Section IV.2: Reduced representatives

Def A word is reduced if it doesn't admit elementary contractions.

Proposition IV.8 $\forall w \in F(S)$ has a unique reduced representative.

Lemma IV.9 w_1, w_2, w_3 words such that $w_1 \nearrow w_2 \searrow w_3$.
Then either $\exists w_2'$ such that $w_1 \searrow w_2' \nearrow w_3$, or $w_1 = w_3$.

Proof Lemma IV.9 $w_1 \nearrow w_2$, $\exists$ words a, b and $x \in S \cup S^{-1}$:
$w_1 = ab$ and $w_2 = axx^{-1}b$.
$w_2 \searrow w_3$: w_3 is obtained from w_2 by removing yy^{-1}, $y \in S \cup S^{-1}$.
xx^{-1} & yy^{-1} overlap in 0, 1, or 2 letters. 3 cases:
0: w_2' be the word obtained from w_1 by removing yy^{-1}.
Then $w_1 \searrow w_2' \nearrow w_3$.
adding xx^{-1}

1: $x = y^{-1}$, so in w_2 we have $xx^{-1}x$ or $x^{-1}xx^{-1}$
w_1 and w_3 are obtained from w_2 by performing the two possible contractions on $xx^{-1}x$ or $x^{-1}xx^{-1}$ $\Rightarrow w_1 = w_3$.

2: $xx^{-1} = yy^{-1} \Rightarrow w_1 = w_3$ □

Proof Proposition IV.8

Elementary contraction reduces length of a word by 2.
So $\forall$ shortest representative of $w \in F(S)$ is reduced.
Suppose that w, w' reduced, $w \sim w'$.
$\Rightarrow \exists$ $w_1, w_2, \dots, w_n$ such that $w_1 = w$, $w_n = w'$
$w_i \nearrow w_{i+1}$ or $w_i \searrow w_{i+1}$. Suppose n is minimal.
$\Rightarrow w_i \neq w_j$ for $i \neq j$, as otherwise remove w_k for $k \in [i, j]$.
Suppose $w_i \nearrow w_{i+1} \searrow w_{i+2} \overset{\text{Lemma IV.9}}{\Longrightarrow}$ either $\exists w_{i+1}'$:
$w_i \searrow w_{i+1}' \nearrow w_{i+2}$ or $w_i = w_{i+2}$ ← not possible by above.
We can perform all contractions before expansions.
Either $w_1 \searrow w_2$ or $w_{n-1} \nearrow w_n$.
length(w_2) < length(w_1) or length(w_{n-1}) < length(w_n)
Contradicts that w_1, w_n are reduced. □