Section V.2: Tietze transformations

Def A Tietze transformation is the following moves applied to a group presentation $\langle x_1,\dots,x_m \mid r_1,\dots,r_n\rangle$:

(T1) permuting generators & relations
(T2) adding or removing e to relations
(T3) performing elementary contraction or expansion to r_i
(T4) inserting a relation r_i or its inverse into one of the r_j or removing it.
(T5) add a generator x_{m+1} and a relation $w(x_1,\dots,x_m)x_{m+1}^{-1}$, which defines x_{m+1} as a word in the old generators, or the reverse of this operation.

Rem (T5) $\exists$ infinitely many possibilities for w
Moves (T1) - (T5) do not change the isomorphism class of the group.

Example $\langle a,b \mid abab^{-1}\rangle \cong \langle b,c \mid cbbc\rangle$:

$$\begin{aligned}
\langle a,b \mid abab^{-1}\rangle &\xrightarrow{(T5)} \langle a,b,c \mid abab^{-1},\ ab^{-1}c^{-1}\rangle\\
&\xrightarrow{(T4)} \langle a,b,c \mid (ab^{-1}c^{-1})^{-1}abab^{-1},\ ab^{-1}c^{-1}\rangle\\
&\xrightarrow{(T4)} \langle a,b,c \mid (ab^{-1}c^{-1})^{-1}ab(ab^{-1}c^{-1})^{-1}ab^{-1},\ ab^{-1}c^{-1}\rangle\\
&\xrightarrow{(T3)} \langle a,b,c \mid cbbc,\ ab^{-1}c^{-1}\rangle\\
&\xrightarrow{(T2)} \langle a,b,c \mid cbbc,\ ab^{-1}c^{-1},\ e\rangle\\
&\xrightarrow{(T4)} \langle a,b,c \mid cbbc,\ ab^{-1}c^{-1},\ (ab^{-1}c^{-1})^{-1}\rangle\\
&\xrightarrow{(T4)} \langle a,b,c \mid cbbc,\ ab^{-1}c^{-1}(ab^{-1}c^{-1})^{-1},\ (ab^{-1}c^{-1})^{-1}\rangle\\
&\xrightarrow{(T3)} \langle a,b,c \mid cbbc,\ e,\ (ab^{-1}c^{-1})^{-1}\rangle\\
&\xrightarrow{(T2)} \langle a,b,c \mid cbbc,\ \underline{cba^{-1}}\rangle\\
&\xrightarrow{(T5)} \langle b,c \mid cbbc\rangle
\end{aligned}$$

Theorem V.14 (Tietze) Any two finite presentations of a group G are convertible into each other by a finite sequence of Tietze transformations.

Lemma V.15 $\langle x_1,\dots,x_m \mid r_1,\dots,r_n\rangle \cong G$
w word in $x_1,\dots,x_m$, trivial in G.
Then $\exists$ a finite sequence of (T2), (T3), (T4) moves taking the presentation to $\langle x_1,\dots,x_m \mid r_1,\dots,r_n, w\rangle$.

Proof By (T2), add relation e. Since $w \sim e$ in G, it can be obtained from e by a sequence of moves (1) & (2) of Proposition V.6. So $\exists$ sequence of moves (T3) & (T4) taking $\langle x_1,\dots,x_m \mid r_1,\dots,r_n\rangle$ to $\langle x_1,\dots,x_m \mid r_1,\dots,r_n, w\rangle$. $\square$

Proof Thm V.14 $\langle x_1,\dots,x_m \mid r_1,\dots,r_n\rangle \cong \langle x_1',\dots,x_{m'}' \mid r_1',\dots,r_{n'}'\rangle \cong G.$

$x_i' = X_i'$ word in $x_1,\dots,x_m$

$x_i = X_i$ word in $x_1',\dots,x_{m'}'$

$m' \times$ (T3) : $\langle x_1,\dots,x_m, x_1',\dots,x_{m'}' \mid r_1,\dots,r_n, X_1'(x_1')^{-1},\dots,X_{m'}'(x_{m'}')^{-1}\rangle$

Since $x_i = X_i$ in G, by Lemma V.15, $\exists$ sequence of (T2), (T3), (T4) moves taking the presentation to

$$\left\langle x_1,\dots,x_m, x_1',\dots,x_{m'}' \;\middle|\; \begin{array}{l} r_1,\dots,r_n \\ X_1'(x_1')^{-1},\dots,X_{m'}'(x_{m'}')^{-1} \\ X_1 x_1^{-1},\dots,X_m x_m^{-1} \end{array}\right\rangle$$

$r_1',\dots,r_{n'}'$ represent trivial words in G. Hence by Lemma V.15, $\exists$ sequence of moves (T2), (T3), (T4) taking the presentation to

$$\left\langle x_1,\dots,x_m, x_1',\dots,x_{m'}' \;\middle|\; \begin{array}{l} r_1,\dots,r_n \\ X_1'(x_1')^{-1},\dots,X_{m'}'(x_{m'}')^{-1} \\ X_1 x_1^{-1},\dots,X_m x_m^{-1} \\ r_1',\dots,r_{n'}' \end{array}\right\rangle$$

Symmetric in $x_1,\dots,x_m$ & $x_1',\dots,x_{m'}'$. By applying (T1) and reversing the above procedure, we obtain

$\langle x_1',\dots,x_{m'}' \mid r_1',\dots,r_{n'}'\rangle$. $\square$