Section II.1: homotopy equivalence

Def X, Y are <u>homotopy equivalent</u> if $\exists\ X \underset{g}{\overset{f}{\rightleftarrows}} Y$ such that $g\circ f \simeq id_X$ and $f\circ g \simeq id_Y$.

Lemma II.8 Homotopy equiv. is an equiv. relation on top. spaces.

Proof. Reflexive: $X \underset{id_X}{\overset{id_X}{\rightleftarrows}} X$ Symmetric: by def

Transitivity: $X \underset{g}{\overset{f}{\rightleftarrows}} Y \underset{k}{\overset{h}{\rightleftarrows}} Z$ $\quad g\underbrace{kh}_{\simeq id_Y}f \underset{\text{Lemma II.6}}{\simeq} gf \simeq id_X$

$hfgk \simeq id_Z$. $\square$

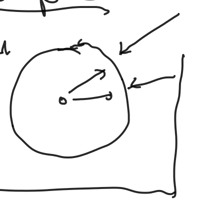
Example. $X \subseteq \mathbb{R}^n$ convex $\Rightarrow \forall x \in X$, $c_x \simeq id_X$ via straight-line homotopy $\Rightarrow X \simeq pt$

Def X is contractible if $X \simeq pt$

e.g. $\mathbb{R}^n, D^n, \Delta^n$ is contractible

Def $A \subseteq X$, $i: A \hookrightarrow X$ embedding
$r: X \to A$ is a homotopy retract if $ri = id_A$ and $ir \simeq id_X$.
(Then $A \simeq X$.)

Example



$$r: \mathbb{R}^n\setminus\{0\} \to S^{n-1}$$
$$x \mapsto \frac{x}{|x|}$$

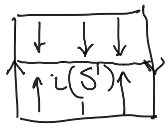
$ri = id_{S^{n-1}}$, $ir \simeq_H id_{S^{n-1}}$

$$H: (\mathbb{R}^n\setminus\{0\})\times I \to \mathbb{R}^n\setminus\{0\}$$
$$(x,t) \mapsto tx + (1-t)\frac{x}{|x|}$$

Well-defined: line segment between x and $\frac{x}{|x|}$ disjoint from $\{0\}$.

$\Rightarrow r$ homot. retract and $S^{n-1} \simeq \mathbb{R}^n\setminus\{0\}$

Example M = Möbius band



$i: S^1 \to M$
$e^{2\pi i x} \mapsto (x, \frac{1}{2})$

$H: M\times I \to M$
$(x,y,t) \mapsto (x, \frac{1-t}{2} + ty)$

Analogously, $S^1\times I \simeq S^1\times\{1\}$, homotopy retract.

N.B. $S^1\times I \simeq S^1 \simeq M$ but $S^1\times I \not\cong M$.

Def $A \subseteq X$, then $f, g: X \to Y$ are <u>homotopic relative to A</u> if $f|_A = g|_A$ and $\exists H: f \simeq g$ such that $H(x,t) = f(x) = g(x)\ \forall x \in A$

$\forall t \in I$.

Lemmas II.4 and II.6 have versions relative to subspaces.