Section VI.6: Normal subgroups and regular covering spaces

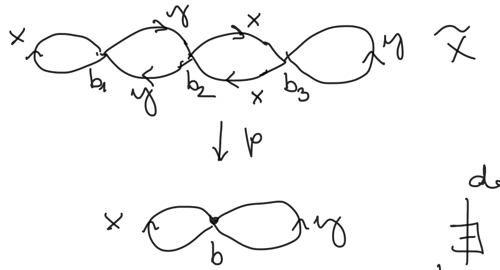
<u>Def</u> $p: \tilde{X} \to X$ covering map. A <u>covering transformation</u> is a homeo. $t: \tilde{X} \to \tilde{X}$ such that $p \circ t = p$.

$$\begin{tikzcd} \tilde{X} \arrow[rr, "t"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

<u>Def</u> A covering map $p: (\tilde{X}, \tilde{b}) \to (X, b)$ is <u>regular</u> if any two points of $p^{-1}(b)$ differ by a covering transformation.

<u>Ex.</u> $\mathbb{R}^2 \to T^2$, $\widetilde{S^1 \vee S^1} \to S^1 \vee S^1$ are regular

<u>Ex.</u>



covering map because $\forall$ vertex in $\tilde{X}$ has one x & one y pointing in
$\underline{\qquad\qquad} \| \underline{\qquad\qquad}$ pointing out.

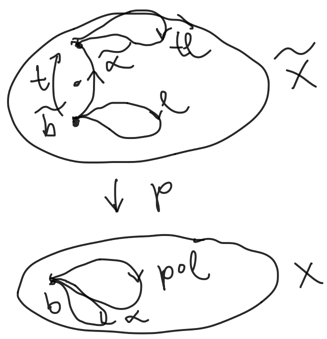
degree$(p) = 3$

$\nexists$ covering transf. of p taking b_1 to b_2 as b_1 has an adjacent x loop, b_2 doesn't.

<u>i.e.</u> x lifts to a loop from b_1, but not from b_2.

<u>Theorem VI.49</u> $p: (\tilde{X}, \tilde{b}) \to (X, b)$ is regular, then $p_* \pi_1(\tilde{X}, \tilde{b})$ is a <u>normal</u> subgroup of $\pi_1(X, b)$.

<u>Proof</u> $h \in p_* \pi_1(\tilde{X}, \tilde{b}) \Rightarrow \exists$ loop l based at $\tilde{b}$ in $\tilde{X}$ s.t. $[p \circ l] = h$.



$g \in \pi_1(X, b)$

<u>Claim:</u> $ghg^{-1} \in p_* \pi_1(\tilde{X}, \tilde{b})$:

$g = [\alpha]$

$\tilde{\alpha}$ lift of α from $\tilde{b}$

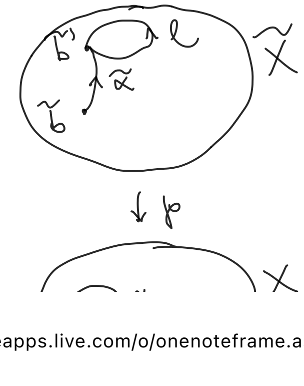
p regular $\Rightarrow \exists$ covering transf. t s.t. $t\tilde{b} = \tilde{\alpha}(1)$

$\tilde{\alpha}(tl)\tilde{\alpha}^{-1}$ lift of $\alpha(p \circ l)\alpha^{-1}$, loop $\Rightarrow$

$ghg^{-1} = [\alpha(p \circ l)\alpha^{-1}] \in p_* \pi_1(\tilde{X}, \tilde{b})$. □

<u>Theorem VI.50</u> $p: (\tilde{X}, \tilde{b}) \to (X, b)$, X locally path-connected. Suppose $p_* \pi_1(\tilde{X}, \tilde{b})$ is normal in $\pi_1(X, b)$. Then p is regular.

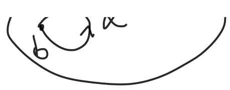
<u>Proof</u>



$\tilde{b}' \in p^{-1}(b)$

$$\begin{tikzcd} & (\tilde{X}, \tilde{b}) \arrow[d, "p"] \\ (\tilde{X}, \tilde{b}') \arrow[ur, dashed, "t"] \arrow[r, "p"] & (X, b) \end{tikzcd}$$

t exists by Theorem VI.29:



check $p_*\pi_1(\tilde{X}, \tilde{b}') \subseteq p_*\pi_1(X, b)$
($\tilde{X}$ is locally path-connected)

$\tilde{\alpha}$ path in $\tilde{X}$ from $\tilde{b}$ to $\tilde{b}'$, $\alpha := p \circ \tilde{\alpha}$ loop based at b.
l loop based at $\tilde{b}'$ $\Rightarrow$ $\tilde{\alpha} l \tilde{\alpha}^{-1}$ loop based at $\tilde{b}$.
So $[\alpha][p \circ l][\alpha^{-1}] \in p_*\pi_1(\tilde{X}, \tilde{b})$, by Cor. VI. 20.
Since $p_*\pi_1(\tilde{X}, \tilde{b}) \triangleleft \pi_1(X, b) \Rightarrow$
$[\alpha]^{-1}[\alpha][p \circ l][\alpha^{-1}][\alpha] = [p \circ l] \in p_*\pi_1(\tilde{X}, \tilde{b})$.
$\Rightarrow$ lift t exists.
Repeat w/ roles of $(\tilde{X}, \tilde{b})$ and $(\tilde{X}, \tilde{b}')$ reversed, get

$$\begin{tikzcd}
(\tilde{X}, \tilde{b}') \arrow[r, "t"] \arrow[dr, "p"'] & (\tilde{X}, \tilde{b}) \arrow[r, "t'"] \arrow[d, "p"] & (\tilde{X}, \tilde{b}') \arrow[dl, "p"] \\
& X &
\end{tikzcd}$$

By uniqueness of lifts, $t't = \mathrm{id}_{\tilde{X}}$. $\Big\}$ $\Rightarrow$ t is a homeo., so a covering transf. mapping $\tilde{b}$ to $\tilde{b}'$. $\square$
Similarly, $tt' = \mathrm{id}_{\tilde{X}'}$.

$\tilde{X}$ is universal cover $\rightsquigarrow$ can realise $\pi_1(X)$ as a group action on $\tilde{X}$, acting via homeomorphisms.