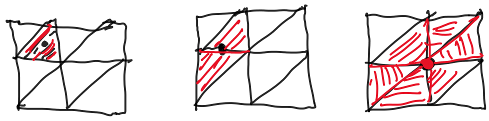
Section II.2: Simplicial approximation - stars

Thm (Simplicial approximation thm.)
K, L simplicial complexes
$f : |K| \to |L|$
$\exists$ subdivision K' of K and a simplicial map
$g : K' \to L$ such that $|g| \simeq f$.

Def K simplicial cx., $x \in |K|$, star of x in $|K|$
$st_K(x) = \bigcup \{ \text{inside}(\sigma) : \sigma \text{ is a simplex of } K \text{ and } x \in \sigma \}$



Lemma II.23 $\forall x \in |K|$, $st_K(x)$ is open in $|K|$.

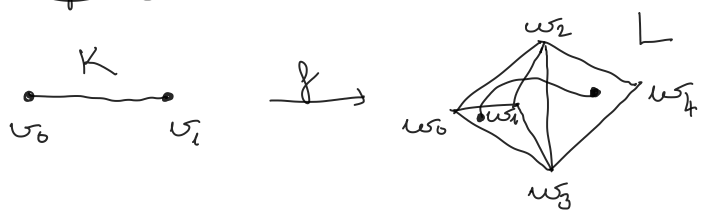
Proof. $|K| \setminus st_K(x) = \bigcup \{ \text{insides}(\sigma) : \sigma \text{ is a simplex of } |K|, x \notin \sigma \}$
$= \bigcup \{ \sigma : \sigma \text{ is a simplex of } K \text{ and } x \notin \sigma \}$

$x \notin \sigma \Rightarrow x \notin \tau$ for $\forall$ face τ of σ

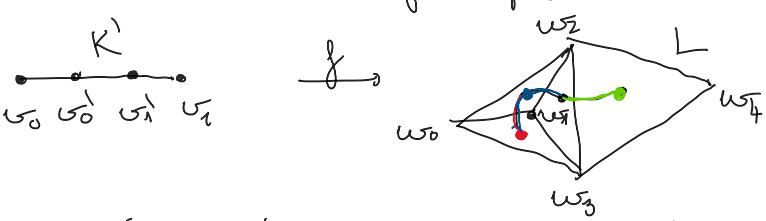
subcomplex of K, so closed $\square$

Proposition II.24 $f : |K| \to |L|$ continuous.
If $\forall v \in V(K)$ $\exists$ $g(v) \in V(L)$ such that $f(st_K(v)) \subseteq st_L(g(v))$, then g is a simplicial map. $V(K) \to V(L)$, and $|g| \simeq f$.

Example



$f(st_K(v_0)) \nsubseteq st_L(v)$ for any $v \in V(L)$.
So condition of Proposition doesn't hold.



$f(st_{K'}(v_0)) \subseteq st_L(w_0) \cap st_L(w_1)$

$f(\mathrm{st}_{K'}(v_0')) \subseteq \mathrm{st}_L(w_1)$

$f(\mathrm{st}_{K'}(v_1')) \subseteq \mathrm{st}_L(w_2)$

$f(\mathrm{st}_{K'}(v_1)) \subseteq \mathrm{st}_L(w_2) \cap \mathrm{st}_L(w_3)$

4 choices for g:

$v_0 \longmapsto$	w_0	w_0	w_1	w_1
$v_0' \longmapsto$	w_1	w_1	w_1	w_1
$v_1' \longmapsto$	w_2	w_2	w_2	w_2
$v_1 \longmapsto$	w_2	w_3	w_2	w_3

Proof of Prop. II.24

Claim. $\sigma = (v_0, \dots, v_n)$ be a simplex of K, $x \in \mathrm{inside}(\sigma)$
Let τ be the simplex of L s.t. $f(x) \in \mathrm{inside}(\tau)$.
Then $g(v_0), \dots, g(v_n)$ are vertices of τ.

Proof claim. $x \in \mathrm{inside}(\sigma) \Rightarrow x \in \mathrm{st}_K(v_i)$ for $\forall i \in \{0, \dots, n\}$
So $f(x) \in f(\mathrm{st}_K(v_i)) \subseteq \mathrm{st}_L(g(v_i)) \Rightarrow \mathrm{inside}(\tau) \subseteq \mathrm{st}_L(g(v_i))$
$\Rightarrow g(v_i) \in V(\tau)$. $\square$

$f \simeq |g|$: $x \in |K|$, $f(x) \in \mathrm{inside}(\tau)$
$x = \sum_{i=0}^n \lambda_i v_i$, where $v_0, \dots, v_n$ are vertices of same simplex and $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$.

$|g|(x) = \sum_{i=0}^n \lambda_i g(v_i)$

$g(v_0), \dots, g(v_n)$ are vertices of τ

Define straight-line homotopy between $f(x)$ and $|g|(x)$.
Well-defined $H: |K| \times I \to |L|$: independent of choice of simplex containing x. Different choices give same $H(x,t)$ $\forall t$. $\square$

Addendum II.28 K, L, f, g as in II.24.
A subcx. of K, B subcx. of L s.t. $f(|A|) \subseteq |B|$.
Then $g(A) \in B$ and $H: |g| \simeq f$ sends $|A|$ to $|B|$ throughout.

Proof $v \in A$, $f(v) \in \mathrm{inside}(\tau)$
Claim $\Rightarrow g(v) \in V(\tau)$
Since $f(v) \in |B| \Rightarrow \tau \subseteq |B| \Rightarrow g(v) \in B$.

$x \in |A|$, let $(v_0, \dots, v_n)$ simplex of K containing x in its inside.

$f(x) \in \mathrm{inside}(\tau')$

$\Rightarrow \tau' \subset B$ because $f(x) \in |B|$. Claim $\Rightarrow g(v_0), \dots, g(v_n)$ vertices of τ' so vertices of B. Straight-line homotopy between f and $|g|$ sends x into τ' throughout, so image of x remains in $|B|$. $\square$