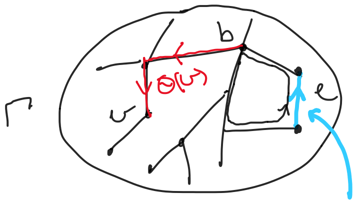
Section IV.4: The fundamental group of a graph - proof

Theorem IV.11 Γ graph, then $\pi_1(\Gamma)$ is free

Proof



T maximal tree, $V(T) = V(\Gamma)$
$b \in V(\Gamma)$ basepoint
$\forall v \in V(\Gamma)$ $\exists!$ embedded path $\theta(v)$ from b to v in T.

Orient edges in $E(\Gamma) \setminus E(T)$.

Claim $\{\theta(\underset{\substack{\uparrow \\ \text{initial} \\ \text{vertex of } e}}{\iota(e)})\, e\, \theta(\underset{\substack{\uparrow \\ \text{terminal} \\ \text{vertex of } e}}{\tau(e)}) : e \in E(\Gamma) \setminus E(T)\}$ is a free generating set of $\pi_1(\Gamma, b)$.

Proof claim Caveat: Γ is not necessarily a simplicial complex.
Let Γ' be obtained from Γ by subdividing each edge into 3.

$\Gamma' \cong \Gamma \Rightarrow \pi_1(\Gamma', b) \cong \pi_1(\Gamma, b)$

exists by universal property of free groups

$$\begin{tikzcd}
F(E(\Gamma)\setminus E(T)) \arrow[r, "\Phi"] & E(\Gamma', b) \\
E(\Gamma)\setminus E(T) \arrow[u, "i"] \arrow[ur]
\end{tikzcd}$$

$e \mapsto \theta(\iota(e))\, e\, \theta(\tau(e))^{-1}$

Φ is an isomorphism: Define inverse Ψ

$[\ell] \in E(\Gamma', b)$ $\quad \ell \mapsto$ a word in $E(\Gamma) \setminus E(T)$: whenever ℓ traverses an edge e in the forwards direction (resp., backwards) we write down e (resp. e^{-1}). Remove all letters corresp. to edges of T. For $\forall$ edge of $\Gamma \setminus T$, remove letters corresp. two outer edges.

Ψ is well-defined: $\ell \sim \ell' \Rightarrow \Psi(\ell) = \Psi(\ell')$

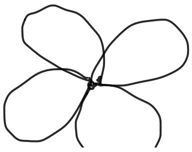
Γ' has no 2-simplices, so only have to check moves (0) and (1).

move (0): removing a repeated vertex from ℓ, then corresp. word in $E(\Gamma')$ is unchanged

move (1): remove an edge and its inverse from $\ell \Rightarrow$ $\Psi(\ell)$ changed by an elementary contraction (if the edge is in the middle of $e \in E(\Gamma) \setminus E(T)$, unchanged o/w).

So $\Psi : E(\Gamma', b) \to F(E(\Gamma) \setminus E(T))$ well-defined, clearly homom. since concatenation of edge loops maps to concatenation of words in $F(E(\Gamma) \setminus E(T))$. $\Psi = \Phi^{-1}$ $\square$

Example

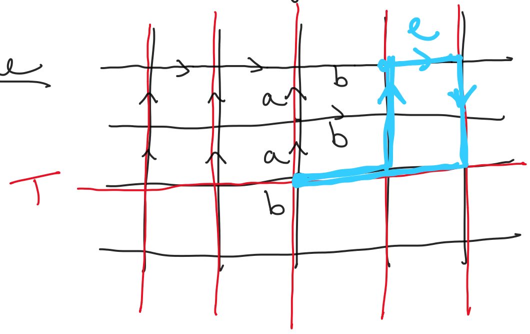


Γ is a graph with one vertex b and four edges e_1, e_2, e_3, e_4.
T maximal tree has $V(T) = \{v\}$

$E(\Gamma) \setminus E(T) = F(\{e_1, \dots, e_4\})$ $E(T) = \emptyset$

i-th generator of $\pi_1(\Gamma, b)$ goes around e_i once.

Example



$V(\Gamma) = \mathbb{Z}^2$

join $(x, y) \in \mathbb{Z}^2$ to $(x \pm 1, y)$ and $(x, y \pm 1)$ with edges.

$b = (0,0)$

T maximal tree: $V(T) = V(\Gamma)$

no embedded loop

$\pi_1(\Gamma) \cong F(E(\Gamma) \setminus E(T))$

orient from left to right

The loop corresp. to the edge from (x, y) to $(x+1, y)$ for $y \neq 0$ is $a^x b^y a b^{-y} a^{-x-1}$.