this way is not coincidental. Using the form (28) above one can see that the map $z \mapsto 1/z$ maps the set of circles and lines to the set of circles and lines. Specifically

- a line through the origin maps to a line through the origin;
- a circle through the origin maps to a line not through the origin (and vice versa);
- a circle not through the origin maps to a circle not through the origin.

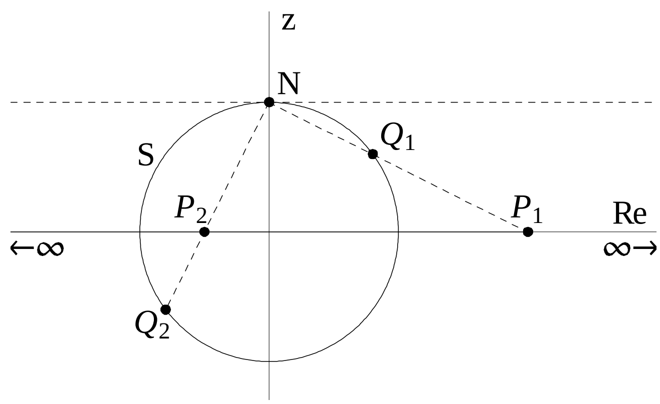
Note that the the map $z \mapsto 1/z$ isn't particularly well-defined on $\mathbb{C}$. It isn't defined at $z = 0$. But note that points near 0 map to distant points, with great modulus and any argument. So it makes sense to think of the image of 0 as a 'infinity' which is 'out there' in all directions (as opposed to $\pm\infty$ which extend the real line). The **extended complex plane** is then $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ and the map $z \mapsto 1/z$ is a bijection of the extended complex plane. Viewed this way, lines can be thought of as 'circles' that pass through infinity and the term 'circline' is used to describe both lines and circles.

The extended complex plane can be naturally identified with a sphere as follows. The sphere $S$, with centre $(0,0,0)$ and radius one, has equation $x^2 + y^2 + z^2 = 1$.

Thinking of $\mathbb{C}$ as the $xy$-plane, every complex number $P = X + Yi$ can be identified with a point $Q$ on $S$ by drawing a line from $(X, Y, 0)$ in the $xy$-plane to the sphere's north pole $N = (0, 0, 1)$; this line intersects the sphere at two points $Q$ and $N$. We define a map $f$ from $\mathbb{C}$ to the sphere $S$ by setting $f(P) = Q$. (i) The image point $f(X + Yi)$ equals

$$\left(\frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{X^2+Y^2-1}{1+X^2+Y^2}\right).$$

(ii) The image of $f$ equals all of $S$ except $N$.

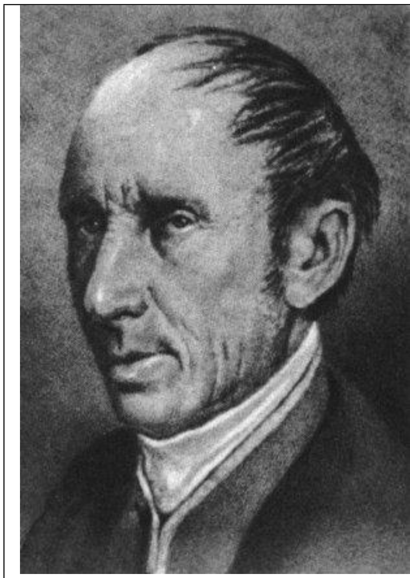



(iii) The inverse map $\pi = f^{-1}$ is called **stereographic projection**. For $(x, y, z) \neq N$, then

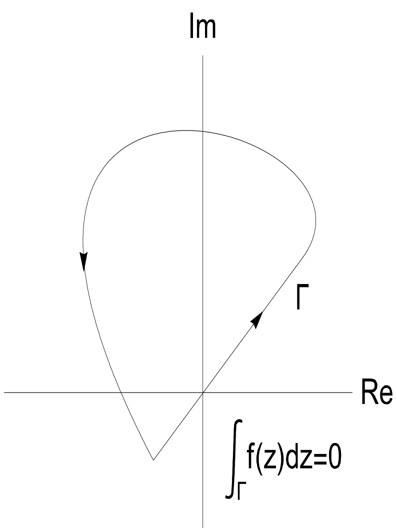
$$\pi(x, y, z) = \frac{x + yi}{1 - z}.$$

- Note that $\pi$ maps points near $N$ to complex numbers with large moduli. It is consequently natural to identify a single 'infinite' point, written $\infty$, with the north pole $N$.
- If we identify $\mathbb{C}_\infty$ via stereographic projection with the sphere $S$, then $S$ is known as the **Riemann sphere**.
- Under stereographic projection, lines in $\mathbb{C}$ correspond to circles on $S$ which pass through $N$, and circles in $\mathbb{C}$ correspond to circles on $S$ which don't pass through $N$.
- Viewed on the Riemann sphere, the map $z \mapsto 1/z$ simply turns the sphere upside down, rotating it about the $x$-axis.

## 7.3 Complex Analysis



**Complex analysis** is a central topic in mathematics and widely regarded as both one of the most aesthetic and harmonius but also highly applicable. Most undergraduates meet analysis for the first time at university in real analysis courses (pure calculus essentially) containing results they might have otherwise considered obvious, or "known" from school, treated in a seemingly pedantic manner and with a focus on the pathological. By and large, a first course in real analysis contains more you-can't-do-that moments than most undergraduates welcome. By comparison, complex analysis is the can-do sibling of real analysis. The main subject of study is the set of *holomorphic functions*, essentially consisting of those complex-valued functions that have a derivative. It turns out that being differentiable in this complex sense is a more demanding requirement; consequently it is possible to build up a richer theory about these functions – for example, holomorphic functions have in fact derivatives of all orders and are defined locally by a Taylor series. But the set of holomorphic functions is still wide enough to include all the important functions you are likely to be interested in. Complex analysis, though, is more than simply another version of real analysis in which the results are cleaner and more positive. The very nature of the subject is different and has a much more *topological* flavour. The first major result of complex analysis is *Cauchy's Theorem*, after Augustin-Louis Cauchy (1789-1857), pictured left. In fact you will see Cauchy's name populating much of a complex analysis course as he almost single-handedly developed the subject. This theorem states that the integral around any closed curve $\Gamma$ (essentially a loop) in the complex plane of a holomorphic function $f(z)$ is zero. If the integrand isn't holomorphic everywhere – like $1/z$, where only $z = 0$ is a problem – then the integral now only depends on whether the loop encloses 0. If not, the answer is still zero; if it wraps around the origin once anticlockwise the answer is $2\pi i$; twice and the answer is $4\pi i$. You might reasonably raise the issue that a real integral often represents something, well, real – e.g. area or arc-length: what meaning can a complex answer have? But by taking the real or imaginary parts of such an answer, real integrals can be determined, often ones that are difficult to calculate by real methods alone. By such means a first course in complex analysis would likely have you determining integrals and infinite sums such as




$$\int_0^\infty \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}; \qquad \int_0^{2\pi} \frac{\mathrm{d}t}{2+\sin t} = \frac{2\pi}{\sqrt{3}}; \qquad \int_0^\infty \frac{\mathrm{d}x}{1+x^n} = \frac{\pi}{n}\mathrm{csc}\left(\frac{\pi}{n}\right); \qquad \sum_1^\infty \frac{1}{n^4} = \frac{\pi^4}{90}.$$

It is usually via complex analysis that one first sees a proof of the Fundamental Theorem of Algebra (Theorem 4). Complex analysis is also widely applicable. In pure mathematics it leads naturally into the study of Riemann surfaces (for example, $x^2+y^2 = 1$, rather than defining a curve in the real plane $\mathbb{R}^2$, defines a two-dimensional surface in four-dimensional $\mathbb{C}^2$) and aspects of algebraic geometry; also it is important in analytic number theory via Riemann's Zeta Function. Complex methods are widely used in the solution of partial differential equations with the links between Laplace's equation and holomorphic functions proving very useful in the study of ideal (inviscid) fluid flow in two dimensions.