Section I.1

<u>Def.</u> A countable <u>graph</u> consists of

- a countable set of <u>vertices</u> V
- a countable set of <u>edges</u> E
- a function $\delta : E \to P(V)$ such that $|\delta(e)| \in \{1, 2\}$ for every $e \in E$.

$\delta(e)$: endpoints of e

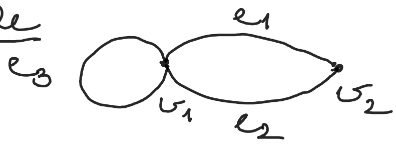
$\Gamma \rightsquigarrow$ topological space

V with discrete topology

$I = [0, 1]$

Take $E \times I$ and identify $e \times \{0\}$ and $e \times \{1\}$ with distinct points of $\delta(e)$ if $|\delta(e)| = 2$ and with the point of $\delta(e)$ if $|\delta(e)| = 1$.

<u>Example</u>



$V = \{v_1, v_2\}$

$E = \{e_1, e_2, e_3\}$

$\delta(e_1) = \{v_1\}$

$\delta(e_2) = \delta(e_3) = \{v_2, v_3\}$

<u>Def</u> An <u>orientation</u> on the graph Γ :

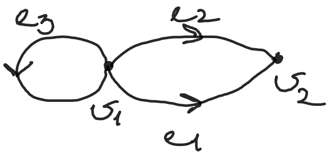
$\iota : E \to V$, $\tau : E \to V$

such that $\delta(e) = \{\iota(e), \tau(e)\}$ for every $e \in E$

$\iota(e)$: initial vertex of e

$\tau(e)$: terminal vertex of e

<u>Ex.</u>



$\iota(e_3) = v_1$ $\tau(e_3) = v_1$

$\iota(e_1) = \iota(e_2) = v_1$ $\tau(e_1) = \tau(e_2) = v_2$

<u>Def</u> G group, $S \subseteq G$ generates G

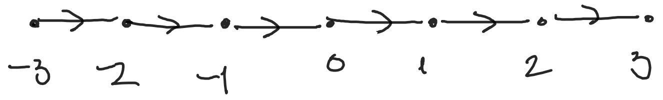
The associated <u>Cayley graph</u> is an oriented graph with $V = G$ and $E = G \times S$

$\iota : G \times S \to G$ $\tau : G \times S \to S$

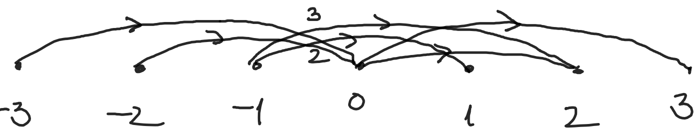
$(g, s) \mapsto g \qquad (g, s) \mapsto gs$

The edge from g to gs is labeled by s.

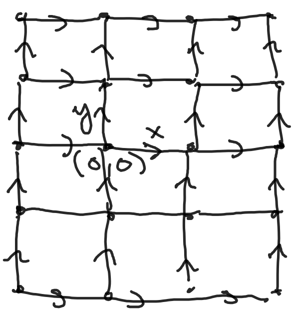
Example $G = \mathbb{Z}$, $S = \{1\}$



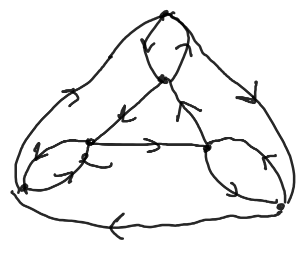
Example $G = \mathbb{Z}$, $S = \{2, 3\}$



Example $G = \mathbb{Z} \times \mathbb{Z}$ $\quad S = \{x = (1,0), y = (0,1)\}$



Example S_3, $S = \{x \in (1\,2\,3), y = (1\,2)\}$



Every $g \in G$ can be written as

$$g = s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_n^{\varepsilon_n}, \quad s_i \in S, \ \varepsilon_i \in \{-1, 1\}$$

Specifies a path from e to g in Γ where we traverse the edges $s_1, \ldots, s_n$ (going backwards edge labeled by s_i if $\varepsilon_i = -1$)

If $g = e$, $s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_n^{\varepsilon_n} = e$, and corresponding path is a loop starting and ending at e.

Proposition I.13. Any two points in a Cayley graph can be joined by a path.