Section V.1: Generators and relations

Group presentations

$D_{2n} = \langle \sigma, \tau \mid \sigma^n = e,\ \tau^2 = e,\ \tau\sigma\tau = \sigma^{-1} \rangle$

$\sigma^n = e,\ \tau^2 = e \Rightarrow \sigma^n \tau^2 = e$

$\tau \sigma^n \tau = \tau e \tau = \tau^2 = e.$

<u>Def</u> $B \subseteq G$ subset. The normal subgroup $\langle\!\langle B \rangle\!\rangle$ generated by B is the intersection of all normal subgroups of G containing B.

<u>Rem</u> Intersection of normal subgroups is a normal subgroup.
$\Rightarrow \langle\!\langle B \rangle\!\rangle \trianglelefteq G$

$\langle\!\langle B \rangle\!\rangle$ smallest normal subgroup containing B: if $N \trianglelefteq G$ and $B \subseteq N$, then $\langle\!\langle B \rangle\!\rangle \subseteq N$.

<u>Proposition V.3</u> The subgroup $\langle\!\langle B \rangle\!\rangle$ consists of all expressions $\prod_{i=1}^{n} g_i b_i^{\varepsilon_i} g_i^{-1}$, where $g_i \in G$, $b_i \in B$, $\varepsilon_i \in \{1, -1\}$, $n \in \mathbb{N}$.

<u>Proof</u> If $B \subseteq N \trianglelefteq G$, then $\forall\, b \in B,\ g \in G$: $gbg^{-1},\ gb^{-1}g^{-1} \in N$
$\Rightarrow \prod_{i=1}^{n} g_i b_i^{\varepsilon_i} g_i^{-1} \in N$.

Let N_0 be the set of $\prod g_i b_i^{\varepsilon_i} g_i^{-1}$. Then $N_0 \subseteq \langle\!\langle B \rangle\!\rangle$.

N_0 is a normal subgroup of G:

Identity: $e \in N_0$

Inverses: $\left(\prod_{i=1}^{n} g_i b_i^{\varepsilon_i} g_i^{-1}\right)^{-1} = \prod_{i=n}^{1} g_i b_i^{-\varepsilon_i} g_i^{-1} \in N_0$.

Closure: clearly product of $\prod g_i b_i^{\varepsilon_i} g_i^{-1} \cdot \prod h_i c_i^{\nu_i} h_i^{-1} \in N_0$.
$h_i \in G$, $c_i \in B$

Normality: $g\left(\prod g_i b_i^{\varepsilon_i} g_i^{-1}\right) g^{-1} = \prod_{i=1}^{n} (gg_i) b_i^{\varepsilon_i} (g_i^{-1} g^{-1}) = \underbrace{\prod (gg_i) b_i^{\varepsilon_i} (gg_i)^{-1}}_{\in N_0}$

So N_0 is the smallest normal subgroup containing B
$\Rightarrow N_0 = \langle\!\langle B \rangle\!\rangle$. □

<u>Def</u> X set, $R \subseteq F(X)$ (relations). The group with <u>presentation</u>
$\langle X \mid R \rangle := F(X) / \langle\!\langle R \rangle\!\rangle$.

We allow relations of the form $w_1 = w_2$, which means $w_1 w_2^{-1} \in R$ ($w_1 w_2^{-1} = e$).

<u>Example</u> $D_{2n} = \langle \sigma, \tau \mid \sigma^n, \tau^2, \tau\sigma\tau\sigma \rangle$

$\tau\sigma^n\tau \in R$: $\tau\sigma^n\tau = (\tau \underbrace{\sigma^n}_{\in R} \tau^{-1}) \tau^2$ in the form of Prop. V.3.

Which relations hold in $G = \langle X \mid R \rangle$?

$w, w' \in F(X)$ represent same element of $G \iff$
$\exists\, y \in \langle\langle R \rangle\rangle$ such that $w' = wy$.
in $F(X)$ (up to elementary contr. and expansions)

<u>Proposition V.6</u> $G = \langle X \mid R \rangle$. Then $w, w' \in F(X)$ repr. same element of $G \iff$ they differ by a finite sequence of the following moves:

(1) elementary contr. or expansion
(2) inserting somewhere into the word $r \in R$ or r^{-1} for $r \in R$.

<u>Proof</u> (1), (2) do not change the elt of G that it represents.
Suppose w, w' repr. same elt. of G.
$\Rightarrow$ $w' = wy$, $y \in \langle\langle R \rangle\rangle \underset{\text{Prop. V.3}}{\Longrightarrow} w' = w \prod g_i r_i^{\varepsilon_i} g_i^{-1}$, $r_i \in R$

$w \xrightarrow{(1)} w g_1 g_1^{-1} \xrightarrow{(2)} w g_1 r_1^{\varepsilon_1} g_1^{-1} \longrightarrow \cdots$
So we can obtain w' from w by moves (1) & (2). $\square$

<u>Example</u> in D_{2n}, $\tau\sigma^n\tau = e$
$\tau\sigma^n\tau \xrightarrow{(2)} \tau\sigma^n\sigma^{-n}\tau \xrightarrow{n\times(1)} \tau\tau \xrightarrow{(2)} \tau^2\tau^{-2} \xrightarrow{2\times(1)} e$.

$\forall$ group G has a presentation:

$$\begin{tikzcd} F(G) \arrow[r, "\phi"] & G \\ G \arrow[u, hook, "i"] \arrow[ur] & \end{tikzcd}$$

$g \mapsto g$

homomorphism ϕ given by universal property of free groups

$R(G) := \ker(\phi) \subseteq F(G)$
first isomorphism thm. (ϕ is surjective),
$G \cong F(G)/R(G)$
normal subgroup of $F(G)$.

If $x_3 = x_1 x_2$ in G, in $F(G)$ $x_3 \neq x_1 x_2$, but $x_3 x_2^{-1} x_1^{-1} \in R(G)$.
$G \cong \langle G \mid R(G) \rangle$

<u>Def</u> The <u>canonical presentation</u> for G is $\langle G \mid R(G) \rangle$.
very inefficient

<u>Def</u> The presentation $\langle X \mid R \rangle$ is <u>finite</u> if X and R are finite. A group is finitely presented if it has a finite presentation.
(<u>e.g.</u> $F(\mathbb{N})$ <u>not</u> finitely presented)

<u>Rem</u> We will show that a group is finitely presented if and only if it is the fundamental group of a finite

simplicial complex.

Lemma V.11 $\langle X | R \rangle$, H groups. Let a function $f : X \to H$ induce a homomorphism $\phi : F(X) \to H$.
This descends to a homomorphism $\langle X | R \rangle \to H \iff \phi(r) = e \ \forall r \in R$

Proof $\phi(r) = e$ is necessary for ϕ to give a homom. since $r \in R$ represents $e \in \langle X | R \rangle$.
Conversely, if $\phi(r) = e \ \forall r \in R \overset{\text{Prop. V.3}}{\Longrightarrow} \forall w \in \langle\langle R \rangle\rangle$ can be written as $\prod w_i r_i^{\varepsilon_i} w_i^{-1}$. Since $\phi(r) = e \ \forall r \in R$, $\phi(w) = e$. Hence ϕ descends to a homom. $F(X) / \langle\langle R \rangle\rangle \to H$. $\square$