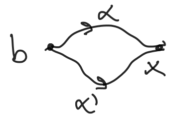
Section VI.4: Construction of covering spaces

<u>Theorem VI.32</u> K path-connected simplicial complex, b vertex of K. Then, for $H \leq \pi_1(K,b)$, $\exists$ based covering $p : (\tilde{K}, \tilde{b}) \to (K, b)$ such that $p_* \pi_1(\tilde{K}, \tilde{b}) = H$.
Moreover, $\tilde{K}$ is a simplicial complex and p is a simplicial map.

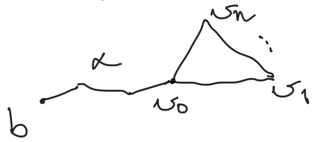
<u>Idea</u>: $p^{-1}(b) \overset{1-1}{\longleftrightarrow}$ right cosets of $p_* \pi_1(\tilde{K}, \tilde{b})$



$p^{-1}(x) \leftrightarrow$ equivalence classes of paths in K from b to x, where $\alpha \sim \alpha' \iff [\alpha' \alpha^{-1}] \in p_* \pi_1(\tilde{K}, \tilde{b})$.

<u>Proof</u> [not examinable]
<u>vertices $V(\tilde{K})$</u>: equivalence class of edge paths starting at b (paths <u>based at b</u>)
$\alpha \sim_H \beta$ <u>H-equivalent</u> $\iff \exists$ loop l based at b s.t. $[l] \in H$
$\alpha \sim l\beta$, where $\sim$ is defined in III.23 (related by elementary expansions and contractions).
$V(\tilde{K}) :=$ set of H-equiv. classes of paths based at b

<u>Simplices of $\tilde{K}$</u>: vertices of $V(\tilde{K})$ form a simplex $\iff$ they have representatives $\alpha(v_0, v_0), \alpha(v_0, v_1), \dots, \alpha(v_0, v_n)$, where α is an edge path from b to v_0 and $(v_0, \dots, v_n)$ is a simplex of K



$\tilde{b} := (b)$ basepoint of $\tilde{K}$

<u>Claim</u> $\tilde{K}$ is a simplicial cx.:
$v \in V(\tilde{K})$ repr. by α ending at say v_0. Then this equivalent to $\alpha(v_0, v_0)$ and so (v) is a simplex of $\tilde{K}$.
Suppose $\{[\alpha(v_0, v_0)], \dots, [\alpha(v_0, v_n)]\}$ are vertices of a simplex of $\tilde{K}$

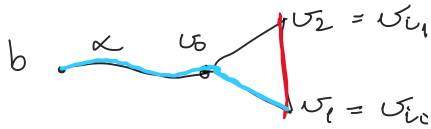
$$\cup\!\!|$$

$$\{[\alpha(v_0, v_{i_0})], \dots, [\alpha(v_0, v_{i_m})]\}$$

$$\shortparallel$$

$$\{[\alpha(v_0, v_{i_0})(v_{i_0}, v_{i_0})], \dots, [\alpha(v_0, v_{i_0})(v_{i_0}, v_{i_m})]\}$$

$\uparrow$ simplex in $\tilde{K}$



<u>Claim</u> $\tilde{K}$ is path-connected
$[\alpha] \in V(\tilde{K})$, $\alpha = (b, b_1, \dots, b_n)$ edge path
Then $[(b)], [(b, b_1)], \dots, [(b, b_1, \dots, b_n)]$ is an edge path in $\tilde{K}$ joining $\tilde{b}$ to $[\alpha]$.

$p : V(\tilde{K}) \to V(K)$
$[\alpha] \longmapsto$ terminal point of α

<u>Claim</u> p is simplicial and its restriction to $\forall$ simplex of K is inj.
$(w_0, \dots, w_n)$ simplex of $\tilde{K}$ $\overset{\text{by Def}}{\Longrightarrow}$ they have representatives

$\alpha(v_0,v_0),\dots,\alpha(v_0,v_n)$, where $(v_0,\dots,v_n)$ span a simplex of K.

Then $p(w_i)=v_i$ $\forall i$, so $(p(w_0),\dots,p(w_n))$ span a simplex of K of the same dim, so p inj. on $\forall$ simplex.

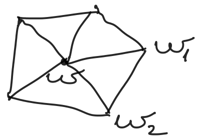
<u>p covering map:</u> $\forall x\in K$, elementary nbhd $U_x := st_K(x)$

$cl(st_K(x))$: closure of $st_K(x)$, subcomplex of K.

<u>Claim</u> $\forall w\in V(\tilde K)$, $p|cl(st_{\tilde K}(w))$ is a simplicial iso. onto $cl(st_K(p(w)))$ $\Longrightarrow$ $p|st_{\tilde K}(w)$ is a homeo. onto $st_K(p(w))$.

$v := p(w)$. Check that $p|V(cl(st_{\tilde K}(w)))$ bijection onto $V(cl(st_K(v)))$

<u>Injective:</u> $w_1\neq w_2\in V(cl(st_{\tilde K}(w)))$



Then w, w_i span a simplex of $\tilde K$

$\Rightarrow$ $w_i=[\alpha_i(v,v_i)]$, $w=[\alpha_i(v,v)]$

$p(w_i)=v_i$ and (v,v_i) span a simplex of K

Since $[\alpha_1(v,v)]=[\alpha_2(v,v)]=w$ $\Rightarrow$ $\alpha_1\sim_H\alpha_2$

So $w_2=[\alpha_1(v,v_2)]$

$w_1\neq w_2\Rightarrow v_1\neq v_2\Rightarrow p(w_1)\neq p(w_2)$, so p is an inj. on vertices of $cl(st_{\tilde K}(w))$.

<u>Surjective:</u> $v_1\in V(cl(st_K(v)))\Rightarrow(v,v_1)$ span a simplex of K.

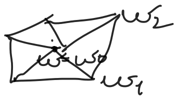
$w=[\alpha]\Rightarrow\alpha(v,v),\alpha(v,v_1)$ span a simplex of $\tilde K$

$\Rightarrow\alpha(v,v_1)$ is a vertex of $cl(st_{\tilde K}(w))$ that maps to v_1.

Suppose $(v_0,\dots,v_n)$ span a simplex in $cl(st_K(v))$

$w_i := cl(st_{\tilde K}(w))\cap p^{-1}(v_i)$

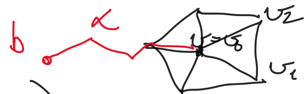
$(w_0,\dots,w_n)$ span a simplex of $\tilde K$



Suppose $v\in\{v_0,\dots,v_n\}$

set $v=v_0$

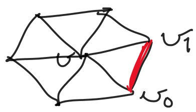
$w=[\alpha]$



$([\alpha(v,v_0)],\dots,[\alpha(v,v_n)])=(w_0,\dots,w_n)$

$\uparrow$ simplex of $\tilde K$

Suppose $v\notin\{v_0,\dots,v_n\}$.



Then $(v, v_0,\dots,v_n)$ span a simplex of K

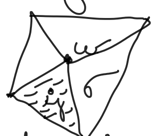
$\Rightarrow(w,w_0,\dots,w_n)$ span a simplex of $\tilde K$ by previous case

$\Rightarrow(w_0,\dots,w_n)$ span a simplex of $\tilde K$.

<u>Claim</u> $\forall y\in\tilde K$, $p|st_{\tilde K}(y)$ is a homeo. onto $st_K(p(y))$

Follows from previous claim when $y\in V(\tilde K)$.

$y\in$ inside(σ), $w\in V(\sigma)$



Then $st_{\tilde K}(w)\supseteq st_{\tilde K}(y)$

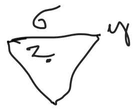
$p|cl(st_{\tilde K}(w))$ is a simplicial iso onto its image $\Rightarrow$ it preserves stars. $\Rightarrow$ $p|_{\mathrm{st}_{\tilde K}(y)}$ is a homeo. onto $\mathrm{st}_K(p(y))$.

$\forall\, x \in K$, the indexing set J in the def. of a covering map is $p^{-1}(x)$. For $y \in J = p^{-1}(x)$, we set $V_y := \mathrm{st}_{\tilde K}(y)$.
We've seen that $p|_{V_y}$ is a homeo. onto U_x.

That p is a covering map follows from:

<u>Claim</u> $y \neq y' \in p^{-1}(x) \Rightarrow V_y \cap V_{y'} = \emptyset$.

Suppose $z \in V_y \cap V_{y'}$. By def. of $\mathrm{st}_{\tilde K}(y)$, $\exists$ simplex $\sigma \ni y$ and $z \in \mathrm{inside}(\sigma) \Rightarrow y \in \mathrm{cl}(V_z)$



Similarly, $y' \in \mathrm{cl}(V_z)$
But $p|_{\mathrm{cl}(V_z)}$ is a homeo. onto its image $\Rightarrow$
$p|_{\mathrm{cl}(V_z)}$ is inj. But $y, y' \in \mathrm{cl}(V_z)$, $p(y) = p(y')$ $\#$

<u>Claim</u> $p_* \pi_1(\tilde K, \tilde b) = H$

l loop based at b, lifts to a loop in $\tilde K \Leftrightarrow [l] \in H$.
We can suppose $l = (b, b_1, \dots, b_{n-1}, b)$ edge loop.
The lift of this is an edge path $[(b)], [(b, b_1)], \dots, [(b, b_1, \dots, b_{n-1}, b)]$
This is a loop $\Leftrightarrow$ $[\underbrace{(b, b_1, \dots, b_{n-1}, b)}_{l}] = [(b)]$ $\Leftrightarrow$
$l \sim_H (b) \Leftrightarrow [l] \in H$. $\square$