Section VI.3: Uniqueness of coverings

<u>Def</u> A space Y is <u>locally path-connected</u> if $\forall y \in Y$ and $\forall$ nbhd V of y $\exists$ open nbhd of y s.t. U path-connected and $U \subset V$.

<u>Rem</u> A simplicial complex is locally path-connected

<u>Thm VI.29</u> (Existence of lifts) Let $p : (\tilde{X}, \tilde{b}) \to (X, b)$ a based covering map, Y loc. path-conn., path-conn., $f : (Y, y_0) \to (X, b)$. Then f has a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{b})$ if and only if $f_* \pi_1(Y, y_0) \subseteq p_* \pi_1(\tilde{X}, \tilde{b})$.

$$\begin{tikzcd} & (\tilde{X}, \tilde{b}) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & (X, b) \end{tikzcd}$$

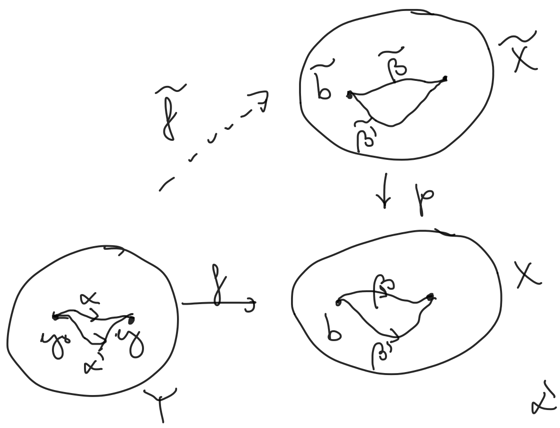
<u>Proof</u>

$$\begin{tikzcd} & \pi_1(\tilde{X}, \tilde{b}) \arrow[d, "p_*"] \\ \pi_1(Y, y_0) \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"] & \pi_1(X, b) \end{tikzcd}$$

$f_* = p_* \circ \tilde{f}_* \Rightarrow \operatorname{im} f_* \in \operatorname{im} p_*$

So condition is necessary.

Suppose $f_* \pi_1(Y, y_0) \subseteq p_* \pi_1(\tilde{X}, \tilde{b})$.



$y \in Y$, α path in Y from y_0 to y

$\beta := f \circ \alpha$

$\tilde{\beta}$ lifts of β from $\tilde{b}$

$\tilde{f}(y) := \tilde{\beta}(1)$

$p \tilde{f}(y) = p \tilde{\beta}(1) = \beta(1) = f \circ \alpha(1) = f(y)$

so diagram commutes.

$\tilde{f}$ is indep. of choice of α:

α' another path from y_0 to y.

$\alpha' \alpha^{-1}$ loop in Y based at y_0

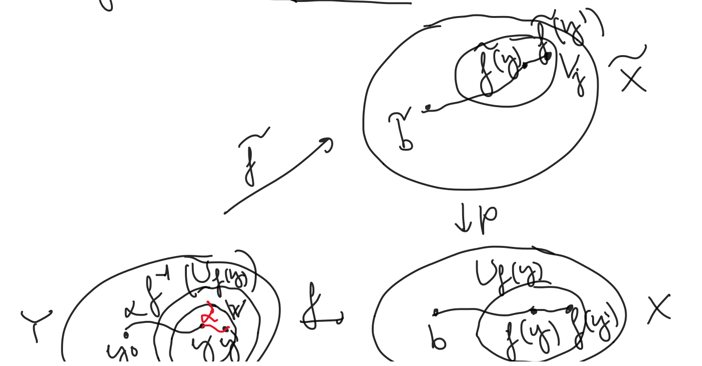
$(f \circ \alpha')(f \circ \alpha^{-1})$ is a loop in X based at b

$[(f \circ \alpha')(f \circ \alpha^{-1})] = f_*([\alpha' \alpha^{-1}]) \in \operatorname{im}(f_*) \underset{\text{assumption}}{\subseteq} \operatorname{im}(p_*)$

$\Rightarrow$ $(f \circ \alpha')(f \circ \alpha^{-1})$ lifts to a loop in $\tilde{X}$ based at $\tilde{b}$.

$\Rightarrow$ $\tilde{\beta}(1) = \tilde{\beta}'(1)$, so $\tilde{f}$ is well-defined

<u>$\tilde{f}$ is continuous:</u>



$U_{f(y)}$ elementary nbhd of $f(y)$

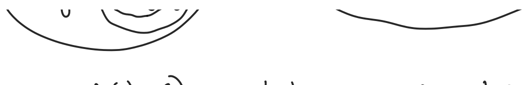
$p|_{V_j} : V_j \to U_{f(y)}$ is a homeo.

$V_j \ni \tilde{f}(y)$

$\exists$ nbhd W of y s.t.

W is path-connected &

$W \in f^{-1}(U_{f(y)})$



$\forall y' \in W$ $\exists$ path α' from y to y' in W
We can define $\tilde{f}(y')$ using $\alpha\alpha'$.

$f(W) \subseteq U_{f(y)} \Rightarrow$ the lift of $f \circ \alpha'$ is $(p|_{V_j})^{-1} \circ (f \circ \alpha')$

$\Rightarrow \tilde{f}$ is continuous at y. $\square$

<u>Def</u>

$$\begin{tikzcd}(\tilde{X}, \tilde{b}) \arrow[rr, "f"] \arrow[dr, "p"'] & & (\tilde{X}', \tilde{b}') \arrow[dl, "p'"] \\ & (X, b) & \end{tikzcd}$$

p, p' based covering maps are <u>equivalent</u> if $\exists$ homeo. $f : (\tilde{X}, \tilde{b}) \to (\tilde{X}', \tilde{b}')$ such that $p' \circ f = p$.

<u>Theorem VI.31</u> (Uniqueness of covering spaces)
X path-conn., locally path conn., $b \in X$ basepoint.
Then $\forall$ $H \leq \pi_1(X, b)$, there is at most one based covering $p : (\tilde{X}, \tilde{b}) \to (X, b)$, up to equivalence, such that $p_* \pi_1(\tilde{X}, \tilde{b}) = H$.

<u>Proof</u> Suppose $p' : (\tilde{X}', \tilde{b}') \to (X, b)$ cover s.t. $p'_* \pi_1(\tilde{X}', \tilde{b}') = H$.

$$\begin{tikzcd}(\tilde{X}', \tilde{b}') \arrow[r, "\tilde{p}'"] \arrow[dr, "p'"'] & (\tilde{X}, \tilde{b}) \arrow[r, "\tilde{p}"] \arrow[d, "p"] & (\tilde{X}', \tilde{b}') \arrow[dl, "p'"] \\ & (X, b) & \end{tikzcd}$$

$\xRightarrow{\text{Thm VI.29}}$ $\exists$ lift $\tilde{p}' : (\tilde{X}', \tilde{b}') \to (X, b)$ of p'
$(Y := \tilde{X}', f = p')$

Similarly, p lifts to $\tilde{p} : (\tilde{X}, \tilde{b}) \to (\tilde{X}', \tilde{b}')$.
$\tilde{p}\tilde{p}'$ is a lift p' $(Y = \tilde{X}', f = p')$
$\xRightarrow{\text{Thm VI.13}}$ $\tilde{p}\tilde{p}' = id_{\tilde{X}'}$ as $id_{\tilde{X}'}$ is also a lift of p'.

So $\tilde{p}'$ is a homeomorphism, and the coverings are equivalent $\square$