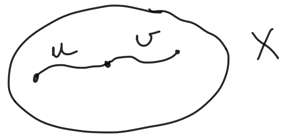
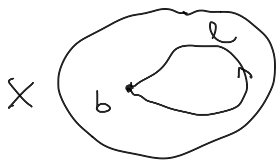
Section III.1: Fundamental group - definition



Def u, v paths in X, $u(1) = v(0)$
Their composition is

$$uv(t) = \begin{cases} u(2t), & t \in [0, \tfrac{1}{2}] \\ v(2t-1), & t \in [\tfrac{1}{2}, 1] \end{cases}$$

Def A loop based at a point $b \in X$ (basepoint) is a path $\ell : I \to X$ such that $\ell(0) = \ell(1) = b$.



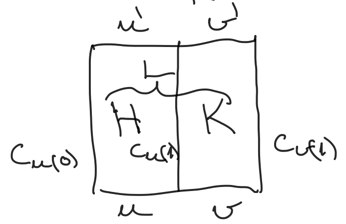
Def / Theorem III.3 The homotopy classes rel ∂I of loops based at b form a group, called the fundamental group of (X, b), denoted by $\pi_1(X, b)$.
If ℓ, ℓ' are loops based at b, and $[\ell], [\ell']$ denote their homotopy classes, then $[\ell][\ell'] := [\ell\ell']$.

Check: Product is well-defined, associative, identity, inverse

Well-defined

Lemma III.6 u, v paths in X, $u(1) = v(0)$
u', v' — " —, $u(i) = u'(i)$, $v(i) = v'(i)$, $i \in \{0, 1\}$
$u \simeq_H u'$, $v \simeq_K v' \Rightarrow uv \simeq u'v'$ rel ∂I.

Proof.



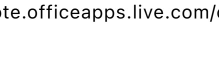
$$L(t, s) = \begin{cases} H(2t, s), & t \in [0, \tfrac{1}{2}] \\ K(2t-1, s), & t \in [\tfrac{1}{2}, 1] \end{cases}$$

□

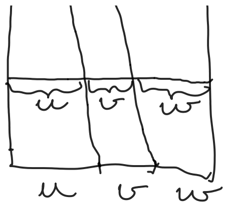
Associativity

Note. Composition is not assoc. on paths due to different parameterisations.

Lemma III.8 u, v, w paths in X, $u(1) = v(0)$, $v(1) = w(0)$
Then $u(vw) \simeq (uv)w$ rel ∂I.



$u(4t)$, $t \in [0, \tfrac{1}{4} - ts]$

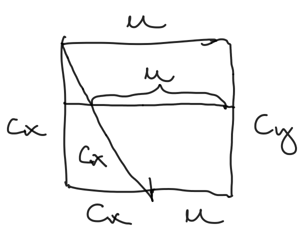


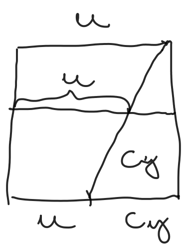
$$H(t,s) = \begin{cases} \ldots \\ v(4t-2+s), & t \in \left[\frac{1}{2}-\frac{1}{4}s, \frac{3}{4}-\frac{1}{4}s\right] \\ w\left(\frac{4t-3+s}{1+s}\right), & t \in \left[\frac{3}{4}-\frac{1}{4}s, 1\right] \end{cases}$$

<u>Identity:</u>

<u>Lemma III.10</u> u path in X, $u(0)=x$, $u(1)=y$
$c_x(t) = x \ \forall t \in I$ const. path
Then $c_x u \simeq u$ rel ∂I
$u c_y \simeq u$ rel ∂I.

<u>Proof.</u>





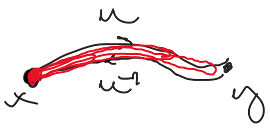
$\square$

Above are continuous by Gluing Lemma (II.5).

<u>Inverses:</u> u path in X, $u(0)=x$, $u(1)=y$
Let u^{-1} be defined by $u^{-1}(t) = u(1-t)$. Then $uu^{-1} \simeq c_x$ rel ∂I and $u^{-1}u \simeq c_y$ rel ∂I.

<u>Proof</u>

$$H(t,s) = \begin{cases} u(2t(1-s)) & t \in [0,\frac{1}{2}] \\ u((2-2t)(1-s)) & t \in [\frac{1}{2},1] \end{cases}$$



$\square$

Completes the proof of Theorem III.3. $\square$