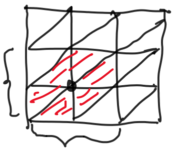
Section II.2: Simplicial approximation - subdivision

Def. Standard metric d on a finite simplicial cx. $|K|$ is
$$d\left(\textstyle\sum_i \lambda_i v_i, \sum_i \lambda_i' v_i\right) = \sum_i |\lambda_i - \lambda_i'|.$$

Def Let K' be a subdivision of K, and let d be the standard metric on $|K|$. The coarseness of K' is
$\sup\{d(x,y) : x, y \text{ belong to the star of the same vertex of } K'\}$
$\left(\begin{array}{l} \text{diam}(A) = \sup\{d(x,y) : x,y \in A\} \\ A \text{ subset of a metric space} \end{array}\right)$ $\qquad \text{diam}(st_{K'}(v'))$, $v' \in V(K')$

Example

$\frac{2}{r}$  $(I \times I)_{(r)}$ has coarseness $\frac{4}{r}$.

$\frac{2}{r}$

II.32

Thm (Simplicial Approximation Thm)
K, L simplicial complexes, K finite
$f : |K| \to |L|$ continuous
Then $\exists\, \delta > 0$: if coarseness of K' (← subdivision of K) $< \delta$, then $\exists$ simplicial map $g : K' \to L$ such that $|g| \simeq f$.

Theorem (Lebesgue Covering Theorem)
X metric space, $\mathcal{U}$ open cover
$\Rightarrow \exists\, \delta > 0 : \forall A \subset X$, $\text{diam}(A) < \delta$, $\exists\, U \in \mathcal{U}$ with $A \subseteq U$.

Proof. Thm II.32
$\{st_L(w) : w \in V(L)\}$ open cover of $|L|$. $\xRightarrow{f \text{ continuous}}$
$\{f^{-1}(st_L(w)) : w \in V(L)\}$ open cover of $|K|$.
Let $\delta > 0$ be as in Lebesgue covering thm.
K' subdivision w/ coarseness $< \delta \Rightarrow st_{K'}(v')$ has diam. $< \delta$
$\forall v' \in V(K')$. $\xRightarrow[\text{Lebesgue}]{}$ $st_{K'}(v') \subseteq f^{-1}(st_L(w))$ for some $w \in V(L)$.
$\Rightarrow f(st_{K'}(v')) \subseteq st_L(w)$. $\xRightarrow[\text{Proposition II.24.}]{}$ $g(v') := w$ works. $\square$

Relative version:

Addendum II.34. $A_1, \dots, A_n$ subcomplexes of K
$B_1, \dots, B_n$ —— " —— L

such that $f(A_i) \subseteq B_i$.

Then $g: V(K') \to V(L)$ given by Theorem II.32 satisfies $|g|(A_i) \subseteq B_i$ and the homotopy between f and $|g|$ sends A_i to B_i throughout.

Proof Apply Addendum II.28. □

Original Simplicial Approx. Thm. follows from:

Proposition II.35 A finite simplicial complex K has subdivisions $K^{(r)}$ for $r \in \mathbb{N}$ such that coarseness of $K^{(r)} \to 0$ as $r \to \infty$.

Barycentric subdivision

Δ^n std simplex, F face with vertices $v_1, \dots, v_r$

$$\text{barycentre}(F) := \frac{v_1 + \dots + v_r}{r}$$

K': vertices are barycentres of faces

$w_1, \dots, w_s$ vertices of K' form a face of K' $\iff$ corresponding faces $F_1, \dots, F_s$ of Δ^n satisfy $F_1 \subset F_2 \subset \dots \subset F_s$ (after re-ordering).

Δ^2

K simplicial cx., then subdividing each simplex as above fit together to barycentric subdivision K' of K.

Def Let $K = (V, \Sigma)$ abstract simpl. cx. Its barycentric subdivision $K^{(1)} = (V', \Sigma')$:

$V' = \Sigma$

$(\sigma_1, \dots, \sigma_n) \in \Sigma' \iff \sigma_1 \subset \sigma_2 \subset \dots \subset \sigma_n$ up to reordering.

$K^{(r)} = (K^{(r-1)})^{(1)}$ recursively.

coarseness$(K^{(r)}) \to 0$ as $r \to \infty$. (no proof).