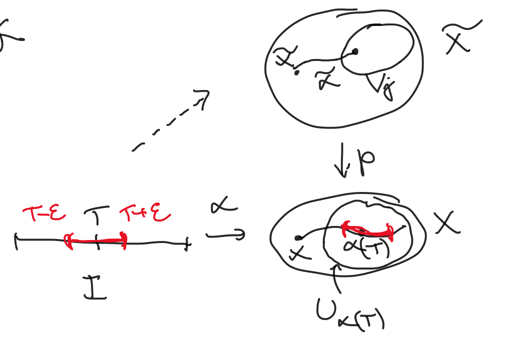
Section VI.1: Path and homotopy lifting

Theorem VI.14 (Path lifting) $p: \tilde{X} \to X$, $\alpha: I \to X$, $\alpha(0) = x$.
Then $\forall\, \tilde{x} \in p^{-1}(x)$ $\exists$ lift $\tilde{\alpha}: I \to \tilde{X}$ of α s.t. $\tilde{\alpha}(0) = \tilde{x}$.

Rem $\tilde{\alpha}$ unique by Theorem VI.13.

Proof



$A := \{t \in I : \exists$ lift of $\alpha|_{[0,t]}$ starting at $\tilde{x}\}$

$0 \in A \Rightarrow A \neq \emptyset$.

$T := \sup(A)$

Let $\varepsilon > 0$ s.t. $\alpha((\varepsilon - T, \varepsilon + T) \cap I) \subseteq U_{\alpha(T)}$

$t := \max\{0, T - \frac{\varepsilon}{2}\}$

$\tilde{\alpha}: [0,t] \to \tilde{X}$ lift of $\alpha|_{[0,t]}$ from $\tilde{x}$

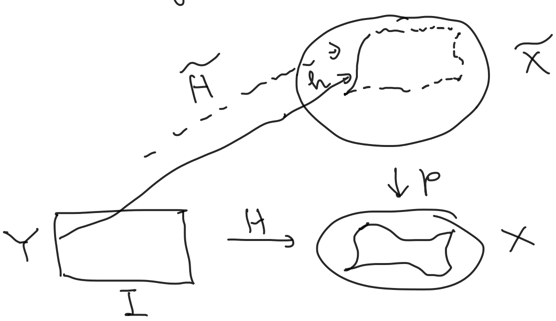
$\tilde{\alpha}(t) \in V_j \subseteq p^{-1}(U_{\alpha(t)})$

$p|_{V_j}: V_j \xrightarrow{\sim} U_{\alpha(t)}$ homeo.

Extend $\tilde{\alpha}$ to $[0, T+\varepsilon) \cap I$ using $(p|_{V_j})^{-1} \circ \alpha|_{(T-\varepsilon, T+\varepsilon) \cap I}$.

Contradicts def. of T, unless $T = 1$. $\square$

Thm VI.16 (Homotopy lifting) $p: \tilde{X} \to X$ covering map,
$H: Y \times I \to X$ map, h is a lift of $H|_{Y \times \{0\}}$ $\Rightarrow$
$\exists!$ lift $\tilde{H}: Y \times I \to \tilde{X}$ s.t. $\tilde{H}|_{Y \times \{0\}} = h$.



Rem Thm VI.14 is the case $Y = \{pt\}$.

Proof (not examinable)

Let $t \mapsto \tilde{H}(y,t)$ be the unique lift of the path $t \mapsto H(y,t)$ from $H(y,0) = h(y)$. (Theorem VI.14, VI.13)

$\tilde{H}$ continuous at $(y,t) \in Y \times I$: Fix y.

H cont. $\Rightarrow$ nbhd $N_t \times (a_t, b_t) \ni (y,t)$ s.t. $H(N_t \times (a_t, b_t)) \subseteq U_{H(y,t)}$.

$\bigcup_{t \in I} (a_t, b_t) = I \xRightarrow[\text{Lebesgue Covering Thm (I.58)}]{} \exists\, n > 0 : [\frac{i}{n}, \frac{i+1}{n}] \subseteq (a_{t_i}, b_{t_i})\ \forall i \in \{0, \dots, n-1\}$.

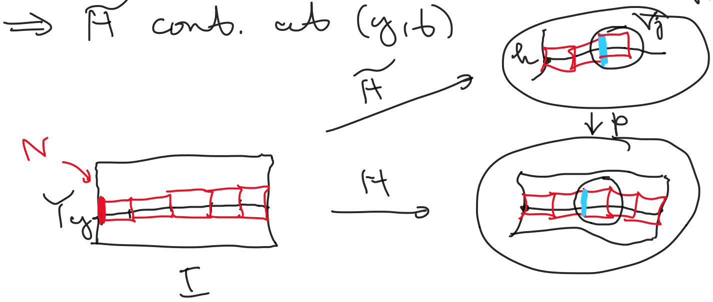
$N := \bigcap_{i=0}^{n-1} N_{t_i}$

We'll show, by induction on i, that $\tilde{H}|_{N\times[\frac{i}{n},\frac{i+1}{n}]}$ is cont., possibly shrinking N.

$H\left(N\times[\frac{i}{n},\frac{i+1}{n}]\right)\in U_{H(y,t_i)}$, $p^{-1}(U_{H(y,t_i)})=\coprod_{j\in J}V_j$, $\tilde{H}(y,\frac{i}{n})\in V_j$.

By induction, $\tilde{H}|_{N\times\{\frac{i}{n}\}}$ is cont. $\Rightarrow \tilde{H}(N\times\{\frac{i}{n}\})\subset V_j$, possibly shrinking N. $\Rightarrow \tilde{H}|_{N\times[\frac{i}{n},\frac{i+1}{n}]}=(p|_{V_j})^{-1}\circ H|_{N\times[\frac{i}{n},\frac{i+1}{n}]}$ cont.

$\Rightarrow \tilde{H}$ cont. at (y,t)



$\square$

<u>Cor</u> $p:(\tilde{X},\tilde{b})\to(X,b)$ based covering map $\Rightarrow$

$p_*:\pi_1(\tilde{X},\tilde{b})\to\pi_1(X,b)$ is injective.

<u>Proof</u> $[\ell]\in\pi_1(\tilde{X},\tilde{b})$, $p_*([\ell])=[p\circ\ell]=e\in\pi_1(X,b)$

$H:I\times I\to X$ homot. rel ∂I from $p\circ\ell$ to c_b.

ℓ lift of $H|_{I\times\{0\}}\underset{\text{Thm. VI.16}}{\Longrightarrow}\exists$ lift $\tilde{H}:I\times I\to\tilde{X}$ of H s.t. $\tilde{H}|_{I\times\{0\}}=\ell$.

$\tilde{H}|_{\{0\}\times I}$, $\tilde{H}|_{\{1\}\times I}$, $\tilde{H}|_{I\times\{1\}}=c_{\tilde{b}}$ since lift of a const. map is const. on a path-connected space

$\Rightarrow \ell\simeq_{\tilde{H}}c_{\tilde{b}}\Rightarrow[\ell]=e\in\pi_1(\tilde{X},\tilde{b})$ and so p_* is inj. $\square$