Section V.3: Push-outs

Def G_0, G_1, G_2 groups, $\phi_1 : G_0 \to G_1$, $\phi_2 : G_0 \to G_2$ homom.
$\langle X_1 | R_1 \rangle$, $\langle X_2 | R_2 \rangle$ canonical presentations of G_1, G_2, resp.
The push-out $G_1 *_{G_0} G_2 := \langle X_1 \cup X_2 \mid R_1 \cup R_2 \cup \{\phi_1(g) = \phi_2(g) : g \in G_0\} \rangle$
assume $X_1 \cap X_2 = \emptyset$

Rem $G_1 *_{G_0} G_2$ depends on ϕ_1, ϕ_2

Rem By Lemma V.11, $X_1 \hookrightarrow X_1 \cup X_2$ and $X_2 \hookrightarrow X_1 \cup X_2$ induce homomorphisms $\alpha_1 : G_1 \to G_1 *_{G_0} G_2$, $\alpha_2 : G_2 \to G_1 *_{G_0} G_2$ such that the following diagram commutes:

$$\begin{tikzcd} G_0 \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & G_1 \arrow[d, "\alpha_1"] \\ G_2 \arrow[r, "\alpha_2"'] & G_1 *_{G_0} G_2 \end{tikzcd}$$

$\alpha_1 \circ \phi_1 = \alpha_2 \circ \phi_2$

Proposition V.19 (Universal property of push-outs)
$G_1 *_{G_0} G_2$ be the push-out of

$$\begin{tikzcd} G_0 \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & G_1 \\ G_2 & \end{tikzcd}$$

Let $\beta_1 : G_1 \to H$, $\beta_2 : G_2 \to H$ are group homom. such that

$$\begin{tikzcd} G_0 \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & G_1 \arrow[d, "\beta_1"] \\ G_2 \arrow[r, "\beta_2"'] & H \end{tikzcd}$$

is commutative (i.e. $\beta_1 \circ \phi_1 = \beta_2 \circ \phi_2$).

Then $\exists!$ homom. $\beta : G_1 *_{G_0} G_2 \to H$ such that

$$\begin{tikzcd} G_0 \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & G_1 \arrow[d, "\alpha_1"] \arrow[ddr, bend left, "\beta_1"] & \\ G_2 \arrow[r, "\alpha_2"'] \arrow[drr, bend right, "\beta_2"'] & G_1 *_{G_0} G_2 \arrow[dr, dashed, "\beta"] & \\ & & H \end{tikzcd}$$

commutative (i.e. $\beta \circ \alpha_1 = \beta_1$, $\beta \circ \alpha_2 = \beta_2$)

Proof $G_1 *_{G_0} G_2$ is generated by $G_1 \cup G_2$.
Let $\beta(g_i) := \beta_i(g_i)$ for $g_i \in G_i$, $i \in \{1, 2\}$.
Forced on us by commutativity ($\beta \circ \alpha_1 = \beta_1$, $\beta \circ \alpha_2 = \beta_2$).
So, if β exists, it is unique.
β well-defined: $\beta(r) = e$ $\forall$ relation r for $G_1 *_{G_0} G_2$.
$\beta(r) = e$ if $r \in R_1 \cup R_2$, since β_1 and β_2 are well-defined
The other type of relation is $\phi_1(g)\phi_2(g)^{-1}$ for $g \in G_0$

But $\beta\phi_1(g)(\beta\phi_2(g))^{-1} = e$ by commutative square $(\beta_1 \circ \phi_1 = \beta_2 \circ \phi_2)$. $\square$

<u>Lemma V.20.</u> $\langle X_1' \mid R_1' \rangle$, $\langle X_2', R_2' \rangle$ presentations of G_1, G_2, resp. $X_1' \cap X_2' = \emptyset$. Then $G_1 *_{G_0} G_2 \cong \underbrace{\langle X_1' \cup X_2' \mid R_1' \cup R_2' \cup \{\phi_1(g) = \phi_2(g) : g \in G_0\} \rangle}_{H}$.

<u>Proof</u> $G := G_1 *_{G_0} G_2$

$G_1 \to \langle X_1' \mid R_1' \rangle$, $G_2 \to \langle X_2' \mid R_2' \rangle$ "identity" maps.

By Lemma V.11, they induce homomorphisms $\beta_1 : G_1 \to H$ and $\beta_2 : G_2 \to H$. Satisfy Prop. V.19 because $\phi_1(g) = \phi_2(g)$ $\forall g \in G_0$. So, by Prop. V.19, $\exists \beta : G \to H$ s.t.:

$$\begin{tikzcd}
G_0 \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & G_1 \arrow[d] \arrow[ddr, bend left, "\beta_1"] & \\
G_2 \arrow[r] \arrow[drr, bend right, "\beta_2"'] & G_1 *_{G_0} G_2 \arrow[dr, dashed, "\beta"] & \\
 & & H \arrow[ul, "\phi"']
\end{tikzcd}$$

commutes. We define an inverse $\phi : H \to G$, as follows:

$\exists$ function $X_i' \to G_i \xrightarrow{\alpha_i} G$

$f : X_1' \cup X_2' \to G$ induces a homom.

$\phi : F(X_1' \cup X_2') \to G$

By Lemma V.11, this descends to a homom. $\phi : H \to G$ because $\phi(r) = e \in H$ $\forall$ relation r in H.

ϕ is inverse of $\beta \Rightarrow G \cong H$. $\square$

<u>Def</u> $G_0 = \{e\}$, then $G_1 *_{G_0} G_2 = G_1 * G_2$ <u>free product</u> of G_1, G_2 (: only depends on G_1, G_2).

<u>Example</u> $\mathbb{Z} * \mathbb{Z} \cong F\langle x, y \rangle$ becase $\mathbb{Z} = \langle x \mid \rangle$, $\mathbb{Z} = \langle y \mid \rangle$, so $\mathbb{Z} * \mathbb{Z} = \langle x, y \mid \rangle$ by Lemma V.20.

<u>Def</u> When $\phi_1 : G_0 \to G_1$ and $\phi_2 : G_0 \to G_1$ are injective, then we say that $G_1 *_{G_0} G_2$ <u>amalgamated free product</u> of G_1 and G_2 along G_0.