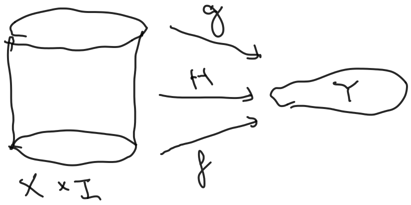
Section II.1: homotopy

<u>Def</u> $f, g : X \to Y$ are <u>homotopic</u> if $\exists\, H : X \times I \to Y$



$H_t(x) := H(x,t)$ for $x \in X, t \in I$

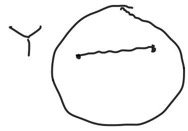
$H_0 = f$, $H_1 = g$

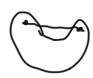
H : homotopy

t : "time"

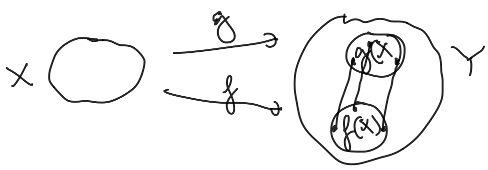
$f \simeq g$

<u>Example</u> $Y \subseteq \mathbb{R}^n$ is convex if $\forall\, y_1, y_2 \in Y$, $\forall\, t \in I$:
$(1-t)y_1 + ty_2 \in Y$.



(otherwise concave )

$f, g : X \to Y$ and Y convex $\Rightarrow f \simeq g$:



$H : X \times I \to Y$

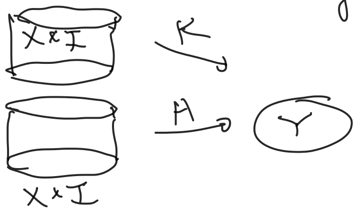
$(x,t) \mapsto (1-t)f(x) + tg(x)$

<u>Lemma II.4</u> Homotopy is an equivalence relation on the set of maps from X to Y.

<u>Proof</u> Reflexive: $f : X \to Y$, $f \simeq f$: $H(x,t) = f(x)\ \forall x \in X, t \in I$.

Symmetric: $H : f \simeq g$, then $\bar{H} : g \simeq f$, where $\bar{H}(x,t) = H(x, 1-t)$.

Transitive: $H : f \simeq g$, $K : g \simeq h \Rightarrow L : f \simeq h$ given by



$$L(x,t) = \begin{cases} H(x, 2t) & \text{if } t \in [0, \tfrac{1}{2}] \\ K(x, 2t-1) & \text{if } t \in [\tfrac{1}{2}, 1] \end{cases}$$

Continuity of L follows from Lemma II.5. $\square$

<u>Lemma II.5</u> If $\{C_1, \dots, C_n\}$ is a finite cover of X by closed sets and $f : X \to Y$ is a function such that $f|_{C_i}$ continuous $\forall i \in \{1, \dots, n\}$, then f is continuous.

<u>Proof</u> f continuous $\Leftrightarrow f^{-1}(C)$ is closed in X $\forall\, C \subseteq Y$ closed.

$$f^{-1}(C) = \bigcup_{i=1}^{n} f^{-1}(C) \cap C_i = \bigcup_{i=1}^{n} \underbrace{(f|_{C_i})(C)}_{\text{closed}} \quad \text{closed.}$$

$\square$

X = point $\quad X \to Y$ map $\leftrightarrow$ points of Y

homotopy $\leftrightarrow$ path

equiv. rel'n $\leftrightarrow$ being connected by a path

is an equivalence relation.
Equivalence classes: <u>path components</u> of Y

Y has 1 path component: <u>path-connected</u>

<u>Lemma II.6</u> $W \xrightarrow{f} X \underset{h}{\overset{g}{\rightrightarrows}} Y \xrightarrow{k} Z$

If $g \simeq h$, then $g \circ f \simeq h \circ f$ and $k \circ g \simeq k \circ h$.

<u>Proof.</u> $H : g \overset{\sim}{\Rightarrow} h$ $\quad k \circ H \overset{\sim}{\Rightarrow} k \circ g \Rightarrow k \circ h$

$H \circ (f \times id_I) : g \circ f \overset{\sim}{\Rightarrow} h \circ f$. □