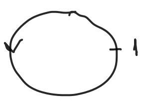
Section III.1: Fundamental group - properties

Example $b := 0 \in \mathbb{R}^n$

$\pi_1(\mathbb{R}^n, b) \cong 1$ because $\forall$ loop. $\simeq c_b$ via straight-line homot.

$S^1 \subseteq \mathbb{C}$, $b := 1$

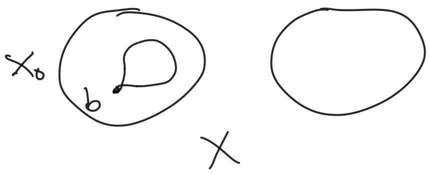


$\pi_1(S^1, b) \cong \mathbb{Z} = \langle \ell \rangle \qquad \ell(t) = e^{2\pi i t}$

- $\forall$ loop $\simeq \ell^n$ for some $n \in \mathbb{Z}$
- $\ell^n \simeq \ell^m \Rightarrow n = m$

Proof later.

Dependence of $\pi_1(X, b)$ on b:

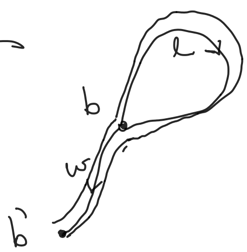


If X_0 is path-component of X containing b, then $\pi_1(X, b) \cong \pi_1(X_0, b)$.

Prop. III.14 If $b, b' \in X$ lie same path-component of X, then $\pi_1(X, b) \cong \pi_1(X, b')$.

We will write $\pi_1(X)$ for the isomorphism class of $\pi_1(X, b)$ if X is path-connected

Proof



$w : I \to X$, $w(0) = b$, $w(1) = b'$

$\ell : I \to X$, $\ell(0) = \ell(1) = b$

$$w_\# : \pi_1(X, b) \to \pi_1(X, b')$$
$$[\ell] \longmapsto [w^{-1} \ell w]$$

well-defined by Lemma III.6

$$\underbrace{w_\#([\ell])\, w_\#([\ell'])}_{\pi_1(X,b)} = [w^{-1} \ell w][w^{-1} \ell' w] = [w^{-1} \ell \underline{w w^{-1}} \ell' w] = [w^{-1} \ell c_b \ell' w] = [w^{-1} \underbrace{\ell \ell'}_{[\ell][\ell']} w] = w_\#([\ell][\ell'])$$

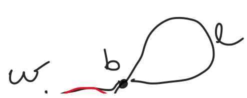
So $w_\#$ is a homomorphism.

Inverse of $w_\#$ is $(w^{-1})_\#$:

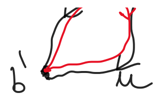
$$(w^{-1})_\#(w_\#([\ell])) = (w^{-1})_\#([w^{-1} \ell w]) = [w w^{-1} \ell w w^{-1}] = [\ell] \qquad \square$$

Rem $w_\#$ depends on w

$u : I \to X$, $u(0) = b$, $u(1) = b'$

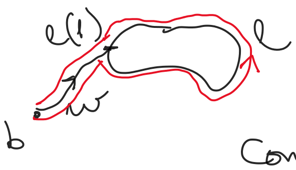


$u_\#^{-1} w_\#([\ell]) = [u w^{-1} \ell w u^{-1}] = [w u^{-1}]^{-1} [\ell] [w u^{-1}]$



conjugation by $[wu^{-1}]$
non-trivial if $[wu^{-1}] \notin Z(\pi_1(X,b))$

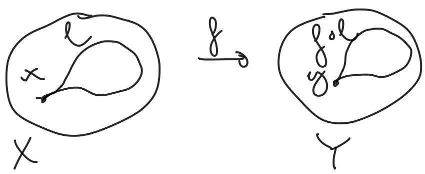
Rem $\ell : S^1 \to X$ unbased loop



$w : I \to X$, $w(0) = b$, $w(1) = \ell(1)$
Obtain a based loop in X by following w, then ℓ, then w^{-1}.
Conjugacy class of this loop is independent of the choice of path w.

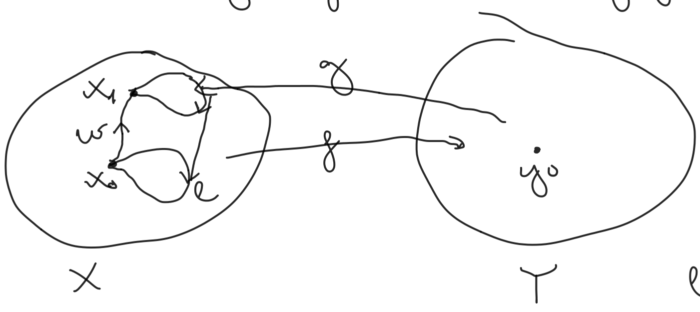
Proposition III.18 $f : (X,x) \to (Y,y)$ $(f(x) = y)$
induces a homomorphism $f_* : \pi_1(X,x) \to \pi_1(Y,y)$, s.t.
(i) $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x)}$
(ii) $g : (Y,y) \to (Z,z)$, then $(gf)_* = g_* f_*$
(iii) if $f \simeq f'$ rel $\{x\}$, then $f_* = f'_*$.

Proof $f_*([\ell]) := [f \circ \ell]$ well-defined by Lemma II.6. (version rel ∂I)
$f \circ (\ell\ell') = (f \circ \ell)(f \circ \ell') \Rightarrow f_*$ homom.
(i), (ii) obvious.
(iii): Lemma II.6, rel. version. □



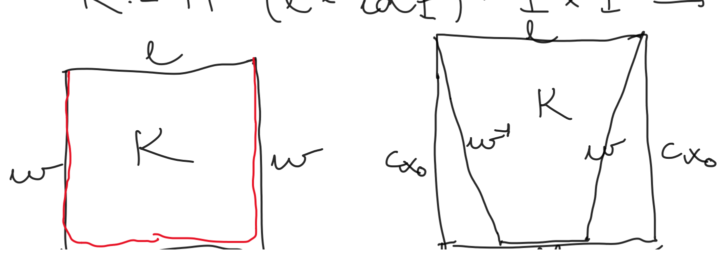
Prop III.19 X, Y path-connected, $X \simeq Y \Rightarrow \pi_1(X) \cong \pi_1(Y)$.

Proof $f : X \to Y$, $g : Y \to X$ homot. equivalences
$gf \simeq \mathrm{id}_X$, $fg \simeq \mathrm{id}_Y$
Issue: f, g & homotopies might not preserve basepoints.
$x_0 \in X$, $y_0 = f(x_0)$, $x_1 = g(y_0)$



$\pi_1(X,x_0) \xrightarrow{f_*} \pi_1(Y,y_0) \xrightarrow{g_*} \pi_1(X,x_1)$
$H : gf \simeq \mathrm{id}_X$.
$w(t) := H(x_0,t)$
$w(0) = x_0$, $w(1) = x_1$.
$\ell : I \to X$, $\ell(0) = \ell(1) = x_0$

$K := H \circ (\ell \times \mathrm{id}_I) : I \times I \to X$



Rescale K to fit onto middle trapezium
Fill in triangles.
$\leadsto$ homotopy between $w^{-1}(gf\ell)w$ and ℓ, rel x_0.

$gf\ell \quad w^{-1} gf\ell\, w \quad \underbrace{\qquad\qquad}_{w_\# g_* f_*([\ell])}$

$w_\# g_* f_* = id_{\pi_1(X, x_0)}$. $\Longrightarrow$ f_* inj., as $w_\#$ iso $\Rightarrow$ g_* surj. $\}\Rightarrow$

By looking at fg $\Rightarrow$ g_* is inj.

$\Rightarrow$ g_* is an iso. □

<u>Cor.</u> $X \simeq *$ (contractible) $\Rightarrow$ $\pi_1(X) \cong 1$.

<u>Def.</u> X is <u>simply-connected</u> if $\pi_1(X) \cong 1$.

<u>Rem</u> $\pi_1(X) \cong 1$ $\not\Rightarrow$ X contractible

<u>e.g.</u> $X = S^2$