Section IV.4: The fundamental group of a graph - trees

Thm IV.11 The fundamental group of a graph is a free group.

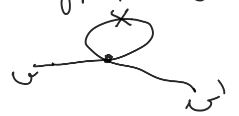
$\Gamma = (V, E)$ $\quad \sigma(e) \subseteq V$, $\quad |\sigma(e)| \in \{1, 2\}$

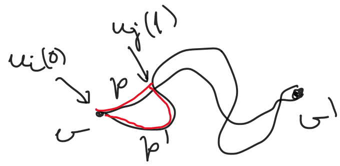
Def. $\Gamma = (V, E, \sigma)$
A subgraph of Γ is a graph with vertices $V' \subseteq V$, $E' \subseteq E$,
$\sigma' = \sigma|_{E'}$, $\sigma(e') \subseteq V'$. $\forall e' \in E'$.

Def An edge path in Γ is a concatenation $u_1 \dots u_n$, where $\forall$ u_i is either a path running along an edge of Γ, or a constant path based at a vertex.
An edge loop is an edge path $u: I \to \Gamma$, where $u(0) = u(1)$.
An edge path $u: I \to \Gamma$ is embedded if u is injective.
— " — loop — " — if u is injective in only two points have the same image, 0 and 1.

Def A tree is a connected graph that contains no embedded edge loop.

Lemma IV.15 In a tree, $\exists!$ edge path between distinct vertices.
Proof Γ tree, Γ connected $\Rightarrow \forall v, v' \in V(\Gamma)$ $\exists$ edge path from v to v'. The shortest such edge path is embedded





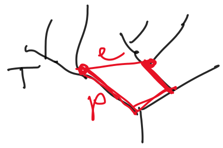
Unique: $p = u_1 \dots u_n$, $p' = u_1' \dots u_n'$
Let $u_i(0)$ be the first point on p where p & p' diverge. Let $u_j(1)$ be the next point on p which lies in p'.
Then the concatenation of $u_i \dots u_j$ with the sub-arc of p' between $u_j(1)$ and $u_i(0)$ is an embedded edge loop # $\square$

Def A maximal tree in a connected graph Γ is a subgraph T of Γ that is a tree, and the addition of any edge of $E(\Gamma) - E(T)$ to T gives a graph that is a not a tree.

Lemma IV.17 Γ connected, T is a subgraph, tree
TFAE:
(i) $V(T) = V(\Gamma)$
(ii) T is maximal

Proof (i) $\Rightarrow$ (ii) $\quad e \in E(\Gamma) \setminus E(T)$
if $|\sigma(e)| = 1$, then e is an edge loop, so $T \cup e$ not a tree.
if $|\sigma(e)| = 2$, then let p be an embedded edge path
in T connecting the vertices of $\sigma(e)$ ...

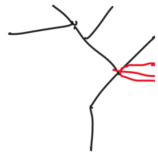
in T connecting the points of $\partial(e)$, $p \cup e$ is an embedded edge loop in $T \cup e$.



$(ii) \Rightarrow (i)$ T maximal. $v \in V(\Gamma) \setminus V(T)$

p shortest edge path from T to v, exists as Γ is connected.

Add first edge of p to T to get a bigger tree.



$\square$

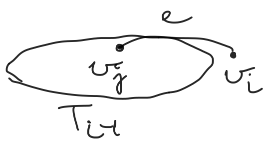
<u>Lemma IV.18</u> $\forall$ connected Γ contains a maximal tree.

$V(\Gamma)$ is countable, pick a total ordering on $V(\Gamma) = \{v_1, v_2, \dots\}$ such that $\forall i \geq 2$, $\exists$ edge from v_i to some v_j for $j < i$.

Construct a sequence of subgraphs $T_1 \subseteq T_2 \subseteq \dots$, $\forall T_i$ tree

$V(T_i) = \{v_1, \dots, v_i\}$

$V(T_1) := \{v_1\}$



$T_i := T_{i-1} \cup e$

no edge loop created $\Rightarrow T_i$ tree

$T := \bigcup T_i$ is a tree: if $\ell \subset T$ is an embedded edge loop $\Rightarrow \exists i : \ell \subset T_i$ as length$(\ell) < \infty$ #

$V(T) = V(\Gamma)$, T is a maximal tree.

$\square$