Section III.2: A simplicial version - isomorphism

<u>Theorem III.27</u> $E(K,b) \cong \pi_1(|K|, b)$

<u>Proof.</u> $I_{(n)}$ triangulation of I s.t. $\forall$ 1-simplex has length $\frac{1}{n}$

edge paths of length $n \longleftrightarrow$ simplicial maps $I_{(n)} \to K$

$\Theta : \{\text{edge loops in } K \text{ based at } b\} \to \{\text{loops in } |K| \text{ based at } b\}$

If α is obtained from β by elementary contraction, then $\Theta(\alpha) \simeq \Theta(\beta)$ rel ∂I. So Θ induces a well-defined map

$\Theta : E(K,b) \to \pi_1(|K|, b)$.

<u>Claim</u>: Θ is an isomorphism.

<u>Homomorphism</u>: α, β edge loops, $\Theta(\alpha\beta) \simeq \Theta(\alpha)\Theta(\beta)$ rel ∂I

<u>Surjective</u>: $\ell : I \to |K|$, $\ell(0) = \ell(1) = b$

$I_{(1)}$ — $I_{(n)}$ is a subdivision

coarseness $(I_{(n)}) \to 0$ as $n \to \infty$

By Simplicial Approx. Thm. II.32 + relative version II.34, $\exists n$ and a simplicial map $\alpha : I_{(n)} \to K$ such that $\ell \simeq \Theta(\alpha)$ rel ∂I. So $\Theta([\alpha]) = \ell$.

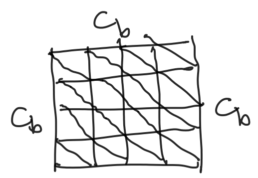
<u>Injective</u>: $\alpha = (b_0, b_1, \dots, b_n)$ loop based at b

$\Theta([\alpha]) = 1 \in \pi_1(|K|, b)$

$\Theta(\alpha) \simeq_H c_b$ rel ∂I

$H : I \times I \to |K|$

$(I \times I)_{(r)}$ triangulation



Simplicial Approx. Thm., $\exists r$

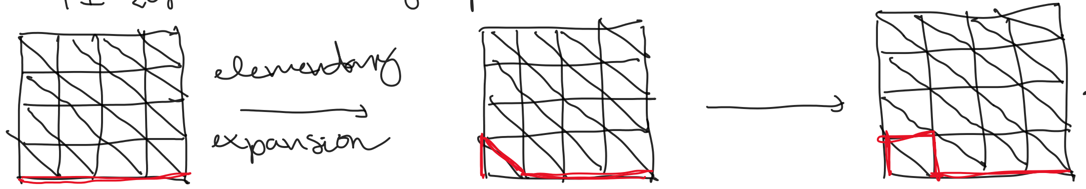
$\exists$ simplicial map $G : (I\times I)_{(r)} \to K$ such that $G \simeq H$ rel

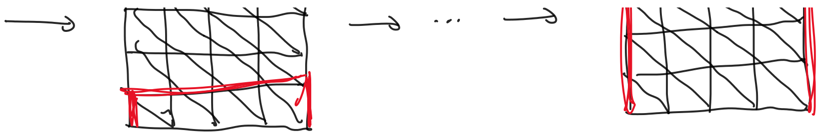
$G(\partial I \times I) = \{b\}$, $G(I \times \{1\}) = \{b\}$ by II.34

when r is a multiple of n, $G(\frac{i}{n}, 0) = b_i$

G sends 1-simplex between $(\frac{i}{n}, 0)$, $(\frac{i+1}{n}, 0)$ to 1-simplex (b_i, b_{i+1}), possible II.34.

$G|_{I\times\{0\}}$ is an edge path that contracts to α.





Get a sequence of elementary expansions from $\overset{\underset{\wr}{\alpha}}{G|_{I\times\{0\}}}$ to $(b, b, \dots, b) \sim (b)$.

$\Rightarrow \alpha \sim (b)$. So Θ is injective. $\square$

$E(K, b)$ is computable

Shows $E(K, b)$ is independent of triangulation $|K|$.

<u>Def</u> K simplicial complex, $b \in V(K)$
$n \geq 0$, <u>n-skeleton</u> of K, $\mathrm{skel}^n(K)$ is the subcomplex consisting of simplices of dimension $\leq n$.

<u>Cor</u> $\pi_1(|K|, b) \cong \pi_1(|\mathrm{skel}^2(K)|, b)$

<u>Proof</u> Def. of $E(K, b)$ involves only simplices of dim ≤ 2.

<u>Cor</u> $\pi_1(S^n) \cong 1$ for $n \geq 2$

<u>Proof</u> Triangulate S^n as the n-skeleton of Δ^{n+1} ($S^2 =$ )

$\mathrm{sk}^2(S^n) = \mathrm{sk}^2(\Delta^{n+1}) \Rightarrow \pi_1(S^n) \cong \pi_1(\Delta^{n+1}) \cong 1$

Δ^{n+1} is contractible $\square$