Section VI.4: The Nielsen-Schreier theorem

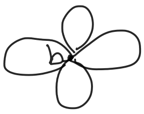
<u>Theorem VI.33</u> K a path-connected simplicial complex, $b \in V(K)$. Then $\exists!$ based covering space, up to equivalence, for $\forall$ subgroup of $\pi_1(K, b)$.

<u>Proof</u> Theorems VI.31, VI.32, Remark VI.28. $\square$

<u>Theorem VI.34</u> (Nielsen-Schreier) Any subgroup of a finitely generated free group is free.

<u>Proof</u> F_n free group on n generators

$X := \vee^n S^1$



$\pi_1(X, b) \cong F_n$ by Cor. V.5

$H \leq F_n \xrightarrow{\text{Theorem VI.32}} \exists$ based cover $p : (\tilde{X}, \tilde{b}) \to (X, b)$ such that $p_* \pi_1(\tilde{X}, \tilde{b}) = H$. $\tilde{X}$ is a simplicial complex

Since p is a local homeo., $\tilde{X}$ has only 0- and 1-simplices, i.e., $\tilde{X}$ is a graph $\xRightarrow{\text{Thm. IV.11}}$ $\pi_1(\tilde{X}, \tilde{b})$ is free. $\square$

By above proof + Rem. IV.19, we can obtain explicit free generating set of H:

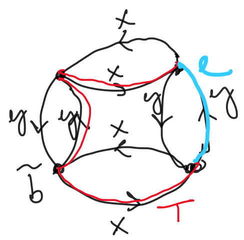
<u>Ex</u> $F_2 = \langle x, y \rangle$, $H = \ker\left(F_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2\right)$

$$x \mapsto (1, 0)$$
$$y \mapsto (0, 1)$$

$X =$ 

$p : (\tilde{X}, \tilde{b}) \to (X, b)$

$$|F_2 : H| = |\underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2}_{\cong F_2/H}| = 4 \Rightarrow \deg(p) = 4 \quad |p^{-1}(b)|$$

$\tilde{X}$ 

A loop l based at b lifts to loop in $\tilde{X}$ $\Leftrightarrow$ it runs over edges labelled x an even number of times and edges y an even number of times

$\Rightarrow p_* \pi_1(\tilde{X}, \tilde{b}) = H$.

T maximal tree

free generators of $\pi_1(\tilde{X}, \tilde{b}) \leftrightarrow E(\tilde{X}) \setminus E(T)$

$e \leftrightarrow x \underbrace{y}_{e} x^{-1} y^{-1}$

Free generating set: $xyx^{-1}y^{-1}$, x^2, y^2, $yxyx^{-1}$, yx^2y^{-1}.

<u>Theorem VI.37</u> G finitely presented group, $H \leq G$ s.t. $|G : H| < \infty$. Then H finitely presented.

<u>Proof.</u> K finite simplicial cx., $b \in V(K)$, s.t. $\pi_1(K, b) \cong G$. Exists by Cor. V.13.

$\exists$ $p : (\tilde{X}, \tilde{b}) \to (X, b)$ s.t. $p_* \pi_1(\tilde{X}, \tilde{b}) = H$ and $\tilde{X}$ is simpl. cx.

$p^{-1}(b) \leftrightarrow$ right cosets of $H \Rightarrow$ since $|G:H| < \infty$, $|p^{-1}(b)| < \infty$

$\Rightarrow \tilde{K}$ finite simplicial cx. $\overset{\text{Cor. V.21}}{\Longrightarrow}$ $\pi_1(\tilde{X}, \tilde{b})$ is finitely presented.

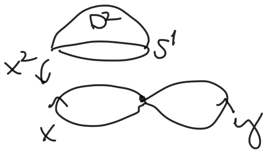
<u>Ex.</u> $G = \mathbb{Z}_2 * \mathbb{Z} \cong \langle x, y \mid x^2 \rangle$

$\phi : G \to \mathbb{Z}_2$, $H := \ker(\phi)$

$x \mapsto 0$

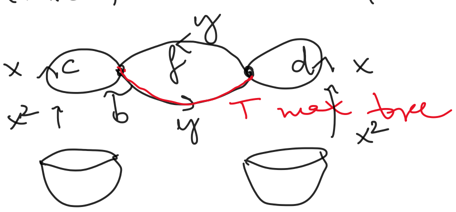
$y \mapsto 1$

$G = \pi_1(K, b)$ as in V.19



$p : (\tilde{K}, \tilde{b}) \to (K, b)$

$p_* \pi_1(\tilde{K}, \tilde{b}) = H$ $\quad |G:H| = |\mathbb{Z}_2| = 2 \Rightarrow |p^{-1}(b)| = 2$

$\tilde{K} =$ 

$E(\tilde{K}) \setminus E(T) = \{c, d, f\}$

$H \cong \langle c, d, f \mid c^2, d^2 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}$

$c = x$, $d = yxy^{-1}$, $f = y^2$