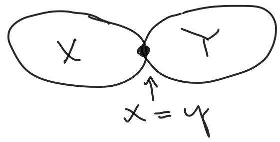
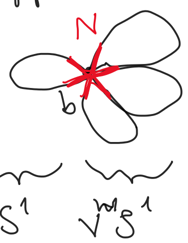
Section V.5: Topological applications

<u>Definition</u> The <u>wedge</u> of based spaces (X,x), (Y,y) is
$(X,x) \vee (Y,y) := X \amalg Y / x \sim y$



$b \in S^1$ basepoint
$\vee^n S^1$ wedge of n copies of (S^1, b), <u>bouquet</u> of n copies of S^1
<u>Cor V.28</u> $\pi_1(\vee^n S^1) \cong F_n$ ← free group on n generators.
<u>Proof</u> By induction on n.
$n=1$: $\pi_1(S^1) \cong F_1 = \mathbb{Z}$ by Theorem III.32.
Suppose $\pi_1(\vee^{n-1} S^1) \cong F_{n-1}$.



N open, connected, star-shaped neighbourhood of b
$K_1 := S^1 \cup N$, $K_2 := \vee^{n-1} S^1 \cup N$, open in $\vee^n S^1$.
$K_1 \cap K_2 = N$ contractible $\Rightarrow \pi_1(N, b) \cong 1$.
By Seifert-van Kampen theorem,
$$\pi_1(\vee^n S^1) \cong \pi_1(S^1) * \pi_1(\vee^{n-1} S^1) \cong \mathbb{Z} * F_{n-1} \cong F_n. \qquad \square$$

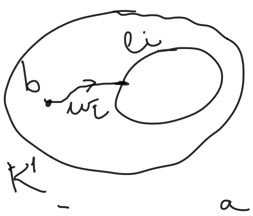
<u>Rem</u> $\vee^n S^1$ is a graph with one vertex and n edges, so Cor. V.28 also follows from Theorem IV.11.

<u>Def</u> X space, $f: S^{n-1} \to X$ map
$X \cup_f D^n := X \amalg D^n / x \sim f(x) \ \forall x \in S^{n-1} = \partial D^n$

<u>Def</u> A finite cell complex is a space X together w/ a filtration $K^0 \subseteq K^1 \subseteq \dots \subseteq K^n = X$, where
(i) K^0 is a finite discrete space
(ii) K^i is obtained from K^{i-1} by attaching finitely many i-cells.

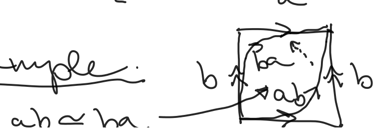
<u>Theorem V.29</u> K connected cell complex.
$l_i: S^1 \to K^1$ be the attaching maps of the 2-cells, $i \in \{1, \dots, n\}$
$b \in K^0$ basepoint
$[l_i]$ be the conjugacy class in $\pi_1(K_1, b)$ repr. by l_i
Then $\pi_1(K, b) \cong \pi_1(K^1, b) / \langle\langle [l_1], \dots, [l_n] \rangle\rangle$

<u>Rem</u>



$l_i' = w_i l_i w_i^{-1}$
$\langle\langle l_1', \dots, l_n' \rangle\rangle = \langle\langle [l_1], \dots, [l_n] \rangle\rangle$
$\pi_1(K^1, b)$ free by Theorem IV.11, so this is a presentation of $\pi_1(K, b)$.

<u>Example</u>:

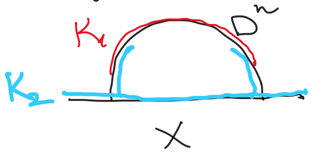


cell decomp. of T^2 with 1 0-cell, 2 1-cells a, b, 1 2-cell.
$K^1 \simeq S^1 \vee S^1$

$\pi_1(T^2) \cong \langle a, b \mid aba^{-1}b^{-1} \rangle$

<u>Proof of Theorem V.29.</u>

$f: S^{n-1} \to X$, $n \geq 2$, $Y := X \cup_f D^n$



$K_1 := \{z \in D^n : |z| < \frac{2}{3}\} \cong \operatorname{int}(D^n)$

$K_2 := \{z \in D^n : |z| > \frac{1}{3}\} \cup X$

$K_1 \cap K_2 = S^{n-1} \times (\frac{1}{3}, \frac{2}{3}) \simeq S^{n-1}$

K_2 path-connected and deformation retracts onto X

$i: X \hookrightarrow K_2$ embedding

$r: K_2 \to X$ given by $r|_X = \mathrm{id}_X$

$r|_{\{z \in D^n : |z| > \frac{1}{3}\}} : \{z \in D^n : |z| > \frac{1}{3}\} \to S^{n-1} \xrightarrow{f} X.$

$z \longmapsto \frac{z}{|z|}$

$ri = \mathrm{id}_X$

$ir \simeq \mathrm{id}_{K_2}$

Seifert-van Kampen $\Rightarrow$ when $n > 2$, $\pi_1(K_1 \cap K_2) \cong \pi_1(S^{n-1}) \cong 1$

$\pi_1(K_1) \cong \pi_1(D^n) \cong 1$, so $\pi_1(X \cup_f D^n) \cong \pi_1(X)$.

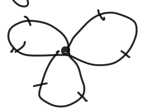
when $n = 2$: $\pi_1(K_1 \cap K_2) = \pi_1(S^1) \cong \mathbb{Z}$, $\pi_1(K_1) = \pi_1(D^2) \cong 1$,

so $\pi_1(X \cup_f D^2) \cong \pi_1(X) / \langle\langle [f] \rangle\rangle$. $\square$

<u>Cor V.34</u> $\forall$ finitely presented group $\langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$ is the fundamental group of a finite cell complex. Moreover, this can be triangulated.

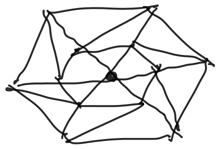
<u>Proof</u> $K^0 = \{pt\}$, $K^1 = \bigvee^m S^1$, $\pi_1(K^1) \cong F_m$

K^1 can be triangulated by subdividing each edge into 3 1-simplices.



Attach 2-cells along words $r_1, \dots, r_n$. Then the resulting space K has $\pi_1(K) \cong \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$ by Theorem V.29.

Triangulate $\forall$ 2-cell:



$\square$

<u>Cor V.36</u> G group, TFAE:

(i) G is finitely presented

(ii) $G \cong \pi_1(K)$, K finite connected simplicial cx.

(iii) $G \cong \pi_1(X)$, X finite connected cell complex

<u>Proof</u> (i) $\Rightarrow$ (ii) V.34.

(ii) $\Rightarrow$ (iii) Because $\forall$ finite simplicial cx. is a finite

call α, (I. 34).
(iii) $\Rightarrow$ (i): Rem. V. 30.