Section VI.5: The universal cover and Cayley graphs

Theorem VI.32 $\Rightarrow$ $\forall$ connected simplicial cx. K has a univ. cover $\tilde{K}$. $p : (\tilde{K}, \tilde{b}) \to (K, b)$

$p^{-1}(b) \overset{1-1}{\longleftrightarrow} \pi_1(K, b)$

G finitely presented group $\xrightarrow{\text{V.19.}}$ 2-complex w/ $\pi_1(K, b) \cong G$.

$G = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$

K has 1 0-cell, m 1-cells, n 2-cells attached along r_i

orient 1-cells, running fwds along it represents x_i

K can be triangulated, so has a universal cover $p : (\tilde{K}, \tilde{b}) \to (K, b)$

Cell str. on K induces a cell str. on $\tilde{K}$: $p^{-1}(b)$ is a discrete set of points, 0-cells of $\tilde{K}$.

Each 1-cell induces a map $D^1 \to K$, can lift it to $\tilde{K}$ from any point of $p^{-1}(b)$. These are the 1-cells of $\tilde{K}$, labelled by $x_1, \dots, x_m$, inherit an orientation from corresp. 1-cell of K.

$\Gamma :=$ 1-skeleton of $\tilde{K}$

interior of a 2-cell in K is an open disk, primage is disjoint union of open disks in $\tilde{K}$

$\rightsquigarrow$ these are the 2-cells of $\tilde{K}$ attached to Γ

Cayley 2-complex of presentation $\langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$.

<u>Proposition VI.40</u> The 1-skeleton of $\tilde{K}$ is the Cayley graph of G with respect to generators $x_1, \dots, x_m$.

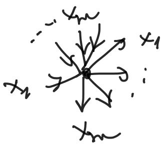
<u>Proof</u> I.5: Cayley graph is defined by taking a vertex for $\forall g \in G$. $\forall x_i$, $\exists$ edge from g to gx_i.

Γ has this form:

VI.19 $\Rightarrow$ $p^{-1}(b) \overset{1-1}{\longleftrightarrow} G$

$\exists$ edges pointing out of b in K corresp. to $x_1, \dots, x_m$, and

——— " ——— into to b in K ——— " ———.



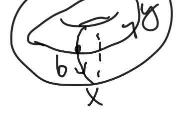
p is a local homeo., we have same picture around $\forall v \in p^{-1}(b)$. $v \leftrightarrow g \in G$

Edge labelled by x_i in $\tilde{K}$ starting from v ends at gx_i:

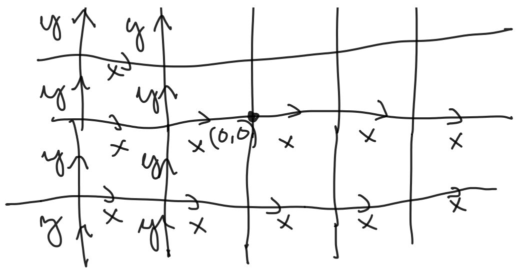
<u>Procedure VI.21</u> $\Rightarrow$ the lift of a loop x_i in K from v has endpoint at a vertex labelled gx_i.

edge labelled x_i connects g to gx_i. $\square$

<u>Ex.</u> $\langle x, y \mid xyx^{-1}y^{-1} \rangle \cong \mathbb{Z} \times \mathbb{Z}$ $\quad K \cong S^1 \times S^1$

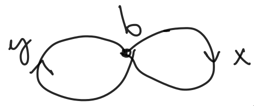


Cayley 2-complex has 1-skeleton:

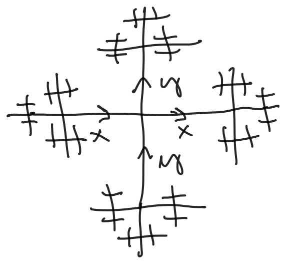


The 2-cells fill in the squares
$\Rightarrow$ Cayley 2-complex is $\mathbb{R}\times\mathbb{R}$.

<u>Ex</u> $F_2 = \langle x, y \mid \rangle$ $K = S^1 \vee S^1$



$\tilde{K} =$



Labelling on the edges of $\tilde{K}$ specifies $p: \tilde{K} \to K$, this is a covering map.
$\tilde{K}$ is a tree, it is simply-connected
Theorem $\underline{\text{IV}}$. 11 and Rem. $\underline{\text{IV}}$. 9.
$\Longrightarrow$ $\tilde{K}$ is the universal cover of K,
So it is the Cayley graph of F w.r.t. $\{x, y\}$.