§1. Introduction.

Primary reference: Hitchin online lecture notes. Notes by Ritter, Segal, Earl also helpful, plus lots of books.

Surfaces are two-dimensional objects, in pictures:

This course discusses three kinds of surfaces:

(A) Topological surfaces: special topological spaces. Lectures 1–4, Hitchin §2.
Highlights: Euler characteristic, orientability, classification of compact surfaces.

(B) Riemann surfaces: generalization of complex analysis. Lectures 5–8, Hitchin §3.
Surfaces with a notion of holomorphic function. Highlights: Riemann–Hurwitz.

(C) Smooth surfaces, surfaces in $\mathbb{R}^3$. Lectures 9–18, Hitchin §4–§5.
Highlights: Gaussian curvature, Gauss–Bonnet Theorem, geodesics, hyperbolic geometry.

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§2. Topological surfaces. §2.1. Background from Topology

A5: Topology is essential for this course. Recall:

Definition A topological space is a set X with a collection $\mathcal{T}$ of subsets of X, called open sets in X, such that:

(i) $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$

(ii) If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$

(iii) If $U_i \in \mathcal{T}$ $\forall i \in I$ then $\bigcup_{i \in I} U_i \in \mathcal{T}$.

- A map $f : X \longrightarrow Y$ of topological spaces is continuous if whenever $V \subseteq Y$ is open then $f^{-1}(V) \subseteq X$ is open.
- X is Hausdorff if whenever $x, y \in X$, $x \neq y$, $\exists$ open $x \in U \subseteq X$, $y \in V \subseteq X$ with $U \cap V = \emptyset$.
- X is compact if every open cover of X has a finite subcover.
- $f : X \longrightarrow Y$ is a homeomorphism if it is continuous and invertible with continuous inverse.
- If X is a topological space and $Y \subseteq X$, the subspace topology on Y is $\mathcal{T}_Y = \{ Y \cap U : U \in \mathcal{T}_X \}$.

You need to be familiar with these ideas, and with constructions such as quotient topological spaces.

§2.2. Topological Surfaces

Definition. A <u>topological surface</u> (or just <u>surface</u>) X is a Hausdorff topological space such that each $x \in X$ has an open neighbourhood $U \subseteq X$ with a homeomorphism $\varphi : U \to V$ to an open subset $V \subseteq \mathbb{R}^2$. The triple (U, V, φ) is a <u>chart</u> on X.

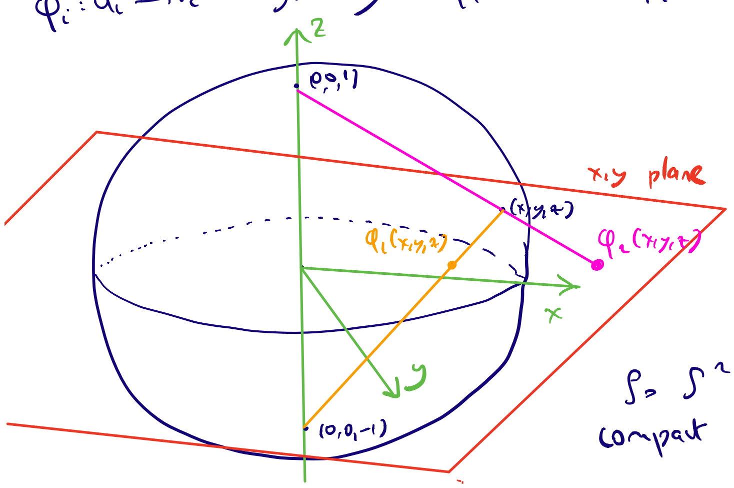
Remarks. (a) In older books, a surface is called <u>closed</u> if it is compact.
(b) It is better to also require X to be <u>second countable</u> or <u>paracompact</u> (global topological conditions). They are automatic for compact surfaces, which is what we mostly care about, so for simplicity we won't worry about this.
(c) More generally, we can define a <u>topological manifold</u> of <u>dimension n</u> to be a (second countable) Hausdorff topological space X locally homeomorphic to $\mathbb{R}^n$. Then surfaces are topological manifolds of dimension 2.
We will see later that compact surfaces can be completely classified. But for $n > 2$, classifying compact topological n-manifolds is very difficult.

Example Any open subset X of $\mathbb{R}^2$ is a topological surface, which can be covered by one chart (U, V, φ) with $U = V = X$, $\varphi = \mathrm{id}$.

Example The 2-sphere is

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Can cover this by two charts (U_1, V_1, φ_1), (U_2, V_2, φ_2), with $U_1 = S^2 \setminus \{(0,0,-1)\}$, $U_2 = S^2 \setminus \{(0,0,1)\}$, $V_1 = V_2 = \mathbb{R}^2$, $\varphi_i : U_i \xrightarrow{\cong} V_i$ given by $\varphi_1(x,y,z) = \frac{1}{1+z}(x,y)$, $\varphi_2(x,y,z) = \frac{1}{1-z}(x,y)$.



Note that you cannot cover S^2 with one chart — you need at least two.

So S^2 is locally homeomorphic to $\mathbb{R}^2$. It is Hausdorff (and second countable), as it is a subspace of $\mathbb{R}^3$ which is both.
So S^2 is a topological surface. It is compact by Heine-Borel, as closed and bounded in $\mathbb{R}^3$.

§2.3. Building surfaces as quotient topological spaces.

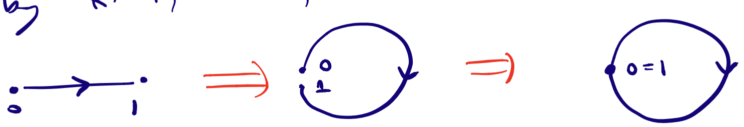
Definition. Let X be a topological space, and $\sim$ an equivalence relation on X. Write $X/\sim$ for the set of $\sim$-equivalence classes $[x]$ of points x in X. Write $\pi: X \longrightarrow X/\sim$ for the surjective projection $\pi: x \mapsto [x]$.

The quotient topology on $X/\sim$ is defined by $U \subseteq X/\sim$ is open iff $\pi^{-1}(U) \subseteq X$ is open in X.

Warning: If $\sim$ is not well chosen, then $X/\sim$ may not be a nice topological space. For example, X Hausdorff does not imply $X/\sim$ Hausdorff (in fact, $X/\sim$ is Hausdorff iff $\{(x,y) \in X \times X : x \sim y\}$ is closed in $X \times X$.

X compact does imply $X/\sim$ compact.

Example 1. Take $X = [0,1]$. Define an equivalence relation $\sim$ on X by $x \sim x$, $0 \sim 1$, $1 \sim 0$. Then $X/\sim$ is the circle.

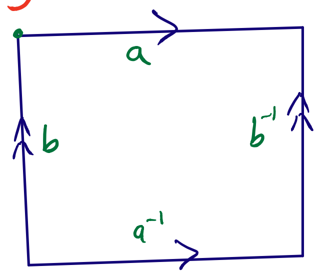


Think of it as gluing 0 to 1.

Example 2. Take $X = [0,1]^2$. Define an equivalence relation $\sim$ on X by $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$, all $x,y \in [0,1]$. (We leave out $(x,y) \sim (x,y)$ and $(x,1) \sim (x,0)$, etc. If we just write some relations, in an equivalence relation, we mean $\sim$ is the weakest equivalence relation including these relations.)

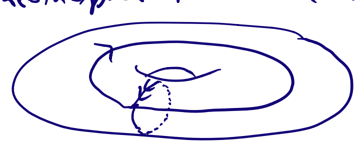
Later we will define notation which describe this diagram as $ab^{-1}a^{-1}b$.

Draw this as:



The same kind of arrows $\rightarrow$, $\twoheadrightarrow$, $\ggg$ etc. mean identify these sides in this direction.

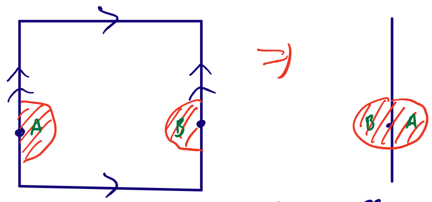
Then $X/\sim$ is a surface homeomorphic to $T^2 = (S^1)^2$.



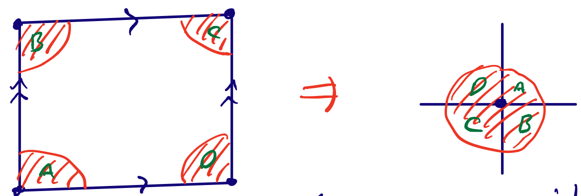
Reason: $X/\sim \; ([0,1]/\sim) \times ([0,1]/\sim) \cong S^1 \times S^1$ by Example 1.

$0 \sim 1$ $\qquad$ $0 \sim 1$

To convince yourself that $X/\sim$ is a surface (locally homeomorphic to $\mathbb{R}^2$)



Point $(0,y)/\sim$ has open neighbourhood homeo. to open ball in $\mathbb{R}^2$



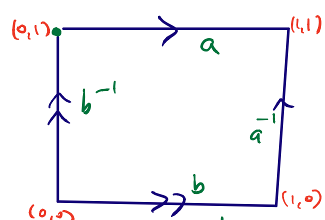
Point $(0,0)/\sim$ has open neighbourhood homeo to a ball in $\mathbb{R}^2$

Note that all four vertices of $X = \square$ are identified in $X/\sim$.

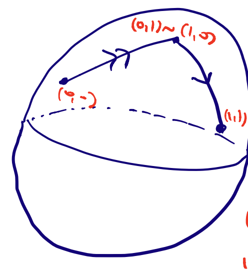
In fact we can make other topological surfaces by identifying sides of $X = [0,1]^2$.

Example 3

$X \cong S^2$.



Later notation: $aa^{-1}bb^{-1}$

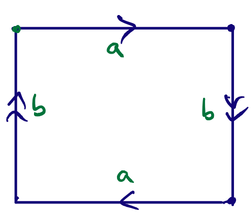


Can do this with paper and sellotape, get a shape like a samosa.

Only 2 vertices get identified, $(0,1) \sim (1,0)$.

Example 4

4 vertices identified in pairs $(0,0) \sim (1,1)$
$(0,1) \sim (1,0)$.



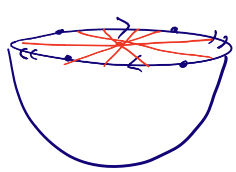
Later notation: $abab$

$X/\sim$ = projective space $\mathbb{RP}^2$.

Non-orientable surface (explained later).

$\mathbb{RP}^2 = S^2/\pm 1 \qquad (x,y,z) \sim (-x,-y,-z)$.

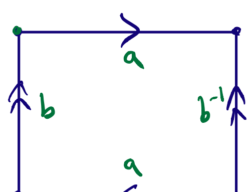
Think of $\mathbb{RP}^2$ as a half-sphere with opposite points on the equator identified.



$\mathbb{RP}^2$ cannot be embedded in $\mathbb{R}^3$ (as not orientable), so it's difficult to visualize.

Example 5

All 4 vertices identified by $\sim$.



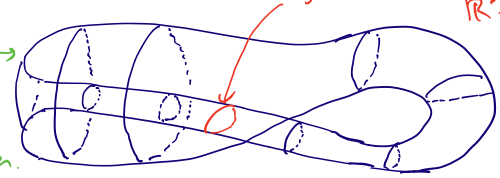
Later notation: $ab^{-1}ab$

$X/\sim$ is the "Klein bottle" K.

Also non-orientable, can't be embedded in $\mathbb{R}^3$. Quite like T^2.

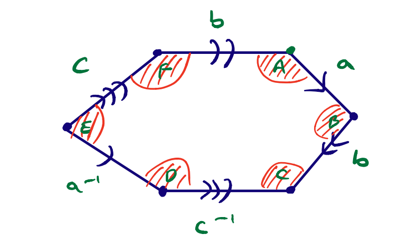
Attempt to draw in $\mathbb{R}^3$ with self-intersection.

self-intersection in $\mathbb{R}^3$.



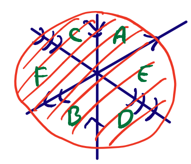
Examples 2–5 are all different (non-homeomorphic) surfaces.

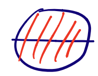
We do not need to work with squares: we can take any polygon X in the plane with an even number of sides, and identify sides in pairs, and $X/\sim$ will be a compact surface.



Later notation: $a\,b\,c^{-1}a^{-1}c\,b$

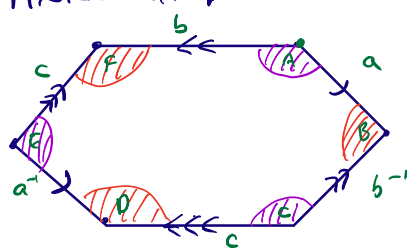
You have to do a little work to decide which subsets of vertices are identified. In this example, all 6 vertices are identified.



Every point in $X/\sim$ looks locally like this  or this  or this 

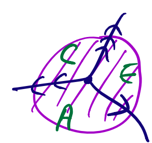
So $X/\sim$ is a surface, which is compact as X is.

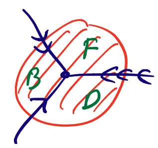
Another example:



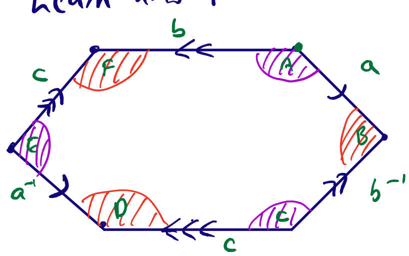
Later notation: $a b^{-1} c a^{-1} c b$.

In this example vertices are identified in two groups of three:

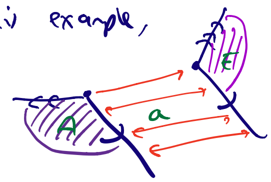




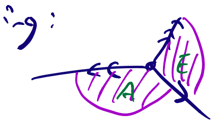
Learn how to do the calculation of which groups of vertices are identified:



in this example, we must identify edges  at vertices A, E;

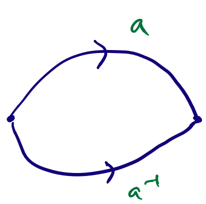


giving:

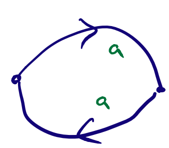


Carry on doing this till it closes up, and you've got one group of vertices identified by $\sim$. Do for all vertex groups.

Also allow curved sides, so can have a 2-gon:



$a a^{-1}$ $\Rightarrow S^2$,

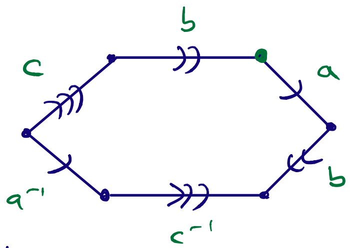


$a a$ $\Rightarrow \mathbb{RP}^2$.

Here is some notation for describing polygons with side identifications: label sides by $a, b, c, \ldots$, or $a^{-1}, b^{-1}, c^{-1}, \ldots$, where identified sides get the same label $a, b, \ldots$, and have $a, b, c, \ldots$ for sides with clockwise arrows, and $a^{-1}, b^{-1}, c^{-1}, \ldots$ for sides with anticlockwise arrows.

Then starting from some vertex (it doesn't matter which), list the labels on the sides in clockwise order.

e.g.



$\Rightarrow$ $abc^{-1}a^{-1}cb$

(or equivalently $c^{-1}a^{-1}cbab$, cyclic permutations don't make any difference).

Labels $a, b, c, \ldots$ appear twice.

This is called a <u>planar model</u> for a surface.

It is a very succinct way to describe surfaces.

Note that different words can describe the same (i.e. homeomorphic) surfaces, and it is not trivial to decide when they do.

B3.2 Geometry of Surfaces Lecture Notes. MT 20. Prof Joyce. Lecture 3.

§2.4. Cellular decompositions and triangulations

Definition For $n = 0,1,2,\dots$ write

$D^n = \{\underline{x} \in \mathbb{R}^n : \|x\| \leq 1\}$ for the closed unit disc,

$\mathring{D}^n = \{\underline{x} \in \mathbb{R}^n : \|x\| < 1\}$ for its interior,

$\partial D^n = \{\underline{x} \in \mathbb{R}^n : \|x\| = 1\}$ for its boundary.

Let X be a compact surface. A cellular decomposition of X is a finite collection of continuous maps, called cells:

(i) maps $v_i : D^0 \longrightarrow X$ called 0-cells or vertices;

(ii) maps $e_j : D^1 \longrightarrow X$ called 1-cells or edges; and

(iii) maps $f_k : D^2 \longrightarrow X$ called 2-cells or faces.

These must satisfy:

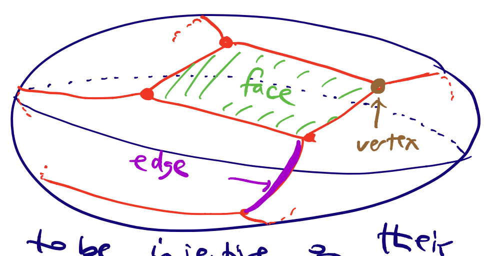
(a) each map restricted to the interior $\mathring{D}^n$ is a homeomorphism with its image in X.

(b) The image of ∂D^n is contained in the images of the cells of dimension $< n$.

(c) X is the disjoint union of the images of the interiors of the cells.

This is essentially the same as a subdivision in Hitchin (subdivision is the subsets $v_i(D^0)$, $e_j(\mathring{D}^1)$, $f_k(\mathring{D}^2)$ but doesn't remember the maps v_i, e_j, f_k).

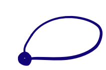
What this means:



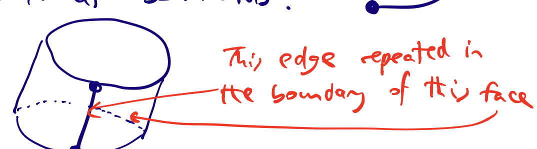
A cellular decomposition / subdivision is a division of X into polygons (the faces), with edges and vertices.

Each edge ends at two vertices.

The maps e_j, f_k are not required to be injective on their boundaries, so an edge can end at the same vertex at both ends.

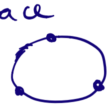


A face can have repeated edges and vertices in its boundary.

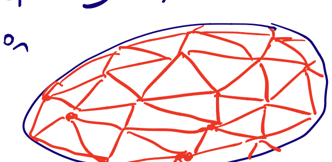


A planar model is equivalent to a cellular decomposition / subdivision with exactly one face.

A triangulation is a cellular decomposition / subdivision in which each face has three edges and three vertices (possibly repeated) in its boundary.



That is, a triangulation is a division of X into closed triangles, with their edges glued in pairs.



Note: for answering finals questions, I don't care if you use cellular decompositions or subdivisions, etc., as long as you get the details right.

Theorem 2.1. Every compact surface X admits a triangulation.

— Proof beyond the scope of the course. Later will comment on constructing triangulations if X is a Riemann surface, or a smooth surface.

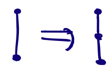
§2.5. The Euler characteristic.

Definition. Let X be a compact surface. Choose a cellular decomposition / subdivision of X, with V vertices, E edges, and F faces. The *Euler characteristic* of X is $\chi(X) = V - E + F$ in $\mathbb{Z}$. Learn this.

Theorem. $\chi(X)$ depends only on X as a topological space, not on the choice of subdivision.

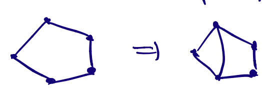
Sketch proof. (Not examinable). Consider two transitions between subdivisions:

(A) divide an edge into two



$V \to V+1$
$E \to E+1$
$F \to F$
$\chi(X)$ unchanged.

(B) divide a face into two



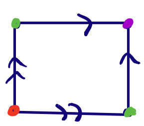
$V \to V$
$E \to E+1$
$F \to F+1$
$\chi(X)$ unchanged.

Claim: any two subdivisions can be linked by a finite sequence of moves (A), (B) and their inverses, plus continuous deformations.
So $\chi(X)$ independent of subdivision.

Alternative: Can define the homology groups $H_i(X)$, depending only on X as a topological space (see C3.1 Algebraic Topology) and show that $\chi(X) = \dim H_0(X) - \dim H_1(X) + \dim H_2(X)$.

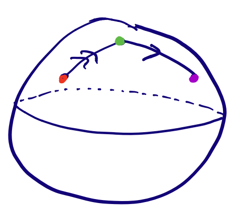
Examples

$X = S^2$

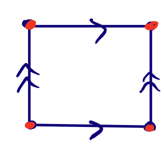


$V = 3$, $E = 2$, $F = 1$

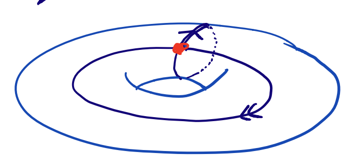
$\chi(S^2) = 2$



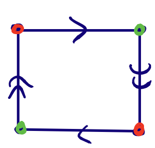
$X = T^2$



$V = 1$
$E = 2$
$F = 1$
$\chi(T^2) = 0.$

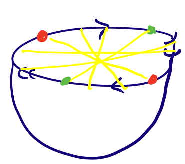


$X = \mathbb{RP}^2$

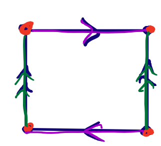


$V = 2$
$E = 2$
$F = 1$

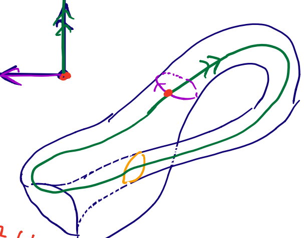
$\chi(\mathbb{RP}^2) = 1$



$X = K$
Klein bottle



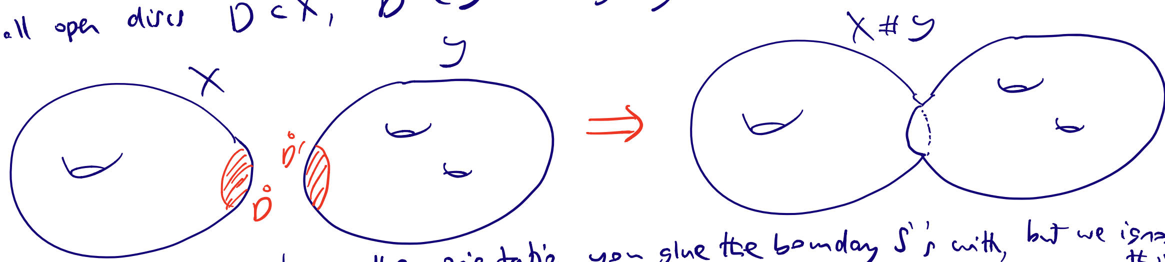
$V = 1$
$E = 2$
$F = 1$
$\chi(K) = 0.$



NB $\chi(T^2) = \chi(K)$ but $T^2 \not\cong K$.

§2.6 Connected sums

Let X, Y be compact, connected surfaces. The connected sum $X \# Y$ is a compact, connected surface obtained by cutting out small open discs $\mathring{D} \subset X$, $\mathring{D}' \subset Y$ and gluing the S^1 boundaries of $X \setminus \mathring{D}$, $Y \setminus \mathring{D}'$.

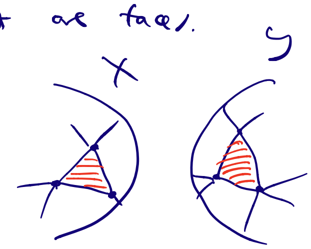


(Technically this depends on the orientation you glue the boundary S^1's with, but we ignore this.)

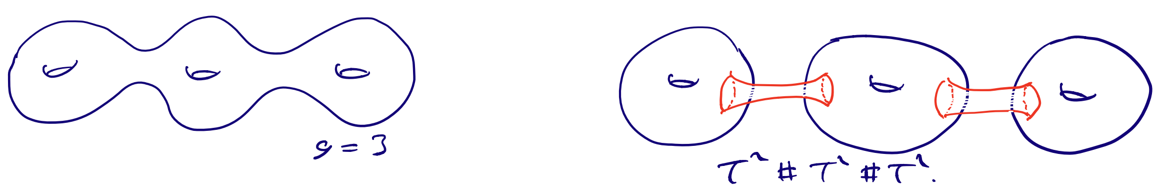
Then $X \# S^2 \cong X$, as cut out a disc and glue back in a disc.

Choose triangulations of X, Y, such that the discs cut out are faces.

Then
$$\left.\begin{aligned} V_{X\#Y} &= V_X + V_Y - 3, \\ E_{X\#Y} &= E_X + E_Y - 3, \\ F_{X\#Y} &= F_X + F_Y - 2 \end{aligned}\right\} \text{ so } \chi(X\#Y) = \chi(X) + \chi(Y) - 2.$$

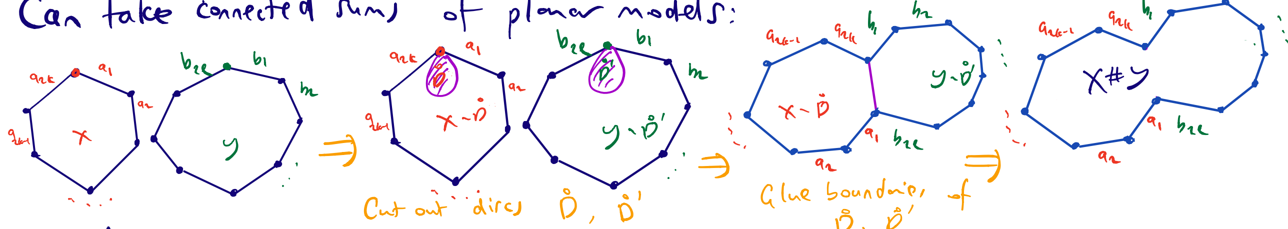


Example The surface Σ_g of genus $g \geq 0$, or 'sphere with g holes', is the multiple connect sum of g copies of T^2 (or S^2 if $g = 0$).



From $\chi(X\#Y) = \chi(X) + \chi(Y) - 2$, $\chi(S^2) = 2$, $\chi(T^2) = 0$, we see by induction that $\chi(\Sigma_g) = 2 - 2g$. Learn this. Note that this distinguishes Σ_g for different g.

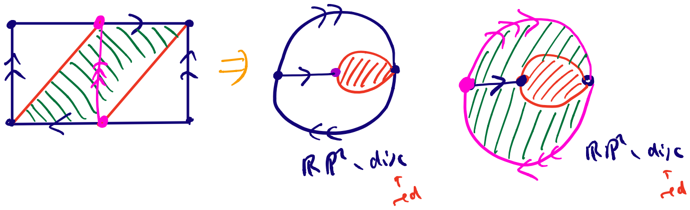
Can take connected sums of planar models:



If X is represented by a word $a_1 \cdots a_{2k}$, and Y by a word $b_1 \cdots b_{2l}$, then $X \# Y$ is represented by $a_1 \cdots a_{2k} b_1 \cdots b_{2l}$.

Example. There is a homeomorphism $\mathbb{RP}^2 \# \mathbb{RP}^2 \cong K$, Klein bottle.

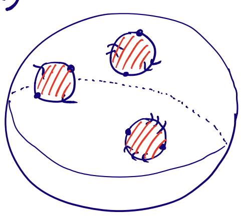
Reason: $K =$



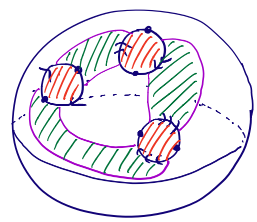
Cut K along the red S^1 into two Möbius strips (white and green regions), which are each $\mathbb{RP}^2 \setminus$ disc.

Example There are homeomorphisms $K \# \mathbb{RP}^2 \cong \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 \cong T^2 \# \mathbb{RP}^2$.

- The first follows from previous example. For the second, $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 = S^2 \# (\mathbb{RP}^2)^3$. Regard this as S^2 with 3 discs removed, with boundaries glued as shown.



Cut along purple lines into two regions, as shown.



Green region is $\mathbb{RP}^2 \setminus$ disc.
White region is $T^2 \setminus$ disc.
So $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 \cong T^2 \# \mathbb{RP}^2$.

Note: $T^2 \# \mathbb{RP}^2 \cong K \# \mathbb{RP}^2$, but $T^2 \not\cong K$, so can't cancel connected sums.

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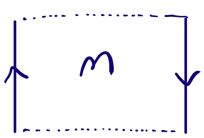
§2.7. Orientation and orientability.

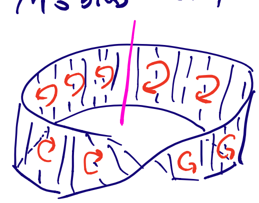
Definition The Möbius strip M is $[0,1] \times (0,1) / \sim$ where $(0,y) \sim (1,1-y)$.



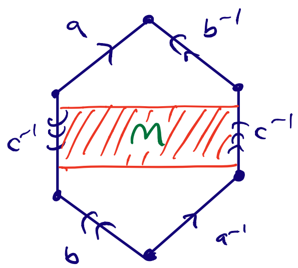
non compact surface.

A surface X is orientable if it does not contain any open subset homeomorphic to M.

Equivalent point of view: An orientation on X is a consistent notion of 'clockwise' everywhere on X. The Möbius strip has no orientation, as if take a notion of clockwise and deform around the loop, it turns into anticlockwise.



If X cannot be oriented, there is some loop in X, deforming round which turns clockwise into anticlockwise, and a neighborhood of this loop is a Möbius strip.



For a planar model, X is orientable iff every pair of glued edges are oriented one clockwise, one anticlockwise, ie in the word like $ab^{-1}c^{-1}a^{-1}bc^{-1}$, each symbol $a, b, c, \dots$ appears once as a and once as a^{-1}.

Since otherwise, a strip drawn from $a \Rightarrow a$ or $a^{-1} \Rightarrow a^{-1}$ is a Möbius strip.

Example.

$S^2 = aa^{-1}bb^{-1}$ is orientable.
$T^2 = ab^{-1}a^{-1}b$ is orientable.
$\mathbb{RP}^2 = abab$ is not orientable.
$K = ab^{-1}ab$ is not orientable.

NB. $\chi(T^2) = \chi(K) = 0$ but can distinguish T^2, K as one is orientable, one is not.

A connect sum $X \# Y$ is orientable iff X and Y are both orientable.

---

§2.8. The classification of surfaces

Theorem 2.2. Let X be a compact, connected surface.

Learn this.

Then either:
(a) X is orientable, and then X is homeomorphic to a surface Σ_g for $g \geq 0$, with $\Sigma_0 \cong S^2$, Σ_g = connected sum of g 'T^2's for $g \geq 1$, and $\chi(X) = 2-2g$.

Or (b) X is not orientable, and then X is homeomorphic to a connected sum of h $\mathbb{RP}^2$, $X \cong \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ for $h \geq 1$, and $\chi(X) = 2-h$.

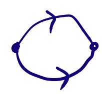
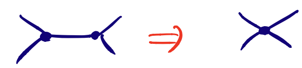
— Note that X is determined by the Euler characteristic $\chi(X)$ and orientability / non-orientability. (Nothing like this happens in higher dimensions.)

Sketch proof. (Not examinable.) See Hitchin notes for more details.

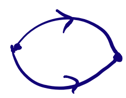
Step 1: X admits a triangulation: Theorem 2.1, §2.4.

Step 2: X admits a planar model.

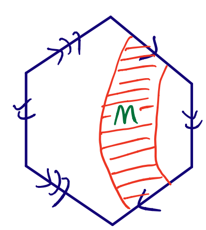
Proof: Take a subdivision of X with minimal # faces. (Subdivisions exist as a triangulation is one.) If > 1 faces, glue two neighbouring faces together along an edge to reduce # faces. So minimal number of faces is 1.

Step 3. X admits a planar model with only 1 vertex, or $X = S^2$ , as otherwise can cut and paste to reduce # vertices, e.g. shrink edge .

Hence: X has a subdivision with 1 vertex, 1 face and n edges, from gluing sides of a $2n$-gon. Then $\chi(X) = 2-n$, so $\chi(X) \leq 2$.

[Special case: $X = S^2$  2 vertices, 1 face, 1 edge, $\chi(X) = 2$. Can't shrink edge to reduce to 1 vertex in this case.]

Step 4. If X is not orientable, planar model has two glued edges with same orientation. Can draw a Möbius strip M joining these. Hence can write $X = Y \# \mathbb{R}P^2$ with $\chi(Y) = \chi(X) + 1$, $n_Y = n_X - 1$, $M = \mathbb{R}P^2 \setminus (\text{disc})$.



Step 5. If X is orientable, show $X = Y \# T^2$, $\chi(Y) = \chi(X) + 2$, $n_Y = n_X - 2$, or $X = S^2$, by cutting and pasting planar models.

Step 6. Induction on n now implies that

$$X = \#_g T^2 \quad \text{if } X \text{ is orientable, and}$$

$$X = (\#_{g \geq 0} T^2) \# (\#_{h \geq 1} \mathbb{R}P^2) \quad \text{if } X \text{ is not orientable.}$$

But $T^2 \# \mathbb{R}P^2 \cong \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$ by Example in §2.6, so $X = \#_{(2g+h)} \mathbb{R}P^2$, and the proof is finished. $\square$.

3. Riemann surfaces.

A Riemann surface is a topological surface X with an extra geometric structure, a complex structure or holomorphic atlas $\mathcal{A}$. This gives a notion of holomorphic function $f: U \to \mathbb{C}$ for $U \subseteq X$ open, and more generally a notion of holomorphic map $f: X \to Y$ for Riemann surfaces X, Y.

A2: Complex Analysis generalizes to Riemann surfaces.

Riemann surfaces are 1-dimensional complex manifolds, and generalize to n-dimensional complex manifolds, locally modelled on $\mathbb{C}^n$.

3.1. The definition of Riemann surface.

Definition Let X be a topological surface. A complex chart on X is a triple (U, V, φ) such that $U \subseteq X$, $V \subseteq \mathbb{C}$ are open and $\varphi : U \to V$ is a homeomorphism.

Aside: as φ is a homeomorphism, $\varphi^{-1}: V \to U$ is also a homeomorphism. Notation varies in different books: can also write charts as (U, φ), or as (V, φ^{-1}), since $V = \varphi(U)$ and $U = \varphi^{-1}(V)$.

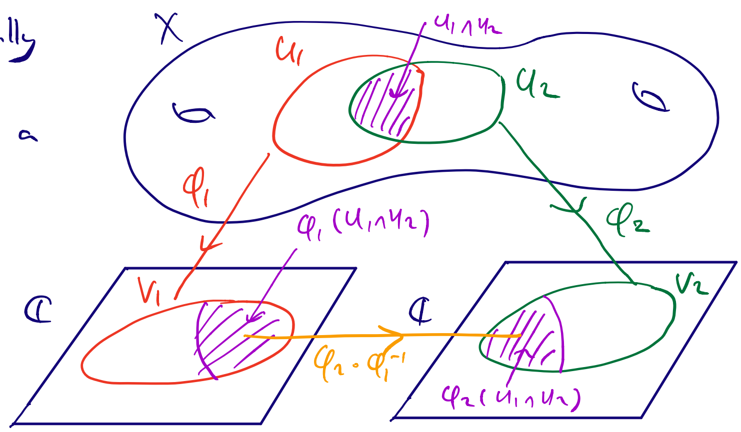
As $\mathbb{C} \cong \mathbb{R}^2$, by definition of surfaces, every $x \in X$ admits a complex chart with $x \in U$. Think of φ as a holomorphic coordinate on $U \subseteq X$.

We call two charts (U_1, V_1, φ_1), (U_2, V_2, φ_2) compatible if $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \longrightarrow \varphi_2(U_1 \cap U_2)$ is a holomorphic map between open subsets of $\mathbb{C}$. [with holomorphic inverse.]

$$\varphi_1(U_1 \cap U_2) \subseteq V_1 \subseteq \mathbb{C}, \qquad \varphi_2(U_1 \cap U_2) \subseteq V_2 \subseteq \mathbb{C}$$

Call $\varphi_2 \circ \varphi_1^{-1}$ the transition function.

Note: $\varphi_2 \circ \varphi_1^{-1}$ is automatically a homeomorphism, so being holomorphic implies it has a holomorphic inverse by a theorem in complex analysis.



A <u>holomorphic atlas</u> $\mathcal{A} = \{ (U_i, V_i, \varphi_i) : i \in I \}$ is a family of complex charts on X, such that:

index set

(i) (U_i, V_i, φ_i) and (U_j, V_j, φ_j) are compatible for all $i,j \in I$.

(ii) $X = \bigcup_{i \in I} U_i$.

A <u>Riemann surface</u> $(X, \mathcal{A})$ is a topological surface X with a holomorphic atlas $\mathcal{A}$. Usually omit $\mathcal{A}$ from the notation, and just say X is a Riemann surface.

[Sometimes one requires $\mathcal{A}$ to be a <u>maximal</u> atlas, i.e. not a proper subset of any other atlas. Any atlas is contained in a unique maximal atlas. This makes the definition a bit more canonical.]

Let X, Y be Riemann surfaces, with holomorphic atlases

$$\mathcal{A} = \{ (U_i, V_i, \varphi_i) : i \in I \}, \quad \mathcal{B} = \{ (U_j', V_j', \varphi_j') : j \in J \}.$$

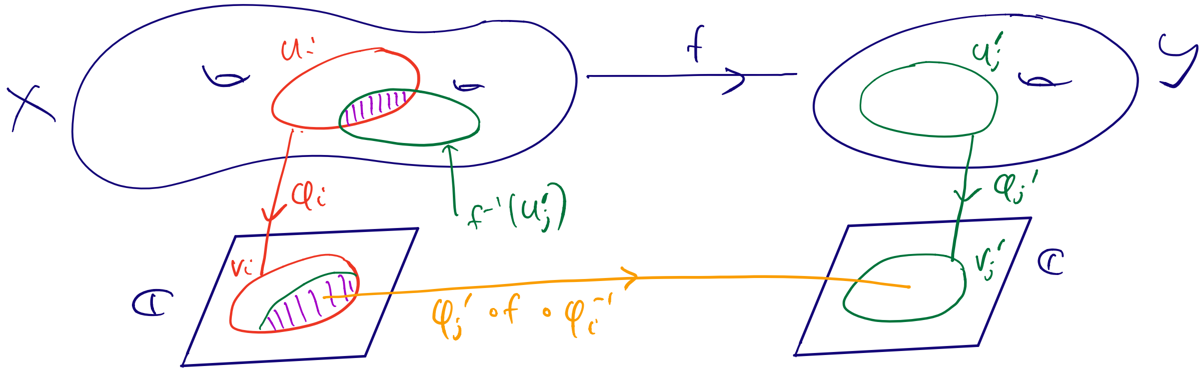
We call a continuous map $f : X \to Y$ <u>holomorphic</u> if for all $i \in I$ and $j \in J$

$$\varphi_i^{-1}(U_i \cap f^{-1}(U_j')) \xrightarrow{\varphi_j' \circ f \circ \varphi_i^{-1}} V_j'$$

is a holomorphic map between open subsets of $\mathbb{C}$.

($\varphi_i^{-1}(U_i \cap f^{-1}(U_j')) \subset V_i$ open, $V_i \subset \mathbb{C}$ open; $V_j' \subset \mathbb{C}$ open.)

That is, f is holomorphic when written in local holomorphic coordinates φ_i on X and φ_j' on Y.



<u>Example</u> $X = \mathbb{C}$ is a Riemann surface, with atlas

$\mathcal{A} = \{(\mathbb{C}, \mathbb{C}, \mathrm{id}_{\mathbb{C}})\}$ with one chart.

<u>Aside:</u> could also take $\mathcal{A} = \{(\mathbb{C}, \mathbb{C}, f)\}$ where $f(x+iy) = x - iy$, or $f(x+iy) = x + 2iy$, or ... These give <u>non-equivalent</u> Riemann surface structures on $X = \mathbb{C}$. The atlas $\mathcal{A}$ in $(X, \mathcal{A})$ is essential data. It is not enough that there exists an atlas, need a particular choice.

<u>Example</u> We make $X = \mathbb{C} \cup \{\infty\}$ into a topological surface homeomorphic to S^2, with open sets $U \subseteq \mathbb{C}$ open, and $U \cup \{\infty\}$ for $U \subseteq \mathbb{C}$, $\mathbb{C} \setminus U$ compact.

Then X is a Riemann surface with atlas $\mathcal{A} = \{(U_1, V_1, \varphi_1), (U_2, V_2, \varphi_2)\}$, where

$$U_1 = \mathbb{C} = X \setminus \{\infty\}, \quad V_1 = \mathbb{C}, \quad \varphi_1 = \mathrm{id}_{\mathbb{C}},$$

$$U_2 = (\mathbb{C} \setminus 0) \cup \{\infty\}, \quad V_2 = \mathbb{C}, \quad \varphi_2(z) = \begin{cases} 1/z, & z \in \mathbb{C} \setminus \{0\}, \\ 0, & z = \infty. \end{cases}$$

The transition function is

$$\varphi_2 \circ \varphi_1^{-1} : \mathbb{C} \setminus 0 \longrightarrow \mathbb{C} \setminus 0$$
$$\varphi_2 \circ \varphi_1^{-1} : z \longmapsto z^{-1},$$

which is holomorphic (no problem with pole z^{-1} as 0 not in domain)

so (U_1, V_1, φ_1), (U_2, V_2, φ_2) are compatible.

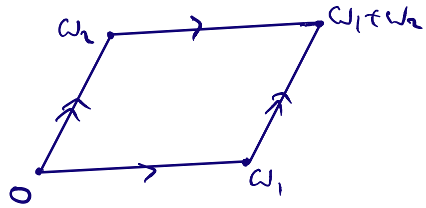
As $X = U_1 \cup U_2$, they cover X, so $(X, \mathcal{A})$ is a Riemann surface.

Also write $\mathbb{C} \cup \{\infty\}$ as $\mathbb{CP}^1$ (complex projective line).

Call $\mathbb{C} \cup \{\infty\}$ the <u>Riemann sphere</u>.

Example 3.1. Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$. Write $\Lambda = \{ m\omega_1 + n\omega_2 : m, n \in \mathbb{Z} \}$, an additive subgroup of $\mathbb{C}$ (a lattice). Take $X = \mathbb{C}/\Lambda$ the quotient group, with the quotient topology.

A fundamental domain for the action of Λ on $\mathbb{C}$ is the parallelogram with vertices $0, \omega_1, \omega_1+\omega_2, \omega_2$.



Then X is obtained by identifying opposite sides of this parallelogram as shown, so topologically $X \cong T^2$, a torus.

If $V \subseteq \mathbb{C}$ is open such that $(V+\lambda)\cap V = \phi$ for all $0 \neq \lambda \in \Lambda$ (this holds if V is a small ball), define $U = \{ v+\Lambda : v \in V \} \subseteq \mathbb{C}/\Lambda$, and $\varphi : U \to V$ to be the inverse of the homeomorphism $\varphi^{-1} : V \to U$, $\varphi^{-1} : v \longmapsto v + \Lambda$.

Then (U, V, φ) is a complex chart on X.

Write $\mathcal{A}$ for the set of all charts of this form.

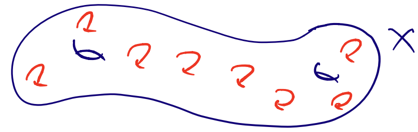
If (U_1, V_1, φ_1), (U_2, V_2, φ_2) are charts of this form, can show the transition function $\varphi_2 \circ \varphi_1^{-1}$ is locally of the form $z \longmapsto z + \lambda$ for $\lambda \in \Lambda$, and is holomorphic.

Thus the charts in $\mathcal{A}$ are pairwise compatible, and they cover X. So $\mathcal{A}$ is a holomorphic atlas, and $X = \mathbb{C}/\Lambda$ is a Riemann surface.

Aside: as a topological surface $X \cong T^2$, but as a Riemann surface, X depends on the lattice Λ: if $\Lambda \neq \alpha \cdot \Lambda'$ for $\alpha \in \mathbb{C}\setminus 0$ then $\mathbb{C}/\Lambda \not\cong \mathbb{C}/\Lambda'$!

Proposition 3.2. Any Riemann surface X is orientable.

Sketch proof. X has a holomorphic atlas $\{(U_i, V_i, \varphi_i) : i \in I\}$.



Recall that X is orientable if it has a consistent notion of 'clockwise'.

On $U_i \subseteq X$ we define 'clockwise' by identifying $U_i \xrightarrow{\varphi_i} V_i \subseteq \mathbb{C}$ and using standard notion of 'clockwise' in $\mathbb{C}$.

For two charts (U_i, V_i, φ_i), (U_j, V_j, φ_j), as $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is holomorphic, it preserves angles, so it preserves the notion of 'clockwise'.

Thus the notions of 'clockwise' on U_i, U_j agree on $U_i \cap U_j$.

As the $U_i : i \in I$ cover X, we have an orientation on X. $\square$

From the classification of surfaces (Th. 2.2) we have:

Corollary 3.3. Any compact, connected Riemann surface X is homeomorphic to a surface Σ_g of genus $g \geq 0$.

---

§3.2. Meromorphic functions

Definition. Let X be a Riemann surface. A *meromorphic function* on X is a holomorphic function
$$f : X \longrightarrow \mathbb{C} \cup \{\infty\}$$
to the Riemann sphere $\mathbb{C} \cup \{\infty\}$, as a Riemann surface.

Remark. We can also consider holomorphic functions $f : X \to \mathbb{C}$, of course. However, if X is a compact Riemann surface, then $|f| : X \longrightarrow [0, \infty)$ attains its maximum, and using the maximum modulus theorem in complex analysis, can show f is constant.

So holomorphic functions $f : X \to \mathbb{C}$ are boring for compact X.

It is a deep theorem, proved in B3.3 using the Riemann–Roch Theorem, that any compact Riemann surface has non-constant meromorphic functions (in fact, an infinite-dimensional family of them).

So they are a good thing to study.

Example Let $p(z)$, $q(z)$ be nonzero complex polynomials with no common factors.

Define $f : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$ by

$$f(z) = \begin{cases} p(z)/q(z), & z \in \mathbb{C},\ q(z) \neq 0 \\ \infty & z \in \mathbb{C},\ q(z) = 0, \end{cases}$$

$$f(\infty) = \begin{cases} \infty, & \deg p > \deg q, \\ 0, & \deg p < \deg q, \\ \dfrac{\text{leading coefficient } p}{\text{leading coefficient } q}, & \deg p = \deg q. \end{cases}$$

Then $f : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$ is a nonconstant meromorphic function. Can show (in B3.3) that every meromorphic function on $\mathbb{C} \cup \{\infty\}$ is of this form.

Aside. Transcendental holomorphic functions such as $e^z : \mathbb{C} \to \mathbb{C}$ do not extend to holomorphic functions $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$.
E.g. $e^{1/z}$ has an 'essential singularity' at $z = 0$. Can't define $f(z) = e^{1/z}$, $z \neq 0$, $f(0) = \infty$, as this would not be continuous $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$.

Example 3.4. Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$, $\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$, and $X = \mathbb{C}/\Lambda$ as in Example 3.1.

Define the <u>Weierstrass $\wp$-function</u> $\wp : \mathbb{C} \longrightarrow \mathbb{C} \cup \{\infty\}$ by

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \omega \in \Lambda} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right), \quad z \in \mathbb{C} \setminus \Lambda,$$

$$\wp(z) = \infty, \quad z \in \Lambda.$$

learn this.

Can prove:

- The sum converges uniformly on compact subsets of $\mathbb{C} \setminus \Lambda$ to a holomorphic function.
- $\wp(z)$ is meromorphic with a double pole at each $\omega \in \Lambda$, $\wp(z) = \frac{1}{(z-\omega)^2} + O(1)$.
- $\wp(z) = \wp(z+\omega)$ for all $z \in \mathbb{C}$ and $\omega \in \Lambda$, it is doubly-periodic.

Hence it descends to a meromorphic function $\wp : \mathbb{C}/\Lambda \longrightarrow \mathbb{C} \cup \{\infty\}$, with one pole at $0 + \Lambda$.

Can use this to build other meromorphic functions on $\mathbb{C}/\Lambda$, e.g. $\wp', \wp'', \frac{1}{\wp + c}, \ldots$

§3.3. Branch points and ramification points.

Recall some facts from complex analysis:

Let $U \subseteq \mathbb{C}$ be open and $f: U \to \mathbb{C}$ be holomorphic. If f is not locally constant, the zeroes of f are isolated in U. Thus the zeroes of $\frac{df}{dz}$ are also isolated.

For any $a \in U$, as f has a Taylor series at a, if f is not locally constant there is a unique $m \geq 1$ such that $\frac{d^m f}{dz^m}(a) \neq 0$ and $\frac{d^k f}{dz^k}(a) = 0$ for $k = 1, \ldots, m-1$.

Definition. Let X, Y be Riemann surfaces, and $f: X \to Y$ a holomorphic map, which is not locally constant on X. Let $x \in X$ with $f(x) = y \in Y$, and choose charts (U, V, φ) on X, (U', V', φ') on Y with $x \in U$, $y \in U'$. Then $\varphi' \circ f \circ \varphi^{-1}: \varphi(U \cap f^{-1}(U')) \to V'$ is holomorphic, and not locally constant. We call x a ramification point, and y a branch point, of f, if $\frac{d}{dz}(\varphi' \circ f \circ \varphi^{-1})\big|_{\varphi(x)} = 0$.

Can check this is independent of the choice of charts.

The ramification index $\nu_f(x)$ of $x \in X$ is the unique $m \geq 1$ with $\left(\frac{d}{dz}\right)^m (\varphi' \circ f \circ \varphi^{-1})\big|_{\varphi(x)} \neq 0$ and $\left(\frac{d}{dz}\right)^k (\varphi' \circ f \circ \varphi^{-1})\big|_{\varphi(x)} = 0$, $k = 1, 2, \ldots, m-1$.

Then $\nu_f(x) > 1$ iff x is a ramification point.

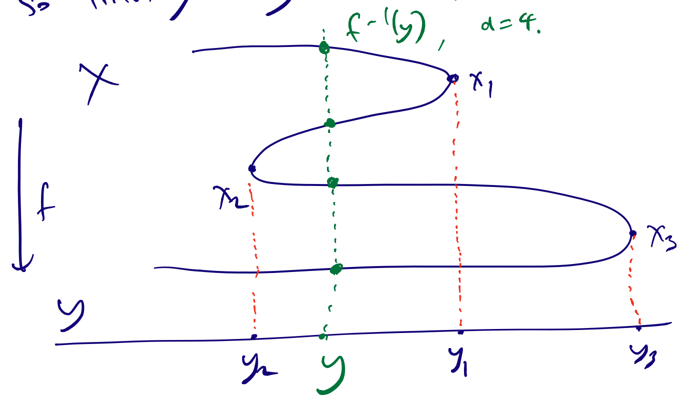
Ramification points are isolated in X, since zeroes of $\frac{df}{dz}$ are isolated. Thus, if X is compact then f has finitely many ramification points.

If $f: X \to Y$ has ramification index m at x with $f(x) = y$, can choose holomorphic coordinates w on X near x and z on Y near y with x at $w = 0$ and y at $z = 0$, such that f maps $w \mapsto z = w^m$.

That is, a ramification point $x \in X$ is a point where f looks locally like the function $w \mapsto w^m$ in holomorphic coordinates, for $m > 1$.

Notice that if y' is close to y then $f^{-1}(y')$ contains m points close to x, $m = \nu_f(x)$, as small $\varepsilon \neq 0$ has m m^{th} roots.

Definition. Let X, Y be compact Riemann surfaces with Y connected, and $f: X \to Y$ be holomorphic and not locally constant. Then f has finitely many ramification points $x_1, \dots, x_k$, and so finitely many branch points $y_1, \dots, y_k$ with $f(x_i) = y_i$. (The y_i need not be distinct.)



There is a number $d = \deg f$ called the <u>degree</u> of f, such that if $y \in Y \setminus \{y_1, \dots, y_k\}$ then $|f^{-1}(y)| = d$. This holds as on $X \setminus \{x_1, \dots, x_k\}$, f looks locally like a holomorphic function $f(z)$ with $\frac{df}{dz} \neq 0$, so it is locally invertible, and locally maps d sheets $\to$ 1 sheet. The number d is locally constant on $Y \setminus \{y_1, \dots, y_k\}$ as X is compact, and globally constant as $Y \setminus \{y_1, \dots, y_k\}$ is connected. So $\deg f$ is well defined.

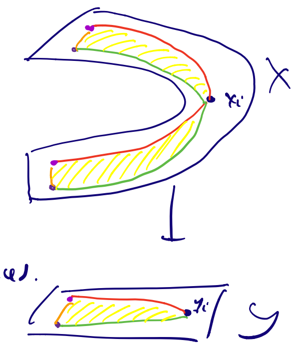
In fact, for all $y \in Y$, we have $d = \sum_{x \in X:\, f(x) = y} \nu_f(x)$. Learn this.

This holds for $y \in Y \setminus \{y_1, \dots, y_k\}$ as $\nu_f(x) = 1$ except at $x_1, \dots, x_k$. At a ramification point x_i, as $y \to y_i$, $\nu_f(x_i)$ points in $f^{-1}(y)$ come together at x_i, so replace $\underbrace{1 + \dots + 1}_{\nu_f(x_i)}$ by $\nu_f(x_i)$ in the sum.

Theorem 3.5. (The Riemann–Hurwitz formula). Let X, Y be compact, connected Riemann surfaces, and $f: X \to Y$ a nonconstant holomorphic map of degree d, with ramification points $x_1, \dots, x_k$. Then
$$\chi(X) = d\,\chi(Y) - \sum_{i=1}^{k} (\nu_f(x_i) - 1).$$

Proof. Let $y_i = f(x_i)$, so $y_1, \dots, y_k$ are the (not necessarily distinct) branch points of f. Choose a triangulation of Y whose vertices include the branch points $y_1, \dots, y_k$ of f.

Over the interiors of edges and faces, f is locally invertible, and maps d points to one point. So we can lift the triangulation of Y to a triangulation of X, in which each edge and face lifts to d edges and faces. Each vertex y lifts to $|f^{-1}(y)|$ points, where $\sum_{x \in f^{-1}(y)} v_f(x) = d$.



Hence $$|f^{-1}(y)| = \sum_{x \in f^{-1}(y)} 1 = d - \sum_{x \in f^{-1}(y)} (v_f(x) - 1).$$

Let the triangulation of Y have V, E, F vertices, edges, faces.
Then the triangulation of X has
$\left(dV - \sum_{i=1}^{k} (v_f(x_i) - 1)\right)$, dE, dF vertices, edges, and faces.

Thus $$\chi(X) = \left(dV - \sum_{i=1}^{k} (v_f(x_i) - 1)\right) - dE + dF$$
$$= d(V - E + F) - \sum_{i=1}^{k} (v_f(x_i) - 1) = d\,\chi(Y) - \sum_{i=1}^{k} (v_f(x_i) - 1)$$ $\square$

Remark: Given a meromorphic function $f: X \to \mathbb{C} \cup \{\infty\} = S^2$, can we use this to construct a triangulation of X — compare Theorem 2.1.

§3.4. An example.

Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$, $\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$ and $X = \mathbb{C}/\Lambda$ as in Example 3.1.

Define $\wp : X \longrightarrow \mathbb{C} \cup \{\infty\}$ as in Example 3.4, by $\wp(z+\Lambda) = \frac{1}{z^2} + \sum_{0 \neq \omega \in \Lambda} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$.

Observe: there is a unique point $0 + \Lambda$ in X with $\wp(0+\Lambda) = \infty$.

As this is a double pole, $\wp(z+\Lambda) = \frac{1}{z^2} + O(1)$.

Using $\frac{1}{z}$ as the coordinate on $\mathbb{C} \cup \{\infty\}$ near ∞, we see that $\wp$ has ramification index 2 at $0 + \Lambda$.

Hence the degree of $\wp$ is $d = \sum_{x \in X : \wp(x) = \infty} \nu_\wp(x) = 2$.

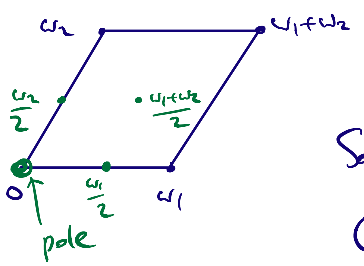
What about other ramification points?

These occur when $\wp'(z) = 0$.

Since $\wp(z) = \wp(-z)$ and $\wp(z) = \wp(\omega_1 + z)$, we have $\wp(\frac{\omega_1}{2} + z) = \wp(\frac{\omega_1}{2} - z)$, i.e. $\wp$ is even around $\frac{\omega_1}{2}$, so $\wp'(\frac{\omega_1}{2}) = 0$.

Hence $\frac{\omega_1}{2} + \Lambda$ is a ramification point of $\wp$, of ramification index $\nu_\wp(\frac{\omega_1}{2} + \Lambda) \geq 2$.

Similarly $\frac{\omega_2}{2} + \Lambda$, $\frac{\omega_1+\omega_2}{2} + \Lambda$ are ramification points.



We have $\chi(X) = 0$ as $X \cong T^2$, $\chi(\mathbb{C} \cup \{\infty\}) = 2$ as $\mathbb{C} \cup \{\infty\} \cong S^2$.

So Riemann–Hurwitz gives

$$\underset{\chi(X)}{0} = \underset{d}{2} \cdot \underset{\chi(Y)}{2} - \underbrace{\sum_{i=1}^{k} (\nu_f(x_i) - 1)}_{\text{This has contributions}}$$

This has contributions 1 from $0+\Lambda$, ≥ 1 from $\frac{\omega_1}{2}+\Lambda$, $\frac{\omega_2}{2}+\Lambda$, $\frac{\omega_1+\omega_2}{2}+\Lambda$ and any other ramification point.

Hence $0+\Lambda$, $\frac{\omega_1}{2}+\Lambda$, $\frac{\omega_2}{2}+\Lambda$, $\frac{\omega_1+\omega_2}{2}+\Lambda$ are the only ramification points, and all have ramification index 2.

Note: can also see ramification index $= 2$ as ramification index $\leq$ degree, by using $d = \sum_{x : f(x) = y} \nu_f(x)$.

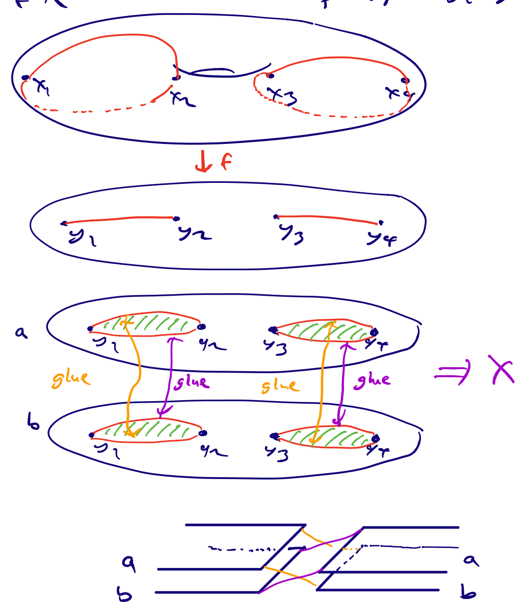
§3.5. Building Riemann surfaces as branched double covers.

In our example, $X = \mathbb{C}/\Lambda$ is a compact Riemann surface with a meromophic map $f: X \longrightarrow \mathbb{C}\cup\{\infty\}$ of degree 2 with ramification points $x_1, \dots, x_4 \in X$ and branch points $y_1 \dots y_4 \in \mathbb{C}\cup\{\infty\}$.

In fact we can build X as a Riemann surface just out of $\mathbb{C}\cup\{\infty\}$ and $y_1, \dots, y_4$.

Choose 'cuts' in $\mathbb{C}\cup\{\infty\}$ from y_1 to y_2 and y_3 to y_4.

Take 2 copies $(\mathbb{C}\cup\{\infty\})_a$, $(\mathbb{C}\cup\{\infty\})_b$, cut both along the line segments $y_1 \longrightarrow y_2$, $y_3 \longrightarrow y_4$, and glue the cut edges together, swapping over the 'a' and 'b' copies.



We can also do this using any even number of ramification points $y_1, \dots, y_{2n}$ in $\mathbb{C}\cup\{\infty\}$. That is, we can define a compact, connected Riemann surface X with a holomorphic map $f: X \longrightarrow \mathbb{C}\cup\{\infty\}$ of degree 2, with branch points at $y_1, \dots, y_{2n}$. This X is unique up to isomorphism.

Such Riemann surfaces are called <u>hyperelliptic</u>.

Here is a more algebraic way to define X. For simplicity take $y_1, \dots, y_{2n-1} \in \mathbb{C}$ and $y_{2n} = \infty$. Define

$$X = \{ (w, x) \in \mathbb{C}^2 : w^2 = (x-y_1)\cdots(x-y_{2n-1}) \} \cup \{(\infty, \infty)\}.$$

Can show that X has the natural structure of a Riemann surface, such that $f: X \longrightarrow \mathbb{C}\cup\{\infty\}$, $f(w,x) = x$, is degree 2 meromorphic with branch points $y_1, \dots, y_{2n}$.

Note that $w = \sqrt{(x-y_1)\cdots(x-y_{2n-1})}$ is also a rational function on X. Hyperelliptic surfaces occur in problems involving <u>hyperelliptic integrals</u> $\displaystyle\int \frac{dx}{\sqrt{(x-y_1)\cdots(x-y_k)}}$.

§4. Smooth surfaces.

§4.1. Abstract smooth surfaces.

We define smooth surfaces as for Riemann surfaces, but replace '$\mathbb{C}$, holomorphic' by '$\mathbb{R}^2$, smooth'.

Definition. Let X be a topological surface. A (smooth) chart on X is a triple (U, V, φ) with $U \subseteq X$, $V \subseteq \mathbb{R}^2$ open and $\varphi : U \to V$ a homeomorphism.

Two charts (U_1, V_1, φ_1), (U_2, V_2, φ_2) are compatible if $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \longrightarrow \varphi_2(U_1 \cap U_2)$ is a smooth map between open subsets of $\mathbb{R}^2$, with smooth inverse.

$$\varphi_1(U_1\cap U_2) \subseteq V_1 \subseteq \mathbb{R}^2, \qquad \varphi_2(U_1\cap U_2) \subseteq V_2 \subseteq \mathbb{R}^2$$

(Here smooth, C^∞, means all partial derivatives $\dfrac{\partial^{k+\ell}(\varphi_2\circ\varphi_1^{-1})}{\partial x^k\,\partial y^\ell}$ exist, all $k, \ell \geq 0$.)

A (smooth) atlas $\mathcal{A} = \{(U_i, V_i, \varphi_i) : i \in I\}$ on X is a family of smooth charts on X, such that (i) (U_i, V_i, φ_i) and (U_j, V_j, φ_j) are compatible, all $i, j \in I$.
(ii) $X = \bigcup_{i\in I} U_i$.

A (smooth) surface $(X, \mathcal{A})$ is a topological surface X with a smooth atlas $\mathcal{A}$.

Can also define smooth maps $f : X \longrightarrow Y$ between smooth surfaces X, Y, by f continuous and $\varphi_j' \circ f \circ \varphi_i^{-1} : \varphi_i(U_i \cap f^{-1}(U_j')) \longrightarrow V_j'$ is smooth between open subsets of $\mathbb{R}^2$, all charts (U_i, V_i, φ_i) on X and (U_j', V_j', φ_j') on Y.

Remark (a) Can generalize the above to smooth manifolds of dimension n by taking $V \subseteq \mathbb{R}^n$ open, not $V \subseteq \mathbb{R}^2$. See C3.3 Differentiable Manifolds.

(b) Every Riemann surface is also a smooth surface, by identifying $\mathbb{C} \cong \mathbb{R}^2$, and the transition functions $\varphi_j \circ \varphi_i^{-1}$ holomorphic $\Rightarrow$ $\varphi_j \circ \varphi_i^{-1}$ smooth, so a holomorphic atlas is also a smooth atlas.

§4.2. Smooth surfaces in $\mathbb{R}^3$.

Definition. A smooth surface in $\mathbb{R}^3$ is a subset $X \subset \mathbb{R}^3$ such that each point $x \in X$ has an open neighbourhood $x \in U \subseteq X$ and a map $\underset{\sim}{r} : V \longrightarrow X \subset \mathbb{R}^3$ from $V \subseteq \mathbb{R}^2$ open, such that:

(i) $\underset{\sim}{r} : V \longrightarrow U$ is a homeomorphism.

(ii) $\underset{\sim}{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ has derivatives of all orders.

(iii) at each $(u,v) \in V$, $\underset{\sim}{r}_u = \frac{\partial \underset{\sim}{r}}{\partial u}$ and $\underset{\sim}{r}_v = \frac{\partial \underset{\sim}{r}}{\partial v}$ are linearly independent in $\mathbb{R}^3$.

We call $\underset{\sim}{r}$ satisfying (i)-(iii) a local parametrization of the surface.

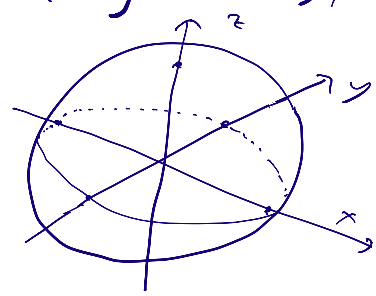
If $X \subset \mathbb{R}^3$ is a smooth surface, define

$$\mathcal{A} = \{ (U, V, \varphi) : \text{ for } U, V, \underset{\sim}{r} \text{ as above, } \varphi = r^{-1} : U \longrightarrow V \}.$$

Then each (U, V, φ) is a chart, and can show that any two such charts are compatible (needs the Inverse Function Theorem.)

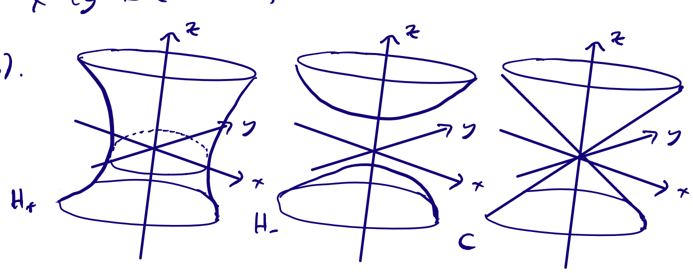
Then $\mathcal{A}$ is a smooth atlas on X, and makes X into an abstract smooth surface.

Example (a) $S^2 = \{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \}$, the unit sphere, is a smooth surface in $\mathbb{R}^3$.



(b) The hyperboloids
$$H_+ = \{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1 \},$$
$$H_- = \{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = -1 \}$$
are smooth surfaces in $\mathbb{R}^3$. But the cone
$$C = \{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0 \}$$ is not:

no chart exists around $(0,0,0)$.



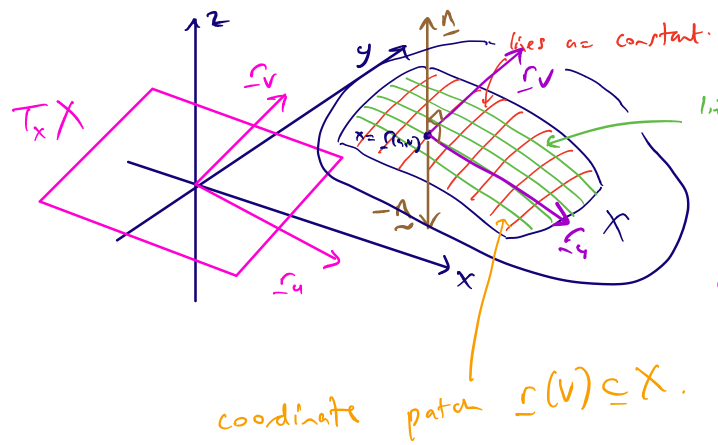
Definition Let X be a surface in $\mathbb{R}^3$, and $x \in X$. Let $\underline{r} : V \longrightarrow X \subset \mathbb{R}^3$ be a local parametrization of X, with $\underline{r}(u,v) = x$. The <u>tangent space</u> $T_x X$ is the vector subspace $T_x X = \langle \underline{r}_u|_{(u,v)}, \underline{r}_v|_{(u,v)} \rangle \subset \mathbb{R}^3$.

linearly independent by definition.

The two <u>unit normals</u> to X at x are $\pm \dfrac{\underline{r}_u \wedge \underline{r}_v}{|\underline{r}_u \wedge \underline{r}_v|} = \pm \underline{n}$.

Then $T_x X \subset \mathbb{R}^3$ and $\{\pm n\}$ are independent of the choice of parametrization $\underline{r}$, since any other parametrization $\hat{\underline{r}}$ is locally of the form $\hat{\underline{r}}(u,v) = \underline{r}(\hat{u}(u,v), \hat{v}(u,v))$ for $(\hat{u},\hat{v}) : (\text{open in } \mathbb{R}^2) \longrightarrow (\text{open in } \mathbb{R}^2)$ smooth and invertible.

Then $\begin{pmatrix} \hat{\underline{r}}_u \\ \hat{\underline{r}}_v \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \underline{r}_u \\ \underline{r}_v \end{pmatrix}$, where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \frac{\partial \hat{u}}{\partial u} & \frac{\partial \hat{u}}{\partial v} \\ \frac{\partial \hat{v}}{\partial u} & \frac{\partial \hat{v}}{\partial v} \end{pmatrix}$ is invertible, as $(\hat{u},\hat{v})$ is smooth and invertible, so $T_x X$ and $\{\pm \underline{n}\}$ are independent of choices.



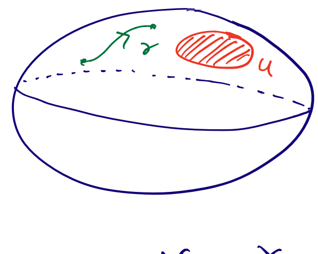
$T_x X = \{ w \in \mathbb{R}^3 : \underline{w} \cdot \underline{n} = 0 \}$

orthogonal subspace to unit normal $\underline{n}$.

B3.2 Geometry of Surfaces Lecture Notes. MT 20. Prof Joyce. Lecture 10.

§4.3. The first fundamental form.

Let X be a surface in $\mathbb{R}^3$. Using the usual notion of distance in $\mathbb{R}^3$, we can define lengths of curves γ in X, and areas of regions $U \subseteq X$. We will do this using the first fundamental form of X.



Note: if X is an abstract smooth surface, not embedded in $\mathbb{R}^3$, we cannot define lengths of curves or areas of regions without choosing an extra structure, a Riemannian metric, which is what the first fundamental form really is.

Definition. Let $X \subset \mathbb{R}^3$ be a smooth surface. Let $\underline{r}: V \to X$ be a smooth parametrization, for $V \subseteq \mathbb{R}^2$ open. Define smooth functions $E, F, G: V \to \mathbb{R}$ by $E = \underline{r}_u \cdot \underline{r}_u$, $F = \underline{r}_u \cdot \underline{r}_v = \underline{r}_v \cdot \underline{r}_u$, $G = \underline{r}_v \cdot \underline{r}_v$. The first fundamental form of X is the expression $g = E\,du^2 + 2F\,du\,dv + G\,dv^2$. This is the quadratic form $Q(\underline{v},\underline{v}) = \underline{v} \cdot \underline{v}$ on $T_x X \subset \mathbb{R}^3$ in the basis $\underline{r}_u, \underline{r}_v$.

We can write $g = (du\ \ dv)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$, for $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ a symmetric, positive definite matrix of functions.

Remark. You probably wonder, "What are $du, dv, du^2, du\,dv, dv^2$"? To do this properly: du, dv are smooth sections of the cotangent bundle $T^*X \to X$, and $g, du^2, du\,dv, dv^2$ are sections of the tensor product $\bigotimes^2 T^*X \to X$. Beyond the scope of this course, explained in C3.3.

For now: $du, du^2, \ldots$ are formal symbols which make sense if you have a choice of local coordinates on X. They behave as you expect under change of coordinates. E.g. if have two coordinates (u,v), (x,y) on X with $u = u(x,y)$, $v = v(x,y)$, then $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$, $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$, "chain rule".

$$du^2 = \left(\frac{\partial u}{\partial x}\right)^2 dx^2 + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\,dx\,dy + \left(\frac{\partial u}{\partial y}\right)^2 dy^2, \text{ etc.}$$

Think of du as "the derivative of u", but without taking partial derivatives w.r.t. particular coordinates. At $x \in X$, du lies in the vector space $T_x^* X = \langle du, dv \rangle$ dual to $T_x X$.

We can use the first fundamental form to compute lengths of curves in X:

<u>Definition.</u> Let $X \subset \mathbb{R}^3$ be a smooth surface, and let $\gamma : [a,b] \longrightarrow X \subset \mathbb{R}^3$ be a smooth curve. Suppose $\underline{r} : V \longrightarrow X$ is a local parametrization of X, and γ factors through $\underline{r}$, i.e. $\gamma(t) = \underline{r}(u(t), v(t))$. Then the length of γ is $\ell(\gamma) = \int_a^b \left|\frac{d\gamma}{dt}\right| dt = \int_a^b \left|\frac{d}{dt}\left(\underline{r}(u(t),v(t))\right)\right| dt$ (*)

$$= \int_a^b \left(\left|\underline{r}_u \frac{du}{dt} + \underline{r}_v \frac{dv}{dt}\right|^2\right)^{1/2} dt = \int_a^b \left(E(u,v)\left(\frac{du}{dt}\right)^2 + 2F(u,v)\frac{du}{dt}\frac{dv}{dt} + G(u,v)\left(\frac{dv}{dt}\right)^2\right)^{1/2} dt.$$

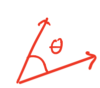
So can write lengths of curves using the first fundamental form $E\,du^2 + 2F\,du\,dv + G\,dv^2$.

Note that formally $\left(E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2\right)^{1/2} dt = \left(E\,du^2 + 2F\,du\,dv + G\,dv^2\right)^{1/2}$, if cancel the 'dt's.

We can also write areas in terms of first fundamental forms:

<u>Definition.</u> Let $X \subset \mathbb{R}^3$ be a smooth surface, of finite area, and suppose X is covered by a single parametrization $\underline{r} : V \longrightarrow X \subset \mathbb{R}^3$.

Then
$$\operatorname{area}(X) = \int_V |\underline{r}_u \wedge \underline{r}_v| \, du\,dv$$



$|\underline{r}_u \wedge \underline{r}_v| = |\underline{r}_u|\,|\underline{r}_v| \sin\theta.$

$\underline{r}_u \cdot \underline{r}_v = |\underline{r}_u|\,|\underline{r}_v| \cos\theta.$

$\sin^2\theta = 1 - \cos^2\theta.$

$$= \int_V \left(|\underline{r}_u|^2 |\underline{r}_v|^2 - |\underline{r}_u \cdot \underline{r}_v|^2\right)^{1/2} du\,dv$$

(**) $$= \int_V \left(EG - F^2\right)^{1/2} du\,dv = \int_V \left(\det\begin{pmatrix} E & F \\ F & G \end{pmatrix}\right)^{1/2} du\,dv.$$

§4.4. Riemannian metrics on abstract surfaces

Let X be a smooth surface with atlas $\mathcal{A} = \{(U_i, V_i, \varphi_i) : i \in I\}$.

Definition. A Riemannian metric g on X is data $E_i\, du^2 + 2F_i\, du\,dv + G_i\, dv^2$ on V_i, $E_i, F_i, G_i : V_i \to \mathbb{R}$ smooth functions, for each $i \in I$, satisfying:

(a) $\begin{pmatrix} E_i & F_i \\ F_i & G_i \end{pmatrix}$ is a positive definite matrix at each point of V_i.

(b) Let $i, j \in I$, and write the map $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ as $(u,v) \mapsto (\tilde{u}, \tilde{v})$. Then

$$E_j\, d\tilde{u}^2 + 2F_j\, d\tilde{u}\, d\tilde{v} + G_j\, d\tilde{v}^2 = E_i\, du^2 + 2F_i\, du\,dv + G_i\, dv^2$$

with $d\tilde{u} = \frac{\partial \tilde{u}}{\partial u} du + \frac{\partial \tilde{u}}{\partial v} dv$ and $d\tilde{v} = \frac{\partial \tilde{v}}{\partial u} du + \frac{\partial \tilde{v}}{\partial v} dv$.

That is, $E_i\big|_{\text{open set}} = E_j \left(\frac{\partial \tilde{u}}{\partial u}\right)^2 + 2F_j \frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{v}}{\partial u} + G_j \left(\frac{\partial \tilde{v}}{\partial u}\right)^2$, etc.

Then we can define lengths of curves $\gamma : (a,b) \to X$ and areas of regions $X' \subset X$, using g, by formulae (*), (**) in §4.3.

Any surface $X \subset \mathbb{R}^3$ has the natural structure of an abstract smooth surface, as in §4.2, and has a natural Riemannian metric, as in §4.3. But Riemannian metrics on surfaces don't have to come from embeddings in $\mathbb{R}^3$ — indeed, some surfaces (e.g. $\mathbb{RP}^2$, K) cannot be embedded in $\mathbb{R}^3$, but every smooth surface admits Riemannian metrics.

Remark. For an abstract surface X, can define a tangent space T_xX at each $x \in X$, with basis $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ if (u,v) are local coordinates on X near x, but can make T_xX coordinate-independent. The cotangent space is $T_x^*X = (T_xX)^*$, with basis du, dv.

Can interpret a Riemannian metric g as giving a positive definite quadratic form $\underline{v} \mapsto g_x(\underline{v}, \underline{v})$ on each tangent space T_xX, varying smoothly with x, which determines the (squared) length of vectors. In the dual basis du, dv, can write $g = E\, du^2 + 2F\, du\, dv + G\, dv^2$.

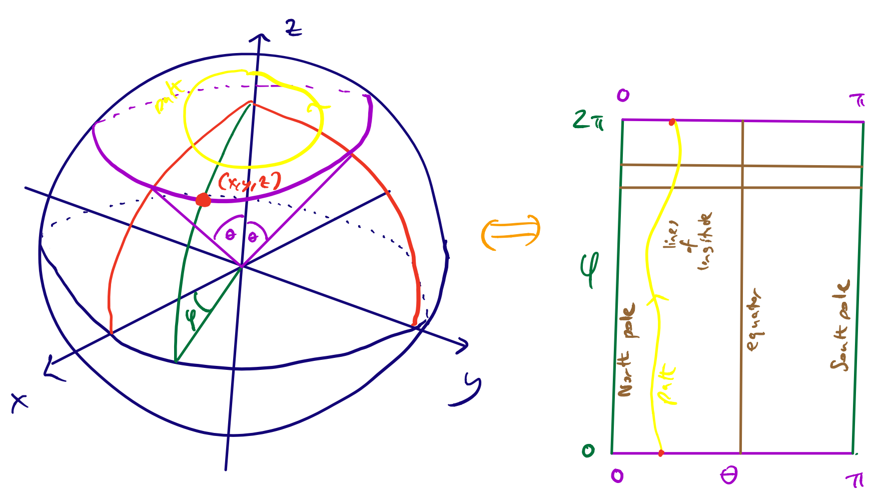
§4.5. Examples of first fundamental forms

Example $X = \{(x,y,z) \in \mathbb{R}^3 : z = 0\}$, the (x,y) plane, has parametrization $\underline{r}(u,v) = (u,v,0)$, $\underline{r}_u = (1,0,0)$, $\underline{r}_v = (0,1,0)$, and first fundamental form (1FF) $g = du^2 + dv^2$.

Example Let $X = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = R^2\}$ be the sphere of radius $R > 0$ in $\mathbb{R}^3$.
Define spherical polar coordinates $\underline{r} : (0,\pi) \times (0,2\pi) \longrightarrow X \subset \mathbb{R}^3$ by

$$\underline{r}(\theta,\varphi) = (R\sin\theta\cos\varphi,\ R\sin\theta\sin\varphi,\ R\cos\theta).$$

Then
$$\underline{r}_\theta = (R\cos\theta\cos\varphi,\ R\cos\theta\sin\varphi,\ -R\sin\theta),$$
$$\underline{r}_\varphi = (-R\sin\theta\sin\varphi,\ R\sin\theta\cos\varphi,\ 0),$$
and 1FF is $g = R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$.



Think of this as a map of the Earth's surface, where $(\theta,\varphi) =$ (latitude, longitude).

The map distorts distances, angles and areas.

Knowing the 1FF $g = R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$ allows you to compute distances, angles, and areas on the Earth's surface, just using the map. You don't need to know how the map is embedded in $\mathbb{R}^3$.

§4.6. Isometric surfaces

Definition Let $X, Y \subset \mathbb{R}^3$ be surfaces in $\mathbb{R}^3$. We call X, Y isometric if there is a smooth homeomorphism $f: X \to Y$ which maps curves $\gamma: (a,b) \to X$ to curves $f\circ\gamma: (a,b) \to Y$ of the same length. That is, isometries preserve lengths of curves.

Greek iso-metry = "same distance".

If (X,g), (Y,h) are abstract smooth surfaces with Riemannian metrics also call $f: X \to Y$ an isometry if it is a smooth homeomorphism which preserves lengths of curves.

Proposition A smooth homeomorphism $f: X \to Y$ is an isometry iff whenever $\underline{r}: V \to X$ is a smooth local parametrization, $V \subseteq \mathbb{R}^2$ open, then $f\circ\underline{r}: V \to Y$ is a smooth local parametrization and $\underline{r}$, $f\circ\underline{r}$ have the same first fundamental form $E\,du^2 + 2F\,du\,dv + G\,dv^2$.

Proof: 'if': obvious as lengths of curves computed using 1FF.

'only if': Fix $(u,v) \in V$ and $(u',v') \in \mathbb{R}^2$. Define $\gamma_\varepsilon: (0,1) \to V$ for $\varepsilon > 0$ small.

by $\gamma_\varepsilon(t) = (u + \varepsilon t u', v + \varepsilon t v')$

Then $$\lim_{\varepsilon\to 0} \frac{(\mathrm{length}(\underline{r}\circ\gamma_\varepsilon))^2}{\varepsilon^2} = E(u,v)u'^2 + 2F(u,v)u'v' + G(u,v)v'^2.$$

Can recover $E(u,v)$, $F(u,v)$, $G(u,v)$ by $(u',v') = (1,0), (0,1), (1,1)$.

If $f\circ\underline{r}$ smooth local parametrization, then f length preserving $\Rightarrow$ f identifies 1FFs. (Can also show $f\circ\underline{r}$ must be a smooth local parametrization using Inverse Function Theorem, as otherwise would have $0 \neq (u',v')$ in $\mathrm{Ker}(df)$.) $\square$

Two surfaces in $\mathbb{R}^3$ can be isometric even if they do not differ by an ambient isometry of $\mathbb{R}^3$.

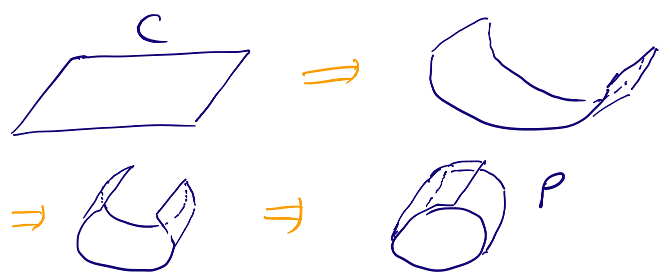
<u>Example.</u> Consider the plane $P = \{(x,y,z) \in \mathbb{R}^3 : z = 0\}$ with parametrization $\underline{r}(u,v) = (u,v,0)$, and the cylinder $C = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1\}$ with parametrization

$\underline{r}'(u,v) = (\cos u, \sin u, v)$.

Both have 1FF du^2+dv^2.

So mapping $\underline{r}(u,v) \longmapsto \underline{r}'(u,v)$ gives a (local) isometry from P to C.

<u>Explanation:</u> Can make the cylinder C by rolling up a piece of paper P; 'rolling up' does not change distances in P.



<u>Remark:</u> It is an important question when a surface $X \subset \mathbb{R}^3$ or (X,g) is locally isometric to the plane P.

We will answer this using <u>Gaussian curvature</u> $K : X \to \mathbb{R}$:

X is locally isometric to $(\mathbb{R}^2, du^2+dv^2)$ iff $K \equiv 0$.

§4.7. The second fundamental form.

Definition Let $X \subset \mathbb{R}^3$ be a smooth surface, and $\underline{r}: V \to X$ be a local parametrization, for $V \subseteq \mathbb{R}^2$ open.

As in §4.2, the unit normal is $\underline{n} = \frac{\underline{r}_u \wedge \underline{r}_v}{|\underline{r}_u \wedge \underline{r}_v|}$

(or minus this: we now choose a sign).

(Choosing a sign is equivalent to choosing an orientation on X.)

The second fundamental form of X is the expression

$$\mathrm{II} = L\,du^2 + 2M\,du\,dv + N\,dv^2,$$

where $L, M, N: V \to \mathbb{R}$ are the smooth functions $L = \underline{r}_{uu} \cdot \underline{n}$, $M = \underline{r}_{uv} \cdot \underline{n}$, $N = \underline{r}_{vv} \cdot \underline{n}$.

Since $\underline{r}_u \cdot \underline{n} = \underline{r}_v \cdot \underline{n} = 0$, by differentiating get alternative expressions

$L = -\underline{r}_u \cdot \underline{n}_u$, $M = -\underline{r}_u \cdot \underline{n}_v = -\underline{r}_v \cdot \underline{n}_u$, $N = -\underline{r}_v \cdot \underline{n}_v$.

Can write $\mathrm{II} = (du \;\; dv)\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$,

$\begin{pmatrix} L & M \\ M & N \end{pmatrix}$ symmetric matrix of functions.

The second fundamental form has the same behaviour under change of coordinates as the first fundamental form does (although $\begin{pmatrix} L & M \\ M & N \end{pmatrix}$ need not be positive definite).

It is a geometric structure of the same kind (section of $\bigotimes^2 T^*X$).

Remark The second fundamental form depends on the embedding $X \hookrightarrow \mathbb{R}^3$. It does not make sense for abstract surfaces (X, g) with Riemannian metrics.

Isometric surfaces in $\mathbb{R}^3$ need not have the same 2FF.

Changing orientation changes the sign of the 2FF.

<u>Example.</u> The plane $P = \{(x,y,z) \in \mathbb{R}^3 : z = 0\}$,

$\underline{r}(u,v) = (u,v,0)$, $\underline{n} = (0,0,1)$, has

1FF $g = du^2 + dv^2$,

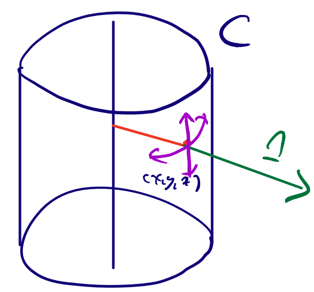
2FF $\mathrm{II} = 0$.

The cylinder $C = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1\}$,

$\underline{r}(u,v) = (\cos u, \sin v, v)$, $\underline{n} = (\cos u, \sin u, 0)$

has 1FF $g = du^2 + dv^2$,

2FF $\mathrm{II} = -du^2$.



P and C are locally isometric but have different 2FF's

<u>Definition</u> Let $X \subset \mathbb{R}^3$ be a smooth surface, and $\underline{r} : V \to X$ be a local parametrization for $V \subseteq \mathbb{R}^2$ open.

Then we have the first fundamental form (1FF) $g = E\,du^2 + 2F\,du\,dv + G\,dv^2$

and second fundamental form (2FF) $\mathrm{II} = L\,du^2 + 2M\,du\,dv + N\,dv^2$.

Fix $x = \underline{r}(u,v)$ in X.

The <u>principal curvatures</u> κ_1, κ_2 of X at x are the solutions of $\det\left(\lambda\begin{pmatrix} E & F \\ F & G\end{pmatrix} - \begin{pmatrix} L & M \\ M & N\end{pmatrix}\right) = 0$.

(unique if require $\kappa_1 \leq \kappa_2$.)

The <u>Gaussian curvature</u> is $K = \kappa_1\kappa_2 = \dfrac{\det\begin{pmatrix} L & M \\ M & N\end{pmatrix}}{\det\begin{pmatrix} E & F \\ F & G\end{pmatrix}} = \dfrac{LN - M^2}{EG - F^2}$.

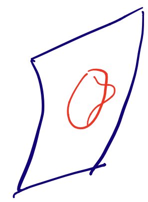
The <u>mean curvature</u> is $H = \kappa_1 + \kappa_2 = \mathrm{Trace}\left(\begin{pmatrix} L & M \\ M & N\end{pmatrix}\begin{pmatrix} E & F \\ F & G\end{pmatrix}^{-1}\right)$.

<u>Warning:</u> conventions differ, some authors write $H = \frac{1}{2}(\kappa_1 + \kappa_2)$.

Both K, H are smooth functions $X \longrightarrow \mathbb{R}$, independent of the choice of local parametrization $\underline{r}$.

We call X a <u>minimal surface</u> if $H \equiv 0$. This is a p.d.e. on X.

A surface is minimal if it has stationary area with fixed boundary. That is, if you deform the surface in its interior a little bit, you don't change the area to first order.
(In general the first order change is $\int_X H \underline{n} \cdot (\text{normal deformation})$.)



→ small deformation of X in normal direction in interior: how does area change?

A bubble spanning a loop of wire is a minimal surface, as surface tension minimizes the area.

A coordinate independent point of view is to regard g, II as quadratic forms on $T_x X$ for $x \in X$, i.e. $g(\underline{u}_1, \underline{u}_2)$, $\mathrm{II}(\underline{u}_1, \underline{u}_2)$ for $\underline{u}_1, \underline{u}_2 \in T_x X$.

The <u>shape operator</u> or <u>Weingarten map</u> is the unique linear map $S : T_x X \longrightarrow T_x X$ such that $\mathrm{II}(\underline{v}_1, \underline{v}_2) = g(S(\underline{v}_1), \underline{v}_2)$ for all $\underline{v}_1, \underline{v}_2$ in $T_x X$.

In coordinates it has matrix $\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1/2} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1/2}$. | This is a symmetric matrix.

Then κ_1, κ_2 are the eigenvalues of S, and $K = \det S$, and $H = \operatorname{Trace} S$.

The <u>principal directions</u> $\underline{v}_1, \underline{v}_2$ in $T_x X$ are the unit eigenvectors associated to the eigenvalues κ_1, κ_2 of S. Then $\underline{v}_1 \perp \underline{v}_2$.

<u>Example.</u> Let $X \subset \mathbb{R}^3$ be a smooth surface, and $x \in X$.

Then $\underline{v}_1, \underline{v}_2, \underline{n}$ at x are an orthonormal basis of $\mathbb{R}^3$.

Apply a rotation / reflection and translation such that

$x = (0,0,0)$, $\underline{v}_1 = (1,0,0)$, $\underline{v}_2 = (0,1,0)$, $\underline{n} = (0,0,1)$.

Then locally near x, can write X as the graph

$\{(x,y, f(x,y)) : (x,y) \in \mathbb{R}^2\}$ of a smooth function $f(x,y)$

with $f(0,0) = f_x(0,0) = f_y(0,0)$.

We have $\underline{r}_x = (1, 0, f_x)$, $\underline{r}_y = (0, 1, f_y)$,

$$\underline{n} = \frac{(-f_x, -f_y, 1)}{(1+f_x^2+f_y^2)^{1/2}}.$$ At x we have

$g = dx^2 + dy^2$, $\mathbb{II} = f_{xx}(0,0)\,dx^2 + 2 f_{xy}(0,0)\,dx\,dy + f_{yy}(0,0)\,dy^2$.

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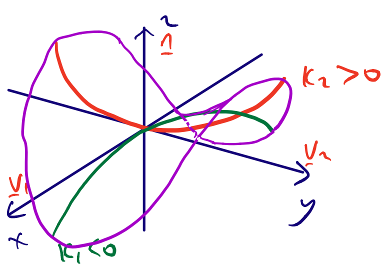
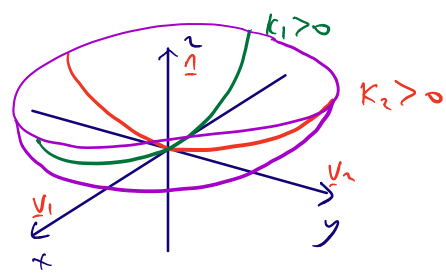
so $S = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$ since S has eigenvalues k_1, k_2 with eigenvectors $(1,0,0), (0,1,0)$.

Thus $f(x,y) = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 y^2 + O(|(x,y)|^3)$.



$K = k_1 k_2 < 0$, negative Gaussian curvature — like a saddle point, or Pringle.

Necessary for $H = 0$, minimal.



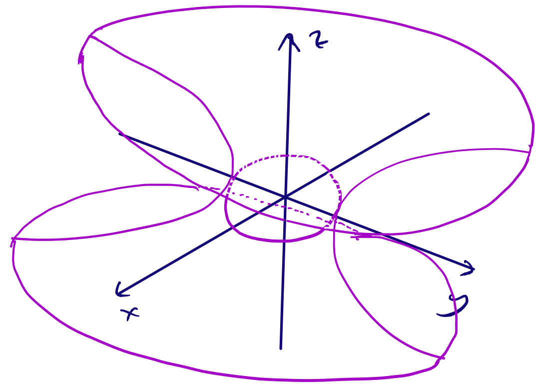
$K = k_1 k_2 > 0$, positive Gaussian curvature.

— Like a paraboloid, sphere, or ellipsoid (rugby ball) locally.

— Locally convex.

— Not minimal.

Example The catenoid is the surface of evolution of the graph $y = \cosh v$ (this is called the catenary). Parametrize it as



$$\underline{r}(u,v) = (\cos u \cosh v, \sin u \cosh v, v).$$

$$\underline{r}_u = (-\sin u \cosh v, \cos u \cosh v, 0)$$

$$\underline{r}_v = (\cos u \sinh v, \sin u \sinh v, 1).$$

$$\underline{n} = (\cos u \operatorname{sech} v, \sin u \operatorname{sech} v, -\tanh v)$$

$$g = \cosh^2 v \, (du^2 + dv^2)$$

$$\mathrm{II} = -du^2 + dv^2, \quad \kappa_1 = \frac{-1}{\cosh^2 v}, \quad v_1 = (-\sin u, \cos u, 0),$$

$$\kappa_2 = \frac{1}{\cosh^2 v}, \quad v_2 = (\cos u \tanh v, \sin u \tanh v, \operatorname{sech} v),$$

$K = \dfrac{-1}{\cosh^4 v}$, $H = 0$. So the catenoid is a minimal surface.

§4.8. Tangential derivatives and the Theorema Egregium

Definition Let $X \subset \mathbb{R}^3$ be a smooth surface, and $\underline{r}: V \to X$ a local parametrization, $(u,v) \in V \subseteq \mathbb{R}^2$ open.
Define smooth functions $\Gamma_{ab}^c : V \to \mathbb{R}$ for $a,b,c \in \{u,v\}$, called Christoffel symbols, by

$$(*)\quad \left\{\begin{aligned} \underline{r}_{uu} &= L\underline{n} + \Gamma_{uu}^u \underline{r}_u + \Gamma_{uu}^v \underline{r}_v \\ \underline{r}_{uv} &= M\underline{n} + \Gamma_{uv}^u \underline{r}_u + \Gamma_{uv}^v \underline{r}_v \\ \underline{r}_{vu} &= M\underline{n} + \Gamma_{vu}^u \underline{r}_u + \Gamma_{vu}^v \underline{r}_v \\ \underline{r}_{vv} &= N\underline{n} + \Gamma_{vv}^u \underline{r}_u + \Gamma_{vv}^v \underline{r}_v \end{aligned}\right.$$

Consistent with definition of L, M, N as $L = \underline{r}_{uu}\cdot\underline{n}$, etc.
As $\underline{r}_{uv} = \underline{r}_{vu}$, we have $\Gamma_{ab}^c = \Gamma_{ba}^c$.

Proposition 4.1. The Γ_{ab}^c depend only on the 1FF $g = E\,du^2 + 2F\,du\,dv + G\,dv^2$. Hence the Γ_{ab}^c are also defined for (X,g) abstract surface with Riemannian metric.

Proof. As $E = \underline{r}_u\cdot\underline{r}_u$, $F = \underline{r}_u\cdot\underline{r}_v$, we have

$$E_u = 2\underline{r}_u\cdot\underline{r}_{uu}, \quad E_v = 2\underline{r}_u\cdot\underline{r}_{uv}, \quad F_u = \underline{r}_{uu}\cdot\underline{r}_v + \underline{r}_u\cdot\underline{r}_{uv}.$$

$$\text{So } \underbrace{E\,\Gamma_{uu}^u + F\,\Gamma_{uu}^v = \underline{r}_u\cdot\underline{r}_{uu}}_{\text{get this by } \underline{r}_u\cdot(\text{1st eqn of }(*)).\ \underline{r}_u\cdot\underline{n}=0.} \underbrace{\quad = \quad}_{\text{from above.}} \tfrac{1}{2}E_u$$

$$F\,\Gamma_{uu}^u + G\,\Gamma_{uu}^v = \underline{r}_v\cdot\underline{r}_{uu} = F_u - \tfrac{1}{2}E_v, \text{ and}$$

$$\begin{pmatrix}\Gamma_{uu}^u \\ \Gamma_{uu}^v\end{pmatrix} = \begin{pmatrix} E & F \\ F & G\end{pmatrix}^{-1}\begin{pmatrix}\tfrac{1}{2}E_u \\ F_u - \tfrac{1}{2}E_v\end{pmatrix}.$$

The proofs for the other Γ_{bc}^a are similar. $\square$

Definition $X \subset \mathbb{R}^3$ smooth surface, $\underset{\sim}{r} : V \to X$ local parametrization. A vector field on X is a smooth map $\underset{\sim}{a} : X \to \mathbb{R}^3$ mapping $x \longrightarrow T_x X \subset \mathbb{R}^3$ for each $x \in X$.

On V, write $\underset{\sim}{a}$ for $\underset{\sim}{a} \circ \underset{\sim}{r} : V \to \mathbb{R}^3$.

We have $\underset{\sim}{a} = \underset{\sim}{a} \circ \underset{\sim}{r} = e\underset{\sim}{r}_u + f\underset{\sim}{r}_v$, $e, f : V \to \mathbb{R}$ smooth.

Also write this as $\underset{\sim}{a} = e\frac{\partial}{\partial u} + f\frac{\partial}{\partial v}$.

Then vector fields also make sense on abstract smooth surfaces X in local coordinates (u,v).

The tangential derivatives of $\underset{\sim}{a}$ are ∇ = "nabla"

$$\nabla_u \underset{\sim}{a} = \underset{\sim}{a}_u - (\underset{\sim}{n} \cdot \underset{\sim}{a}_u)\underset{\sim}{n} = \underset{\sim}{a}_u + (\underset{\sim}{n}_u \cdot \underset{\sim}{a})\underset{\sim}{n}$$

$$\nabla_v \underset{\sim}{a} = \underset{\sim}{a}_v - (\underset{\sim}{n} \cdot \underset{\sim}{a}_v)\underset{\sim}{n} = \underset{\sim}{a}_v + (\underset{\sim}{n}_v \cdot \underset{\sim}{a})\underset{\sim}{n}$$

That is, they are the orthogonal projections of $\underset{\sim}{a}_u, \underset{\sim}{a}_v$ to the tangent spaces $T_x X$, the derivatives of $\underset{\sim}{a}$ in the tangent directions.

Can also write them as $\nabla_{\frac{\partial}{\partial u}} \underset{\sim}{a}$, $\nabla_{\frac{\partial}{\partial v}} \underset{\sim}{a}$, i.e. we differentiate the vector field $\underset{\sim}{a} = e\frac{\partial}{\partial u} + f\frac{\partial}{\partial v}$ in the directions of vector fields $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$.

From $(*)$ (beginning of lecture) we see that

$$\nabla_u \underset{\sim}{a} = e_u \underset{\sim}{r}_u + f_u \underset{\sim}{r}_v + e\Gamma^u_{uu}\underset{\sim}{r}_u + e\Gamma^v_{uu}\underset{\sim}{r}_v + f\Gamma^u_{vu}\underset{\sim}{r}_u + f\Gamma^v_{vu}\underset{\sim}{r}_v$$

$$\nabla_v \underset{\sim}{a} = e_v \underset{\sim}{r}_u + f_v \underset{\sim}{r}_v + e\Gamma^u_{uv}\underset{\sim}{r}_u + e\Gamma^v_{uv}\underset{\sim}{r}_v + f\Gamma^u_{vv}\underset{\sim}{r}_u + f\Gamma^v_{vv}\underset{\sim}{r}_v.$$

That is, for $\underset{\sim}{a} = e\frac{\partial}{\partial u} + f\frac{\partial}{\partial v}$

$$\nabla_{\frac{\partial}{\partial u}} \underset{\sim}{a} = e_u\frac{\partial}{\partial u} + f_u\frac{\partial}{\partial v} + e\Gamma^u_{uu}\frac{\partial}{\partial u} + e\Gamma^v_{uu}\frac{\partial}{\partial v} + f\Gamma^u_{vu}\frac{\partial}{\partial u} + f\Gamma^v_{vu}\frac{\partial}{\partial v}$$

$$\nabla_{\frac{\partial}{\partial v}} \underset{\sim}{a} = e_v\frac{\partial}{\partial u} + f_v\frac{\partial}{\partial v} + e\Gamma^u_{uv}\frac{\partial}{\partial u} + e\Gamma^v_{uv}\frac{\partial}{\partial v} + f\Gamma^u_{vv}\frac{\partial}{\partial u} + f\Gamma^v_{vv}\frac{\partial}{\partial v}.$$

By Proposition 4.1, the Γ^a_{bc} depend only on the 1FF, and are also defined on general (X, g). Hence ∇_u, ∇_v also make sense on a surface X with Riemannian metric g.

This defines a structure ∇ ("nabla") called the <u>Levi-Civita connection</u> on (X, g). It differentiates a vector field on X in the direction of another vector field, where vector fields are of the form $e\frac{\partial}{\partial u} + f\frac{\partial}{\partial v}$ in coordinates (u,v) on X.

Aside. Let X be a surface, and $\underline{a}, \underline{b}$ be vector fields on X. Would like to define $\nabla_{\underline{b}}\underline{a}$, the derivative of $\underline{a}$ in direction $\underline{b}$. Heuristically, at $x \in X$, would like

$$\nabla_{\underline{b}}\underline{a} = \lim_{\varepsilon \to 0} \frac{\underline{a}|_{x+\varepsilon \underline{b}} - \underline{a}|_x}{\varepsilon}.$$

However, $\underline{a}|_{x+\varepsilon \underline{b}} \in T_{x+\varepsilon \underline{b}}X$, $\underline{a}|_x \in T_x X$ lie in <u>different vector spaces</u>, so we can't subtract them.

Heuristically, the job a connection does is identify nearby tangent spaces $T_x X \cong T_y X$ for x, y close in X, so can make sense of this.

Proposition 4.2 For any vector field $\underline{a}$ on $X \subset \mathbb{R}^3$ we have

$$(**) \quad \nabla_v \nabla_u \underline{a} - \nabla_u \nabla_v \underline{a} = \frac{LN - M^2}{\sqrt{EG-F^2}} \underline{n} \wedge \underline{a} = K\sqrt{EG-F^2}\, \underline{n} \wedge \underline{a}.$$

Proof. $\nabla_v \nabla_u \underline{a} = \underline{a}_{vu} - (\underline{n} \cdot \underline{a}_{vu})\underline{n} + (\underline{n}_u \cdot \underline{a})\underline{n}_v$.

So $\nabla_v \nabla_u \underline{a} - \nabla_u \nabla_v \underline{a} = (\underline{n}_u \cdot \underline{a})\underline{n}_v - (\underline{n}_v \cdot \underline{a})\underline{n}_u$
$= (\underline{n}_u \wedge \underline{n}_v) \wedge \underline{a}$.

Write $\underline{n}_u \wedge \underline{n}_v = \lambda \underline{n}$. Then

$\lambda \underline{n} \cdot (\underline{r}_u \wedge \underline{r}_v) = (\underline{n}_u \wedge \underline{n}_v) \cdot (\underline{r}_u \wedge \underline{r}_v) = (\underline{n}_u \cdot \underline{r}_u)(\underline{n}_v \cdot \underline{r}_v) - (\underline{n}_u \cdot \underline{r}_v)(\underline{n}_v \cdot \underline{r}_u) = LN - M^2$.

Also $\underline{n} \cdot (\underline{r}_u \wedge \underline{r}_v) = \sqrt{EG - F^2}$, so $\lambda = \dfrac{LN - M^2}{\sqrt{EG-F^2}}$. $\square$

Note: the r.h.s. of $(**)$ involves no derivatives of $\underline{a}$. $(**)$ concerns <u>commuting partial derivatives</u> using ∇.

Corollary 4.3. (Gauss' Theorema Egregium).
The Gaussian curvature K of a surface X can be written solely in terms of the 1FF $E\,du^2 + 2F\,du\,dv + G\,dv^2$ and its first and second derivatives. (Could give explicit formula, but it's complicated.)

Proof. Proposition 4.2 implies that $a \mapsto (\nabla_v \nabla_u - \nabla_u \nabla_v) a$ is $K\sqrt{EG-F^2}$. (rotation by $90°$). But ∇_u, ∇_v depend only on 1FF by Prop. 4.1. Rotation by $90°$ also depends on 1FF. (Actually it also depends on orientation, but orientation is used to determine order of u, v in $\nabla_v \nabla_u - \nabla_u \nabla_v$, so overall formula is orientation-independent.) So $K\sqrt{EG-F^2}$ only depends on 1FF, and so K does. Each ∇_u, ∇_v depends on 1FF and first derivatives, but taking second derivatives $\nabla_v \nabla_u, \nabla_u \nabla_v$ includes an extra derivative of 1FF. $\square$

This implies that Gaussian curvature K is also defined for surface X with a Riemannian metric g.

§4.9. Geodesic curvature and geodesics

Definition. Let $X \subset \mathbb{R}^3$ be an (oriented) smooth surface, and $\gamma : [a,b] \to X$ be a smooth curve parametrized by arc-length s, i.e. $\left|\frac{d}{ds}\gamma(s)\right| = 1$.

Set $\underline{t} = \gamma' = \frac{d\gamma}{ds}$ and $\underline{t}' = \frac{d^2\gamma}{ds^2}$.

Then $\underline{t}$ is the unit tangent vector to γ at $\gamma(s)$.

The geodesic curvature $k_g : [a,b] \to \mathbb{R}$ of γ is

$k_g = \underline{t}' \cdot (\underline{n} \wedge \underline{t})$, where $\underline{n}$ is the unit normal to X at $\gamma(s)$.

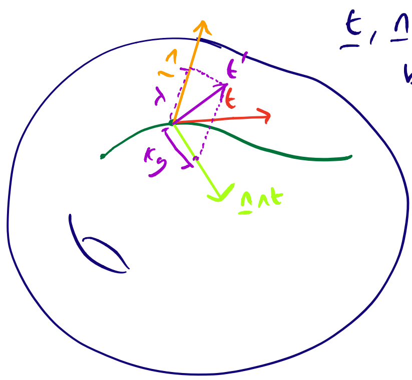
(Need orientation on X to choose the sign of $\underline{n}$).

We call γ a geodesic if $k_g = 0$. This is an o.d.e. on γ.

The geodesic equation $k_g = 0$ is equivalent to

$\underline{t}' = \frac{d^2\gamma}{ds^2} = \lambda \underline{n}$, some $\lambda : [a,b] \to \mathbb{R}$.

That is, the acceleration of $\gamma(s)$, as a moving point, is normal to X.



$\underline{t}, \underline{n}, \underline{n}\wedge\underline{t}$ are an orthonormal basis. Also $\underline{t}\cdot\underline{t}' = 0$ as $|\underline{t}|^2 = 1$. So

$\underline{t}' = \lambda \underline{n} + k_g\, \underline{n}\wedge\underline{t}$,

some $\lambda, k_g \in \mathbb{R}$.

Condition $k_g = 0$ $\iff$ $\underline{t}' = \lambda \underline{n}$.

The geodesic equation $k_g = 0$ appears in mechanics: the motion of a small ball rolling on a frictionless surface without gravity, so the only force is normal to the surface.

Geodesics are locally length-minimizing. That is, if $a \leq t_1 < t_2 \leq b$ with $|t_1 - t_2|$ small, then $\gamma([t_1,t_2])$ is the shortest path from $\gamma(t_1)$ to $\gamma(t_2)$ in X.

Let $\underline{r}: V \longrightarrow X$ be a local parametrization, and write $\gamma(s) = \underline{r}(u(s), v(s))$. Then $\underline{t} = u'\underline{r}_u + v'\underline{r}_v$,

$$\underline{t}' = u''\underline{r}_u + v''\underline{r}_v + (u')^2\underline{r}_{uu} + 2u'v'\underline{r}_{uv} + (v')^2\underline{r}_{vv}$$

$$= u''\underline{r}_u + v''\underline{r}_v + \left((u')^2 L + 2u'v' M + (v')^2 N\right)\underline{n}$$

by (*) in §4.8

$$+ (u')^2\left(\Gamma^u_{uu}\underline{r}_u + \Gamma^v_{uu}\underline{r}_v\right) + 2(u'v')\left(\Gamma^u_{uv}\underline{r}_u + \Gamma^v_{uv}\underline{r}_v\right) + (v')^2\left(\Gamma^u_{vv}\underline{r}_u + \Gamma^v_{vv}\underline{r}_v\right)$$

Write $\underline{n} \wedge \underline{t} = f\underline{r}_u + g\underline{r}_v$. Then f, g depend only on 1FF, u', v', and orientation, as $\underline{n}\wedge\underline{t}$ is rotation of $\underline{t}$ by $90°$, and

$$\kappa_g = (Ef + Fg)\left(u'' + (u')^2\Gamma^u_{uu} + 2u'v'\Gamma^u_{uv} + (v')^2\Gamma^u_{vv}\right) + (Ff + Gg)\left(v'' + (u')^2\Gamma^v_{uu} + 2u'v'\Gamma^v_{uv} + (v')^2\Gamma^v_{vv}\right).$$

Hence by Proposition 4.1, the geodesic curvature κ_g of γ depends only on γ, 1FF and the orientation of X. (Orientation determines the sign.) Thus κ_g is also well-defined for curves in surfaces (X, g) with Riemannian metrics.

Proposition 4.4. Let $X \subset \mathbb{R}^3$ be a smooth surface with local parametrization $\underline{r}: V \longrightarrow X$. A curve $\gamma(s) = \underline{r}(u(s), v(s))$ parametrized by arc-length s is a geodesic iff

$$\frac{d}{ds}\left(Eu' + Fv'\right) = \frac{1}{2}\left(E_u(u')^2 + 2F_u u'v' + G_u(v')^2\right),$$

$$\frac{d}{ds}\left(Fu' + Fv'\right) = \frac{1}{2}\left(E_v(u')^2 + 2F_v u'v' + G_v(v')^2\right).$$

Proof. We have $\underline{t} = \underline{r}_u u' + \underline{r}_v v'$, and γ is a geodesic iff $\underline{t}'$ is normal, i.e. $\underline{t}'\cdot\underline{r}_u = \underline{t}'\cdot\underline{r}_v = 0$. But $\underline{t}'\cdot\underline{r}_u = (\underline{t}\cdot\underline{r}_u)' - \underline{t}\cdot\underline{r}_u'$, so the first equation is $(\underline{t}\cdot\underline{r}_u)' = \underline{t}\cdot\underline{r}_u'$, i.e. $\frac{d}{ds}\left((u'\underline{r}_u + v'\underline{r}_v)\cdot\underline{r}_u\right) = (u'\underline{r}_u + v'\underline{r}_v)\cdot(u'\underline{r}_{uu} + v'\underline{r}_{uv})$

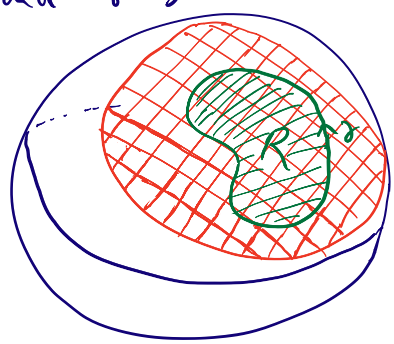
i.e. $\frac{d}{ds}\left(Eu' + Fv'\right) = \frac{1}{2}\left(E_u(u')^2 + 2F_u u'v' + G_u(v')^2\right)$, as $E_u = (\underline{r}_u\cdot\underline{r}_u)_u = 2\underline{r}_u\cdot\underline{r}_{uu}$, etc. The second equation is similar. $\square$

These geodesic equations also make sense on (X, g) surface with Riemannian metric.

Can use them and results on o.d.e.s to show that given a point x in X and a direction at x, there is a unique geodesic through x in this direction.

§4.10. The Gauss–Bonnet Theorem

Theorem 4.5. (Local Gauss–Bonnet.)
Let $X \subset \mathbb{R}^3$ be an oriented smooth surface, and $\underline{r}: V \to X$ be a local parametrization. Let $\gamma: S^1 \to \underline{r}(V) \subseteq X$ be a smooth curve parametrized by arc-length s, which is the boundary of a compact disc-shaped region R in $\underline{r}(V) \subseteq X$, and γ goes anticlockwise around R.



Then $\int_\gamma \kappa_g \, ds = 2\pi - \int_R K \, dA$, where κ_g is the geodesic curvature of γ, and K the Gaussian curvature of X, and $\int \cdots ds$, $\int \cdots dA$ are integration w.r.t. arc-length and area.

Proof. (Not examinable.)
Write $R = \underline{r}(D)$ for $D \subset V$ a closed disc-shaped region with smooth boundary ∂D, so $\gamma(S^1) = \underline{r}(\partial D)$.
Parametrize ∂D by arc-length s in X, writing $(u(s), v(s)) \in \partial D$.
Let $P, Q: V \to \mathbb{R}$ be smooth. Then Green's formula gives

$$(+) \quad \int_{\partial D} \big(P(u(s),v(s))\, u' + Q(u(s),v(s))\, v' \big)\, ds = \int_D (Q_u - P_v)\, du\, dv.$$

Choose a unit length tangent vector field $\underline{e}$ on $\underline{r}(V)$, for instance $\underline{e} = \underline{r}_u / \sqrt{E}$. Then $\nabla_u \underline{e}$, $\nabla_v \underline{e}$ are tangent vector fields orthogonal to $\underline{e}$, so there are smooth functions $P, Q: V \to \mathbb{R}$ with $\nabla_u \underline{e} = P\, \underline{n} \wedge \underline{e}$, $\nabla_v \underline{e} = Q\, \underline{n} \wedge \underline{e}$.
Let $\underline{t}$ be the unit tangent to γ, and write $\underline{t} = \cos\theta\, \underline{e} + \sin\theta\, \underline{n} \wedge \underline{e}$, for smooth $\theta: \partial D \to \frac{\mathbb{R}}{2\pi\mathbb{Z}}$, so θ is the angle between $\underline{t}$ and $\underline{e}$.

Then the geodesic curvature is $k_g = \underline{t}' \cdot (\underline{n} \wedge \underline{t})$

$$= \Big[\, \theta'(-\sin\theta\, \underline{e} + \cos\theta\, \underline{n}\wedge\underline{e}) + \cos\theta\,(u' \underbrace{\nabla_u \underline{e}}_{P\,\underline{n}\wedge\underline{e}} + v' \underbrace{\nabla_v \underline{e}}_{Q\,\underline{n}\wedge\underline{e}}) + \sin\theta\, \underline{n}\wedge(u' \underbrace{\nabla_u \underline{e}}_{P\,\underline{n}\wedge\underline{e}} + v' \underbrace{\nabla_v \underline{e}}_{Q\,\underline{n}\wedge\underline{e}}) \Big]$$

$$\cdot \big[\, \underline{n} \wedge (\cos\theta\, \underline{e} + \sin\theta\, \underline{n}\wedge\underline{e}) \big]$$

$= \theta' + Pu' + Qv'$. Hence the l.h.s. of (†) is

$$\int_{\partial D} (k_g - \theta')\, ds = \int_\gamma k_g\, ds - 2\pi,$$ since θ increases from 0 to 2π round $S^1 = \partial D$.

For the r.h.s. of (†), we have

$$\nabla_v \nabla_u \underline{e} = \nabla_v (P\, \underline{n}\wedge\underline{e}) = P_v\, \underline{n}\wedge\underline{e} + P\, \underline{n} \wedge \nabla_v \underline{e} = P_v\, \underline{n}\wedge\underline{e} + PQ\, \underline{n}\wedge(\underline{n}\wedge\underline{e}).$$

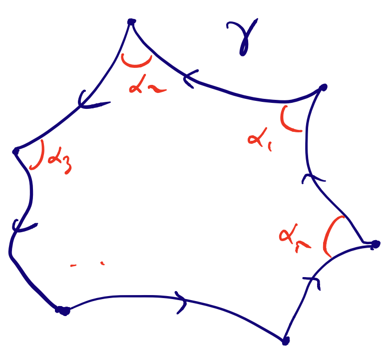
Similarly $\nabla_u \nabla_v \underline{e} = Q_u\, \underline{n}\wedge\underline{e} + PQ\, \underline{n}\wedge(\underline{n}\wedge\underline{e})$, so

$\nabla_v \nabla_u \underline{e} - \nabla_u \nabla_v \underline{e} = (P_v - Q_u)\, \underline{n}\wedge\underline{e} = K\sqrt{EG-F^2}\, \underline{n}\wedge\underline{e}$ by Prop. 4.2.

Thus the r.h.s. of (†) is $\int_D K\sqrt{EG-F^2}\, du\, dv = \int_D K\, dA$. The theorem follows. $\square$

Remark. (a) Theorem 4.5 also holds for a smooth disc in a surface X with Riemannian metric g, not embedded in $\mathbb{R}^3$. We have shown k_g and K are defined then, as they depend on the 1FF.

(b) We can extend Theorem 4.5 to allow γ piecewise-smooth – a curvilinear polygon with n vertices, with internal angles $\alpha_1, \dots, \alpha_n$. Then we get



$$\int_\gamma k_g\, ds = 2\pi - (\pi - \alpha_1) - \dots - (\pi - \alpha_n) - \int_R K\, dA \qquad (\dagger\dagger)$$

$$= (2-n)\pi + \alpha_1 + \dots + \alpha_n - \int_R K\, dA,$$

since smoothing off a corner 

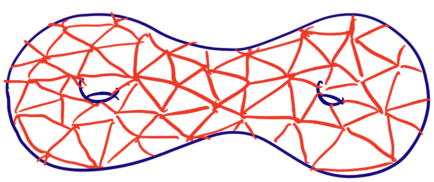
adds $\pi - \alpha$ to $\int_\gamma k_g\, dA$ in the limit.

Note: nice case if the sides are geodesics, so $k_g = 0$, and $(n-2)\pi + \int_R K\, dA = \alpha_1 + \dots + \alpha_n$. If $K = 0$, get usual formula for angles of polygon in $\mathbb{R}^2$.

Theorem 4.6. (Gauss–Bonnet). Let (X, g) be a compact smooth surface with Riemannian metric g and Gaussian curvature κ. Then $\int_X \kappa \, dA = 2\pi \chi(X)$.

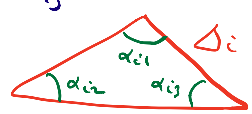
— This says the total curvature $\int_X \kappa \, dA$ is a topological invariant
— it is independent of the metric g.

Proof. Choose a triangulation of X into smooth triangles $\Delta_1, \dots, \Delta_{2n}$, each lying in a coordinate neighborhood of X.



There are V vertices, $E = 3n$ edges, and $F = 2n$ faces, so $\chi(X) = V - E + F$
$= V - n$.

Write α_{ij} for the internal angles of Δ_i, $j = 1, 2, 3$.



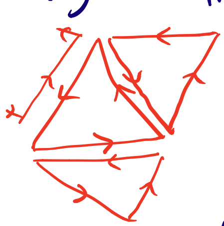
Then (††) (two slides ago) gives

$$\int_{\partial \Delta_i} \kappa_g \, ds = -\pi + \alpha_{i1} + \alpha_{i2} + \alpha_{i3} - \int_{\Delta_i} \kappa \, dA.$$

Adding over $i = 1, \dots, 2n$ gives

$$\sum_{i=1}^{2n} \int_{\partial \Delta_i} \kappa_g \, ds = -2\pi n + \sum_{i=1}^{2n} \sum_{j=1}^{3} \alpha_{ij} - \int_X \kappa \, dA.$$

Each edge is the boundary of two triangles $\Delta_{i_1}, \Delta_{i_2}$, and as the edges are oriented in opposite directions, κ_g has opposite signs



so the integral of κ_g along pairs of edges cancel, giving $\sum_{i=1}^{2n} \int_{\partial \Delta_i} \kappa_g \, ds = 0$. Also $\sum_{i,j} \alpha_{ij} = 2\pi V$, as at each vertex the internal angles sum to 2π. Hence

$$0 = \sum_{i=1}^{2n} \int_{\partial \Delta_i} \kappa_g \, ds = -2\pi n + 2\pi V - \int_X \kappa \, dA = 2\pi(V - n) - \int_X \kappa \, dA$$
$$= 2\pi \chi(X) - \int_X \kappa \, dA. \qquad \square$$

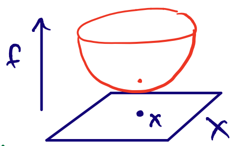
§4.11. Critical points and the Euler characteristic

Definition. Let X be a smooth surface and $f: X \to \mathbb{R}$ a smooth function. A point $x \in X$ is a critical point if $f_u = f_v = 0$ in local coordinates (u,v).

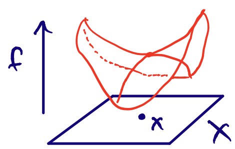
The Hessian of f at x is $\mathrm{Hess}\, f = \begin{pmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{pmatrix}$.

The critical point is nondegenerate if $\det \mathrm{Hess}\, f \neq 0$.

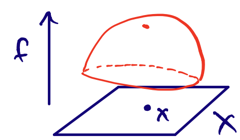
Nondegenerate critical points are isolated, so there are only finitely many if X is compact. They are divided into local minima, saddle points, and local maxima, if $\mathrm{Hess}\, f$ has signature $(+,+)$, $(+,-)$, $(-,-)$, respectively.



local minimum



saddle point

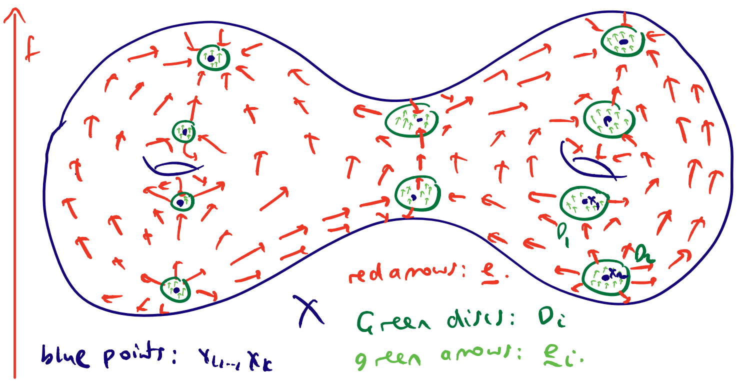


local maximum.

Theorem 4.7. Let X be a compact smooth surface, and $f: X \to \mathbb{R}$ a smooth function with only nondegenerate critical points. Then $\chi(X) = (\#\text{ local min}) - (\#\text{ saddle points}) + (\#\text{ local max})$.

Proof. We adapt the proof of the Gauss–Bonnet Theorem in §4.10. Choose a Riemannian metric g on X. Define a vector $\underset{\sim}{a} = \nabla f$ on X, with zeroes only at the critical points of f. Write $x_1, \dots, x_k$ for the critical points. Choose a small disc D_i round x_i, $i = 1, \dots, k$. On $X \setminus (D_1 \cup \dots \cup D_k)$, define $\underset{\sim}{e} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$, a unit vector field.

On each D_i, choose a unit vector field $\underset{\sim}{e}_i$.



The proof of Theorem 4.5 also shows: if $R \subset X$ is a region with smooth boundary $\gamma = \partial R$, and $\underline{e}$ is a unit vector field on R, then

$$\int_\gamma (\kappa_g - \theta')\, ds = -\int_R \kappa\, dA,$$ where θ is the angle between $\underline{t} = \frac{d\gamma}{ds}$ and $\underline{e}$.

This holds for all closed regions R, not just discs.

Set $\gamma_i = \partial D_i$, and write $\overline{\gamma_i}$ for γ_i with the opposite orientation.

We have, for $R = X \setminus (D_1 \cup \dots \cup D_k)$

$$\sum_{i=1}^{k} \int_{\overline{\gamma_i}} (\kappa_g - \theta')\, ds = -\int_{X \setminus (D_1 \cup \dots \cup D_k)} \kappa\, dA,$$

---

and for $R = D_i$

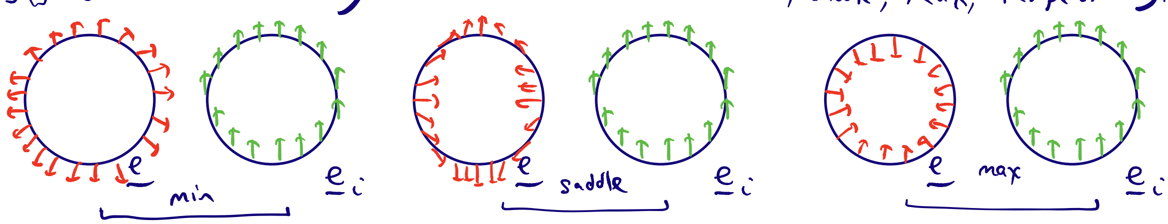
$$\int_{\gamma_i} (\kappa_g - \theta_i')\, ds = -\int_{D_i} \kappa\, dA,$$

where θ, θ_i are the angles between $\underline{t}$ and $\underline{e}, \underline{e}_i$.

As $\overline{\gamma_i}$ has opposite orientation to γ_i, $\int_{\overline{\gamma_i}} (\kappa_g - \theta')\, ds = -\int_{\gamma_i} (\kappa_g - \theta')\, ds$.

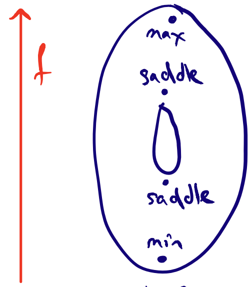
Adding gives $\displaystyle\sum_{i=1}^{k} \int_{\gamma_i} (\theta' - \theta_i')\, ds = -\int_X \kappa\, dA = -2\pi\, \chi(X)$ by Gauss–Bonnet.

Now $\theta - \theta_i$ increases by $-2\pi, 2\pi, -2\pi$ for a min, saddle, max, respectively.



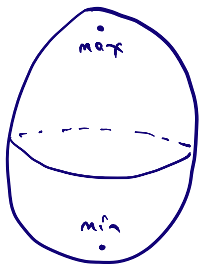
Hence $2\pi(-\#\text{min} + \#\text{saddle} - \#\text{max}) = -2\pi\, \chi(X)$. $\square$

Examples.



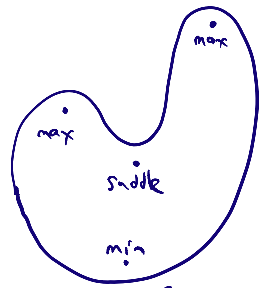
$X \cong T^2$

$\chi(X) = 1 - 2 + 1 = 0$



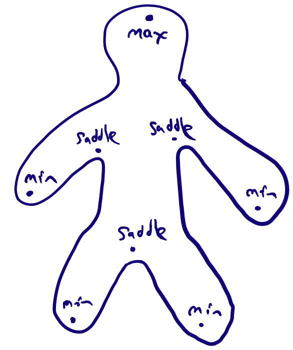
$X \cong S^2$

$\chi(X) = 1 - 0 + 1 = 2.$



$X \cong S^2$

$\chi(X) = 1 - 1 + 2 = 2.$



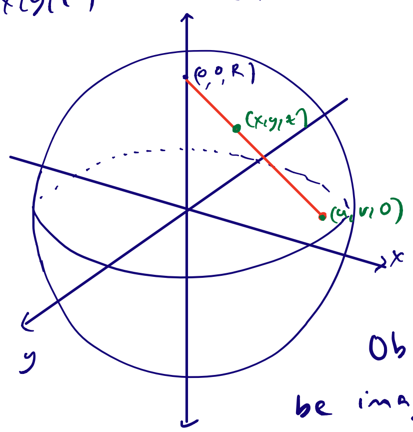
$X \cong S^2$

$\chi(X) = 4 - 3 + 1 = 2.$

The "Hairy Ball Theorem" says that if you have a 2-sphere S^2 with "hair" all over it (e.g. a hamster) you can't comb the hair so it lies flat everywhere. That is, there are no vector fields $\underline{a}$ on S^2 with no zeroes, as then a variant of Theorem 4.7 would give $\chi(S^2) = 0$, contradicting $\chi(S^2) = 2$. Thinking of $\underline{a}$ as the wind velocity on the surface of the Earth, there must be some point with no wind.

§5. The hyperbolic plane

Example. Let $S^2_R = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=R^2\}$. Define coordinates (u,v) on $S^2_R \setminus (R,0,0)$ such that $(0,0,R)$, (x,y,z) and $(u,v,0)$ are collinear. This gives



$$\underline{r}(u,v) = \left(\frac{2R^2u}{R^2+r^2}, \frac{2R^2v}{R^2+r^2}, \frac{Rr^2-R^3}{R^2+r^2}\right),$$

where $r^2=u^2+v^2$. The first fundamental form is $g = \dfrac{4R^4}{(R^2+r^2)^2}(du^2+dv^2)$. The principal curvatures are $\kappa_1=\kappa_2=\frac{1}{R}$, so $K=\frac{1}{R^2}$.

Observe that g still makes sense if we take R to be imaginary, so $R^2<0$.

If $R=i=\sqrt{-1}$, this gives

$$g = \frac{4}{(1-r^2)^2}(du^2+dv^2), \quad \text{with } K=-1.$$

As this blows up when $r=1$, we set

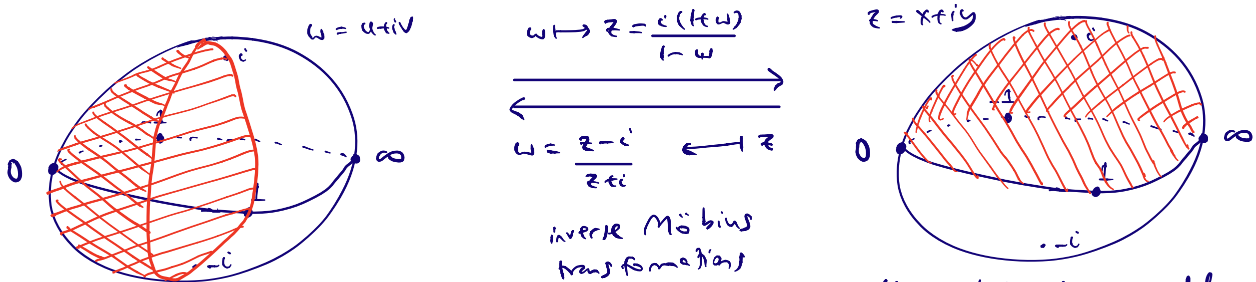
$$X = \{(u,v)\in\mathbb{R}^2 : u^2+v^2<1\}, \text{ with metric}$$

$$g = \frac{4}{(1-u^2-v^2)^2}(du^2+dv^2).$$

This is the <u>Poincaré disc model of the hyperbolic plane</u>. It has Gaussian curvature $K=-1$. It is an abstract surface X with Riemannian metric g, which can be thought of as the "sphere with radius i".

An alternative model is the <u>upper half plane model</u> $\mathcal{H}=\{(x,y)\in\mathbb{R}^2 : y>0\}$, with metric $g=\dfrac{dx^2+dy^2}{y^2}$. The isometric transformation between them is $u+iv = \dfrac{x+iy-i}{x+iy+i}$, or $x+iy = \dfrac{i(1+u+iv)}{1-(u+iv)}$.

We will usually work with the upper half plane model, as it is simpler. It is helpful to write $z = x+iy$ and consider $\mathcal{H} \subset \mathbb{C} \subset \mathbb{C}\cup\{\infty\}$



Unit disc model = unit disc in $\mathbb{C}\cup\{\infty\}$: an open hemisphere.

Upper half plane model: also open hemisphere in $\mathbb{C}\cup\{\infty\}$

Theorem 5.1. Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a real 2×2 matrix with $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$. Then $z \mapsto \dfrac{az+b}{cz+d}$ and $z \mapsto \dfrac{b-a\bar{z}}{d-c\bar{z}}$ are isometries of $\mathcal{H}$. All isometries are of this form.

Proof. See Hitchin §5. $\square$

Note that $\mathbb{R}^\times = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : 0 \neq a \in \mathbb{R}\right\}$ acts trivially, so the isometry group is $\mathbb{Z}_2 \ltimes PGL(2,\mathbb{R})$, where $PGL(2,\mathbb{R}) = \dfrac{GL(2,\mathbb{R})}{\mathbb{R}^\times}$ is 3-dimensional. The unit sphere S^2 also has a 3-dimensional isometry group $O(3) = \mathbb{Z}_2 \ltimes SO(3)$.

Dimension count for $SO(3)$: 2 dim for axis, 1 dim for angle of rotation.

§5.1. Geodesics in the hyperbolic plane.

Let $\gamma(s) = (x(s), y(s))$ be a geodesic in $\left(\mathcal{H}, \dfrac{dx^2+dy^2}{y^2}\right)$.

Then $\underline{t} = (x', y')$ is a unit vector, so $(x')^2 + (y')^2 = y^2$.

Also γ satisfies the geodesic equations in Prop. 4.4. These reduce to

$$\frac{d}{ds}\left(\frac{x'}{y^2}\right) = 0, \qquad \frac{d}{ds}\left(\frac{y'}{y^2}\right) = \frac{(x')^2 y' + (y')^3}{y^3}.$$

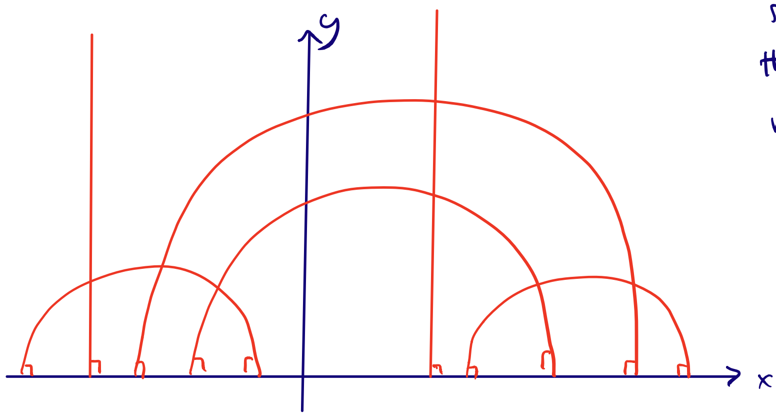
So $x' = cy^2$, and $(x')^2 + (y')^2 = y^2$ gives $y' = \sqrt{y^2 - c^2 y^4}$.

$$\frac{dy}{dx} = \frac{y'}{x'} = \sqrt{\frac{y^2 - c^2y^4}{c^2y^4}}, \qquad \frac{cy\,dy}{\sqrt{1-c^2y^2}} = dx.$$

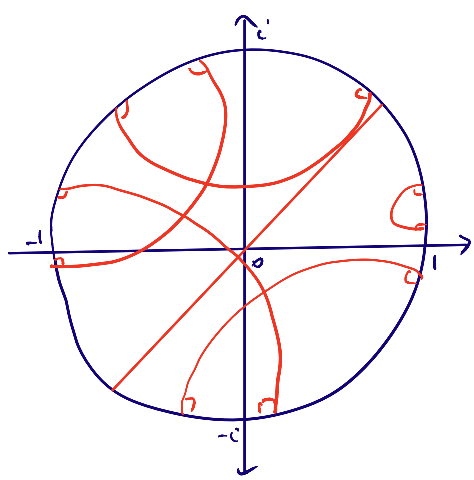
This integrates to $\quad -c^{-1}\sqrt{1-c^2y^2} = x-a,$

i.e. $\quad (x-a)^2 + y^2 = \frac{1}{c^2}.$

Geodesics are semicircles centred on the x-axis, plus vertical half-lines (the case $c=0$, which reduces to $x=a$.).

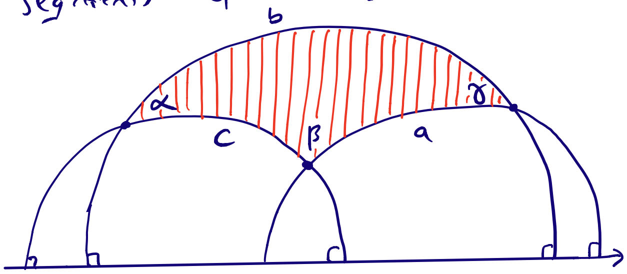


In the unit disc model, geodesics are arcs of circles meeting the unit circle at right angles, plus diameters of the unit disc (straight lines through $(0,0)$).



§5.2. Hyperbolic triangles.

Consider a hyperbolic triangle $\triangle$ in $\mathcal{H}$, with sides segments of 3 geodesics. Let α, β, γ be the internal angles at the vertices.



N.B. Angles in (H, g) are the same as angles measured in $\mathbb{R}^2$ in the coordinates (u,v), since $g = (\text{function}).(du^2 + dv^2)$, and multiplying by function doesn't change angles.

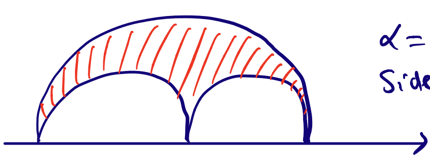
Then piecewise-smooth local Gauss-Bonnet gives

$$0 = (2-3)\pi + \alpha + \beta + \gamma - \int_\triangle K \, dA.$$

$\uparrow$ As $k_g = 0$ for geodesics. $\uparrow K \equiv -1$.

Hence $\text{Area}(\triangle) = \pi - (\alpha+\beta+\gamma)$.

Note that $\text{Area}(\triangle) \leq \pi$ even for arbitrarily large $\triangle$.



$\alpha = \beta = \gamma = 0$.
Side lengths infinite.
$\text{Area}(\triangle) = \pi$.

Also $\alpha + \beta + \gamma < \pi$.
(In the Euclidean plane $\alpha + \beta + \gamma = \pi$.)

Let a, b, c be the lengths of the sides opposite α, β, γ.

The <u>hyperbolic cosine rule</u> says that

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma.$$

If a,b,c v. small $\cosh a \approx 1 + a^2$, $\sinh a \approx a$

$$1 + c^2 \approx (1+a^2)(1+b^2) - ab\cos\gamma \iff c^2 \approx a^2 + b^2 - ab\cos\gamma.$$

So, recover usual cosine rule in limit $a,b,c \to 0$.

The <u>hyperbolic sine rule</u> says that $\dfrac{\sin\alpha}{\sinh a} = \dfrac{\sin\beta}{\sinh b} = \dfrac{\sin\gamma}{\sinh c}$

(If a,b,c small $\Rightarrow$) $\dfrac{\sin\alpha}{a} \approx \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$. usual sine rule.

Lots of geometry in $\mathbb{R}^2$ has nice extensions to $\mathcal{H}$. Tends to involve hyperbolic trigonometric functions sinh, cosh, tanh.

§5.3. The uniformization Theorem.

Let X be a Riemann surface, with complex structure J.
Then X is also a smooth surface, so we can consider Riemannian metrics g on X. Now X has tangent spaces $T_xX \cong \mathbb{R}^2$ for $x \in X$. The complex structure on X makes T_xX into a 1-dimensional complex vector space $T_xX \cong \mathbb{C}$.
So multiplication by i gives rotation by $90°$ in T_xX.
The metric g gives an inner product on T_xX. There is a natural compatibility condition: the notions of rotation by $90°$ for J, g should be the same.
Then we say that (X, J) and (X, g) <u>have the same conformal structure</u>. (Conformal = notion of angle.)

If $z = x+iy$ is a complex coordinate, this happens if $g = E(dx^2+dy^2)$, i.e. $E = G$ and $F = 0$.

<u>Theorem 5.2</u> (Uniformization Theorem.)
Every compact, connected Riemann surface (X, J) has a Riemannian metric g compatible with its conformal structure, with constant Gaussian curvature $K = 1, 0,$ or -1.

From Gauss–Bonnet we see that $K=1$ if $\chi(X) > 0$, i.e. $g=0$; $K=0$ if $\chi(X)=0$, i.e. $g=1$, and $K=-1$ if $\chi(X)<0$, i.e. $g>1$.

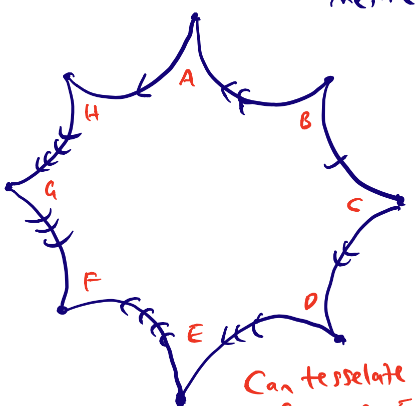
$K=1 \Rightarrow X = (S^2, g)$ with round metric
$K=0 \Rightarrow X = \mathbb{R}^2/\Lambda$, with Euclidean metric on $\mathbb{R}^2$.
$K=-1 \Rightarrow X = \mathcal{H}/\Gamma$, for Γ an infinite group of isometries of $\mathcal{H}$, acting freely on $\mathcal{H}$. So can understand Riemann surfaces of genus $g>1$ using hyperbolic geometry.

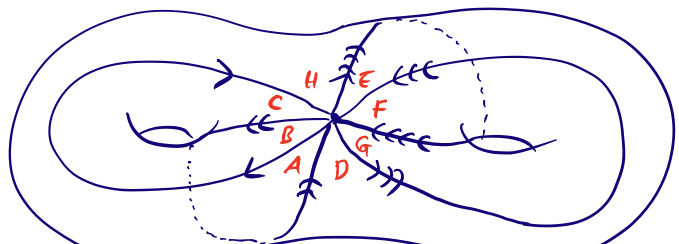
Can get them by gluing sides of a polygon in $\mathcal{H}$ with geodesic sides: the hyperbolic version of planar models.

__Example:__ Take a regular octagon in the hyperbolic plane, with geodesic sides, and internal angles $45°$. Local Gauss–Bonnet gives

$$\int_{\text{octagon}} k \, dA = (2-8)\pi + \frac{\pi}{4} + \dots + \pi/4 = -4\pi = 2\pi(-2).$$

Gluing sides as shown gives a genus 2 surface with a hyperbolic metric. $V=1$, $E=4$, $F=1$ $\Rightarrow$ $\chi(X) = -2$, $g=2$.

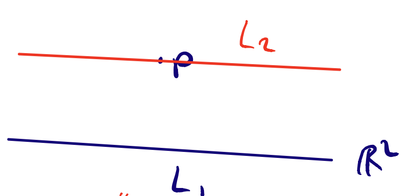


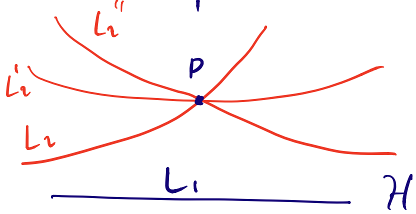


Can tesselate $\mathcal{H}$ by octagons like this, acted on by Γ with $\mathcal{H}/\Gamma = X$. Compare Escher pictures.

The hyperbolic plane was historically important in the development of mathematics. Euclid studied geometry in $\mathbb{R}^2$ starting from axioms. His final axiom was called the "parallel postulate":

- two lines L_1, L_2 are __parallel__ if $L_1 \cap L_2 = \emptyset$.
- given a line L_1 and a point $p \in \mathbb{R}^2$ not on L_1, there is a unique line L_2 through p parallel to L_1.





Euclid seemed to be embarrassed about the parallel postulate, and avoided using it as much as possible. It was a long-standing problem to prove the parallel postulate from the other axioms.

The hyperbolic plane satisfies all Euclid's axioms except the parallel postulate: there are many lines L_2 through p not intersecting L_1.

This eventually led to a reassessment of what geometry was.