Section IV.3: The universal property

<u>Theorem IV.10</u> S set, G group, $f: S \to G$ function $\Rightarrow$
$\exists!$ homomorphism $\phi: F(S) \to G$ such that the following diagram commutes

$\uparrow$

$\phi \circ i = f$

$$\begin{tikzcd} & F(S) \arrow[d, dashed, "\phi"] \\ S \arrow[ur, "i"] \arrow[r, "f"] & G \end{tikzcd}$$

<u>Proof</u> Existence of ϕ: $w = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, $x_i \in S$, $\varepsilon_i \in \{1, -1\}$

$\phi(w) := f(x_1)^{\varepsilon_1} \cdots f(x_n)^{\varepsilon_n}$.

well-defined: $w \sim w' \overset{?}{\Rightarrow} \phi(w) = \phi(w')$

$w \searrow w'$ $\quad w = w_1 x x^{-1} w_2$ or $w_1 x^{-1} x w_2$, where $x \in S$

$w' = w_1 w_2$

$$\phi(w) = \left\{ \begin{matrix} \phi(w_1) f(x) f(x)^{-1} \phi(w_2) \\ \phi(w_1) f(x)^{-1} f(x) \phi(w_2) \end{matrix} \right\} = \phi(w_1)\phi(w_2) = \phi(w').$$

ϕ homomorphism ✓

uniqueness: any homom. is determined by its values on a set of generators, and $\underbrace{\phi|_S}_{f}$ is unique.

$\square$

$f: S \to G$ <u>induces</u> the homomorphism $\phi: F(S) \to G$.