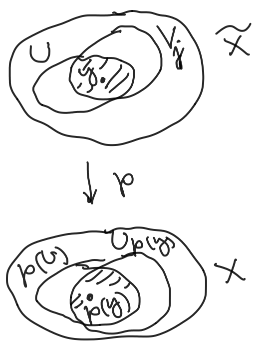
Section VI.1: Properties

Proposition VI.9 $p : \tilde{X} \to X$ covering map.

1) $p(U)$ is open for $\forall\, U \subseteq \tilde{X}$ open
2) $\forall\, x_1, x_2 \in X : |p^{-1}(x_1)| = |p^{-1}(x_2)|$
3) p is surjective
4) p is a quotient map.

Proof 1)



$y \in U$

$p^{-1}(U_{p(y)}) = \coprod_{j \in J} V_j \;,\; y \in V_j$

$p|_{V_j} : V_j \to U_{p(y)}$ homeo. $\Rightarrow$

$p(V_j \cap U)$ is open in X.

$p(y) \in p(V_j \cap U) \subseteq p(U) \Rightarrow p(U)$ open

2) $|p^{-1}(x)|$ is locally const. on X as $p^{-1}(U_x) = \coprod_{j \in J} V_j$, so same J for $\forall\, x' \in U_x$. X connected $\Rightarrow |p^{-1}(x)|$ const.

3) $\tilde{X} \neq \emptyset \Rightarrow \exists\, x \in X : p^{-1}(x) \neq \emptyset \overset{2)}{\Longrightarrow} p^{-1}(x) \neq \emptyset \;\forall\, x \in X$, p surj.

4) A surjective open map is a quotient map. $\square$

Def The *degree* of $p : \tilde{X} \to X$ is $|p^{-1}(x)|$ for any $x \in X$.

Def

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

$\tilde{f}$ is a *lift* of f if $p\tilde{f} = f$

Ex

$$\begin{tikzcd} & \mathbb{R} \arrow[d] & t \arrow[d, mapsto] \\ I \arrow[ur, dashed] \arrow[r] & S^1 & e^{2\pi i t} \end{tikzcd}$$

$t \mapsto t$

$t \longmapsto e^{2\pi i t}$

Ex

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] & t \arrow[d, mapsto] \\ S^1 \arrow[ur, dashed, "\nexists \tilde{f}"] \arrow[r, "id"] & S^1 & e^{2\pi i t} \end{tikzcd}$$

By contradiction, suppose $\tilde{f}$ exists, if $\tilde{f}(1) = n \in \mathbb{Z}$, then apply π_1

$$\begin{tikzcd} & \pi_1(\mathbb{R}, n) = 1 \arrow[d, "p_*"] \\ \pi_1(S^1, 1) \arrow[ur, dashed, "\tilde{f}_*"] \arrow[r] & \pi_1(S^1, 1) \cong \mathbb{Z} \end{tikzcd}$$

$\pi_1(S^1,1) \cong \mathbb{Z}$

$p_* \circ \tilde{f}_* = id_{\mathbb{Z}}$ #

factors through $\pi_1(\mathbb{R}, n) = 1$.

Theorem VI.13 (Uniqueness of lifts)

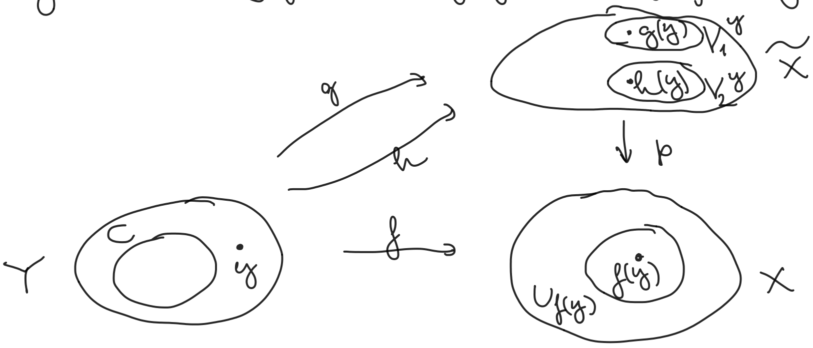
$\alpha \dashrightarrow \tilde{X}$ If $\alpha(y_0) = h(y_0)$, then $\alpha = h$.

$\downarrow p$

$y_0 \in Y \xrightarrow{f} X$

connected

Proof. $C := \{ y \in Y : g(y) = h(y) \} \ni y_0 \Rightarrow C \neq \emptyset.$



C closed: $y \in Y \setminus C \Rightarrow V_1^y \cap V_2^y = \emptyset \Rightarrow g^{-1}(V_1^y) \cap h^{-1}(V_2^y) \subseteq Y \setminus C$ open (if $z \in g^{-1}(V_1^y) \cap h^{-1}(V_2^y) \cap C$, then $g(z) = h(z) \in V_1^y \cap V_2^y$).

C open: $y \in C \Rightarrow V_1^y = V_2^y$

$z \in g^{-1}(V_1^y) \cap h^{-1}(V_1^y)$, $\underbrace{p \circ g}_{f}(z) = \underbrace{p \circ h}_{f}(z) \overset{p|_{V_1^y} \text{ inj.}}{\Longrightarrow} g(z) = h(z)$

$\Rightarrow y \in g^{-1}(V_1^y) \cap h^{-1}(V_2^y)$ open subset of C.

$C \neq \emptyset$, open, closed, Y connected $\Rightarrow C = Y$. □