Section V.4: The Seifert-van Kampen theorem

Theorem V.24 (Seifert - van Kampen)
K space, $K_1, K_2 \subseteq K$ subspaces are open in K
$K_1, K_2, K_1 \cap K_2$ are path-connected.
$b \in K_1 \cap K_2$ basepoint, $\iota_1 : K_1 \cap K_2 \hookrightarrow K_1$, $\iota_2 : K_1 \cap K_2 \hookrightarrow K_2$ embeddings
push-out:

$$\begin{tikzcd} \pi_1(K_1 \cap K_2, b) \arrow[r, "\iota_{1*}"] \arrow[d, "\iota_{2*}"'] & \pi_1(K_1, b) \arrow[d] \\ \pi_1(K_2, b) \arrow[r] & G \end{tikzcd}$$

Then $G \cong \pi_1(K, b)$. Moreover, $\pi_1(K_1, b) \to G \cong \pi_1(K, b)$, $\pi_1(K_2, b) \to G \cong \pi_1(K, b)$ are the maps induced by the inclusions $K_1 \hookrightarrow K$ and $K_2 \hookrightarrow K$, resp.

Alternatively:

Theorem V.25 Let $K, K_1, K_2, \iota_1, \iota_2, b$ be as above.
Let $\langle X_1, R_1 \rangle$, $\langle X_2 | R_2 \rangle$ be presentations of $\pi_1(X_1, b)$, $\pi_1(X_2, b)$, resp.
Then

$$\pi_1(K, b) \cong \langle X_1 \cup X_2 \mid R_1 \cup R_2 \cup \{\iota_{1*}(g) = \iota_{2*}(g) : g \in \pi_1(K_1 \cap K_2, b)\} \rangle$$

$X_1 \cap X_2 = \emptyset$

The homomorphism $\langle X_i | R_i \rangle \to \pi_1(K, b)$ induced by inclusion of generators $X_i \hookrightarrow X_1 \cup X_2$ for $i \in \{1, 2\}$ agrees with the maps induced by $K_i \hookrightarrow K$.

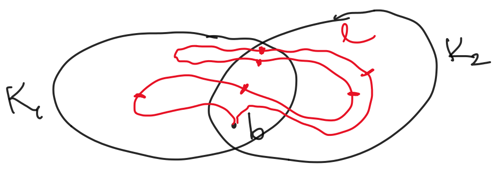
Proof

$$\begin{tikzcd} \pi_1(K_1 \cap K_2, b) \arrow[r, "\iota_{1*}"] \arrow[d, "\iota_{2*}"'] & \pi_1(K_1, b) \arrow[d] \arrow[ddr, bend left, "(K_1 \hookrightarrow K)_*"] & \\ \pi_1(K_2, b) \arrow[r] \arrow[drr, bend right, "(K_2 \hookrightarrow K)_*"'] & G \arrow[dr, dashed, "\beta"] & \\ & & \pi_1(K, b) \end{tikzcd}$$

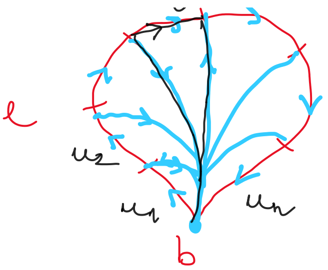
By universal property of push-outs (V.19), $\exists$ homomorphism $\beta : G \to \pi_1(K, b)$ that makes above diagram commutative.

Claim β is an isomorphism

β surjective: $\ell : I \to K$ loop, $\ell(0) = \ell(1) = b$



$\{\ell^{-1}(K_1), \ell^{-1}(K_2)\}$ open cover of I
$\xRightarrow{\text{Lebesgue Covering Thm (II.33)}}$ $\exists n$: $I_{(n)}$ subdivision of I such that $\forall$ simplex of $I_{(n)}$



maps to K_1 or K_2 under l.

$\forall x \in K$: choose a path $\Theta(x)$ from b to x such that $\Theta(x) \subseteq K_i$ if $x \in K_i$ and $\Theta(b) = c_b$ (if $x \in K_1 \cap K_2$, then $\Theta(x) \subseteq K_1 \cap K_2$).
This is possible because $K_1, K_2, K_1 \cap K_2$ are path-connected.

$u_i := l|_{[\frac{i}{n}, \frac{i+1}{n}]}$, $l = u_1 \cdots u_n$

$\overline{u_i} := \Theta(u_i(0))\, u_i\, \Theta(u_i(1))^{-1}$

$l \simeq \overline{u_1} \cdots \overline{u_n}$ and $\overline{u_i} \in K_1$ or K_2 $\forall i \in \{1, \dots, n\}$.

<u>β is injective</u> $g \in G$, $\beta(g) = e \in \pi_1(K, b)$
Need: $g = e$

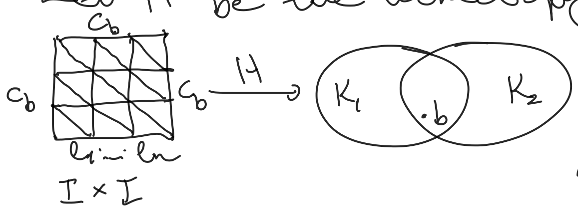
g is a product of generators in $X_1 \cup X_2$

So $\beta(g)$ is a composition of loops $l_1 \cdots l_n$, where $l_i \in \pi_1(K_1, b)$ or $\pi_1(K_2, b)$

<u>Need</u>: $l_1 \cdots l_n \simeq c_b$ rel. $\partial I \Rightarrow g = e$.

We will find a sequence of moves (1) and (2) of V.6 to the trivial word in G.

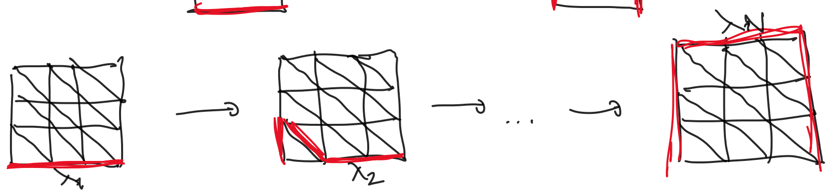
Let H be the homotopy from $l_1 \cdots l_n$ to c_b.



$\{H^{-1}(K_1), H^{-1}(K_2)\}$ open cover of $I \times I$

$\xRightarrow{\text{Lebesgue Covering Thm.}}$ $\exists r$: $\forall$ simplex of $(I \times I)_{(r)}$ maps to K_1 or K_2.

Recall: $\exists$ sequence of elementary expansions in $(I \times I)_{(r)}$ taking $I \times \{0\}$ to $(\partial I \times I) \cup (I \times \{1\})$, realising homotopy of $l_1 \cdots l_n$ to c_b.



Obtain a sequence $\lambda_1, \dots, \lambda_N$ of based loops.

$\lambda_1 = l_1 \cdots l_n$, $\lambda_N = c_b$

Homotopy from λ_i to λ_{i+1} is supported in WLOG K_1.

$\lambda_i|_\sigma$, σ is a 1-simplex of $(I \times I)_{(r)}$, lies in K_1 or K_2, label $\lambda_i|_\sigma$ with 1 or 2 respectively.

If $\lambda_i|_\sigma \subseteq K_1 \cap K_2$, we have a choice.

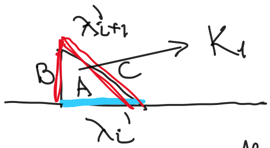
$\overline{\lambda_i|_\sigma} := \Theta(\lambda_i(\sigma(0)))\, \lambda_i|_\sigma\, \Theta(\lambda_i(\sigma(1)))^{-1}$
loop based at b.

If $\lambda_i|_\sigma$ is labeled j, then $\overline{\lambda_i|_\sigma} \subseteq K_j$, $j \in \{1,2\}$.

$\lambda_i' := \prod_{1\text{-simplex } \sigma} \overline{\lambda_i|_\sigma}$, $\lambda_i' \simeq \lambda_i$ rel ∂I.

If $\overline{\lambda_i|_\sigma}$ is labelled j, we write it as a word in X_j.
Product of these words over 1-simplices σ by w_i.
w_i represents λ_i'.

Homotopy from λ_i to λ_{i+1} induces a homotopy from λ_i' to λ_{i+1}', supported in K_1.



<u>Claim</u> w_{i+1} is obtained from w_i using moves (1) and (2).

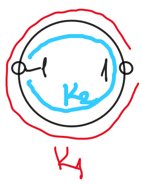
If $\lambda_i'|_A$, or $\lambda_{i+1}'|_B$, or $\lambda_{i+1}'|_C$ is labelled 2, then, using relation $i_{1*}(g) = i_{2*}(g)$ $\forall g \in \pi_1(K_1 \cap K_2, b)$, we can rewrite it as a word in X_1.
$\lambda_{i+1}'|_B \, \lambda_{i+1}'|_C \simeq \lambda_i'|_A$ rel ∂I, so corresponding words in X_1 can be connected using moves (1) and (2) in $\langle X_1 | R_1 \rangle$.

So w_{i+1} can be obtained from w_i as well using moves (1) and (2) in G.

Since $w_1 = g$, $w_N = e \in G \Rightarrow g = e$, so β is injective.

$\pi_1(K_i, b) \to \underset{G}{\overset{\cong}{\pi_1(K,b)}}$ push-out map is induced by the embedding $K_i \hookrightarrow K$. $\square$

<u>Example</u> $K = S^1$, $K_1 = S^1 \setminus \{1\}$, $K_2 = S^1 \setminus \{-1\}$



$K_1 \cap K_2$ <u>not</u> path-connected
$\pi_1(K_1) = \{e\}$, $\pi_1(K_2) = \{e\}$, $\pi_1(K) \cong \mathbb{Z}$
so $\pi_1(K) \ncong \pi_1(K_1) *_{\pi_1(K_1 \cap K_2)} \pi_1(K_2)$.

K_1, K_2 open also necessary.

True if K simplicial cx., K_1, K_2 are closed subcomplexes, $K_1, K_2, K_1 \cap K_2$ path-connected.