Section VI.2: The inverse image of the basepoint

$p : (\tilde{X}, \tilde{b}) \to (X, b)$ based covering map

$l, l' : I \to X$ loops based at b

$\tilde{l}, \tilde{l'}$ lifts from $\tilde{b}$

If $l \simeq l'$ rel $\partial I \xRightarrow{\text{Thm VI.16}} \tilde{l} \simeq \tilde{l'}$ rel $\partial I \Rightarrow \tilde{l}(1) = \tilde{l'}(1)$.

$$\lambda : \pi_1(X, b) \to p^{-1}(b)$$
$$[l] \longmapsto \tilde{l}(1)$$

Proposition VI.16 $g_1, g_2 \in \pi_1(X, b)$, $\lambda(g_1) = \lambda(g_2) \iff$ g_1, g_2 belong to same right coset of $p_* \pi_1(\tilde{X}, \tilde{b})$.
This induces a bijection between right cosets of $p_* \pi_1(\tilde{X}, \tilde{b})$ and $p^{-1}(b)$

Proof $g_i = [l_i]$. If $\tilde{l}_1(1) = \tilde{l}_2(1)$, then $\tilde{l}_1 \tilde{l}_2^{-1}$ is a loop based at $\tilde{b}$.

$[l_1][l_2]^{-1} = p_*[\tilde{l}_1 \tilde{l}_2^{-1}] \in p_* \pi_1(\tilde{X}, \tilde{b}) \Rightarrow g_1, g_2$ belong to same right coset of $p_* \pi_1(\tilde{X}, \tilde{b})$.

Conversely, suppose $[l_1][l_2]^{-1} \in p_* \pi_1(\tilde{X}, \tilde{b})$.
Then $l_1 l_2^{-1} \simeq_H p \circ l$ rel ∂I for a loop l in $\tilde{X}$ based at $\tilde{b}$.
H lifts to a homotopy rel ∂I between l and a lift of $\widetilde{l_1 l_2^{-1}}$.
So $l_1 l_2^{-1}$ lifts to a loop based at $\tilde{b}$. The first half of $\widetilde{l_1 l_2^{-1}}$ is $\tilde{l}_1$. The second half is $\tilde{l}_2^{-1}$ because its reverse is the lift of l_2 from $\tilde{b}$ $\Rightarrow \tilde{l}_1(1) = \tilde{l}_2(1)$.

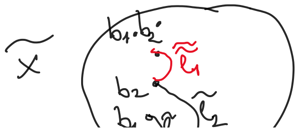
So λ : right cosets of $p_* \pi_1(\tilde{X}, \tilde{b}) \to p^{-1}(b)$ inj.
λ surjective: $\tilde{X}$ path-connected $\Rightarrow \forall x \in p^{-1}(b)\ \exists u : I \to \tilde{X}$: $u(0) = \tilde{b}$, $u(1) = x$. Then $p \circ u$ is a loop in X that lifts to u, so $\lambda([p \circ u]) = x$. □

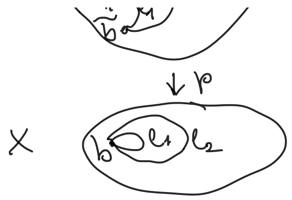
Corollary VI.20 A loop l in X based at b lifts to a loop based at $\tilde{b} \iff [l] \in p_* \pi_1(\tilde{X}, \tilde{b})$.

Proof l lifts to a loop based at $\tilde{b} \iff \lambda([l]) = \tilde{b}$.
But $\tilde{b}$ corresponds to the identity coset of $p_* \pi_1(\tilde{X}, \tilde{b})$
$\iff [l] \in p_* \pi_1(\tilde{X}, \tilde{b})$. □

Procedure VI.21 Suppose $p_* \pi_1(\tilde{X}, \tilde{b})$ is a normal subgroup of $\pi_1(X, b)$. Let $b_1, b_2 \in p^{-1}(b)$. These correspond to elts in $\pi_1(X, b) / p_* \pi_1(\tilde{X}, \tilde{b})$. We wish to find the point of $p^{-1}(b)$ (called $b_1 \cdot b_2$) corresp. to the product of these two elts.

$\tilde{l}_1, \tilde{l}_2$ paths from $\tilde{b}$ to b_1, b_2 resp. in $\tilde{X}$
$l_i := p \circ \tilde{l}_i$ loop based at b $i = 1, 2$
$\lambda([l_i]) = b_i$





$\lambda([l_1][l_2])$: lift $l_1 l_2$ to path from b_1 and $b_1 \cdot b_2$ is its endpoint ($b_1 \cdot b_2 = \widetilde{l_1 l_2}(1)$)
Alternatively, the second half of $\widetilde{l_1 l_2}$ is the lift of l_2 from b_1.

<u>Def</u> When $\tilde{X}$ is simply-connected, a based covering map $p : (\tilde{X}, \tilde{b}) \to (X, b)$ is known as the <u>universal cover</u> of X.
Then $p^{-1}(b) \overset{1-1}{\longleftrightarrow} \pi_1(X, b)$, gives a method for computing $\pi_1(X)$.

<u>Theorem VI.23</u> $\pi_1(S^1) \cong \mathbb{Z}$

<u>Proof</u> $\mathbb{R} \xrightarrow{p} S^1$, $t \mapsto e^{2\pi i t}$ covering map, $\pi_1(\mathbb{R}) = 1$, so universal cover.

$\pi_1(S^1, 1) \overset{1-1}{\longleftrightarrow} p^{-1}(1) = \mathbb{Z}$

Check group structure on $p^{-1}(1) \xrightarrow{\sim} \pi_1(S^1, 1)$ agrees w/ group structure on $\mathbb{Z}$.
$n_1, n_2 \in \mathbb{Z}$
$n_1 \cdot n_2$ using Procedure VI.21 : $\tilde{l}_2$ path from 0 to n_2, let $l_2 = p \circ \tilde{l}_2$. Then lift of l_2 from n_1 ends at $n_1 + n_2$.
Hence $\lambda : \pi_1(S^1, 1) \to \mathbb{Z}$ is an isomorphism. □

<u>Example</u>. $\mathbb{R} \times \mathbb{R} \to S^1 \times S^1$, $(x, y) \mapsto (e^{2\pi i x}, e^{2\pi i y})$ universal cover of $S^1 \times S^1$.

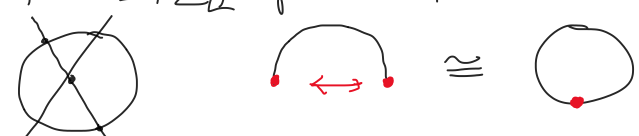
$b := (1, 1) \in S^1 \times S^1$, $\tilde{b} := (0, 0) \in \mathbb{R} \times \mathbb{R}$

$p^{-1}(b) = \mathbb{Z} \times \mathbb{Z} \subset \mathbb{R} \times \mathbb{R}$. $\pi_1(S^1 \times S^1) \overset{1-1}{\longleftrightarrow} \mathbb{Z} \times \mathbb{Z}$

Using procedure VI.23, this is an isomorphism.
$\Rightarrow \pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$ (c.f. V.17).

<u>Thm VI.26</u> $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$ if $n > 1$, and $\cong \mathbb{Z}$ if $n = 1$.

<u>Proof</u> $\mathbb{RP}^1 \cong S^1$
$\Rightarrow \pi_1(\mathbb{RP}^1) \cong \mathbb{Z}$



$n > 1$ $p : S^n \to \mathbb{RP}^n$ 2-to-1
$v \mapsto \langle v \rangle$ line in $\mathbb{R}^{n+1}$

$\pi_1(S^n) = 1$ if $n > 1$ $|p^{-1}(b)| = 2 \Rightarrow |\pi_1(\mathbb{RP}^n)| = 2$
$\Rightarrow \pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$.