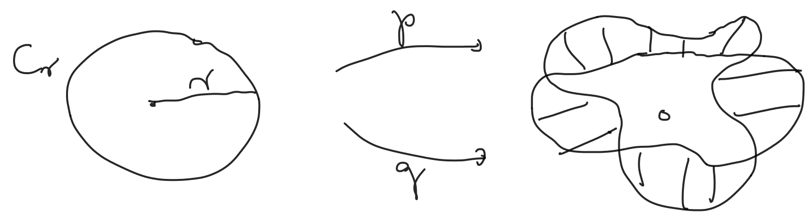
Section III.4: The fundamental theorem of algebra

Thm $p(z) \in \mathbb{C}[z]$ non-constant $\Rightarrow \exists z \in \mathbb{C} : p(z) = 0$

Proof. $p(z) = a_n z^n + \dots + a_0$, $a_n \neq 0$, $n > 0$

$C_r = \{ z \in \mathbb{C} : |z| = r \}$, r large real number

$k := \frac{p(r)}{r^n}$, $q(z) = k z^n$, then $p(r) = q(r)$



Claim $r \gg 0 \Rightarrow p|_{C_r}$, $q|_{C_r}$, and the straight-line homotopy between them avoid $0 \in \mathbb{C}$.

Proof Claim If not, $\exists z \in C_r$, $t \in I$,

$(1-t)p(z) + t\,q(z) = 0$.

$$(1-t)(a_n z^n + \dots + a_0) + t\left(\frac{a_n |z|^n + \dots + a_0}{|z|^n}\right) z^n = 0$$

$(|z| = r)$

$$\Leftrightarrow \quad a_n z^n + \dots + a_0 = t\left(a_{n-1} z^{n-1} + \dots + a_0 - a_{n-1}\frac{z^n}{|z|} - \dots - a_0 \frac{z^n}{|z|^n}\right)$$

left-hand side has order $|z|^n$, right-hand side has order $|z|^{n-1}$. Hence $|t| \to \infty$ as $|z| \to \infty$. So no solution for $t \in I$ when $r \gg 0$. ☐

So $p|_{C_r} \simeq q|_{C_r}$ in $\mathbb{C} \setminus \{0\}$ rel $\{r\}$.

Suppose p has no root in $\mathbb{C}$. $\Rightarrow p : \mathbb{C} \to \mathbb{C} \setminus \{0\}$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "p"] & \mathbb{C}\setminus\{0\} \\ C_r \arrow[u, "i"] \arrow[ur, "p|_{C_r}"'] & \end{tikzcd}$$

commutative (i.e. $p|_{C_r} = p \circ i$)

$$\begin{tikzcd} 0 = \pi_1(\mathbb{C}, r) \arrow[r, "p_*"] & \pi_1(\mathbb{C}\setminus\{0\}, r) \cong \mathbb{Z} \\ \mathbb{Z} \cong \pi_1(C_r, r) \arrow[u, "i_*"] \arrow[ur] & \end{tikzcd}$$

retracts to S^1

$\pi_1(C_r, r) \cong S^1$

$(p|_{C_r})_* = (q|_{C_r})_*$ because $p \simeq q$ rel r by Claim

$q([\ell]) = [\ell^n] \neq 0 \in \pi_1(\mathbb{C} \setminus \{0\}, r)$ #

generator of $\pi_1(C_r, r)$

$\Rightarrow (p|_{C_r})_* = p_* \circ i_* = 0$ ☐