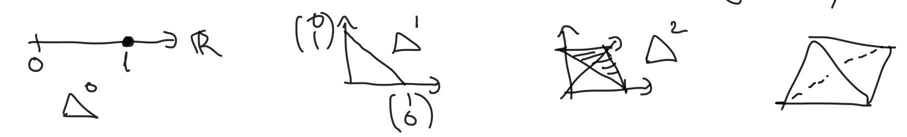
Section I.2

<u>Def</u> <u>standard n-simplex</u> $\Delta^n := \left\{ \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^{n+1} : x_i \geq 0 ,\ \sum_{i=0}^{n} x_i = 1 \right\}$



regular tetrahedron

$\dim(\Delta^n) = n$

$\text{inside}(\Delta^n) = \left\{ \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \in \Delta^n : x_i > 0 \ \forall i \right\},\ n > 0$

$\text{inside}(\Delta^0) = \Delta^0$

$V(\Delta^n) = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$ vertices of Δ^n

$A \subset \{0, \dots, n\} \rightsquigarrow$ face $\left\{ \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \in \Delta^n : x_i = 0 \ \forall i \notin A \right\}$

$A \neq \emptyset$

$V(\Delta^n)$ basis of $\mathbb{R}^{n+1} \Rightarrow \forall$ map $f : V(\Delta^n) \to \mathbb{R}^m$ uniquely extends to a linear map $\mathbb{R}^{n+1} \to \mathbb{R}^m$, whose restriction to Δ^n is called the <u>affine extension</u> of f.

<u>Def</u> A <u>face inclusion</u> is the affine extension of an injection $V(\Delta^m) \to V(\Delta^n)$, $m < n$.

There are 6 face inclusions $\Delta^1 \to \Delta^2$, corresponding to injections $\{0, 1\} \to \{0, 1, 2\}$.

<u>Def</u> An <u>abstract simplicial complex</u> (V, Σ)

V set of vertices

Σ set of finite subsets of V (simplices)

(i) $\{v\} \in \Sigma \quad \forall v \in V$

(ii) $\emptyset \neq \tau \subset \sigma \in \Sigma \Rightarrow \tau \in \Sigma$

<u>Def</u> The <u>topological realisation</u> of $K = (V, \Sigma)$ is a top. space $|K|$ obtain as follows:

(i) $\forall \sigma \in \Sigma$, take a copy Δ_σ of $\Delta^{|\sigma| - 1}$

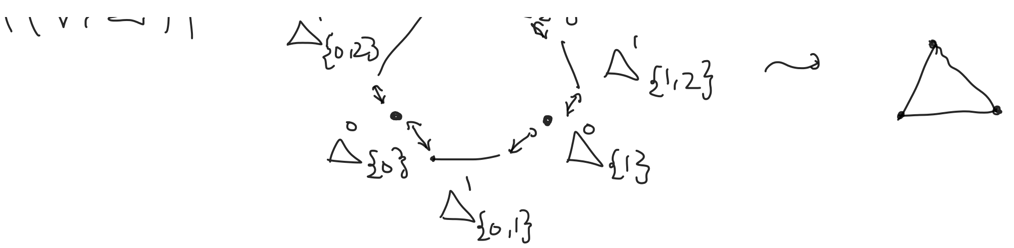
$V(\Delta_\sigma) \leftrightarrow \sigma$ bijection

$\coprod_{\sigma \in \Sigma} \Delta_\sigma$

(ii) $\sigma, \tau \in \Sigma$, $\sigma \subset \tau$, we identify Δ_σ and Δ_τ via the face identification $i : \sigma \to \tau$.

<u>Example</u> $V = \{0, 1, 2\}$, $\Sigma = \{\{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{0,2\}\}$

$|(V, \Sigma)|$, $\cdot\, \Delta^0_{\{2\}}$

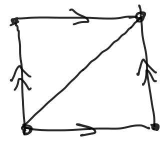


$x \in |K|$ lies in the inside of a unique simplex $\sigma = (v_0, \dots, v_n)$

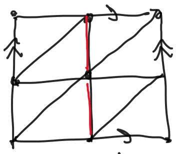
$x = \sum_{i=0}^{n} \lambda_i v_i \qquad \lambda_i > 0, \ \sum_i \lambda_i = 1.$

<u>Def</u> A <u>triangulation</u> of a top. space X is a homeomorphism $|K| \to X$ for a simplicial complex K.

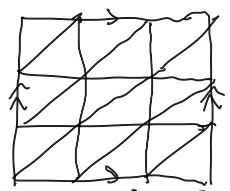
<u>Example</u>



only one vertex
not a triang. of T^2!



red edges connect same pair of vertices
not a triang. of T^2!



triang. of T^2

<u>Example</u> A graph with no loops or double edges is a simplicial complex.

<u>Def</u> A <u>subcomplex</u> of (V, Σ) is a simpl. cx. (V', Σ') such that $V' \subseteq V$, $\Sigma' \subseteq \Sigma$.

<u>Def</u> A <u>simplicial map</u> $(V_1, \Sigma_1) \to (V_2, \Sigma_2)$ is a function $f : V_1 \to V_2$ such that $\forall \sigma_1 \in \Sigma_1$, $f(\sigma_1) \in \Sigma_2$.

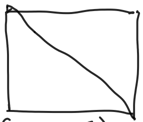
f is a <u>simplicial isomorphism</u> if it has a simplicial inverse.

$f : K_1 \to K_2$ induces $|f| : |K_1| \to |K_2|$ by setting $|f|\big|_{V(K_1)} := f$ and extending to each simplex using affine extension.

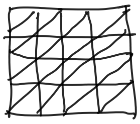
<u>Def</u> A <u>subdivision</u> of K is a simplicial cx. K' + homeo. $h : |K'| \to |K|$ such that $\forall \sigma'$ simplex of K',

$h(\sigma')$ lies entirely in a simplex of $|K|$ and the restriction of h to σ' is affine.

Example



$(I \times I)_{(1)}$



$(I \times I)_{(r)}$ subdivision of $(I \times I)_{(1)}$.