Prelims Analysis III

Marc Lackenby

Trinity Term 2024

Definition of the integral of a function

Main goal. Define

$$\int_a^b f(x)dx.$$

Definition of the integral of a function

Main goal. Define
$$\int_a^b f(x)dx.$$

Approach 1. Integration is ‘anti-differentation’.

Definition of the integral of a function

Main goal. Define

$$\int_a^b f(x)dx.$$

Approach 1. Integration is 'anti-differentation'.

Approach 2. Integration is 'area under the curve'.

Definition of the integral of a function

Main goal. Define

$$\int_a^b f(x)dx.$$

Approach 1. Integration is 'anti-differentation'.

Approach 2. Integration is 'area under the curve'.

An advantage of approach 2 is that we can deduce results like

$$f(x) \leq g(x) \quad \forall x \in [a, b] \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Definition of the integral of a function

Main goal. Define

$$\int_a^b f(x)dx.$$

Approach 1. Integration is 'anti-differentation'.

Approach 2. Integration is 'area under the curve'.

An advantage of approach 2 is that we can deduce results like

$$f(x) \leq g(x) \quad \forall x \in [a, b] \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Definition of the integral of a function

Main goal. Define

$$\int_a^b f(x)dx.$$

Approach 1. Integration is 'anti-differentation'.

Approach 2. Integration is 'area under the curve'.

An advantage of approach 2 is that we can deduce results like

$$f(x) \leq g(x) \quad \forall x \in [a,b] \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

So it is Approach 2 that we will follow.

Definition of the integral of a function

Main goal. Define
$$\int_a^b f(x)dx.$$

Approach 1. Integration is 'anti-differentation'.

Approach 2. Integration is 'area under the curve'.

An advantage of approach 2 is that we can deduce results like

$$f(x) \leq g(x) \quad \forall x \in [a,b] \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

So it is Approach 2 that we will follow.

The fact that integration and differentation are 'inverses' will become a theorem called the Fundamental Theorem of Calculus (that needs some extra hypotheses!)

The integral

We would like to define the integral of a function f to be the ‘area’ under the graph of f.

The integral

We would like to define the integral of a function f to be the 'area' under the graph of f.

But what do we mean by 'area'?

The integral

We would like to define the integral of a function f to be the 'area' under the graph of f.

But what do we mean by 'area'?

There are several different approaches to this, most notably:

1. Riemann integration / Darboux integration
2. Lebesgue integration

The integral

We would like to define the integral of a function f to be the 'area' under the graph of f.

But what do we mean by 'area'?

There are several different approaches to this, most notably:

1. Riemann integration / Darboux integration $\leftarrow$ we'll do this;
2. Lebesgue integration

The integral

We would like to define the integral of a function f to be the 'area' under the graph of f.

But what do we mean by 'area'?

There are several different approaches to this, most notably:

1. Riemann integration / Darboux integration $\leftarrow$ we'll do this;
2. Lebesgue integration

Not every function will be integrable!

The integral

We would like to define the integral of a function f to be the 'area' under the graph of f.

But what do we mean by 'area'?

There are several different approaches to this, most notably:

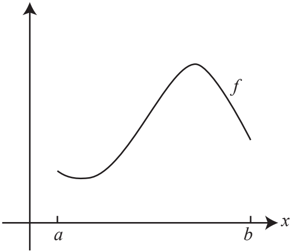
1. Riemann integration / Darboux integration $\leftarrow$ we'll do this;
2. Lebesgue integration

Not every function will be integrable!

But once we've defined integration, we'll prove that every continuous function on a closed bounded interval is integrable.

How do we define 'area under the curve'?

Let $f : [a, b] \to \mathbb{R}$ be bounded function.

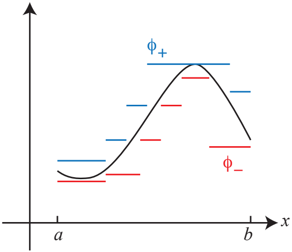


How do we define 'area under the curve'?

Let $f : [a, b] \to \mathbb{R}$ be bounded function.

We consider step functions ϕ_- and ϕ_+ satisfying

$$\phi_- \leq f \leq \phi_+.$$

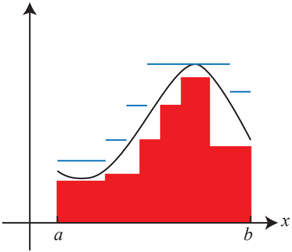


How do we define 'area under the curve'?

Let $f : [a, b] \to \mathbb{R}$ be bounded function.

We consider step functions ϕ_- and ϕ_+ satisfying

$$\phi_- \leq f \leq \phi_+.$$



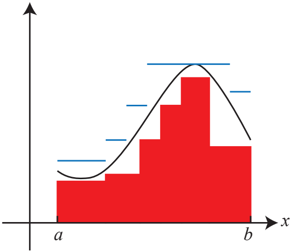
We'll first define the 'integral' $I(\phi)$ of a step function ϕ.

How do we define 'area under the curve'?

Let $f : [a, b] \to \mathbb{R}$ be bounded function.

We consider step functions ϕ_- and ϕ_+ satisfying

$$\phi_- \leq f \leq \phi_+.$$



We'll first define the 'integral' $I(\phi)$ of a step function ϕ.

Then we'll consider all steps functions $\phi_- \leq f$ and all step functions $\phi_+ \geq f$. We'll say that f is integrable if

$$\sup_{\phi_-} I(\phi_-) = \inf_{\phi_+} I(\phi_+).$$

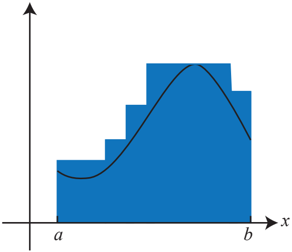
Then we'll define $\int_a^b f$ to be this common value of the sup and the inf.

How do we define 'area under the curve'?

Let $f : [a, b] \to \mathbb{R}$ be bounded function.

We consider step functions ϕ_- and ϕ_+ satisfying

$$\phi_- \leq f \leq \phi_+.$$



We'll first define the 'integral' $I(\phi)$ of a step function ϕ.

Then we'll consider all steps functions $\phi_- \leq f$ and all step functions $\phi_+ \geq f$. We'll say that f is integrable if

$$\sup_{\phi_-} I(\phi_-) = \inf_{\phi_+} I(\phi_+).$$

Then we'll define $\int_a^b f$ to be this common value of the sup and the inf.

Chapter 1A: The definition of integration

Step functions

Definition. Let $[a, b]$ be an interval. A function $\phi : [a, b] \to \mathbb{R}$ is called a step function if there is a finite sequence $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ such that ϕ is constant on each open interval (x_{i-1}, x_i).

Step functions

Definition. Let $[a, b]$ be an interval. A function $\phi : [a, b] \to \mathbb{R}$ is called a step function if there is a finite sequence $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ such that ϕ is constant on each open interval (x_{i-1}, x_i).

We call a sequence $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ a partition $\mathcal{P}$, and we say that ϕ is a step function adapted to $\mathcal{P}$.

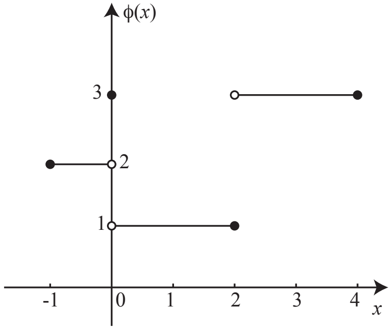
Step functions

Definition. Let $[a, b]$ be an interval. A function $\phi : [a, b] \to \mathbb{R}$ is called a step function if there is a finite sequence $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ such that ϕ is constant on each open interval (x_{i-1}, x_i).

We call a sequence $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ a partition $\mathcal{P}$, and we say that ϕ is a step function adapted to $\mathcal{P}$.

A partition $\mathcal{P}'$ given by $a = x'_0 \leq \cdots \leq x'_{n'} \leq b$ is refinement of $\mathcal{P}$ if every x_i is an x'_j for some j.

An example



$$\phi(x) = \begin{cases} 2 & \text{if } -1 \leq x < 0; \\ 3 & \text{if } x = 0; \\ 1 & \text{if } 0 < x \leq 2; \\ 3 & \text{if } 2 < x \leq 4. \end{cases}$$

Simple properties of step functions

Lemma 1.3. We have the following facts about partitions:

Simple properties of step functions

Lemma 1.3. We have the following facts about partitions:

1. Suppose that ϕ is a step function adapted to $\mathcal{P}$, and if $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then ϕ is also a step function adapted to $\mathcal{P}'$.

Simple properties of step functions

Lemma 1.3. We have the following facts about partitions:

1. Suppose that ϕ is a step function adapted to $\mathcal{P}$, and if $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then ϕ is also a step function adapted to $\mathcal{P}'$.
2. If $\mathcal{P}_1, \mathcal{P}_2$ are two partitions then there is a common refinement of both of them.

Simple properties of step functions

Lemma 1.3. We have the following facts about partitions:

1. Suppose that ϕ is a step function adapted to $\mathcal{P}$, and if $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then ϕ is also a step function adapted to $\mathcal{P}'$.
2. If $\mathcal{P}_1, \mathcal{P}_2$ are two partitions then there is a common refinement of both of them.
3. If ϕ_1, ϕ_2 are step functions then so are $\max(\phi_1, \phi_2)$, $\phi_1 + \phi_2$ and $\lambda\phi_i$ for any scalar λ.

Indicator functions

If $X \subset \mathbb{R}$ is a set, the indicator function of X is the function $\mathbf{1}_X$ taking the value 1 for $x \in X$ and 0 elsewhere.

Indicator functions

If $X \subset \mathbb{R}$ is a set, the indicator function of X is the function $\mathbf{1}_X$ taking the value 1 for $x \in X$ and 0 elsewhere.

Lemma 1.4. A function $\phi : [a, b] \to \mathbb{R}$ is a step function if and only if it is a finite linear combination of indicator functions of intervals (open and closed).

Indicator functions

If $X \subset \mathbb{R}$ is a set, the indicator function of X is the function $\mathbf{1}_X$ taking the value 1 for $x \in X$ and 0 elsewhere.

Lemma 1.4. A function $\phi : [a, b] \to \mathbb{R}$ is a step function if and only if it is a finite linear combination of indicator functions of intervals (open and closed).

Proof.

Indicator functions

If $X \subset \mathbb{R}$ is a set, the indicator function of X is the function $\mathbf{1}_X$ taking the value 1 for $x \in X$ and 0 elsewhere.

Lemma 1.4. A function $\phi : [a, b] \to \mathbb{R}$ is a step function if and only if it is a finite linear combination of indicator functions of intervals (open and closed).

Proof. Suppose first that ϕ is a step function adapted to some partition $\mathcal{P}$, $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$. Then ϕ can be written as a weighted sum of the functions $\mathbf{1}_{(x_{i-1}, x_i)}$ (each an indicator function of an open interval) and the functions $\mathbf{1}_{\{x_i\}}$ (each an indicator function of a closed interval containing a single point).

Indicator functions

If $X \subset \mathbb{R}$ is a set, the indicator function of X is the function $\mathbf{1}_X$ taking the value 1 for $x \in X$ and 0 elsewhere.

Lemma 1.4. A function $\phi : [a, b] \to \mathbb{R}$ is a step function if and only if it is a finite linear combination of indicator functions of intervals (open and closed).

Proof. Suppose first that ϕ is a step function adapted to some partition $\mathcal{P}$, $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$. Then ϕ can be written as a weighted sum of the functions $\mathbf{1}_{(x_{i-1}, x_i)}$ (each an indicator function of an open interval) and the functions $\mathbf{1}_{\{x_i\}}$ (each an indicator function of a closed interval containing a single point).

Conversely, the indicator function of any interval is a step function, and hence so is any finite linear combination of these by Lemma 1.3. $\square$

Indicator functions

If $X \subset \mathbb{R}$ is a set, the indicator function of X is the function $\mathbf{1}_X$ taking the value 1 for $x \in X$ and 0 elsewhere.

Lemma 1.4. A function $\phi : [a, b] \to \mathbb{R}$ is a step function if and only if it is a finite linear combination of indicator functions of intervals (open and closed).

Proof. Suppose first that ϕ is a step function adapted to some partition $\mathcal{P}$, $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$. Then ϕ can be written as a weighted sum of the functions $\mathbf{1}_{(x_{i-1}, x_i)}$ (each an indicator function of an open interval) and the functions $\mathbf{1}_{\{x_i\}}$ (each an indicator function of a closed interval containing a single point).

Conversely, the indicator function of any interval is a step function, and hence so is any finite linear combination of these by Lemma 1.3. □

In particular, the step functions on $[a, b]$ form a vector space, which we occasionally denote by $\mathscr{L}_{\text{step}}[a, b]$.

I of a step function

Definition. Let ϕ be a step function adapted to some partition $\mathcal{P}$, and suppose that $\phi(x) = c_i$ on the interval (x_{i-1}, x_i). Then we define

$$I(\phi) = \sum_{i=1}^{n} c_i(x_i - x_{i-1}).$$

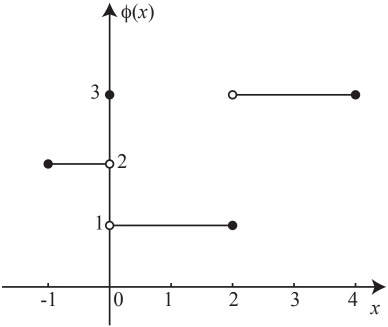
I of a step function

Definition. Let ϕ be a step function adapted to some partition $\mathcal{P}$, and suppose that $\phi(x) = c_i$ on the interval (x_{i-1}, x_i). Then we define

$$I(\phi) = \sum_{i=1}^{n} c_i(x_i - x_{i-1}).$$

We call this $I(\phi)$ rather than $\int_a^b \phi$, because we are going to define $\int_a^b f$ for a class of functions f much more general than step functions. It will then be a theorem that $I(\phi) = \int_a^b \phi$, rather than simply a definition.

An example



$$I(\phi) = (2 \times 1) + (1 \times 2) + (3 \times 2) = 10.$$

I does not depend on the partition

I does not depend on the partition

Our notation suggests that $I(\phi)$ depends only on ϕ, but its definition depended also on the partition $\mathcal{P}$:

$$I(\phi) = \sum_{i=1}^{n} c_i(x_i - x_{i-1}).$$

I does not depend on the partition

Our notation suggests that $I(\phi)$ depends only on ϕ, but its definition depended also on the partition $\mathcal{P}$:

$$I(\phi) = \sum_{i=1}^{n} c_i(x_i - x_{i-1}).$$

In fact, it does not matter which partition one chooses. Write:

$$I(\phi; \mathcal{P}) = \sum_{i=1}^{n} c_i(x_i - x_{i-1}).$$

I does not depend on the partition

Our notation suggests that $I(\phi)$ depends only on ϕ, but its definition depended also on the partition $\mathcal{P}$:

$$I(\phi) = \sum_{i=1}^{n} c_i(x_i - x_{i-1}).$$

In fact, it does not matter which partition one chooses. Write:

$$I(\phi; \mathcal{P}) = \sum_{i=1}^{n} c_i(x_i - x_{i-1}).$$

Then one may easily check that

$$I(\phi; \mathcal{P}) = I(\phi; \mathcal{P}')$$

for any refinement $\mathcal{P}'$ of $\mathcal{P}$.

I does not depend on the partition

Our notation suggests that $I(\phi)$ depends only on ϕ, but its definition depended also on the partition $\mathcal{P}$:

$$I(\phi) = \sum_{i=1}^{n} c_i(x_i - x_{i-1}).$$

In fact, it does not matter which partition one chooses. Write:

$$I(\phi; \mathcal{P}) = \sum_{i=1}^{n} c_i(x_i - x_{i-1}).$$

Then one may easily check that

$$I(\phi; \mathcal{P}) = I(\phi; \mathcal{P}')$$

for any refinement $\mathcal{P}'$ of $\mathcal{P}$.
Now if ϕ is a step function adapted to both $\mathcal{P}_1$ and $\mathcal{P}_2$ then they have a common refinement $\mathcal{P}'$ and so

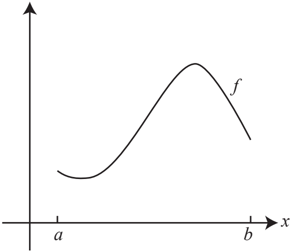
$$I(\phi; \mathcal{P}_1) = I(\phi; \mathcal{P}') = I(\phi; \mathcal{P}_2).$$

Linearity of I

Lemma 1.6. The map $I : \mathscr{L}_{\text{step}}[a, b] \to \mathbb{R}$ is linear:
$I(\lambda\phi_1 + \mu\phi_2) = \lambda I(\phi_1) + \mu I(\phi_2)$.

Majorants and minorants

Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

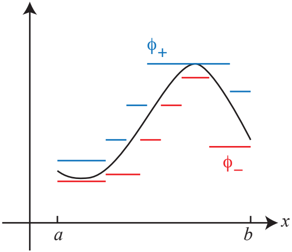


Majorants and minorants

Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

We say that a step function ϕ_- is a minorant for f if $f \geq \phi_-$ pointwise.

We say that a step function ϕ_+ is a majorant for f if $f \leq \phi_+$ pointwise.



Definition of the integral

Definition. A function f is integrable if

$$\sup_{\phi_-} I(\phi_-) = \inf_{\phi_+} I(\phi_+),$$

where the sup is over all minorants $\phi_- \le f$, and the inf is over all majorants $\phi_+ \ge f$.

Definition of the integral

Definition. A function f is integrable if

$$\sup_{\phi_-} I(\phi_-) = \inf_{\phi_+} I(\phi_+),$$

where the sup is over all minorants $\phi_- \leq f$, and the inf is over all majorants $\phi_+ \geq f$.

We define the integral $\int_a^b f$ to be the common value of the sup and the inf.

Definition of the integral

Definition. A function f is integrable if

$$\sup_{\phi_-} I(\phi_-) = \inf_{\phi_+} I(\phi_+),$$

where the sup is over all minorants $\phi_- \leq f$, and the inf is over all majorants $\phi_+ \geq f$.

We define the integral $\int_a^b f$ to be the common value of the sup and the inf.

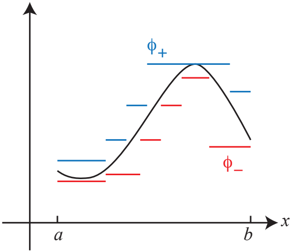
We note that the sup and inf exist for any bounded function f.

Initial observations

Initial observations

For any majorant ϕ_+ and minorant ϕ_- for f, we have

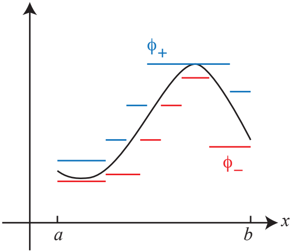
$$I(\phi_-) \leq I(\phi_+).$$



Initial observations

For any majorant ϕ_+ and minorant ϕ_- for f, we have

$$I(\phi_-) \leq I(\phi_+).$$



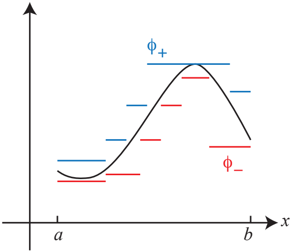
Hence, it is always the case that

$$\sup_{\phi_-} I(\phi_-) \leq \inf_{\phi_+} I(\phi_+).$$

Initial observations

For any majorant ϕ_+ and minorant ϕ_- for f, we have

$$I(\phi_-) \leq I(\phi_+).$$



Hence, it is always the case that

$$\sup_{\phi_-} I(\phi_-) \leq \inf_{\phi_+} I(\phi_+).$$

It follows that when f is integrable, then

$$I(\phi_-) \leq \int_a^b f \leq I(\phi_+)$$

whenever $\phi_- \leq f \leq \phi_+$ are a minorant and majorant.

Minor remarks

1. If a function f is only defined on an open interval (a, b), then we say that it is integrable if an arbitrary extension of it to $[a, b]$ is.

Minor remarks

1. If a function f is only defined on an open interval (a, b), then we say that it is integrable if an arbitrary extension of it to $[a, b]$ is.
2. Integrals are often written using the dx notation. For example, $\int_0^1 x^2 dx$. This means the same as $\int_0^1 f$, where $f(x) = x^2$.

An important lemma

Lemma 1.8. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then the following are equivalent:

An important lemma

Lemma 1.8. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then the following are equivalent:

(i) f is integrable;

An important lemma

Lemma 1.8. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then the following are equivalent:

(i) f is integrable;

(ii) for every $\epsilon > 0$, there is a majorant ϕ_+ and a minorant ϕ_- for f such that $I(\phi_+) - I(\phi_-) < \epsilon$.

Proof of $(i) \Rightarrow (ii)$

Proof of $(i) \Rightarrow (ii)$

Suppose first that f is integrable. Let $\epsilon > 0$.

Proof of $(i) \Rightarrow (ii)$

Suppose first that f is integrable. Let $\epsilon > 0$.

Then by the approximation property for sup and inf, there is a minorant ϕ_- such that

$$I(\phi_-) > \sup_{\phi_-} I(\phi_-) - (\epsilon/2)$$

and a majorant ϕ_+ such that

$$I(\phi_+) < \inf_{\phi_+} I(\phi_+) + (\epsilon/2).$$

Proof of $(i) \Rightarrow (ii)$

Suppose first that f is integrable. Let $\epsilon > 0$.

Then by the approximation property for sup and inf, there is a minorant ϕ_- such that

$$I(\phi_-) > \sup_{\phi_-} I(\phi_-) - (\epsilon/2)$$

and a majorant ϕ_+ such that

$$I(\phi_+) < \inf_{\phi_+} I(\phi_+) + (\epsilon/2).$$

Since the sup and inf are assumed to be equal, we deduce that

$$I(\phi_+) - I(\phi_-) < \epsilon.$$

Proof of $(ii) \Rightarrow (i)$

Proof of $(ii) \Rightarrow (i)$

Now suppose that

(ii) for every $\epsilon > 0$, there is a majorant ϕ_+ and a minorant ϕ_- for f such that $I(\phi_+) - I(\phi_-) < \epsilon$.

Proof of $(ii) \Rightarrow (i)$

Now suppose that

(ii) for every $\epsilon > 0$, there is a majorant ϕ_+
and a minorant ϕ_- for f such that $I(\phi_+) - I(\phi_-) < \epsilon$.

Let $\epsilon > 0$ be arbitrary, and let ϕ_+ and ϕ_- be the majorant and minorant provided by (ii). Then

$$I(\phi_+) < I(\phi_-) + \epsilon \leq \sup_{\phi_-} I(\phi_-) + \epsilon.$$

Proof of $(ii) \Rightarrow (i)$

Now suppose that

(ii) for every $\epsilon > 0$, there is a majorant ϕ_+
and a minorant ϕ_- for f such that $I(\phi_+) - I(\phi_-) < \epsilon$.

Let $\epsilon > 0$ be arbitrary, and let ϕ_+ and ϕ_- be the majorant and minorant provided by (ii). Then

$$I(\phi_+) < I(\phi_-) + \epsilon \leq \sup_{\phi_-} I(\phi_-) + \epsilon.$$

So, taking the infimum over all majorants, we deduce that

$$\inf_{\phi_+} I(\phi_+) < \sup_{\phi_-} I(\phi_-) + \epsilon.$$

Proof of $(ii) \Rightarrow (i)$

Now suppose that

(ii) for every $\epsilon > 0$, there is a majorant ϕ_+
and a minorant ϕ_- for f such that $I(\phi_+) - I(\phi_-) < \epsilon$.

Let $\epsilon > 0$ be arbitrary, and let ϕ_+ and ϕ_- be the majorant and minorant provided by (ii). Then

$$I(\phi_+) < I(\phi_-) + \epsilon \leq \sup_{\phi_-} I(\phi_-) + \epsilon.$$

So, taking the infimum over all majorants, we deduce that

$$\inf_{\phi_+} I(\phi_+) < \sup_{\phi_-} I(\phi_-) + \epsilon.$$

Therefore, $\inf_{\phi_+} I(\phi_+)$ is squeezed between $\sup_{\phi_-} I(\phi_-)$ and $\sup_{\phi_-} I(\phi_-) + \epsilon$.

Proof of $(ii) \Rightarrow (i)$

Now suppose that

(ii) for every $\epsilon > 0$, there is a majorant ϕ_+
and a minorant ϕ_- for f such that $I(\phi_+) - I(\phi_-) < \epsilon$.

Let $\epsilon > 0$ be arbitrary, and let ϕ_+ and ϕ_- be the majorant and minorant provided by (ii). Then

$$I(\phi_+) < I(\phi_-) + \epsilon \leq \sup_{\phi_-} I(\phi_-) + \epsilon.$$

So, taking the infimum over all majorants, we deduce that

$$\inf_{\phi_+} I(\phi_+) < \sup_{\phi_-} I(\phi_-) + \epsilon.$$

Therefore, $\inf_{\phi_+} I(\phi_+)$ is squeezed between $\sup_{\phi_-} I(\phi_-)$ and $\sup_{\phi_-} I(\phi_-) + \epsilon$.
Since $\epsilon > 0$ was arbitrary, we deduce that inf and sup must be equal. In other words, f is integrable.

Estimating the integral

Estimating the integral

Once we know that f is integrable, then any majorant ϕ_+ and ϕ_- as in (ii) gives an approximation to the integral.

Estimating the integral

Once we know that f is integrable, then any majorant ϕ_+ and ϕ_- as in (ii) gives an approximation to the integral.

This is because

$$I(\phi_-) \leq \sup_{\phi_-} I(\phi_-) = \int_a^b f = \inf_{\phi_+} I(\phi_+) \leq I(\phi_+).$$

Estimating the integral

Once we know that f is integrable, then any majorant ϕ_+ and ϕ_- as in (ii) gives an approximation to the integral.

This is because

$$I(\phi_-) \leq \sup_{\phi_-} I(\phi_-) = \int_a^b f = \inf_{\phi_+} I(\phi_+) \leq I(\phi_+).$$

So, $\int_a^b f$ is squeezed between $I(\phi_-)$ and $I(\phi_+)$, which differ by less than ϵ.

An example

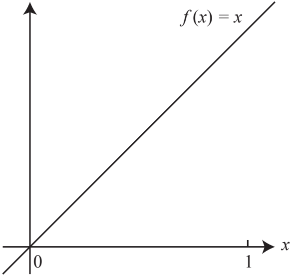
Example The function $f(x) = x$ is integrable on $[0, 1]$, and

$$\int_0^1 x \; dx = \frac{1}{2}.$$

An example

Example The function $f(x) = x$ is integrable on $[0, 1]$, and

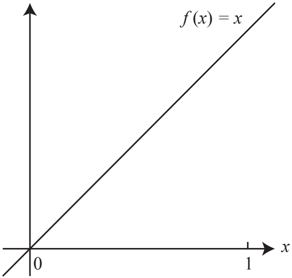
$$\int_0^1 x\ dx = \frac{1}{2}.$$



An example

Example The function $f(x) = x$ is integrable on $[0, 1]$, and

$$\int_0^1 x\, dx = \frac{1}{2}.$$

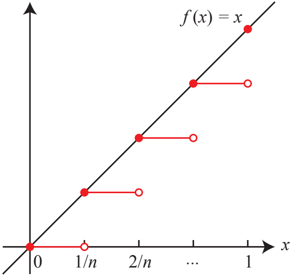


Proof. Let n be an integer to be specified later, and set

An example

Example The function $f(x) = x$ is integrable on $[0,1]$, and

$$\int_0^1 x\, dx = \frac{1}{2}.$$



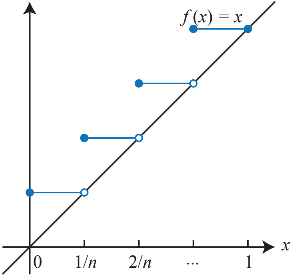
Proof. Let n be an integer to be specified later, and set

$$\phi_-(x) = \frac{i}{n} \text{ for } \frac{i}{n} \leq x < \frac{i+1}{n},\ \ i = 0, 1, \dots, n-1.$$

An example

Example The function $f(x) = x$ is integrable on $[0, 1]$, and

$$\int_0^1 x \, dx = \frac{1}{2}.$$



Proof. Let n be an integer to be specified later, and set

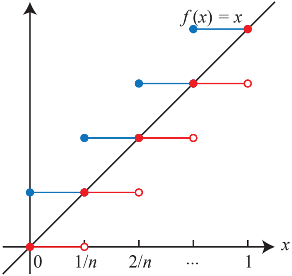
$$\phi_-(x) = \frac{i}{n} \text{ for } \frac{i}{n} \leq x < \frac{i+1}{n}, \quad i = 0, 1, \ldots, n-1.$$

$$\phi_+(x) = \frac{j}{n} \text{ for } \frac{j-1}{n} \leq x < \frac{j}{n}, \quad j = 1, \ldots, n.$$

An example

Example The function $f(x) = x$ is integrable on $[0, 1]$, and

$$\int_0^1 x\, dx = \frac{1}{2}.$$



Proof. Let n be an integer to be specified later, and set

$$\phi_-(x) = \frac{i}{n} \text{ for } \frac{i}{n} \leq x < \frac{i+1}{n}, \quad i = 0, 1, \ldots, n-1.$$

$$\phi_+(x) = \frac{j}{n} \text{ for } \frac{j-1}{n} \leq x < \frac{j}{n}, \; j = 1, \ldots, n.$$

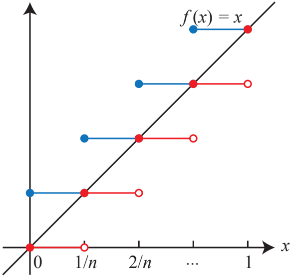
Proof

We have

$$I(\phi_-) = \sum_{i=0}^{n-1} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{2}(1 - \frac{1}{n})$$

and

$$I(\phi_+) = \sum_{j=1}^{n} \frac{j}{n} \cdot \frac{1}{n} = \frac{1}{2}(1 + \frac{1}{n}).$$



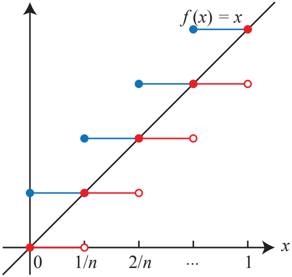
Proof

We have

$$I(\phi_-) = \sum_{i=0}^{n-1} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{2}(1 - \frac{1}{n})$$

and

$$I(\phi_+) = \sum_{j=1}^{n} \frac{j}{n} \cdot \frac{1}{n} = \frac{1}{2}(1 + \frac{1}{n}).$$



So, by Lemma 1.8, f is integrable.

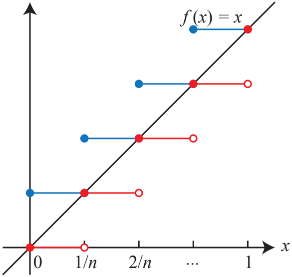
Proof

We have

$$I(\phi_-) = \sum_{i=0}^{n-1} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{2}(1 - \frac{1}{n})$$

and

$$I(\phi_+) = \sum_{j=1}^{n} \frac{j}{n} \cdot \frac{1}{n} = \frac{1}{2}(1 + \frac{1}{n}).$$



So, by Lemma 1.8, f is integrable.

Moreover, the integral of f must lie between $\frac{1}{2}(1 - \frac{1}{n})$ and $\frac{1}{2}(1 + \frac{1}{n})$.

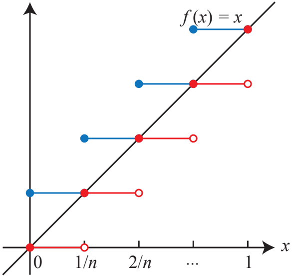
Proof

We have

$$I(\phi_-) = \sum_{i=0}^{n-1} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{2}(1 - \frac{1}{n})$$

and

$$I(\phi_+) = \sum_{j=1}^{n} \frac{j}{n} \cdot \frac{1}{n} = \frac{1}{2}(1 + \frac{1}{n}).$$



So, by Lemma 1.8, f is integrable.

Moreover, the integral of f must lie between $\frac{1}{2}(1 - \frac{1}{n})$ and $\frac{1}{2}(1 + \frac{1}{n})$.

Since n was arbitrary, the integral must be $\frac{1}{2}$.

The integral of a step function

The integral of a step function

Proposition 1.10. Suppose that ϕ is a step function on $[a, b]$. Then ϕ is integrable, and $\int_a^b \phi = I(\phi)$.

The integral of a step function

Proposition 1.10. Suppose that ϕ is a step function on $[a, b]$. Then ϕ is integrable, and $\int_a^b \phi = I(\phi)$.

Proof. Take $\phi_- = \phi_+ = \phi$, and the result is immediate. $\square$

Not all functions are integrable

Example The function $f : [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

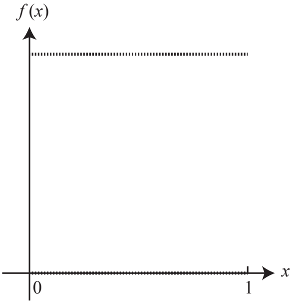
is not integrable.

Not all functions are integrable

Example The function $f : [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not integrable.



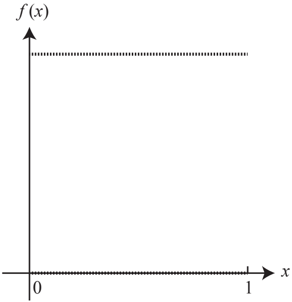
Not all functions are integrable

Example The function $f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not integrable.

Proof.



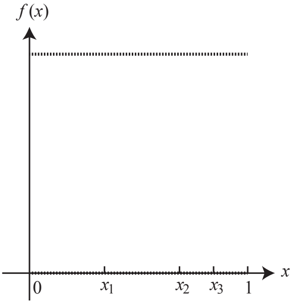
Not all functions are integrable

Example The function $f : [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not integrable.

Proof.



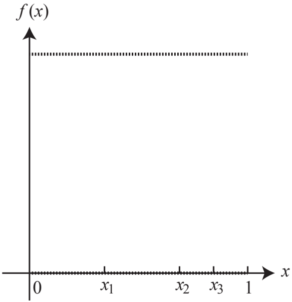
Not all functions are integrable

Example The function $f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not integrable.

Proof. Any open interval contains both rational and irrational points.



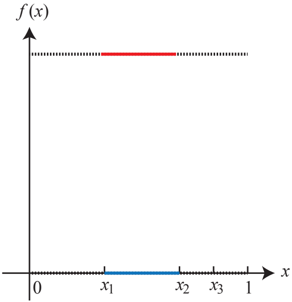
Not all functions are integrable

Example The function $f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not integrable.

Proof. Any open interval contains both rational and irrational points.



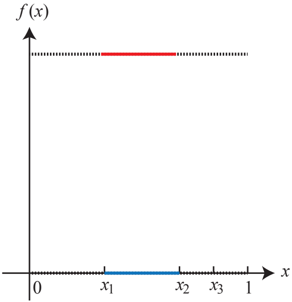
So any step function ϕ_+ majorising f must satisfy $\phi_+(x) \geq 1$ except possibly the finitely many points of the partition. So, $I(\phi_+) \geq 1$.

Not all functions are integrable

<ins>Example</ins> The function $f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not integrable.



<ins>Proof.</ins> Any open interval contains both rational and irrational points.

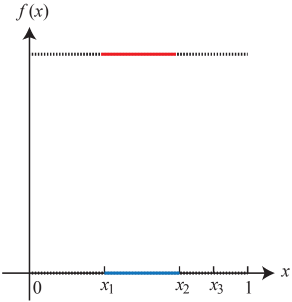
So any step function ϕ_+ majorising f must satisfy $\phi_+(x) \geq 1$ except possibly the finitely many points of the partition. So, $I(\phi_+) \geq 1$.
Similarly, any minorant ϕ_- satisfies $\phi_-(x) \leq 0$ except possibly the finitely many points of the partition. So $I(\phi_-) \leq 0$.

Not all functions are integrable

Example The function $f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not integrable.



Proof. Any open interval contains both rational and irrational points.

So any step function ϕ_+ majorising f must satisfy $\phi_+(x) \geq 1$ except possibly the finitely many points of the partition. So, $I(\phi_+) \geq 1$.
Similarly, any minorant ϕ_- satisfies $\phi_-(x) \leq 0$ except possibly the finitely many points of the partition. So $I(\phi_-) \leq 0$.
So, f is not integrable.

Chapter 1B: Basic theorems about the integral

Monotonicity of the integral

Proposition 1.18(ii). If $f, g : [a, b] \to \mathbb{R}$ are integrable and $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f \leq \int_a^b g.$$

Monotonicity of the integral

Proposition 1.18(ii). If $f, g : [a, b] \to \mathbb{R}$ are integrable and $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f \leq \int_a^b g.$$

Proof.

$$\int_a^b f = \sup_{\phi_-} I(\phi_-)$$

where the supremum over all minorants ϕ_- for f.

Monotonicity of the integral

Proposition 1.18(ii). If $f, g : [a, b] \to \mathbb{R}$ are integrable and $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f \leq \int_a^b g.$$

Proof.

$$\int_a^b f = \sup_{\phi_-} I(\phi_-)$$

where the supremum over all minorants ϕ_- for f.

But any minorant ϕ_- for f is a minorant for g. $\square$

Restricting to a subinterval

Proposition 1.13. Suppose that f is integrable on $[a, b]$. Then, for any c with $a < c < b$, f is Riemann integrable on $[a, c]$ and on $[c, b]$. Moreover $\int_a^b f = \int_c^b f + \int_a^c f$.

Restricting to a subinterval

Proposition 1.13. Suppose that f is integrable on $[a, b]$. Then, for any c with $a < c < b$, f is Riemann integrable on $[a, c]$ and on $[c, b]$. Moreover $\int_a^b f = \int_c^b f + \int_a^c f$.

Corollary 1.14. Suppose that $f : [a, b] \to \mathbb{R}$ is integrable, and that $[c, d] \subset [a, b]$. Then f is integrable on $[c, d]$.

Proof

Proof

Let M be a bound for f, thus $|f(x)| \leq M$ everywhere.

Proof

Let M be a bound for f, thus $|f(x)| \leq M$ everywhere.

In this proof it is convenient to assume that

1. all partitions of $[a, b]$ include the point c;
2. all minorants take the value $-M$ at c, and all majorants the value M.

By refining partitions if necessary, this makes no difference to any computations involving $I(\phi_-), I(\phi_+)$.

Proof

Let M be a bound for f, thus $|f(x)| \leq M$ everywhere.

In this proof it is convenient to assume that

1. all partitions of $[a, b]$ include the point c;
2. all minorants take the value $-M$ at c, and all majorants the value M.

By refining partitions if necessary, this makes no difference to any computations involving $I(\phi_-), I(\phi_+)$.

Now observe that a minorant ϕ_- of f on $[a, b]$ is precisely the same thing as a minorant $\phi_-^{(1)}$ of f on $[a, c]$ juxtaposed with a minorant $\phi_-^{(2)}$ of f on $[c, b]$, and that $I(\phi_-) = I(\phi_-^{(1)}) + I(\phi_-^{(2)})$. A similar comment applies to majorants.

Proof

Let M be a bound for f, thus $|f(x)| \leq M$ everywhere.

In this proof it is convenient to assume that

1. all partitions of $[a, b]$ include the point c;
2. all minorants take the value $-M$ at c, and all majorants the value M.

By refining partitions if necessary, this makes no difference to any computations involving $I(\phi_-), I(\phi_+)$.

Now observe that a minorant ϕ_- of f on $[a, b]$ is precisely the same thing as a minorant $\phi_-^{(1)}$ of f on $[a, c]$ juxtaposed with a minorant $\phi_-^{(2)}$ of f on $[c, b]$, and that $I(\phi_-) = I(\phi_-^{(1)}) + I(\phi_-^{(2)})$. A similar comment applies to majorants. So,

$$\sup_{\phi_-} I(\phi_-) = \sup_{\phi_-^{(1)}} I(\phi_-^{(1)}) + \sup_{\phi_-^{(2)}} I(\phi_-^{(2)})$$

$$\inf_{\phi_+} I(\phi_+) = \inf_{\phi_+^{(1)}} I(\phi_+^{(1)}) + \inf_{\phi_+^{(2)}} I(\phi_+^{(2)}).$$

Proof (continued)

Since f is integrable, $\sup_{\phi_-} I(\phi_-) = \inf_{\phi_+} I(\phi_+)$.

□

Proof (continued)

Since f is integrable, $\sup_{\phi_-} I(\phi_-) = \inf_{\phi_+} I(\phi_+)$. So,

$$\sup_{\phi_-^{(1)}} I(\phi_-^{(1)}) + \sup_{\phi_-^{(2)}} I(\phi_-^{(2)}) = \inf_{\phi_+^{(1)}} I(\phi_+^{(1)}) + \inf_{\phi_+^{(2)}} I(\phi_+^{(2)}).$$

$\square$

Proof (continued)

Since f is integrable, $\sup_{\phi_-} I(\phi_-) = \inf_{\phi_+} I(\phi_+)$. So,

$$\sup_{\phi_-^{(1)}} I(\phi_-^{(1)}) + \sup_{\phi_-^{(2)}} I(\phi_-^{(2)}) = \inf_{\phi_+^{(1)}} I(\phi_+^{(1)}) + \inf_{\phi_+^{(2)}} I(\phi_+^{(2)}).$$

Also,

$$\sup_{\phi_-^{(i)}} I(\phi_-^{(i)}) \leq \inf_{\phi_+^{(i)}} I(\phi_+^{(i)})$$

for $i = 1, 2$.

$\square$

Proof (continued)

Since f is integrable, $\sup_{\phi_-} I(\phi_-) = \inf_{\phi_+} I(\phi_+)$. So,

$$\sup_{\phi_-^{(1)}} I(\phi_-^{(1)}) + \sup_{\phi_-^{(2)}} I(\phi_-^{(2)}) = \inf_{\phi_+^{(1)}} I(\phi_+^{(1)}) + \inf_{\phi_+^{(2)}} I(\phi_+^{(2)}).$$

Also,

$$\sup_{\phi_-^{(i)}} I(\phi_-^{(i)}) \leq \inf_{\phi_+^{(i)}} I(\phi_+^{(i)})$$

for $i = 1, 2$. So,

$$\sup_{\phi_-^{(i)}} I(\phi_-^{(i)}) = \inf_{\phi_+^{(i)}} I(\phi_+^{(i)})$$

for $i = 1, 2$.

$\square$

Proof (continued)

Since f is integrable, $\sup_{\phi_-} I(\phi_-) = \inf_{\phi_+} I(\phi_+)$. So,

$$\sup_{\phi_-^{(1)}} I(\phi_-^{(1)}) + \sup_{\phi_-^{(2)}} I(\phi_-^{(2)}) = \inf_{\phi_+^{(1)}} I(\phi_+^{(1)}) + \inf_{\phi_+^{(2)}} I(\phi_+^{(2)}).$$

Also,

$$\sup_{\phi_-^{(i)}} I(\phi_-^{(i)}) \leq \inf_{\phi_+^{(i)}} I(\phi_+^{(i)})$$

for $i = 1, 2$. So,

$$\sup_{\phi_-^{(i)}} I(\phi_-^{(i)}) = \inf_{\phi_+^{(i)}} I(\phi_+^{(i)})$$

for $i = 1, 2$. (Here, we used the fact that if $x \leq x'$, $y \leq y'$ and $x + y = x' + y'$ then $x = x'$ and $y = y'$.)

$\square$

Proof (continued)

Since f is integrable, $\sup_{\phi_-} I(\phi_-) = \inf_{\phi_+} I(\phi_+)$. So,

$$\sup_{\phi_-^{(1)}} I(\phi_-^{(1)}) + \sup_{\phi_-^{(2)}} I(\phi_-^{(2)}) = \inf_{\phi_+^{(1)}} I(\phi_+^{(1)}) + \inf_{\phi_+^{(2)}} I(\phi_+^{(2)}).$$

Also,

$$\sup_{\phi_-^{(i)}} I(\phi_-^{(i)}) \leq \inf_{\phi_+^{(i)}} I(\phi_+^{(i)})$$

for $i = 1, 2$. So,

$$\sup_{\phi_-^{(i)}} I(\phi_-^{(i)}) = \inf_{\phi_+^{(i)}} I(\phi_+^{(i)})$$

for $i = 1, 2$. (Here, we used the fact that if $x \leq x'$, $y \leq y'$ and $x + y = x' + y'$ then $x = x'$ and $y = y'$.)

Thus f is indeed integrable on $[a, c]$ and on $[c, b]$, and $\int_a^b f = \int_a^c f + \int_c^b f$. $\square$

Linearity of the integral

Proposition 1.15. If f, g are integrable on $[a, b]$ then so is $\lambda f + \mu g$ for any $\lambda, \mu \in \mathbb{R}$. Moreover

$$\int_a^b (\lambda f + \mu g) = \lambda \int_a^b f + \mu \int_a^b g.$$

Linearity of the integral

Proposition 1.15. If f, g are integrable on $[a, b]$ then so is $\lambda f + \mu g$ for any $\lambda, \mu \in \mathbb{R}$. Moreover

$$\int_a^b (\lambda f + \mu g) = \lambda \int_a^b f + \mu \int_a^b g.$$

Proof. This follows from two simpler claims:

1. λf is integrable and $\int_a^b \lambda f = \lambda \int_a^b f$
2. $f + g$ is integrable and $\int_a^b f + g = \int_a^b f + \int_a^b g$.

Proof (continued)

Proof (continued)

Suppose first that $\lambda > 0$. Let $\epsilon > 0$. We know that there is a minorant ϕ_- and majorant ϕ_+ for f such that

$$I(\phi_+) - I(\phi_-) < \epsilon/\lambda.$$

Proof (continued)

Suppose first that $\lambda > 0$. Let $\epsilon > 0$. We know that there is a minorant ϕ_- and majorant ϕ_+ for f such that

$$I(\phi_+) - I(\phi_-) < \epsilon/\lambda.$$

Hence, $\lambda\phi_-$ and $\lambda\phi_+$ are minorants and majorants for λf satisfying

$$I(\lambda\phi_+) - I(\lambda\phi_-) < \epsilon.$$

Proof (continued)

Suppose first that $\lambda > 0$. Let $\epsilon > 0$. We know that there is a minorant ϕ_- and majorant ϕ_+ for f such that

$$I(\phi_+) - I(\phi_-) < \epsilon/\lambda.$$

Hence, $\lambda\phi_-$ and $\lambda\phi_+$ are minorants and majorants for λf satisfying

$$I(\lambda\phi_+) - I(\lambda\phi_-) < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we deduce that λf is integrable. Also,

$$\int_a^b \lambda f \leq I(\lambda\phi_+) = \lambda I(\phi_+) \leq \lambda \int_a^b f + \epsilon.$$

Proof (continued)

Suppose first that $\lambda > 0$. Let $\epsilon > 0$. We know that there is a minorant ϕ_- and majorant ϕ_+ for f such that

$$I(\phi_+) - I(\phi_-) < \epsilon/\lambda.$$

Hence, $\lambda\phi_-$ and $\lambda\phi_+$ are minorants and majorants for λf satisfying

$$I(\lambda\phi_+) - I(\lambda\phi_-) < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we deduce that λf is integrable. Also,

$$\int_a^b \lambda f \leq I(\lambda\phi_+) = \lambda I(\phi_+) \leq \lambda \int_a^b f + \epsilon.$$

Similarly

$$\int_a^b \lambda f \geq \lambda \int_a^b f - \epsilon.$$

Proof (continued)

Suppose first that $\lambda > 0$. Let $\epsilon > 0$. We know that there is a minorant ϕ_- and majorant ϕ_+ for f such that

$$I(\phi_+) - I(\phi_-) < \epsilon/\lambda.$$

Hence, $\lambda\phi_-$ and $\lambda\phi_+$ are minorants and majorants for λf satisfying

$$I(\lambda\phi_+) - I(\lambda\phi_-) < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we deduce that λf is integrable. Also,

$$\int_a^b \lambda f \leq I(\lambda\phi_+) = \lambda I(\phi_+) \leq \lambda \int_a^b f + \epsilon.$$

Similarly

$$\int_a^b \lambda f \geq \lambda \int_a^b f - \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we deduce that $\int_a^b \lambda f = \lambda \int_a^b f$.

Proof (continued)

Proof (continued)

Now suppose that $\lambda < 0$. Again we know that there is a minorant ϕ_- and majorant ϕ_+ for f such that

$$I(\phi_+) - I(\phi_-) < \epsilon/|\lambda|.$$

Proof (continued)

Now suppose that $\lambda < 0$. Again we know that there is a minorant ϕ_- and majorant ϕ_+ for f such that

$$I(\phi_+) - I(\phi_-) < \epsilon/|\lambda|.$$

Hence, $\lambda\phi_+$ and $\lambda\phi_-$ are minorants and majorants for λf satisfying

$$I(\lambda\phi_-) - I(\lambda\phi_+) < \epsilon.$$

Proof (continued)

Now suppose that $\lambda < 0$. Again we know that there is a minorant ϕ_- and majorant ϕ_+ for f such that

$$I(\phi_+) - I(\phi_-) < \epsilon/|\lambda|.$$

Hence, $\lambda\phi_+$ and $\lambda\phi_-$ are minorants and majorants for λf satisfying

$$I(\lambda\phi_-) - I(\lambda\phi_+) < \epsilon.$$

Now repeat as before.

Proof (continued)

Now suppose that $\lambda < 0$. Again we know that there is a minorant ϕ_- and majorant ϕ_+ for f such that

$$I(\phi_+) - I(\phi_-) < \epsilon/|\lambda|.$$

Hence, $\lambda\phi_+$ and $\lambda\phi_-$ are minorants and majorants for λf satisfying

$$I(\lambda\phi_-) - I(\lambda\phi_+) < \epsilon.$$

Now repeat as before.

Finally, $\lambda = 0$ is easy because λf is then a step function, and its integral is 0.

Proof (continued)

We now want to show that $f + g$ is integrable and

$$\int_a^b f + g = \int_a^b f + \int_a^b g.$$

Proof (continued)

We now want to show that $f + g$ is integrable and

$$\int_a^b f + g = \int_a^b f + \int_a^b g.$$

Let $\epsilon > 0$. We know that there is a minorant ϕ_- and majorant ϕ_+ for f such that

$$I(\phi_+) - I(\phi_-) < \epsilon/2.$$

Proof (continued)

We now want to show that $f + g$ is integrable and

$$\int_a^b f + g = \int_a^b f + \int_a^b g.$$

Let $\epsilon > 0$. We know that there is a minorant ϕ_- and majorant ϕ_+ for f such that

$$I(\phi_+) - I(\phi_-) < \epsilon/2.$$

We also know that there is a minorant ψ_- and majorant ψ_+ for g such that

$$I(\psi_+) - I(\psi_-) < \epsilon/2.$$

Proof (continued)

We now want to show that $f+g$ is integrable and

$$\int_a^b f+g=\int_a^b f+\int_a^b g.$$

Let $\epsilon>0$. We know that there is a minorant ϕ_- and majorant ϕ_+ for f such that

$$I(\phi_+)-I(\phi_-)<\epsilon/2.$$

We also know that there is a minorant ψ_- and majorant ψ_+ for g such that

$$I(\psi_+)-I(\psi_-)<\epsilon/2.$$

Hence, $\phi_-+\psi_-$ and $\phi_++\psi_+$ are minorants and majorants for $f+g$ satisfying

$$I(\phi_++\psi_+)-I(\phi_-+\psi_-)<\epsilon.$$

Proof (continued)

We now want to show that $f + g$ is integrable and

$$\int_a^b f + g = \int_a^b f + \int_a^b g.$$

Let $\epsilon > 0$. We know that there is a minorant ϕ_- and majorant ϕ_+ for f such that

$$I(\phi_+) - I(\phi_-) < \epsilon/2.$$

We also know that there is a minorant ψ_- and majorant ψ_+ for g such that

$$I(\psi_+) - I(\psi_-) < \epsilon/2.$$

Hence, $\phi_- + \psi_-$ and $\phi_+ + \psi_+$ are minorants and majorants for $f + g$ satisfying

$$I(\phi_+ + \psi_+) - I(\phi_- + \psi_-) < \epsilon.$$

Hence, $f + g$ is integrable.

Proof (continued)

We now want to show that $f+g$ is integrable and

$$\int_a^b f+g = \int_a^b f + \int_a^b g.$$

Let $\epsilon > 0$. We know that there is a minorant ϕ_- and majorant ϕ_+ for f such that

$$I(\phi_+) - I(\phi_-) < \epsilon/2.$$

We also know that there is a minorant ψ_- and majorant ψ_+ for g such that

$$I(\psi_+) - I(\psi_-) < \epsilon/2.$$

Hence, $\phi_- + \psi_-$ and $\phi_+ + \psi_+$ are minorants and majorants for $f+g$ satisfying

$$I(\phi_+ + \psi_+) - I(\phi_- + \psi_-) < \epsilon.$$

Hence, $f+g$ is integrable. As before, $\int_a^b f+g$ is within ϵ of $\int_a^b f + \int_a^b g$.

Proof (continued)

We now want to show that $f+g$ is integrable and

$$\int_a^b f+g=\int_a^b f+\int_a^b g.$$

Let $\epsilon>0$. We know that there is a minorant ϕ_- and majorant ϕ_+ for f such that

$$I(\phi_+)-I(\phi_-)<\epsilon/2.$$

We also know that there is a minorant ψ_- and majorant ψ_+ for g such that

$$I(\psi_+)-I(\psi_-)<\epsilon/2.$$

Hence, $\phi_-+\psi_-$ and $\phi_++\psi_+$ are minorants and majorants for $f+g$ satisfying

$$I(\phi_++\psi_+)-I(\phi_-+\psi_-)<\epsilon.$$

Hence, $f+g$ is integrable. As before, $\int_a^b f+g$ is within ϵ of $\int_a^b f+\int_a^b g$. □

Changing a function at finitely many points

Changing a function at finitely many points

Corollary 1.16. If f is integrable on $[a, b]$, and if $\tilde{f}$ differs from f in finitely many points, then $\tilde{f}$ is also integrable.

Changing a function at finitely many points

Corollary 1.16. If f is integrable on $[a, b]$, and if $\tilde{f}$ differs from f in finitely many points, then $\tilde{f}$ is also integrable.

Proof.

Changing a function at finitely many points

Corollary 1.16. If f is integrable on $[a, b]$, and if $\tilde{f}$ differs from f in finitely many points, then $\tilde{f}$ is also integrable.

Proof.
The function $\tilde{f} - f$ is zero except at finitely many points. Suppose that these points are $x_1, \ldots, x_{n-1}$.

Changing a function at finitely many points

Corollary 1.16. If f is integrable on $[a, b]$, and if $\tilde{f}$ differs from f in finitely many points, then $\tilde{f}$ is also integrable.

Proof.
The function $\tilde{f} - f$ is zero except at finitely many points. Suppose that these points are $x_1, \dots, x_{n-1}$.
Then $\tilde{f} - f$ is a step function adapted to the partition
$a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$.

Changing a function at finitely many points

Corollary 1.16. If f is integrable on $[a, b]$, and if $\tilde{f}$ differs from f in finitely many points, then $\tilde{f}$ is also integrable.

Proof.
The function $\tilde{f} - f$ is zero except at finitely many points. Suppose that these points are $x_1, \ldots, x_{n-1}$.
Then $\tilde{f} - f$ is a step function adapted to the partition
$a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$.
By Proposition 1.10, $\tilde{f} - f$ is integrable.

Changing a function at finitely many points

Corollary 1.16. If f is integrable on $[a, b]$, and if $\tilde{f}$ differs from f in finitely many points, then $\tilde{f}$ is also integrable.

Proof.
The function $\tilde{f} - f$ is zero except at finitely many points. Suppose that these points are $x_1, \ldots, x_{n-1}$.
Then $\tilde{f} - f$ is a step function adapted to the partition
$a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$.
By Proposition 1.10, $\tilde{f} - f$ is integrable.
Hence so is $\tilde{f} = (\tilde{f} - f) + f$, by Proposition 1.15. $\square$

The maximum and minimum of integrable functions

The maximum and minimum of integrable functions

Proposition 1.17. Suppose that f and g are integrable on $[a, b]$. Then $\max(f, g)$ and $\min(f, g)$ are both Riemann integrable, as is $|f|$.

The maximum and minimum of integrable functions

Proposition 1.17. Suppose that f and g are integrable on $[a, b]$. Then $\max(f, g)$ and $\min(f, g)$ are both Riemann integrable, as is $|f|$.

Proof.

The maximum and minimum of integrable functions

Proposition 1.17. Suppose that f and g are integrable on $[a, b]$. Then $\max(f, g)$ and $\min(f, g)$ are both Riemann integrable, as is $|f|$.

Proof. We have

$$\begin{aligned} \max(f, g) &= g + \max(f - g, 0) \\ \min(h, 0) &= -\max(-h, 0) \\ |h| &= \max(h, 0) - \min(h, 0). \end{aligned}$$

The maximum and minimum of integrable functions

Proposition 1.17. Suppose that f and g are integrable on $[a, b]$. Then $\max(f, g)$ and $\min(f, g)$ are both Riemann integrable, as is $|f|$.

Proof. We have

$$\begin{aligned} \max(f, g) &= g + \max(f - g, 0) \\ \min(h, 0) &= -\max(-h, 0) \\ |h| &= \max(h, 0) - \min(h, 0). \end{aligned}$$

Using these relations and Proposition 1.15, it is enough to prove that if f is integrable on $[a, b]$, then so is $\max(f, 0)$.

Proof (continued)

Claim. If f is integrable on $[a, b]$, then so is $\max(f, 0)$

Proof (continued)

Claim. If f is integrable on $[a, b]$, then so is $\max(f, 0)$

Now the function $x \mapsto \max(x, 0)$ is order-preserving:
if $x \leq y$ then $\max(x, 0) \leq \max(y, 0)$

Proof (continued)

Claim. If f is integrable on $[a, b]$, then so is $\max(f, 0)$

Now the function $x \mapsto \max(x, 0)$ is order-preserving:
if $x \leq y$ then $\max(x, 0) \leq \max(y, 0)$
and non-expanding:
$|\max(x, 0) - \max(y, 0)| \leq |x - y|$,
as can be established by an easy case-check, according to the signs of x, y.

Proof (continued)

Claim. If f is integrable on $[a, b]$, then so is $\max(f, 0)$

Now the function $x \mapsto \max(x, 0)$ is order-preserving:
if $x \leq y$ then $\max(x, 0) \leq \max(y, 0)$
and non-expanding:
$|\max(x, 0) - \max(y, 0)| \leq |x - y|$,
as can be established by an easy case-check, according to the signs of x, y.
It follows that if $\phi_- \leq f \leq \phi_+$ are minorant and majorant for f then

$$\max(\phi_-, 0) \leq \max(f, 0) \leq \max(\phi_+, 0)$$

are minorant and majorant for $\max(f, 0)$ (it is obvious that they are both step functions).

Proof (continued)

Claim. If f is integrable on $[a, b]$, then so is $\max(f, 0)$

Now the function $x \mapsto \max(x, 0)$ is order-preserving:
if $x \leq y$ then $\max(x, 0) \leq \max(y, 0)$
and non-expanding:
$|\max(x, 0) - \max(y, 0)| \leq |x - y|$,
as can be established by an easy case-check, according to the signs of x, y.
It follows that if $\phi_- \leq f \leq \phi_+$ are minorant and majorant for f then

$$\max(\phi_-, 0) \leq \max(f, 0) \leq \max(\phi_+, 0)$$

are minorant and majorant for $\max(f, 0)$ (it is obvious that they are both step functions). Moreover,

$$I(\max(\phi_+, 0)) - I(\max(\phi_-, 0)) \leq I(\phi_+) - I(\phi_-).$$

Proof (continued)

Claim. If f is integrable on $[a, b]$, then so is $\max(f, 0)$

Now the function $x \mapsto \max(x, 0)$ is order-preserving:
if $x \leq y$ then $\max(x, 0) \leq \max(y, 0)$
and non-expanding:
$|\max(x, 0) - \max(y, 0)| \leq |x - y|$,
as can be established by an easy case-check, according to the signs of x, y.
It follows that if $\phi_- \leq f \leq \phi_+$ are minorant and majorant for f then

$$\max(\phi_-, 0) \leq \max(f, 0) \leq \max(\phi_+, 0)$$

are minorant and majorant for $\max(f, 0)$ (it is obvious that they are both step functions). Moreover,

$$I(\max(\phi_+, 0)) - I(\max(\phi_-, 0)) \leq I(\phi_+) - I(\phi_-).$$

Since f is integrable, this can be made arbitrarily small. □

Bounds on the integral

Proposition 1.18. Suppose that f is integrable on $[a, b]$.

Bounds on the integral

Proposition 1.18. Suppose that f is integrable on $[a, b]$.

(i) We have
$(b-a)\inf_{x\in[a,b]} f(x) \leq \int_a^b f \leq (b-a)\sup_{x\in[a,b]} f(x)$.

Bounds on the integral

Proposition 1.18. Suppose that f is integrable on $[a, b]$.

(i) We have
$(b-a)\inf_{x\in[a,b]} f(x) \leq \int_a^b f \leq (b-a)\sup_{x\in[a,b]} f(x)$.

(ii) If g is another integrable function on $[a, b]$, and if $f \leq g$ pointwise, then $\int_a^b f \leq \int_a^b g$.

Bounds on the integral

Proposition 1.18. Suppose that f is integrable on $[a,b]$.

(i) We have $(b-a)\inf_{x\in[a,b]} f(x) \leq \int_a^b f \leq (b-a)\sup_{x\in[a,b]} f(x)$.

(ii) If g is another integrable function on $[a,b]$, and if $f \leq g$ pointwise, then $\int_a^b f \leq \int_a^b g$.

(iii) $|\int_a^b f| \leq \int_a^b |f|$.

(i) The constant function $\phi_-(x) = \inf_{x\in[a,b]} f(x)$ is a minorant for f on $[a,b]$, whilst $\phi_+(x) = \sup_{x\in[a,b]} f(x)$ is a majorant. Thus

$$(b-a)\inf_{x\in[a,b]} f(x) = I(\phi_-) \leq \sup_{\phi_-} I(\phi_-) \leq \int_a^b f,$$

and similarly for the upper bound.

Bounds on the integral

Proposition 1.18. Suppose that f is integrable on $[a, b]$.

(i) We have
$(b-a)\inf_{x\in[a,b]} f(x) \leq \int_a^b f \leq (b-a)\sup_{x\in[a,b]} f(x)$.

(ii) If g is another integrable function on $[a, b]$, and if $f \leq g$ pointwise, then $\int_a^b f \leq \int_a^b g$.

(iii) $|\int_a^b f| \leq \int_a^b |f|$.

(i) The constant function $\phi_-(x) = \inf_{x\in[a,b]} f(x)$ is a minorant for f on $[a, b]$, whilst $\phi_+(x) = \sup_{x\in[a,b]} f(x)$ is a majorant. Thus

$$(b-a)\inf_{x\in[a,b]} f(x) = I(\phi_-) \leq \sup_{\phi_-} I(\phi_-) \leq \int_a^b f,$$

and similarly for the upper bound.

(ii) Applying (i) to $g - f$ gives $\int_a^b (g-f) \geq 0$, from which the result is immediate from linearity of the integral.

Bounds on the integral

Proposition 1.18. Suppose that f is integrable on $[a, b]$.

(i) We have
$(b-a)\inf_{x\in[a,b]} f(x) \leq \int_a^b f \leq (b-a)\sup_{x\in[a,b]} f(x)$.

(ii) If g is another integrable function on $[a, b]$, and if $f \leq g$ pointwise, then $\int_a^b f \leq \int_a^b g$.

(iii) $|\int_a^b f| \leq \int_a^b |f|$.

(i) The constant function $\phi_-(x) = \inf_{x\in[a,b]} f(x)$ is a minorant for f on $[a, b]$, whilst $\phi_+(x) = \sup_{x\in[a,b]} f(x)$ is a majorant. Thus

$$(b-a)\inf_{x\in[a,b]} f(x) = I(\phi_-) \leq \sup_{\phi_-} I(\phi_-) \leq \int_a^b f,$$

and similarly for the upper bound.

(ii) Applying (i) to $g - f$ gives $\int_a^b (g-f) \geq 0$, from which the result is immediate from linearity of the integral.

(iii) Apply (ii) to f and $|f|$, and also to $-f$ and $|f|$, obtaining $\pm\int_a^b f \leq \int_a^b |f|$.

The product of two integrable functions

The product of two integrable functions

Proposition 1.19. Suppose that $f, g : [a, b] \to \mathbb{R}$ are two integrable functions. Then their product fg is integrable.

Proof. Write $f = f_+ - f_-$, where $f_+ = \max(f, 0)$ and $f_- = -\min(f, 0)$, and similarly for g.

The product of two integrable functions

Proposition 1.19. Suppose that $f, g : [a, b] \to \mathbb{R}$ are two integrable functions. Then their product fg is integrable.

Proof. Write $f = f_+ - f_-$, where $f_+ = \max(f, 0)$ and $f_- = -\min(f, 0)$, and similarly for g.
Then $fg = f_+g_+ - f_-g_+ - f_+g_- + f_-g_-$,

The product of two integrable functions

Proposition 1.19. Suppose that $f, g : [a, b] \to \mathbb{R}$ are two integrable functions. Then their product fg is integrable.

Proof. Write $f = f_+ - f_-$, where $f_+ = \max(f, 0)$ and $f_- = -\min(f, 0)$, and similarly for g.
Then $fg = f_+g_+ - f_-g_+ - f_+g_- + f_-g_-$,
and so it suffices to prove the statement for non-negative functions such as $f_\pm, g_\pm$.

The product of two integrable functions

Proposition 1.19. Suppose that $f, g : [a, b] \to \mathbb{R}$ are two integrable functions. Then their product fg is integrable.

Proof. Write $f = f_+ - f_-$, where $f_+ = \max(f, 0)$ and
$f_- = -\min(f, 0)$, and similarly for g.
Then $fg = f_+g_+ - f_-g_+ - f_+g_- + f_-g_-$,
and so it suffices to prove the statement for non-negative functions such as $f_\pm, g_\pm$.
Suppose, then, that $f, g \geq 0$.

Proof (continued)

Suppose, then, that $f, g \geq 0$.

Proof (continued)

Suppose, then, that $f, g \geq 0$.
Let $\varepsilon > 0$, and let $\phi_- \leq f \leq \phi_+$, $\psi_- \leq g \leq \psi_+$ be minorants and majorants for f, g with $I(\phi_+) - I(\phi_-), I(\psi_+) - I(\psi_-) \leq \varepsilon$.

Proof (continued)

Suppose, then, that $f, g \geq 0$.
Let $\varepsilon > 0$, and let $\phi_- \leq f \leq \phi_+$, $\psi_- \leq g \leq \psi_+$ be minorants and majorants for f, g with $I(\phi_+) - I(\phi_-), I(\psi_+) - I(\psi_-) \leq \varepsilon$.
Replacing ϕ_- with $\max(\phi_-, 0)$ if necessary (and similarly for ψ_-), we may assume that $\phi_-, \psi_- \geq 0$ pointwise.

Proof (continued)

Suppose, then, that $f, g \geq 0$.
Let $\varepsilon > 0$, and let $\phi_- \leq f \leq \phi_+$, $\psi_- \leq g \leq \psi_+$ be minorants and majorants for f, g with $I(\phi_+) - I(\phi_-), I(\psi_+) - I(\psi_-) \leq \varepsilon$.
Replacing ϕ_- with $\max(\phi_-, 0)$ if necessary (and similarly for ψ_-), we may assume that $\phi_-, \psi_- \geq 0$ pointwise.
Replacing ϕ_+ with $\min(\phi_+, M)$, where
$M = \max\{\sup_{[a,b]} f, \sup_{[a,b]} g\}$ (and similarly for ψ_+) we may assume that $\phi_+, \psi_+ \leq M$ pointwise.

Proof (continued)

Suppose, then, that $f, g \geq 0$.
Let $\varepsilon > 0$, and let $\phi_- \leq f \leq \phi_+$, $\psi_- \leq g \leq \psi_+$ be minorants and majorants for f, g with $I(\phi_+) - I(\phi_-), I(\psi_+) - I(\psi_-) \leq \varepsilon$.
Replacing ϕ_- with $\max(\phi_-, 0)$ if necessary (and similarly for ψ_-), we may assume that $\phi_-, \psi_- \geq 0$ pointwise.
Replacing ϕ_+ with $\min(\phi_+, M)$, where $M = \max\{\sup_{[a,b]} f, \sup_{[a,b]} g\}$ (and similarly for ψ_+) we may assume that $\phi_+, \psi_+ \leq M$ pointwise.
By refining partitions if necessary, we may assume that all of these step functions are adapted to the same partition $\mathcal{P}$.

Proof (continued)

Suppose, then, that $f, g \geq 0$.
Let $\varepsilon > 0$, and let $\phi_- \leq f \leq \phi_+$, $\psi_- \leq g \leq \psi_+$ be minorants and majorants for f, g with $I(\phi_+) - I(\phi_-), I(\psi_+) - I(\psi_-) \leq \varepsilon$.
Replacing ϕ_- with $\max(\phi_-, 0)$ if necessary (and similarly for ψ_-), we may assume that $\phi_-, \psi_- \geq 0$ pointwise.
Replacing ϕ_+ with $\min(\phi_+, M)$, where $M = \max\{\sup_{[a,b]} f, \sup_{[a,b]} g\}$ (and similarly for ψ_+) we may assume that $\phi_+, \psi_+ \leq M$ pointwise.
By refining partitions if necessary, we may assume that all of these step functions are adapted to the same partition $\mathcal{P}$.
Now observe that $\phi_-\psi_-, \phi_+\psi_+$ are both step functions and that $\phi_-\psi_- \leq fg \leq \phi_+\psi_+$ pointwise.

Proof (continued)

$\phi_-\psi_-, \phi_+\psi_+$ are both step functions and $\phi_-\psi_- \leq fg \leq \phi_+\psi_+$ pointwise.

Proof (continued)

$\phi_-\psi_-, \phi_+\psi_+$ are both step functions and $\phi_-\psi_- \leq fg \leq \phi_+\psi_+$ pointwise.

Moreover, if $0 \leq u, v, u', v' \leq M$ and $u \leq u'$, $v \leq v'$ then we have

$$u'v' - uv = (u' - u)v' + (v' - v)u \leq M(u' - u + v' - v).$$

Applying this on each interval of the partition $\mathcal{P}$, with $u = \phi_-$, $u' = \phi_+$, $v = \psi_-$, $v' = \psi_+$, we have

Proof (continued)

$\phi_-\psi_-, \phi_+\psi_+$ are both step functions and $\phi_-\psi_- \le fg \le \phi_+\psi_+$ pointwise.

Moreover, if $0 \le u, v, u', v' \le M$ and $u \le u'$, $v \le v'$ then we have

$$u'v' - uv = (u' - u)v' + (v' - v)u \le M(u' - u + v' - v).$$

Applying this on each interval of the partition $\mathcal{P}$, with $u = \phi_-$, $u' = \phi_+$, $v = \psi_-$, $v' = \psi_+$, we have

$$I(\phi_+\psi_+) - I(\phi_-\psi_-) \le M(I(\phi_+) - I(\phi_-) + I(\psi_+) - I(\psi_-)) \le 2\varepsilon M.$$

Proof (continued)

$\phi_-\psi_-, \phi_+\psi_+$ are both step functions and $\phi_-\psi_- \leq fg \leq \phi_+\psi_+$ pointwise.

Moreover, if $0 \leq u, v, u', v' \leq M$ and $u \leq u'$, $v \leq v'$ then we have

$$u'v' - uv = (u' - u)v' + (v' - v)u \leq M(u' - u + v' - v).$$

Applying this on each interval of the partition $\mathcal{P}$, with $u = \phi_-$, $u' = \phi_+$, $v = \psi_-$, $v' = \psi_+$, we have

$$I(\phi_+\psi_+) - I(\phi_-\psi_-) \leq M(I(\phi_+) - I(\phi_-) + I(\psi_+) - I(\psi_-)) \leq 2\varepsilon M.$$

Since $\varepsilon > 0$ was arbitrary, the result follows. □

Chapter 2A: Integrating a continuous function

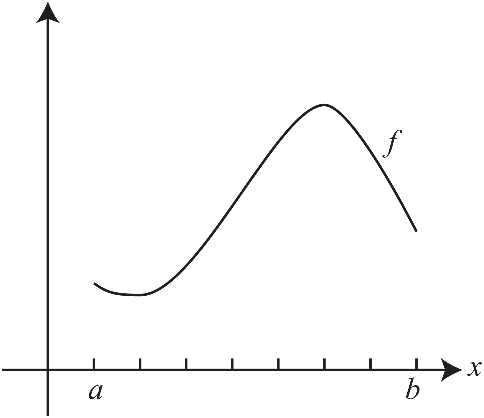
Continuous functions are integrable

Continuous functions are integrable

Theorem 2.1. Any continuous function $f : [a, b] \to \mathbb{R}$ is integrable.

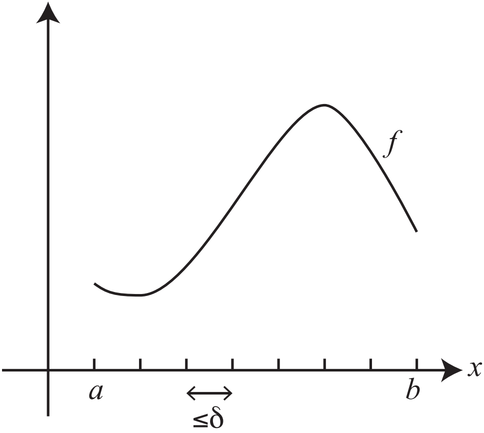
Continuous functions are integrable

Theorem 2.1. Any continuous function $f : [a, b] \to \mathbb{R}$ is integrable.



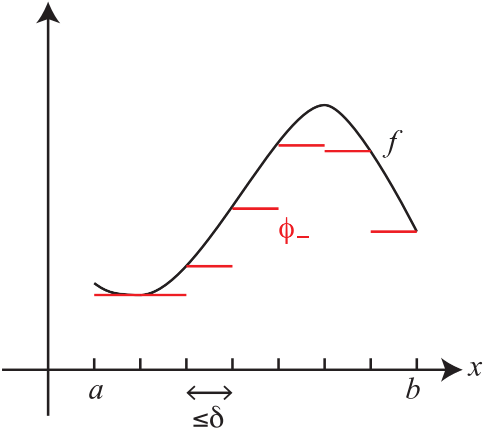
Continuous functions are integrable

Theorem 2.1. Any continuous function $f : [a, b] \to \mathbb{R}$ is integrable.



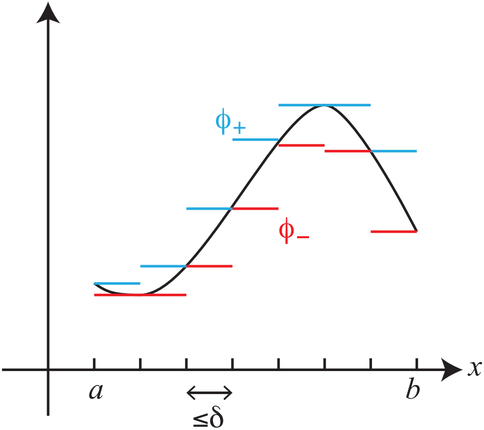
Continuous functions are integrable

Theorem 2.1. Any continuous function $f : [a, b] \to \mathbb{R}$ is integrable.



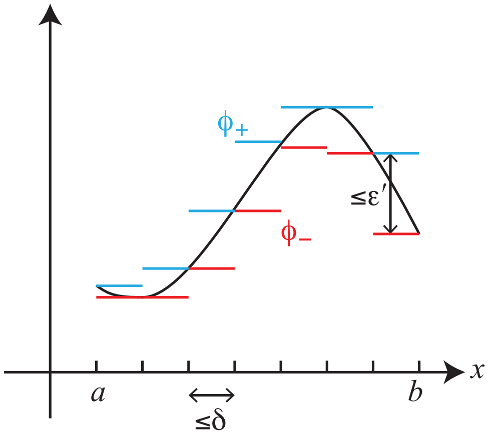
Continuous functions are integrable

Theorem 2.1. Any continuous function $f : [a, b] \to \mathbb{R}$ is integrable.



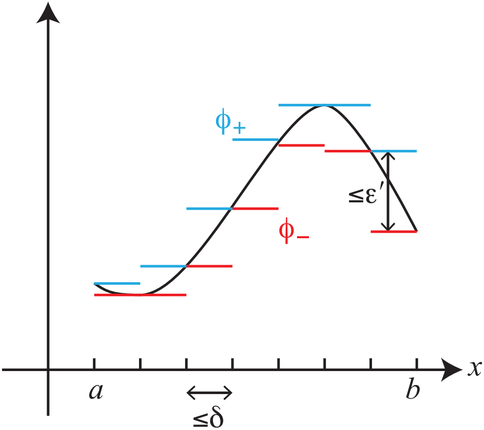
Continuous functions are integrable

Theorem 2.1. Any continuous function $f : [a, b] \to \mathbb{R}$ is integrable.



Continuous functions are integrable

Theorem 2.1. Any continuous function $f : [a, b] \to \mathbb{R}$ is integrable.



$$I(\phi_+) - I(\phi_-) \leq (b-a)\epsilon'.$$

Proof

Proof

Let $\mathcal{P}$ be a partition of $[a, b]$, $a = x_0 < x_1 < \cdots < x_n = b$. The mesh of $\mathcal{P}$ is defined to be $\max_i(x_i - x_{i-1})$.

Proof

Let $\mathcal{P}$ be a partition of $[a, b]$, $a = x_0 < x_1 < \cdots < x_n = b$. The mesh of $\mathcal{P}$ is defined to be $\max_i(x_i - x_{i-1})$.

We want to show that, for any $\epsilon > 0$, there is a minorant ϕ_- and a majorant ϕ_+ such that $I(\phi_+) - I(\phi_-) < \epsilon$.

Proof

Let $\mathcal{P}$ be a partition of $[a, b]$, $a = x_0 < x_1 < \cdots < x_n = b$. The mesh of $\mathcal{P}$ is defined to be $\max_i(x_i - x_{i-1})$.

We want to show that, for any $\epsilon > 0$, there is a minorant ϕ_- and a majorant ϕ_+ such that $I(\phi_+) - I(\phi_-) < \epsilon$.

It is theorem from Analysis 2 that any continuous function $f : [a, b] \to \mathbb{R}$ on a closed bounded interval is uniformly continuous ie

Proof

Let $\mathcal{P}$ be a partition of $[a, b]$, $a = x_0 < x_1 < \cdots < x_n = b$. The mesh of $\mathcal{P}$ is defined to be $\max_i(x_i - x_{i-1})$.

We want to show that, for any $\epsilon > 0$, there is a minorant ϕ_- and a majorant ϕ_+ such that $I(\phi_+) - I(\phi_-) < \epsilon$.

It is theorem from Analysis 2 that any continuous function $f : [a, b] \to \mathbb{R}$ on a closed bounded interval is uniformly continuous ie

For all $\epsilon' > 0$, there is a $\delta > 0$ such that for all $x, y \in [a, b]$,

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon'.$$

Proof

Let $\mathcal{P}$ be a partition of $[a, b]$, $a = x_0 < x_1 < \cdots < x_n = b$. The mesh of $\mathcal{P}$ is defined to be $\max_i(x_i - x_{i-1})$.

We want to show that, for any $\epsilon > 0$, there is a minorant ϕ_- and a majorant ϕ_+ such that $I(\phi_+) - I(\phi_-) < \epsilon$.

It is theorem from Analysis 2 that any continuous function $f : [a, b] \to \mathbb{R}$ on a closed bounded interval is uniformly continuous ie

For all $\epsilon' > 0$, there is a $\delta > 0$ such that for all $x, y \in [a, b]$,

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon'.$$

We'll set $\epsilon' = \epsilon/(b - a)$.

Proof (continued)

Proof (continued)

Pick a partition $\mathcal{P}$ with mesh $< \delta$.

Proof (continued)

Pick a partition $\mathcal{P}$ with mesh $< \delta$.

Let ϕ_+ be the step function whose value on (x_{i-1}, x_i) is $\sup_{x \in [x_{i-1}, x_i]} f(x)$.

Proof (continued)

Pick a partition $\mathcal{P}$ with mesh $< \delta$.

Let ϕ_+ be the step function whose value on (x_{i-1}, x_i) is $\sup_{x \in [x_{i-1}, x_i]} f(x)$. Define ϕ_+ at the points x_i of the partition to be $f(x_i)$.

Proof (continued)

Pick a partition $\mathcal{P}$ with mesh $< \delta$.

Let ϕ_+ be the step function whose value on (x_{i-1}, x_i) is $\sup_{x \in [x_{i-1}, x_i]} f(x)$. Define ϕ_+ at the points x_i of the partition to be $f(x_i)$.

Let ϕ_- be the step function whose value on (x_{i-1}, x_i) is $\inf_{x \in [x_{i-1}, x_i]} f(x)$ and which takes the value $f(x_i)$ at the points x_i.

Proof (continued)

Pick a partition $\mathcal{P}$ with mesh $< \delta$.

Let ϕ_+ be the step function whose value on (x_{i-1}, x_i) is $\sup_{x \in [x_{i-1}, x_i]} f(x)$. Define ϕ_+ at the points x_i of the partition to be $f(x_i)$.

Let ϕ_- be the step function whose value on (x_{i-1}, x_i) is $\inf_{x \in [x_{i-1}, x_i]} f(x)$ and which takes the value $f(x_i)$ at the points x_i.

By construction, ϕ_+ is a majorant for f and ϕ_- is a minorant.

Proof (continued)

Pick a partition $\mathcal{P}$ with mesh $< \delta$.

Let ϕ_+ be the step function whose value on (x_{i-1}, x_i) is $\sup_{x \in [x_{i-1}, x_i]} f(x)$. Define ϕ_+ at the points x_i of the partition to be $f(x_i)$.

Let ϕ_- be the step function whose value on (x_{i-1}, x_i) is $\inf_{x \in [x_{i-1}, x_i]} f(x)$ and which takes the value $f(x_i)$ at the points x_i.

By construction, ϕ_+ is a majorant for f and ϕ_- is a minorant.

Since a continuous function on a closed interval attains its bounds, there are $\xi_-, \xi_+ \in [x_{i-1}, x_i]$ such that $\sup_{x \in [x_{i-1}, x_i]} f(x) = f(\xi_+)$ and $\inf_{x \in [x_{i-1}, x_i]} f(x) = f(\xi_-)$.

Proof (continued)

Pick a partition $\mathcal{P}$ with mesh $< \delta$.

Let ϕ_+ be the step function whose value on (x_{i-1}, x_i) is $\sup_{x \in [x_{i-1}, x_i]} f(x)$. Define ϕ_+ at the points x_i of the partition to be $f(x_i)$.

Let ϕ_- be the step function whose value on (x_{i-1}, x_i) is $\inf_{x \in [x_{i-1}, x_i]} f(x)$ and which takes the value $f(x_i)$ at the points x_i.

By construction, ϕ_+ is a majorant for f and ϕ_- is a minorant.

Since a continuous function on a closed interval attains its bounds, there are $\xi_-, \xi_+ \in [x_{i-1}, x_i]$ such that $\sup_{x \in [x_{i-1}, x_i]} f(x) = f(\xi_+)$ and $\inf_{x \in [x_{i-1}, x_i]} f(x) = f(\xi_-)$.

For $x \in (x_{i-1}, x_i)$ we have $\phi_+(x) - \phi_-(x) = f(\xi_+) - f(\xi_-) < \epsilon'$.

Proof (continued)

Pick a partition $\mathcal{P}$ with mesh $< \delta$.

Let ϕ_+ be the step function whose value on (x_{i-1}, x_i) is $\sup_{x \in [x_{i-1}, x_i]} f(x)$. Define ϕ_+ at the points x_i of the partition to be $f(x_i)$.

Let ϕ_- be the step function whose value on (x_{i-1}, x_i) is $\inf_{x \in [x_{i-1}, x_i]} f(x)$ and which takes the value $f(x_i)$ at the points x_i.

By construction, ϕ_+ is a majorant for f and ϕ_- is a minorant.

Since a continuous function on a closed interval attains its bounds, there are $\xi_-, \xi_+ \in [x_{i-1}, x_i]$ such that $\sup_{x \in [x_{i-1}, x_i]} f(x) = f(\xi_+)$ and $\inf_{x \in [x_{i-1}, x_i]} f(x) = f(\xi_-)$.

For $x \in (x_{i-1}, x_i)$ we have $\phi_+(x) - \phi_-(x) = f(\xi_+) - f(\xi_-) < \epsilon'$.

Therefore $\phi_+(x) - \phi_-(x) < \epsilon'$ for all $x \in [a, b]$, including the points x_i.

Proof (continued)

Pick a partition $\mathcal{P}$ with mesh $< \delta$.

Let ϕ_+ be the step function whose value on (x_{i-1}, x_i) is $\sup_{x \in [x_{i-1}, x_i]} f(x)$. Define ϕ_+ at the points x_i of the partition to be $f(x_i)$.

Let ϕ_- be the step function whose value on (x_{i-1}, x_i) is $\inf_{x \in [x_{i-1}, x_i]} f(x)$ and which takes the value $f(x_i)$ at the points x_i.

By construction, ϕ_+ is a majorant for f and ϕ_- is a minorant.

Since a continuous function on a closed interval attains its bounds, there are $\xi_-, \xi_+ \in [x_{i-1}, x_i]$ such that $\sup_{x \in [x_{i-1}, x_i]} f(x) = f(\xi_+)$ and $\inf_{x \in [x_{i-1}, x_i]} f(x) = f(\xi_-)$.

For $x \in (x_{i-1}, x_i)$ we have $\phi_+(x) - \phi_-(x) = f(\xi_+) - f(\xi_-) < \epsilon'$.

Therefore $\phi_+(x) - \phi_-(x) < \epsilon'$ for all $x \in [a, b]$, including the points x_i.

It follows that $I(\phi_+) - I(\phi_-) < \epsilon'(b - a) = \epsilon$. $\square$

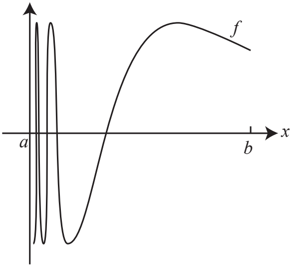
Continuity on an open interval

Continuity on an open interval

Theorem 2.2. Any continuous bounded function $f : (a, b) \to \mathbb{R}$ is integrable.

Continuity on an open interval

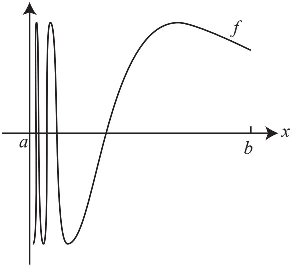
Theorem 2.2. Any continuous bounded function $f : (a, b) \to \mathbb{R}$ is integrable.



Continuity on an open interval

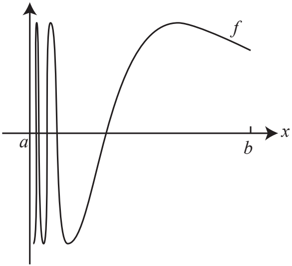
Theorem 2.2. Any continuous bounded function $f : (a, b) \to \mathbb{R}$ is integrable.

Proof.



Continuity on an open interval

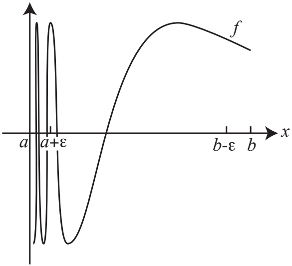
Theorem 2.2. Any continuous bounded function $f : (a, b) \to \mathbb{R}$ is integrable.



Proof. Let $\epsilon > 0$.

Continuity on an open interval

Theorem 2.2. Any continuous bounded function $f : (a, b) \to \mathbb{R}$ is integrable.

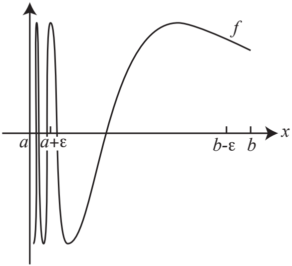


Proof. Let $\epsilon > 0$.

We know that $f|_{[a+\epsilon, b-\epsilon]}$ is integrable.

Continuity on an open interval

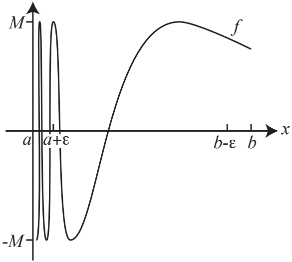
Theorem 2.2. Any continuous bounded function $f : (a, b) \to \mathbb{R}$ is integrable.



Proof. Let $\epsilon > 0$.

We know that $f|_{[a+\epsilon, b-\epsilon]}$ is integrable. So it has a majorant ϕ_+ and minorant ϕ_- such that $I(\phi_+) - I(\phi_-) < \epsilon$.

Continuity on an open interval



Theorem 2.2. Any continuous bounded function $f : (a, b) \to \mathbb{R}$ is integrable.

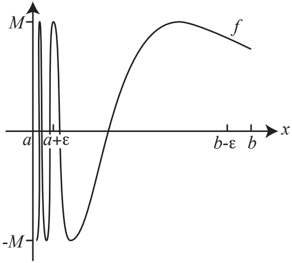
Proof. Let $\epsilon > 0$.

We know that $f|_{[a+\epsilon, b-\epsilon]}$ is integrable. So it has a majorant ϕ_+ and minorant ϕ_- such that $I(\phi_+) - I(\phi_-) < \epsilon$.

We are assuming that f is bounded, say $-M \leq f(x) \leq M$ for all $x \in (a, b)$.

Continuity on an open interval

Theorem 2.2. Any continuous bounded function $f : (a, b) \to \mathbb{R}$ is integrable.



Proof. Let $\epsilon > 0$.

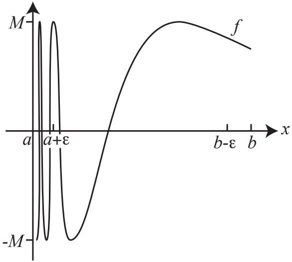
We know that $f|_{[a+\epsilon, b-\epsilon]}$ is integrable. So it has a majorant ϕ_+ and minorant ϕ_- such that $I(\phi_+) - I(\phi_-) < \epsilon$.

We are assuming that f is bounded, say $-M \le f(x) \le M$ for all $x \in (a, b)$.

Extend ϕ_+ to a step function $\tilde{\phi}_+$ on $[a, b]$ by defining it to be M on $[a, a+\epsilon)$ and $(b-\epsilon, b]$.

Continuity on an open interval

Theorem 2.2. Any continuous bounded function $f : (a, b) \to \mathbb{R}$ is integrable.



Proof. Let $\epsilon > 0$.

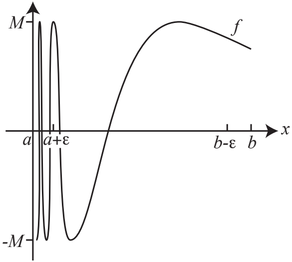
We know that $f|_{[a+\epsilon, b-\epsilon]}$ is integrable. So it has a majorant ϕ_+ and minorant ϕ_- such that $I(\phi_+) - I(\phi_-) < \epsilon$.

We are assuming that f is bounded, say $-M \leq f(x) \leq M$ for all $x \in (a, b)$.

Extend ϕ_+ to a step function $\tilde{\phi}_+$ on $[a, b]$ by defining it to be M on $[a, a + \epsilon)$ and $(b - \epsilon, b]$. Define $\tilde{\phi}_-$ similarly using $-M$.

Continuity on an open interval

Theorem 2.2. Any continuous bounded function $f : (a, b) \to \mathbb{R}$ is integrable.



Proof. Let $\epsilon > 0$.

We know that $f|_{[a+\epsilon, b-\epsilon]}$ is integrable. So it has a majorant ϕ_+ and minorant ϕ_- such that $I(\phi_+) - I(\phi_-) < \epsilon$.

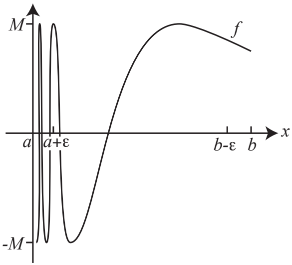
We are assuming that f is bounded, say $-M \leq f(x) \leq M$ for all $x \in (a, b)$.

Extend ϕ_+ to a step function $\tilde{\phi}_+$ on $[a, b]$ by defining it to be M on $[a, a+\epsilon)$ and $(b-\epsilon, b]$. Define $\tilde{\phi}_-$ similarly using $-M$.

Then $\tilde{\phi}_-$ and $\tilde{\phi}_+$ are a minorant and majorant for f and they satisfy

$$I(\tilde{\phi}_+) - I(\tilde{\phi}_-) < \epsilon + 2M \cdot 2\epsilon.$$

Continuity on an open interval



Theorem 2.2. Any continuous bounded function $f : (a, b) \to \mathbb{R}$ is integrable.

Proof. Let $\epsilon > 0$.

We know that $f|_{[a+\epsilon, b-\epsilon]}$ is integrable. So it has a majorant ϕ_+ and minorant ϕ_- such that $I(\phi_+) - I(\phi_-) < \epsilon$.

We are assuming that f is bounded, say $-M \leq f(x) \leq M$ for all $x \in (a, b)$.

Extend ϕ_+ to a step function $\tilde{\phi}_+$ on $[a, b]$ by defining it to be M on $[a, a+\epsilon)$ and $(b-\epsilon, b]$. Define $\tilde{\phi}_-$ similarly using $-M$.

Then $\tilde{\phi}_-$ and $\tilde{\phi}_+$ are a minorant and majorant for f and they satisfy

$$I(\tilde{\phi}_+) - I(\tilde{\phi}_-) < \epsilon + 2M \cdot 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, f is integrable. $\square$

Integrating a non-negative continuous function

Integrating a non-negative continuous function

Lemma 2.3. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function with $f \geq 0$ pointwise and $\int_a^b f = 0$. Then $f(x) = 0$ for $x \in [a, b]$.

Integrating a non-negative continuous function

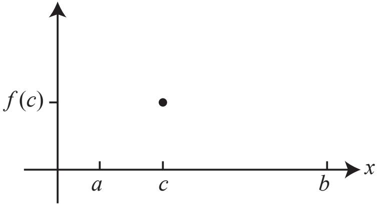
Lemma 2.3. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function with $f \geq 0$ pointwise and $\int_a^b f = 0$. Then $f(x) = 0$ for $x \in [a, b]$.

Proof.

Integrating a non-negative continuous function

Lemma 2.3. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function with $f \geq 0$ pointwise and $\int_a^b f = 0$. Then $f(x) = 0$ for $x \in [a, b]$.

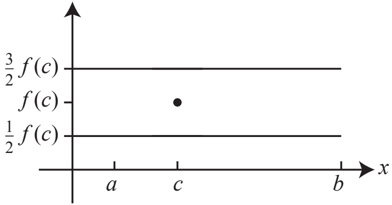
Proof. Suppose not. Then there is some point $c \in [a, b]$ with $f(c) > 0$.



Integrating a non-negative continuous function

Lemma 2.3. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function with $f \geq 0$ pointwise and $\int_a^b f = 0$. Then $f(x) = 0$ for $x \in [a, b]$.

Proof. Suppose not. Then there is some point $c \in [a, b]$ with $f(c) > 0$.

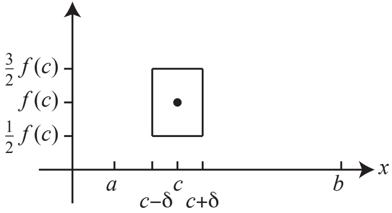


Since f is continuous, there is some $\delta > 0$ such that if $|x - c| \leq \delta$ then $|f(x) - f(c)| \leq f(c)/2$, and hence $f(x) \geq f(c)/2$.

Integrating a non-negative continuous function

Lemma 2.3. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function with $f \geq 0$ pointwise and $\int_a^b f = 0$. Then $f(x) = 0$ for $x \in [a, b]$.

Proof. Suppose not. Then there is some point $c \in [a, b]$ with $f(c) > 0$.

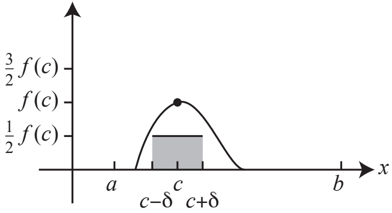


Since f is continuous, there is some $\delta > 0$ such that if $|x - c| \leq \delta$ then $|f(x) - f(c)| \leq f(c)/2$, and hence $f(x) \geq f(c)/2$.

Integrating a non-negative continuous function

Lemma 2.3. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function with $f \geq 0$ pointwise and $\int_a^b f = 0$. Then $f(x) = 0$ for $x \in [a, b]$.

Proof. Suppose not. Then there is some point $c \in [a, b]$ with $f(c) > 0$.

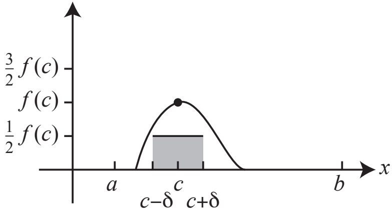


Since f is continuous, there is some $\delta > 0$ such that if $|x - c| \leq \delta$ then $|f(x) - f(c)| \leq f(c)/2$, and hence $f(x) \geq f(c)/2$.

Integrating a non-negative continuous function

Lemma 2.3. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function with $f \geq 0$ pointwise and $\int_a^b f = 0$. Then $f(x) = 0$ for $x \in [a, b]$.

Proof. Suppose not. Then there is some point $c \in [a, b]$ with $f(c) > 0$.

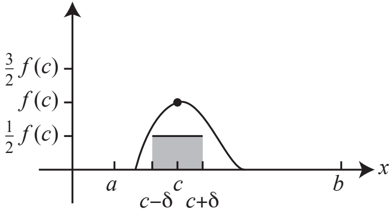


Since f is continuous, there is some $\delta > 0$ such that if $|x - c| \leq \delta$ then $|f(x) - f(c)| \leq f(c)/2$, and hence $f(x) \geq f(c)/2$. The set of all $x \in [a, b]$ with $|x - c| \leq \delta$ is a subinterval $I \subset [a, b]$ with length at least $\min(b - a, \delta)$, and so

Integrating a non-negative continuous function

Lemma 2.3. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function with $f \geq 0$ pointwise and $\int_a^b f = 0$. Then $f(x) = 0$ for $x \in [a, b]$.

Proof. Suppose not. Then there is some point $c \in [a, b]$ with $f(c) > 0$.



Since f is continuous, there is some $\delta > 0$ such that if $|x - c| \leq \delta$ then $|f(x) - f(c)| \leq f(c)/2$, and hence $f(x) \geq f(c)/2$.
The set of all $x \in [a, b]$ with $|x - c| \leq \delta$ is a subinterval $I \subset [a, b]$ with length at least $\min(b - a, \delta)$, and so

$$\int f \geq \int_I f \geq \frac{f(c)}{2} \min(b - a, \delta) > 0.$$

Chapter 2B: Mean values, monotone functions

A first mean value theorem

Proposition 2.4. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there is some $c \in [a, b]$ such that $\int_a^b f = (b - a)f(c)$.

A first mean value theorem

Proposition 2.4. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there is some $c \in [a, b]$ such that $\int_a^b f = (b - a)f(c)$.

When $a \neq b$, this is

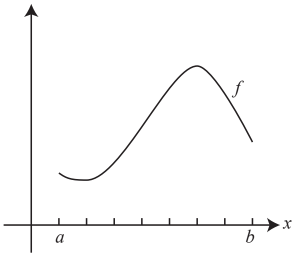
$$\frac{1}{b-a}\int_a^b f = f(c).$$

A first mean value theorem

Proposition 2.4. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there is some $c \in [a, b]$ such that $\int_a^b f = (b - a)f(c)$.

When $a \neq b$, this is

$$\frac{1}{b-a}\int_a^b f = f(c).$$

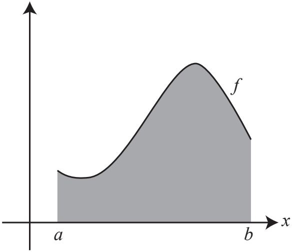


A first mean value theorem

Proposition 2.4. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there is some $c \in [a, b]$ such that $\int_a^b f = (b - a)f(c)$.

When $a \neq b$, this is

$$\frac{1}{b-a}\int_a^b f = f(c).$$

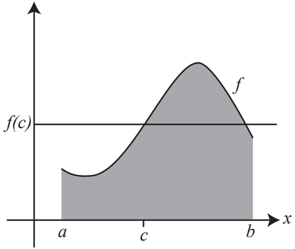


A first mean value theorem

Proposition 2.4. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there is some $c \in [a, b]$ such that $\int_a^b f = (b - a)f(c)$.

When $a \neq b$, this is

$$\frac{1}{b-a}\int_a^b f = f(c).$$

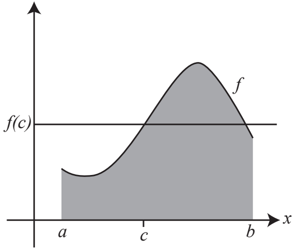


A first mean value theorem

Proposition 2.4. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there is some $c \in [a, b]$ such that $\int_a^b f = (b - a)f(c)$.

When $a \neq b$, this is

$$\frac{1}{b-a}\int_a^b f = f(c).$$



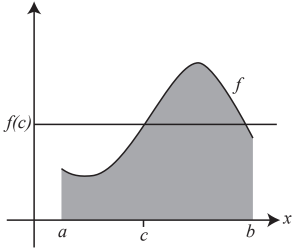
Proof.

A first mean value theorem

Proposition 2.4. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there is some $c \in [a, b]$ such that $\int_a^b f = (b - a)f(c)$.

When $a \neq b$, this is

$$\frac{1}{b-a}\int_a^b f = f(c).$$



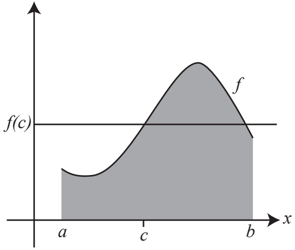
Proof. Since f is continuous, it attains its maximum M and its minimum m.

A first mean value theorem

Proposition 2.4. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there is some $c \in [a, b]$ such that $\int_a^b f = (b - a)f(c)$.

When $a \neq b$, this is

$$\frac{1}{b-a}\int_a^b f = f(c).$$



Proof. Since f is continuous, it attains its maximum M and its minimum m.
By Proposition 1.18 (i), $m(b - a) \leq \int_a^b f \leq M(b - a)$, which implies that

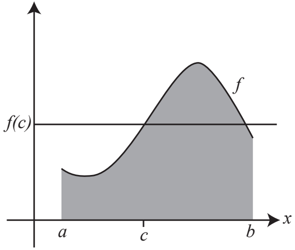
$$m \leq \frac{1}{b-a}\int_a^b f \leq M.$$

A first mean value theorem

Proposition 2.4. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there is some $c \in [a, b]$ such that $\int_a^b f = (b - a)f(c)$.

When $a \neq b$, this is

$$\frac{1}{b-a}\int_a^b f = f(c).$$



Proof. Since f is continuous, it attains its maximum M and its minimum m.
By Proposition 1.18 (i), $m(b - a) \leq \int_a^b f \leq M(b - a)$, which implies that

$$m \leq \frac{1}{b-a}\int_a^b f \leq M.$$

By the intermediate value theorem, f attains every value in $[m, M]$, and in particular there is some c such that $f(c) = \frac{1}{b-a}\int_a^b f$. □

A second mean value theorem

Proposition 2.5. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, and that $w : [a, b] \to \mathbb{R}$ is a nonnegative integrable function. Then there is some $c \in [a, b]$ such that

$$\int_a^b fw = f(c) \int_a^b w.$$

A second mean value theorem

Proposition 2.5. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, and that $w : [a, b] \to \mathbb{R}$ is a nonnegative integrable function. Then there is some $c \in [a, b]$ such that

$$\int_a^b fw = f(c) \int_a^b w.$$

Proof.

A second mean value theorem

Proposition 2.5. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, and that $w : [a, b] \to \mathbb{R}$ is a nonnegative integrable function. Then there is some $c \in [a, b]$ such that

$$\int_a^b fw = f(c) \int_a^b w.$$

Proof. Note that fw is indeed integrable.

A second mean value theorem

Proposition 2.5. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, and that $w : [a, b] \to \mathbb{R}$ is a nonnegative integrable function. Then there is some $c \in [a, b]$ such that

$$\int_a^b fw = f(c) \int_a^b w.$$

Proof. Note that fw is indeed integrable.

Write M, m for the maximum and minimum of f respectively. Then

$mw \leq fw \leq Mw$, and so

A second mean value theorem

Proposition 2.5. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, and that $w : [a, b] \to \mathbb{R}$ is a nonnegative integrable function. Then there is some $c \in [a, b]$ such that

$$\int_a^b fw = f(c) \int_a^b w.$$

Proof. Note that fw is indeed integrable.

Write M, m for the maximum and minimum of f respectively. Then

$$mw \leq fw \leq Mw, \quad \text{and so} \quad m \int_a^b w \leq \int_a^b fw \leq M \int_a^b w.$$

A second mean value theorem

Proposition 2.5. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, and that $w : [a, b] \to \mathbb{R}$ is a nonnegative integrable function. Then there is some $c \in [a, b]$ such that

$$\int_a^b fw = f(c) \int_a^b w.$$

Proof. Note that fw is indeed integrable.

Write M, m for the maximum and minimum of f respectively. Then

$$mw \le fw \le Mw, \quad \text{and so} \quad m \int_a^b w \le \int_a^b fw \le M \int_a^b w.$$

If $\int_a^b w = 0$ then the result is trivial;

A second mean value theorem

Proposition 2.5. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, and that $w : [a, b] \to \mathbb{R}$ is a nonnegative integrable function. Then there is some $c \in [a, b]$ such that

$$\int_a^b fw = f(c) \int_a^b w.$$

Proof. Note that fw is indeed integrable.

Write M, m for the maximum and minimum of f respectively. Then

$$mw \leq fw \leq Mw, \quad \text{and so} \quad m \int_a^b w \leq \int_a^b fw \leq M \int_a^b w.$$

If $\int_a^b w = 0$ then the result is trivial; otherwise,

$$m \leq \frac{\int_a^b fw}{\int_a^b w} \leq M.$$

A second mean value theorem

Proposition 2.5. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, and that $w : [a, b] \to \mathbb{R}$ is a nonnegative integrable function. Then there is some $c \in [a, b]$ such that

$$\int_a^b fw = f(c) \int_a^b w.$$

Proof. Note that fw is indeed integrable.

Write M, m for the maximum and minimum of f respectively. Then

$$mw \leq fw \leq Mw, \quad \text{and so} \quad m \int_a^b w \leq \int_a^b fw \leq M \int_a^b w.$$

If $\int_a^b w = 0$ then the result is trivial; otherwise,

$m \leq \dfrac{\int_a^b fw}{\int_a^b w} \leq M$. So, by IVT, there is a $c \in [a, b]$ s.t. $f(c) = \dfrac{\int_a^b fw}{\int_a^b w}$.

Monotone functions

A function $f : [a, b] \to \mathbb{R}$ is monotone if it increasing (ie $x \leq y \Rightarrow f(x) \leq f(y)$) or decreasing.

Monotone functions

A function $f : [a, b] \to \mathbb{R}$ is monotone if it increasing (ie $x \leq y \Rightarrow f(x) \leq f(y)$) or decreasing.

Theorem 2.6. Any monotone function $f : [a, b] \to \mathbb{R}$ is integrable.

Monotone functions

A function $f : [a, b] \to \mathbb{R}$ is monotone if it increasing (ie $x \leq y \Rightarrow f(x) \leq f(y)$) or decreasing.

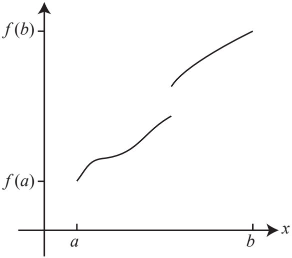
Theorem 2.6. Any monotone function $f : [a, b] \to \mathbb{R}$ is integrable.

Proof.

Monotone functions

A function $f : [a, b] \to \mathbb{R}$ is monotone if it increasing (ie $x \le y \Rightarrow f(x) \le f(y)$) or decreasing.

Theorem 2.6. Any monotone function $f : [a, b] \to \mathbb{R}$ is integrable.

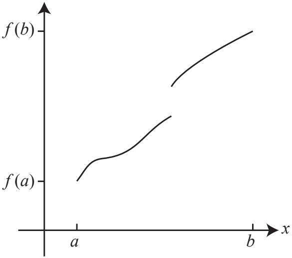


Proof.
By replacing f with $-f$ if necessary we may suppose that f is increasing.

Monotone functions

A function $f : [a, b] \to \mathbb{R}$ is monotone if it increasing (ie $x \leq y \Rightarrow f(x) \leq f(y)$) or decreasing.

Theorem 2.6. Any monotone function $f : [a, b] \to \mathbb{R}$ is integrable.

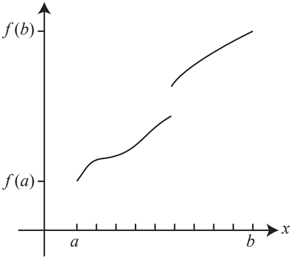


Proof.
By replacing f with $-f$ if necessary we may suppose that f is increasing.
Since $f(a) \leq f(x) \leq f(b)$, f is automatically bounded.

Monotone functions

A function $f : [a, b] \to \mathbb{R}$ is monotone if it increasing (ie $x \leq y \Rightarrow f(x) \leq f(y)$) or decreasing.

Theorem 2.6. Any monotone function $f : [a, b] \to \mathbb{R}$ is integrable.

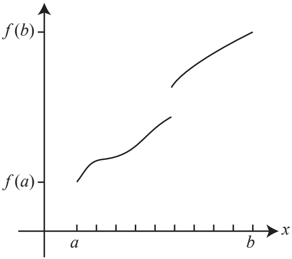


Proof.
By replacing f with $-f$ if necessary we may suppose that f is increasing.
Since $f(a) \leq f(x) \leq f(b)$, f is automatically bounded.

Let n be a positive integer, and consider the partition $\mathcal{P}$ of $[a, b]$ into n equal parts:

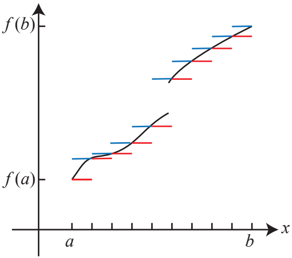
$$a = x_0 \leq x_1 \leq \cdots \leq x_n = b.$$

Proof (continued)



Proof (continued)

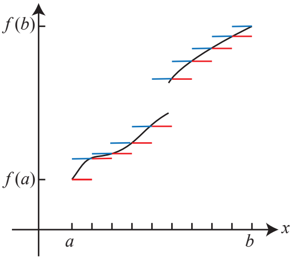
On (x_{i-1}, x_i), define $\phi_+(x) = f(x_i)$ and $\phi_-(x) = f(x_{i-1})$.



Proof (continued)

On (x_{i-1}, x_i), define $\phi_+(x) = f(x_i)$ and $\phi_-(x) = f(x_{i-1})$.

Define $\phi_-(x_i) = f(x_i)$ and $\phi_+(x_i) = f(x_i)$.

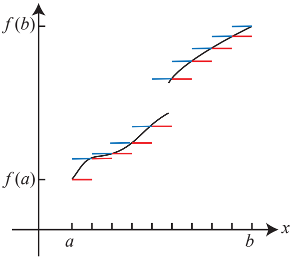


Proof (continued)

On (x_{i-1}, x_i), define $\phi_+(x) = f(x_i)$ and $\phi_-(x) = f(x_{i-1})$.

Define $\phi_-(x_i) = f(x_i)$ and $\phi_+(x_i) = f(x_i)$.

Then ϕ_+ is a majorant for f and ϕ_- is a minorant.

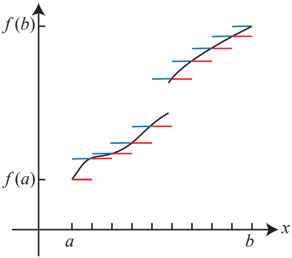


Proof (continued)

On (x_{i-1}, x_i), define $\phi_+(x) = f(x_i)$ and $\phi_-(x) = f(x_{i-1})$.

Define $\phi_-(x_i) = f(x_i)$ and $\phi_+(x_i) = f(x_i)$.

Then ϕ_+ is a majorant for f and ϕ_- is a minorant.



$$I(\phi_+) - I(\phi_-) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1})$$

$$= \frac{b-a}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))$$

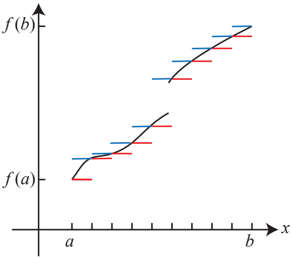
$$= \frac{1}{n}(b-a)(f(b) - f(a)).$$

Proof (continued)

On (x_{i-1}, x_i), define $\phi_+(x) = f(x_i)$ and $\phi_-(x) = f(x_{i-1})$.

Define $\phi_-(x_i) = f(x_i)$ and $\phi_+(x_i) = f(x_i)$.

Then ϕ_+ is a majorant for f and ϕ_- is a minorant.



$$I(\phi_+) - I(\phi_-) = \sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))(x_i - x_{i-1})$$
$$= \frac{b-a}{n}\sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))$$
$$= \frac{1}{n}(b-a)(f(b) - f(a)).$$

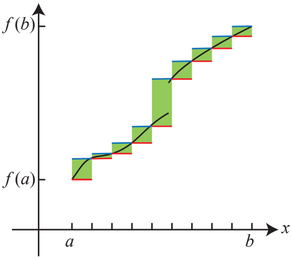
Taking n large, this can be made as small as desired. □

Proof (continued)

On (x_{i-1}, x_i), define $\phi_+(x) = f(x_i)$ and $\phi_-(x) = f(x_{i-1})$.

Define $\phi_-(x_i) = f(x_i)$ and $\phi_+(x_i) = f(x_i)$.

Then ϕ_+ is a majorant for f and ϕ_- is a minorant.



$$I(\phi_+) - I(\phi_-) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1})$$
$$= \frac{b-a}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))$$
$$= \frac{1}{n}(b-a)(f(b) - f(a)).$$

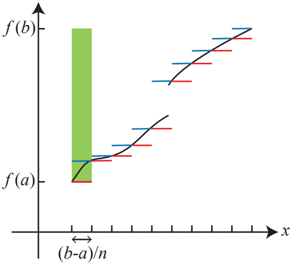
Taking n large, this can be made as small as desired. □

Proof (continued)

On (x_{i-1}, x_i), define $\phi_+(x) = f(x_i)$ and $\phi_-(x) = f(x_{i-1})$.

Define $\phi_-(x_i) = f(x_i)$ and $\phi_+(x_i) = f(x_i)$.

Then ϕ_+ is a majorant for f and ϕ_- is a minorant.



$$I(\phi_+) - I(\phi_-) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1})$$

$$= \frac{b-a}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))$$

$$= \frac{1}{n}(b-a)(f(b) - f(a)).$$

Taking n large, this can be made as small as desired. $\square$

Chapter 3A: Riemann sums

Riemann sums

Riemann sums

If $\mathcal{P}$ is a partition and $f : [a, b] \to \mathbb{R}$ is a function then by a Riemann sum adapted to $\mathcal{P}$ we mean an expression of the form

$$\Sigma(f; \mathcal{P}, \vec{\xi}) = \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}),$$

where $\vec{\xi} = (\xi_1, \ldots, \xi_n)$ and $\xi_j \in [x_{j-1}, x_j]$.

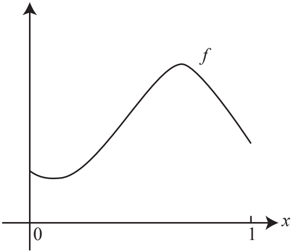
Riemann sums

If $\mathcal{P}$ is a partition and $f : [a, b] \to \mathbb{R}$ is a function then by a Riemann sum adapted to $\mathcal{P}$ we mean an expression of the form

$$\Sigma(f; \mathcal{P}, \vec{\xi}) = \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}),$$

where $\vec{\xi} = (\xi_1, \ldots, \xi_n)$ and $\xi_j \in [x_{j-1}, x_j]$.

Example.



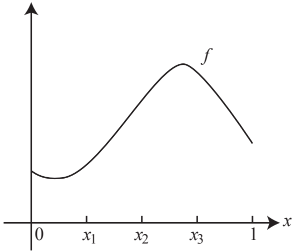
Riemann sums

If $\mathcal{P}$ is a partition and $f : [a, b] \to \mathbb{R}$ is a function then by a Riemann sum adapted to $\mathcal{P}$ we mean an expression of the form

$$\Sigma(f; \mathcal{P}, \vec{\xi}) = \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}),$$

where $\vec{\xi} = (\xi_1, \ldots, \xi_n)$ and $\xi_j \in [x_{j-1}, x_j]$.

Example. Suppose that $\mathcal{P}$ is a partition of $[0, 1]$ into n equal parts, so $x_i = i/n$.



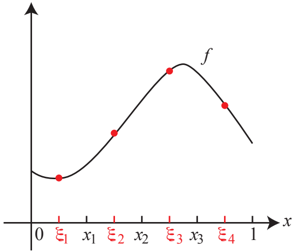
Riemann sums

If $\mathcal{P}$ is a partition and $f : [a, b] \to \mathbb{R}$ is a function then by a Riemann sum adapted to $\mathcal{P}$ we mean an expression of the form

$$\Sigma(f; \mathcal{P}, \vec{\xi}) = \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}),$$

where $\vec{\xi} = (\xi_1, \ldots, \xi_n)$ and $\xi_j \in [x_{j-1}, x_j]$.

Example. Suppose that $\mathcal{P}$ is a partition of $[0, 1]$ into n equal parts, so $x_i = i/n$. Take $\xi_j = (j - \frac{1}{2})/n$.



Riemann sums

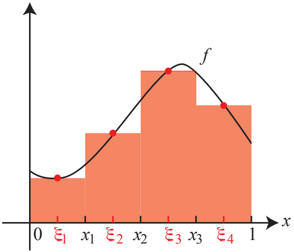
If $\mathcal{P}$ is a partition and $f : [a, b] \to \mathbb{R}$ is a function then by a Riemann sum adapted to $\mathcal{P}$ we mean an expression of the form

$$\Sigma(f; \mathcal{P}, \vec{\xi}) = \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}),$$

where $\vec{\xi} = (\xi_1, \ldots, \xi_n)$ and $\xi_j \in [x_{j-1}, x_j]$.

Example. Suppose that $\mathcal{P}$ is a partition of $[0, 1]$ into n equal parts, so $x_i = i/n$. Take $\xi_j = (j - \frac{1}{2})/n$. Then

$$\Sigma(f; \mathcal{P}, \vec{\xi}) = \frac{1}{n} \sum_{j=1}^{n} f((j - \tfrac{1}{2})/n).$$



Riemann sums and the integral

Riemann sums and the integral

Riemann sums and the integral

Proposition 3.1.

Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$.

Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$. For each i, let $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ be a Riemann sum adapted to $\mathcal{P}^{(i)}$.

Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$. For each i, let $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ be a Riemann sum adapted to $\mathcal{P}^{(i)}$. Suppose that there is some constant c such that, no matter how $\vec{\xi}^{(i)}$ is chosen, $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) \to c$.

Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$. For each i, let $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ be a Riemann sum adapted to $\mathcal{P}^{(i)}$. Suppose that there is some constant c such that, no matter how $\vec{\xi}^{(i)}$ is chosen, $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) \to c$. Then f is integrable and $c = \int_a^b f$.

Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$. For each i, let $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ be a Riemann sum adapted to $\mathcal{P}^{(i)}$. Suppose that there is some constant c such that, no matter how $\vec{\xi}^{(i)}$ is chosen, $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) \to c$. Then f is integrable and $c = \int_a^b f$.

Recall that the mesh $\mathrm{mesh}(\mathcal{P})$ of a partition is the length of the longest subinterval in $\mathcal{P}$.

Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$. For each i, let $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ be a Riemann sum adapted to $\mathcal{P}^{(i)}$. Suppose that there is some constant c such that, no matter how $\vec{\xi}^{(i)}$ is chosen, $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) \to c$. Then f is integrable and $c = \int_a^b f$.

Recall that the mesh $\mathrm{mesh}(\mathcal{P})$ of a partition is the length of the longest subinterval in $\mathcal{P}$.

Proposition 3.2.

Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$. For each i, let $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ be a Riemann sum adapted to $\mathcal{P}^{(i)}$. Suppose that there is some constant c such that, no matter how $\vec{\xi}^{(i)}$ is chosen, $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) \to c$. Then f is integrable and $c = \int_a^b f$.

Recall that the mesh $\mathrm{mesh}(\mathcal{P})$ of a partition is the length of the longest subinterval in $\mathcal{P}$.

Proposition 3.2. Let $\mathcal{P}^{(i)}$, $i = 1, 2, \ldots$ be a sequence of partitions satisfying $\mathrm{mesh}(\mathcal{P}^{(i)}) \to 0$.

Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$. For each i, let $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ be a Riemann sum adapted to $\mathcal{P}^{(i)}$. Suppose that there is some constant c such that, no matter how $\vec{\xi}^{(i)}$ is chosen, $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) \to c$. Then f is integrable and $c = \int_a^b f$.

Recall that the mesh $\text{mesh}(\mathcal{P})$ of a partition is the length of the longest subinterval in $\mathcal{P}$.

Proposition 3.2. Let $\mathcal{P}^{(i)}$, $i = 1, 2, \ldots$ be a sequence of partitions satisfying $\text{mesh}(\mathcal{P}^{(i)}) \to 0$. Suppose that f is integrable.

Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$. For each i, let $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ be a Riemann sum adapted to $\mathcal{P}^{(i)}$. Suppose that there is some constant c such that, no matter how $\vec{\xi}^{(i)}$ is chosen, $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) \to c$. Then f is integrable and $c = \int_a^b f$.

Recall that the mesh $\operatorname{mesh}(\mathcal{P})$ of a partition is the length of the longest subinterval in $\mathcal{P}$.

Proposition 3.2. Let $\mathcal{P}^{(i)}$, $i = 1, 2, \ldots$ be a sequence of partitions satisfying $\operatorname{mesh}(\mathcal{P}^{(i)}) \to 0$. Suppose that f is integrable. Then $\lim_{i\to\infty} \Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) = \int_a^b f$, no matter what choice of $\vec{\xi}^{(i)}$ we make.

An equivalent definition of the integral

An equivalent definition of the integral

Proposition 3.3.

An equivalent definition of the integral

Proposition 3.3. Let $f : [a, b] \to \mathbb{R}$ be a function.

An equivalent definition of the integral

Proposition 3.3. Let $f : [a, b] \to \mathbb{R}$ be a function.
Let $\mathcal{P}^{(i)}$, $i = 1, 2, \dots$ be a sequence of partitions with $\text{mesh}(\mathcal{P}^{(i)}) \to 0$.

An equivalent definition of the integral

Proposition 3.3. Let $f : [a, b] \to \mathbb{R}$ be a function.
Let $\mathcal{P}^{(i)}$, $i = 1, 2, \ldots$ be a sequence of partitions with $\text{mesh}(\mathcal{P}^{(i)}) \to 0$.
Then f is integrable if and only if $\lim_{i\to\infty} \Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ is equal to some constant c, independently of the choice of $\vec{\xi^{(i)}}$.

An equivalent definition of the integral

Proposition 3.3. Let $f : [a, b] \to \mathbb{R}$ be a function.
Let $\mathcal{P}^{(i)}$, $i = 1, 2, \ldots$ be a sequence of partitions with $\text{mesh}(\mathcal{P}^{(i)}) \to 0$.
Then f is integrable if and only if $\lim_{i\to\infty} \Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ is equal to some constant c, independently of the choice of $\vec{\xi}^{(i)}$.
If this is so, then $\int_a^b f = c$.

An equivalent definition of the integral

Proposition 3.3. Let $f : [a, b] \to \mathbb{R}$ be a function.
Let $\mathcal{P}^{(i)}$, $i = 1, 2, \ldots$ be a sequence of partitions with $\mathrm{mesh}(\mathcal{P}^{(i)}) \to 0$.
Then f is integrable if and only if $\lim_{i\to\infty} \Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ is equal to some constant c, independently of the choice of $\vec{\xi^{(i)}}$.
If this is so, then $\int_a^b f = c$.

It is important that the limit must exist for any choice of $\vec{\xi}^{(i)}$.

Further remarks

It is important that the limit must exist for any choice of $\vec{\xi}^{(i)}$.

Further remarks

It is important that the limit must exist for any choice of $\vec{\xi}^{(i)}$.

Example. Suppose, for example, that $[a, b] = [0, 1]$ and that $\mathcal{P}^{(i)}$ is the partition into i equal parts, thus $x_j^{(i)} = \frac{j}{i}$ for $j = 1, \ldots, i$.

Further remarks

It is important that the limit must exist for any choice of $\vec{\xi}^{(i)}$.

Example. Suppose, for example, that $[a, b] = [0, 1]$ and that $\mathcal{P}^{(i)}$ is the partition into i equal parts, thus $x_j^{(i)} = \frac{j}{i}$ for $j = 1, \ldots, i$.

Take $\xi_j^{(i)} = \frac{j}{i}$;

Further remarks

It is important that the limit must exist for any choice of $\vec{\xi}^{(i)}$.

Example. Suppose, for example, that $[a,b]=[0,1]$ and that $\mathcal{P}^{(i)}$ is the partition into i equal parts, thus $x_j^{(i)}=\frac{j}{i}$ for $j=1,\ldots,i$. Take $\xi_j^{(i)}=\frac{j}{i}$; then the Riemann sum $\Sigma(f,\mathcal{P}^{(i)},\vec{\xi}^{(i)})$ is equal to

$$S_i(f):=\frac{1}{i}\sum_{j=1}^{i} f(\frac{j}{i}).$$

Further remarks

It is important that the limit must exist for any choice of $\vec{\xi}^{(i)}$.

Example. Suppose, for example, that $[a, b] = [0, 1]$ and that $\mathcal{P}^{(i)}$ is the partition into i equal parts, thus $x_j^{(i)} = \frac{j}{i}$ for $j = 1, \ldots, i$.
Take $\xi_j^{(i)} = \frac{j}{i}$; then the Riemann sum $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ is equal to

$$S_i(f) := \frac{1}{i} \sum_{j=1}^{i} f(\frac{j}{i}).$$

By Proposition 3.2, if f is integrable then

$$S_i(f) \to \int_a^b f.$$

Further remarks

It is important that the limit must exist for any choice of $\vec{\xi}^{(i)}$.

Example. Suppose, for example, that $[a, b] = [0, 1]$ and that $\mathcal{P}^{(i)}$ is the partition into i equal parts, thus $x_j^{(i)} = \frac{j}{i}$ for $j = 1, \ldots, i$. Take $\xi_j^{(i)} = \frac{j}{i}$; then the Riemann sum $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ is equal to

$$S_i(f) := \frac{1}{i} \sum_{j=1}^{i} f(\frac{j}{i}).$$

By Proposition 3.2, if f is integrable then

$$S_i(f) \to \int_a^b f.$$

However, the converse is not true.

Further remarks

It is important that the limit must exist for any choice of $\vec{\xi}^{(i)}$.

Example. Suppose, for example, that $[a, b] = [0, 1]$ and that $\mathcal{P}^{(i)}$ is the partition into i equal parts, thus $x_j^{(i)} = \frac{j}{i}$ for $j = 1, \dots, i$. Take $\xi_j^{(i)} = \frac{j}{i}$; then the Riemann sum $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ is equal to

$$S_i(f) := \frac{1}{i} \sum_{j=1}^{i} f(\frac{j}{i}).$$

By Proposition 3.2, if f is integrable then

$$S_i(f) \to \int_a^b f.$$

However, the converse is not true. Consider, for example, the function f introduced in the first chapter, with $f(x) = 1$ for $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise.

Further remarks

It is important that the limit must exist for any choice of $\vec{\xi}^{(i)}$.

Example. Suppose, for example, that $[a, b] = [0, 1]$ and that $\mathcal{P}^{(i)}$ is the partition into i equal parts, thus $x_j^{(i)} = \frac{j}{i}$ for $j = 1, \dots, i$. Take $\xi_j^{(i)} = \frac{j}{i}$; then the Riemann sum $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ is equal to

$$S_i(f) := \frac{1}{i} \sum_{j=1}^{i} f(\frac{j}{i}).$$

By Proposition 3.2, if f is integrable then

$$S_i(f) \to \int_a^b f.$$

However, the converse is not true. Consider, for example, the function f introduced in the first chapter, with $f(x) = 1$ for $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise. This function is not integrable. However,

$$S_i(f) = 1 \qquad \text{for all } i.$$

Chapter 3B: Riemann sums (proofs)

Riemann sums and the integral

Proposition 3.1.

Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$.

Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$. For each i, let $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ be a Riemann sum adapted to $\mathcal{P}^{(i)}$.

Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$. For each i, let $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ be a Riemann sum adapted to $\mathcal{P}^{(i)}$. Suppose that there is some constant c such that, no matter how $\vec{\xi}^{(i)}$ is chosen, $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) \to c$.

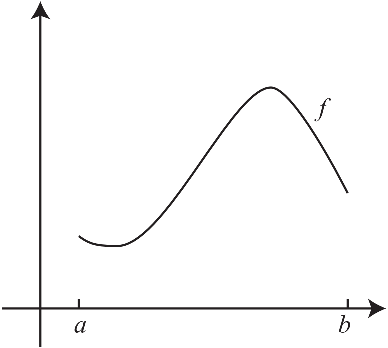
Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$. For each i, let $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ be a Riemann sum adapted to $\mathcal{P}^{(i)}$. Suppose that there is some constant c such that, no matter how $\vec{\xi}^{(i)}$ is chosen, $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) \to c$. Then f is integrable and $c = \int_a^b f$.

Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$. For each i, let $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ be a Riemann sum adapted to $\mathcal{P}^{(i)}$. Suppose that there is some constant c such that, no matter how $\vec{\xi}^{(i)}$ is chosen, $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) \to c$. Then f is integrable and $c = \int_a^b f$.

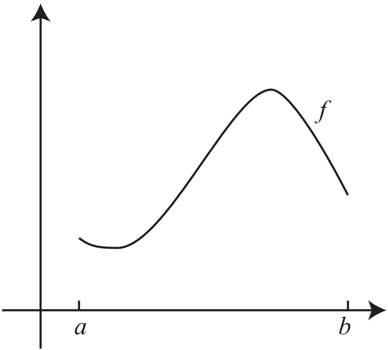
Proof.



Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$. For each i, let $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ be a Riemann sum adapted to $\mathcal{P}^{(i)}$. Suppose that there is some constant c such that, no matter how $\vec{\xi}^{(i)}$ is chosen, $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) \to c$. Then f is integrable and $c = \int_a^b f$.

Proof. Let $\epsilon > 0$.

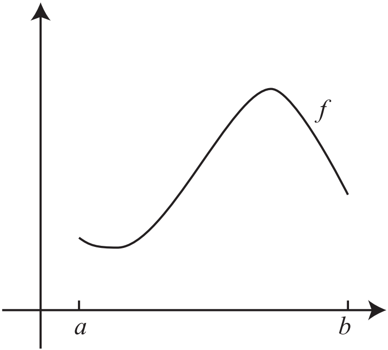


Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$. For each i, let $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ be a Riemann sum adapted to $\mathcal{P}^{(i)}$. Suppose that there is some constant c such that, no matter how $\vec{\xi}^{(i)}$ is chosen, $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) \to c$. Then f is integrable and $c = \int_a^b f$.

Proof. Let $\epsilon > 0$.
We will show that there is a majorant ϕ_+ and minorant ϕ_- such that

$$I(\phi_+) < c + \epsilon(b-a) + \epsilon$$
$$I(\phi_-) > c - \epsilon(b-a) - \epsilon.$$



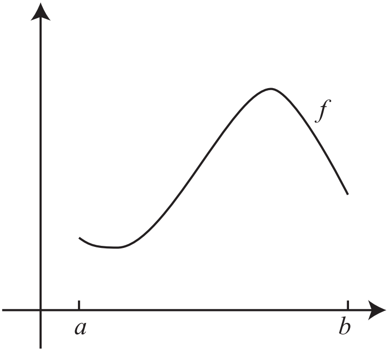
Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$. For each i, let $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ be a Riemann sum adapted to $\mathcal{P}^{(i)}$. Suppose that there is some constant c such that, no matter how $\vec{\xi}^{(i)}$ is chosen, $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) \to c$. Then f is integrable and $c = \int_a^b f$.

Proof. Let $\epsilon > 0$.
We will show that there is a majorant ϕ_+ and minorant ϕ_- such that

$$I(\phi_+) < c + \epsilon(b-a) + \epsilon$$
$$I(\phi_-) > c - \epsilon(b-a) - \epsilon.$$

Let i be chosen so that $\Sigma(f; \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) \leq c + \varepsilon$, no matter which $\vec{\xi}^{(i)}$ is chosen.



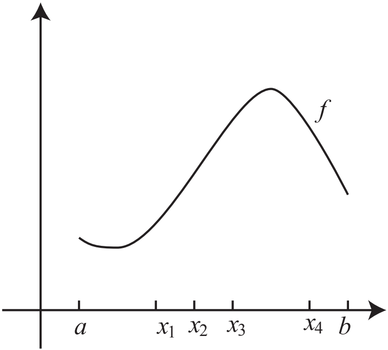
Riemann sums and the integral

Proposition 3.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$. For each i, let $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ be a Riemann sum adapted to $\mathcal{P}^{(i)}$. Suppose that there is some constant c such that, no matter how $\vec{\xi}^{(i)}$ is chosen, $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) \to c$. Then f is integrable and $c = \int_a^b f$.

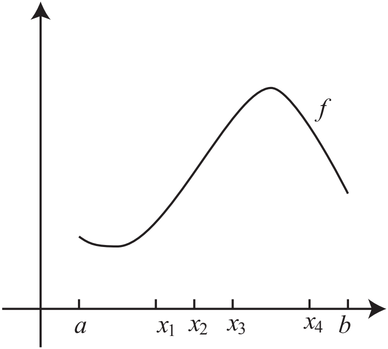
Proof. Let $\epsilon > 0$.
We will show that there is a majorant ϕ_+ and minorant ϕ_- such that

$$I(\phi_+) < c + \epsilon(b - a) + \epsilon$$
$$I(\phi_-) > c - \epsilon(b - a) - \epsilon.$$

Let i be chosen so that $\Sigma(f; \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) \leq c + \varepsilon$, no matter which $\vec{\xi}^{(i)}$ is chosen.
Write $\mathcal{P} = \mathcal{P}^{(i)}$.

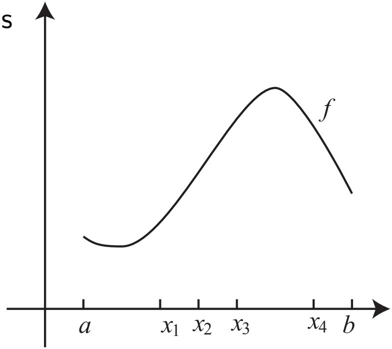


Proof (continued)



Proof (continued)

Write $\mathcal{P} = \mathcal{P}^{(i)}$, and suppose that $\mathcal{P}$ is $a = x_0 \leq \cdots \leq x_n = b$.

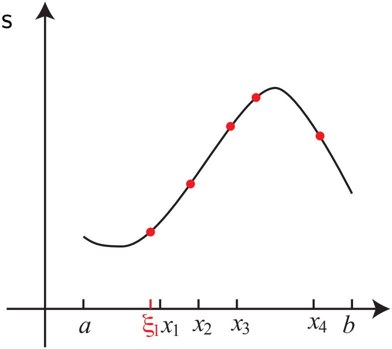


Proof (continued)

Write $\mathcal{P} = \mathcal{P}^{(i)}$, and suppose that $\mathcal{P}$ is $a = x_0 \leq \cdots \leq x_n = b$.

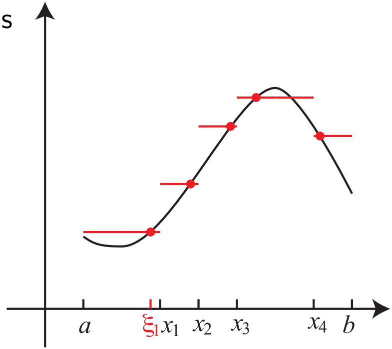
For each j, choose some point $\xi_j \in [x_{j-1}, x_j]$ such that $f(\xi_j) \geq \sup_{x \in [x_{j-1}, x_j]} f(x) - \varepsilon$.

(Note that f does not necessarily attain its supremum on this interval.)



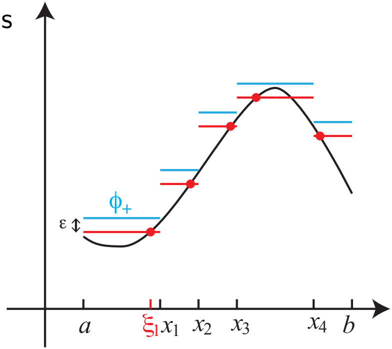
Proof (continued)

Write $\mathcal{P} = \mathcal{P}^{(i)}$, and suppose that $\mathcal{P}$ is $a = x_0 \leq \cdots \leq x_n = b$.
For each j, choose some point $\xi_j \in [x_{j-1}, x_j]$ such that $f(\xi_j) \geq \sup_{x \in [x_{j-1}, x_j]} f(x) - \varepsilon$.
(Note that f does not necessarily attain its supremum on this interval.)



Proof (continued)

Write $\mathcal{P} = \mathcal{P}^{(i)}$, and suppose that $\mathcal{P}$ is $a = x_0 \leq \cdots \leq x_n = b$.
For each j, choose some point $\xi_j \in [x_{j-1}, x_j]$ such that $f(\xi_j) \geq \sup_{x \in [x_{j-1}, x_j]} f(x) - \varepsilon$.
(Note that f does not necessarily attain its supremum on this interval.)
Let ϕ_+ be a step function taking the value $f(\xi_j) + \varepsilon$ on (x_{j-1}, x_j), and with $\phi_+(x_j) = f(x_j)$.



Proof (continued)

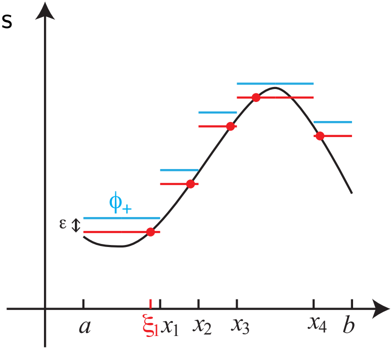
Write $\mathcal{P} = \mathcal{P}^{(i)}$, and suppose that $\mathcal{P}$ is $a = x_0 \leq \cdots \leq x_n = b$.

For each j, choose some point $\xi_j \in [x_{j-1}, x_j]$ such that $f(\xi_j) \geq \sup_{x \in [x_{j-1}, x_j]} f(x) - \varepsilon$.

(Note that f does not necessarily attain its supremum on this interval.)

Let ϕ_+ be a step function taking the value $f(\xi_j) + \varepsilon$ on (x_{j-1}, x_j), and with $\phi_+(x_j) = f(x_j)$.

Then ϕ_+ is a majorant for f.

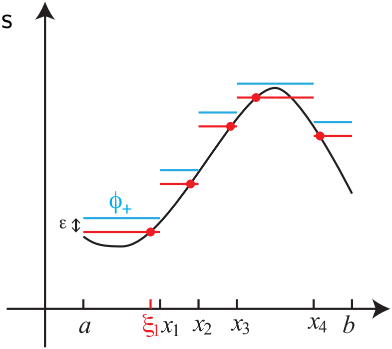


Proof (continued)

Write $\mathcal{P} = \mathcal{P}^{(i)}$, and suppose that $\mathcal{P}$ is $a = x_0 \leq \cdots \leq x_n = b$.

For each j, choose some point $\xi_j \in [x_{j-1}, x_j]$ such that $f(\xi_j) \geq \sup_{x \in [x_{j-1}, x_j]} f(x) - \varepsilon$. (Note that f does not necessarily attain its supremum on this interval.)

Let ϕ_+ be a step function taking the value $f(\xi_j) + \varepsilon$ on (x_{j-1}, x_j), and with $\phi_+(x_j) = f(x_j)$.

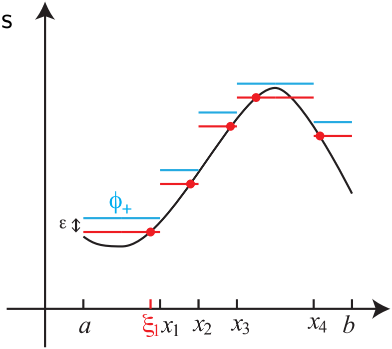


Then ϕ_+ is a majorant for f. It is easy to see that

$$I(\phi_+) = \varepsilon(b - a) + \Sigma(f; \mathcal{P}, \vec{\xi})$$

Proof (continued)

Write $\mathcal{P} = \mathcal{P}^{(i)}$, and suppose that $\mathcal{P}$ is $a = x_0 \leq \cdots \leq x_n = b$.
For each j, choose some point $\xi_j \in [x_{j-1}, x_j]$ such that $f(\xi_j) \geq \sup_{x \in [x_{j-1}, x_j]} f(x) - \varepsilon$.
(Note that f does not necessarily attain its supremum on this interval.)
Let ϕ_+ be a step function taking the value $f(\xi_j) + \varepsilon$ on (x_{j-1}, x_j), and with $\phi_+(x_j) = f(x_j)$.

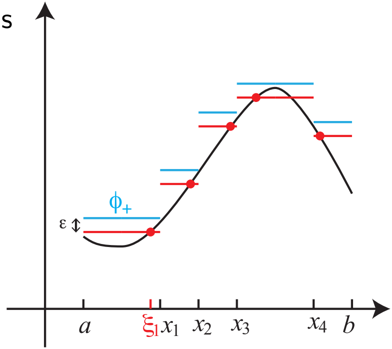


Then ϕ_+ is a majorant for f. It is easy to see that

$$I(\phi_+) = \varepsilon(b-a) + \Sigma(f; \mathcal{P}, \vec{\xi}) \leq \varepsilon(b-a) + c + \varepsilon.$$

Proof (continued)

Write $\mathcal{P} = \mathcal{P}^{(i)}$, and suppose that $\mathcal{P}$ is $a = x_0 \leq \cdots \leq x_n = b$.
For each j, choose some point $\xi_j \in [x_{j-1}, x_j]$ such that $f(\xi_j) \geq \sup_{x \in [x_{j-1}, x_j]} f(x) - \varepsilon$. (Note that f does not necessarily attain its supremum on this interval.)
Let ϕ_+ be a step function taking the value $f(\xi_j) + \varepsilon$ on (x_{j-1}, x_j), and with $\phi_+(x_j) = f(x_j)$.



Then ϕ_+ is a majorant for f. It is easy to see that

$$I(\phi_+) = \varepsilon(b - a) + \Sigma(f; \mathcal{P}, \vec{\xi}) \leq \varepsilon(b - a) + c + \varepsilon.$$

SImilarly, there is a minorant ϕ_- such that

$$I(\phi_-) \geq c - \varepsilon(b - a) - \varepsilon.$$

Riemann sums and the integral

Proposition 3.2.

Riemann sums and the integral

Proposition 3.2. Let $\mathcal{P}^{(i)}$, $i = 1, 2, \ldots$ be a sequence of partitions satisfying $\text{mesh}(\mathcal{P}^{(i)}) \to 0$.

Riemann sums and the integral

Proposition 3.2. Let $\mathcal{P}^{(i)}$, $i = 1, 2, \ldots$ be a sequence of partitions satisfying $\text{mesh}(\mathcal{P}^{(i)}) \to 0$. Suppose that f is integrable.

Riemann sums and the integral

Proposition 3.2. Let $\mathcal{P}^{(i)}$, $i = 1, 2, \ldots$ be a sequence of partitions satisfying $\mathrm{mesh}(\mathcal{P}^{(i)}) \to 0$. Suppose that f is integrable. Then $\lim_{i\to\infty} \Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) = \int_a^b f$, no matter what choice of $\vec{\xi}^{(i)}$ we make.

Optimal majorants and minorants

Optimal majorants and minorants

Let $\mathcal{P} : a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ be a partition.

Optimal majorants and minorants

Let $\mathcal{P} : a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ be a partition.

The optimal majorant $\phi_+^{\mathcal{P}}$ for f relative to $\mathcal{P}$ is defined by

Optimal majorants and minorants

Let $\mathcal{P} : a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ be a partition.

The optimal majorant $\phi_+^{\mathcal{P}}$ for f relative to $\mathcal{P}$ is defined by

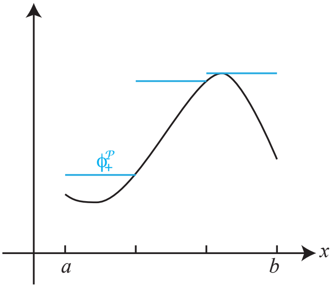
$$\phi_+^{\mathcal{P}} := \begin{cases} \sup_{x \in (x_{i-1}, x_i)} f(x) & \text{on } (x_{i-1}, x_i) \\ f(x_i) & \text{at the points } x_i. \end{cases}$$

Optimal majorants and minorants

Let $\mathcal{P} : a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ be a partition.

The optimal majorant $\phi_+^{\mathcal{P}}$ for f relative to $\mathcal{P}$ is defined by

$$\phi_+^{\mathcal{P}} := \begin{cases} \sup_{x \in (x_{i-1}, x_i)} f(x) & \text{on } (x_{i-1}, x_i) \\ f(x_i) & \text{at the points } x_i. \end{cases}$$

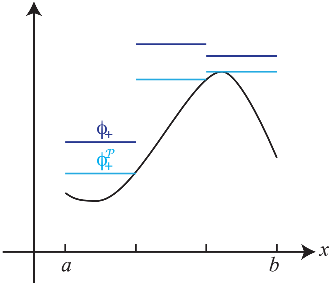


Optimal majorants and minorants

Let $\mathcal{P} : a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ be a partition.

The optimal majorant $\phi_+^{\mathcal{P}}$ for f relative to $\mathcal{P}$ is defined by

$$\phi_+^{\mathcal{P}} := \begin{cases} \sup_{x \in (x_{i-1}, x_i)} f(x) & \text{on } (x_{i-1}, x_i) \\ f(x_i) & \text{at the points } x_i. \end{cases}$$



If ϕ_+ is any majorant for f adapted to $\mathcal{P}$, then $I(\phi_+^{\mathcal{P}}) \leq I(\phi_+)$.

Optimal majorants and minorants

If ϕ_+ is any majorant for f adapted to $\mathcal{P}$, then $I(\phi_+^{\mathcal{P}}) \leq I(\phi_+)$.

Optimal majorants and minorants

If ϕ_+ is any majorant for f adapted to $\mathcal{P}$, then $I(\phi_+^{\mathcal{P}}) \leq I(\phi_+)$.
Similarly, $I(\phi_-^{\mathcal{P}}) \geq I(\phi_-)$, and so

$$I(\phi_+^{\mathcal{P}}) - I(\phi_-^{\mathcal{P}}) \leq I(\phi_+) - I(\phi_-).$$

Optimal majorants and minorants

If ϕ_+ is any majorant for f adapted to $\mathcal{P}$, then $I(\phi_+^{\mathcal{P}}) \leq I(\phi_+)$. Similarly, $I(\phi_-^{\mathcal{P}}) \geq I(\phi_-)$, and so

$$I(\phi_+^{\mathcal{P}}) - I(\phi_-^{\mathcal{P}}) \leq I(\phi_+) - I(\phi_-).$$

Therefore, f is integrable if and only if for every $\varepsilon > 0$, there is a partition $\mathcal{P}$ such $I(\phi_+^{\mathcal{P}}) - I(\phi_-^{\mathcal{P}}) < \varepsilon$.

Riemann sums and the integral

Proposition 3.2. Let $\mathcal{P}^{(i)}$, $i = 1, 2, \ldots$ be a sequence of partitions satisfying $\text{mesh}(\mathcal{P}^{(i)}) \to 0$. Suppose that f is integrable. Then $\lim_{i\to\infty} \Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) = \int_a^b f$, no matter what choice of $\vec{\xi}^{(i)}$ we make.

Proof of Proposition 3.2.

Proof of Proposition 3.2.

Let $\varepsilon > 0$.

Proof of Proposition 3.2.

Let $\varepsilon > 0$.
Since f is integrable, there is a partition
$\mathcal{P} : a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ such that $I(\phi_+^{\mathcal{P}}) - I(\phi_-^{\mathcal{P}}) < \varepsilon$.

Proof of Proposition 3.2.

Let $\varepsilon > 0$.
Since f is integrable, there is a partition
$\mathcal{P} : a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ such that $I(\phi_+^{\mathcal{P}}) - I(\phi_-^{\mathcal{P}}) < \varepsilon$.
In particular, since $I(\phi_-) \leq \int_a^b f$ for any minorant ϕ_-,

$$I(\phi_+^{\mathcal{P}}) \leq \int_a^b f + \varepsilon.$$

Proof of Proposition 3.2.

Let $\varepsilon > 0$.
Since f is integrable, there is a partition
$\mathcal{P} : a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ such that $I(\phi_+^{\mathcal{P}}) - I(\phi_-^{\mathcal{P}}) < \varepsilon$.
In particular, since $I(\phi_-) \leq \int_a^b f$ for any minorant ϕ_-,

$$I(\phi_+^{\mathcal{P}}) \leq \int_a^b f + \varepsilon.$$

Set $\delta := \varepsilon / nM$ where $|f(x)| \leq M$ for all $x \in [a, b]$.

Proof of Proposition 3.2.

Let $\varepsilon > 0$.
Since f is integrable, there is a partition
$\mathcal{P} : a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ such that $I(\phi_+^{\mathcal{P}}) - I(\phi_-^{\mathcal{P}}) < \varepsilon$.
In particular, since $I(\phi_-) \leq \int_a^b f$ for any minorant ϕ_-,

$$I(\phi_+^{\mathcal{P}}) \leq \int_a^b f + \varepsilon.$$

Set $\delta := \varepsilon / nM$ where $|f(x)| \leq M$ for all $x \in [a, b]$.
Let $\mathcal{P}' : a = x'_0 \leq x'_1 \leq \cdots \leq x'_{n'} = b$ be any partition with $\operatorname{mesh}(\mathcal{P}') \leq \delta$.

Proof of Proposition 3.2.

Let $\varepsilon > 0$.
Since f is integrable, there is a partition
$\mathcal{P} : a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ such that $I(\phi_+^{\mathcal{P}}) - I(\phi_-^{\mathcal{P}}) < \varepsilon$.
In particular, since $I(\phi_-) \leq \int_a^b f$ for any minorant ϕ_-,

$$I(\phi_+^{\mathcal{P}}) \leq \int_a^b f + \varepsilon.$$

Set $\delta := \varepsilon / nM$ where $|f(x)| \leq M$ for all $x \in [a, b]$.
Let $\mathcal{P}' : a = x_0' \leq x_1' \leq \cdots \leq x_{n'}' = b$ be any partition with mesh$(\mathcal{P}') \leq \delta$.
We will show that for any Riemann sum $\Sigma(f, \mathcal{P}', \vec{\xi'})$,

$$\int_a^b f - 5\varepsilon \leq \Sigma(f, \mathcal{P}', \vec{\xi'}) \leq \int_a^b f + 5\varepsilon.$$

This will conclude the proof.

Proof of Proposition 3.2 (continued)

$$\Sigma(f, \mathcal{P}', \vec{\xi'})$$

Proof of Proposition 3.2 (continued)

$$\Sigma(f, \mathcal{P}', \vec{\xi'}) = \sum_{j=1}^{n'} f(\xi'_j)(x'_j - x'_{j-1})$$

Proof of Proposition 3.2 (continued)

$$\Sigma(f, \mathcal{P}', \vec{\xi}') = \sum_{j=1}^{n'} f(\xi_j')(x_j' - x_{j-1}') = I(\psi),$$

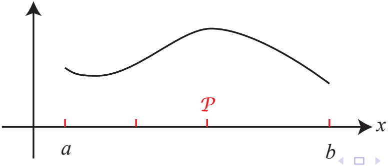
where the step function ψ is defined to be $f(\xi_j')$ on (x_{j-1}', x_j') and $f(x_j')$ at the x_j'.

Proof of Proposition 3.2 (continued)

$$\Sigma(f, \mathcal{P}', \vec{\xi'}) = \sum_{j=1}^{n'} f(\xi'_j)(x'_j - x'_{j-1}) = I(\psi),$$

where the step function ψ is defined to be $f(\xi'_j)$ on (x'_{j-1}, x'_j) and $f(x'_j)$ at the x'_j.

Let us compare ψ and the optimal majorant $\phi_+^{\mathcal{P}}$.



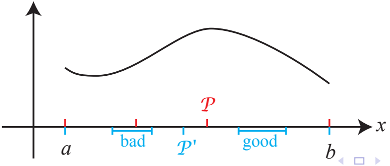
Proof of Proposition 3.2 (continued)

$$\Sigma(f, \mathcal{P}', \vec{\xi'}) = \sum_{j=1}^{n'} f(\xi'_j)(x'_j - x'_{j-1}) = I(\psi),$$

where the step function ψ is defined to be $f(\xi'_j)$ on (x'_{j-1}, x'_j) and $f(x'_j)$ at the x'_j.

Let us compare ψ and the optimal majorant $\phi_+^{\mathcal{P}}$.

Say that j is *good* if $[x'_{j-1}, x'_j] \subset (x_{i-1}, x_i)$ for some i.



Proof of Proposition 3.2 (continued)

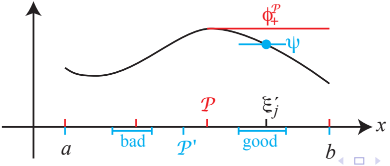
$$\Sigma(f, \mathcal{P}', \vec{\xi'}) = \sum_{j=1}^{n'} f(\xi'_j)(x'_j - x'_{j-1}) = I(\psi),$$

where the step function ψ is defined to be $f(\xi'_j)$ on (x'_{j-1}, x'_j) and $f(x'_j)$ at the x'_j.

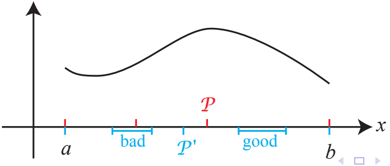
Let us compare ψ and the optimal majorant $\phi_+^{\mathcal{P}}$.

Say that j is *good* if $[x'_{j-1}, x'_j] \subset (x_{i-1}, x_i)$ for some i.
If j is good then, for $t \in (x'_{j-1}, x'_j)$,

$$\psi(t) = f(\xi'_j) \leq \sup_{x \in [x'_{j-1}, x'_j]} f(x) \leq \sup_{x \in (x_{i-1}, x_i)} f(x) = \phi_+^{\mathcal{P}}(t).$$



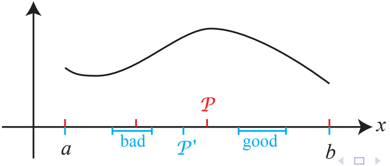
Proof of Proposition 3.2 (continued)



Proof of Proposition 3.2 (continued)

If j is bad (i.e. not good) then we cannot assert such a bound, but we do have the trivial bound

$$\psi(t) \leq \phi_+^{\mathcal{P}}(t) + 2M.$$

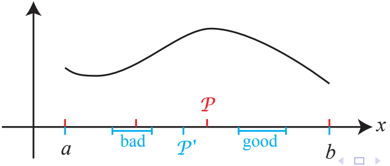


Proof of Proposition 3.2 (continued)

If j is bad (i.e. not good) then we cannot assert such a bound, but we do have the trivial bound

$$\psi(t) \leq \phi_+^{\mathcal{P}}(t) + 2M.$$

Now if j is bad then we have $x_i \in [x'_{j-1}, x'_j]$ for some i.

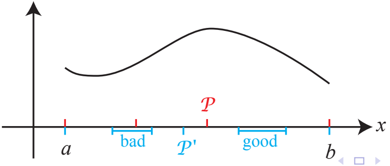


Proof of Proposition 3.2 (continued)

If j is bad (i.e. not good) then we cannot assert such a bound, but we do have the trivial bound

$$\psi(t) \leq \phi_+^{\mathcal{P}}(t) + 2M.$$

Now if j is bad then we have $x_i \in [x'_{j-1}, x'_j]$ for some i. No x_i can belong to more than two intervals $[x'_{j-1}, x'_j]$, so there cannot be more than $2n$ bad values of j.

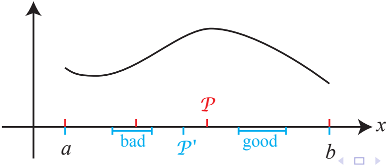


Proof of Proposition 3.2 (continued)

If j is bad (i.e. not good) then we cannot assert such a bound, but we do have the trivial bound

$$\psi(t) \leq \phi_+^{\mathcal{P}}(t) + 2M.$$

Now if j is bad then we have $x_i \in [x'_{j-1}, x'_j]$ for some i. No x_i can belong to more than two intervals $[x'_{j-1}, x'_j]$, so there cannot be more than $2n$ bad values of j. Therefore the total length of the corresponding intervals (x'_{j-1}, x'_j) is at most $2\delta n = 2\varepsilon/M$.



Proof of Proposition 3.2 (continued)

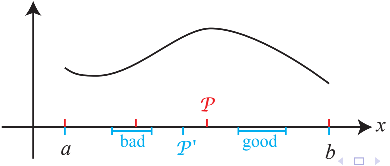
If j is bad (i.e. not good) then we cannot assert such a bound, but we do have the trivial bound

$$\psi(t) \leq \phi_+^{\mathcal{P}}(t) + 2M.$$

Now if j is bad then we have $x_i \in [x'_{j-1}, x'_j]$ for some i. No x_i can belong to more than two intervals $[x'_{j-1}, x'_j]$, so there cannot be more than $2n$ bad values of j. Therefore the total length of the corresponding intervals (x'_{j-1}, x'_j) is at most $2\delta n = 2\varepsilon/M$.

Considering both the good and bad intervals,

$$\Sigma(f, \mathcal{P}', \vec{\xi}') = I(\psi) \leq I(\phi_+^{\mathcal{P}}) + 2M \cdot \frac{2\varepsilon}{M} = I(\phi_+^{\mathcal{P}}) + 4\varepsilon$$



Proof of Proposition 3.2 (continued)

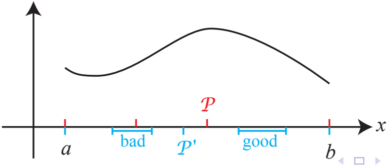
If j is bad (i.e. not good) then we cannot assert such a bound, but we do have the trivial bound

$$\psi(t) \leq \phi_+^{\mathcal{P}}(t) + 2M.$$

Now if j is bad then we have $x_i \in [x'_{j-1}, x'_j]$ for some i. No x_i can belong to more than two intervals $[x'_{j-1}, x'_j]$, so there cannot be more than $2n$ bad values of j. Therefore the total length of the corresponding intervals (x'_{j-1}, x'_j) is at most $2\delta n = 2\varepsilon/M$.

Considering both the good and bad intervals,

$$\Sigma(f, \mathcal{P}', \vec{\xi}') = I(\psi) \leq I(\phi_+^{\mathcal{P}}) + 2M \cdot \frac{2\varepsilon}{M} = I(\phi_+^{\mathcal{P}}) + 4\varepsilon \leq \int_a^b f + 5\varepsilon.$$



Proof of Proposition 3.2 (continued)

If j is bad (i.e. not good) then we cannot assert such a bound, but we do have the trivial bound

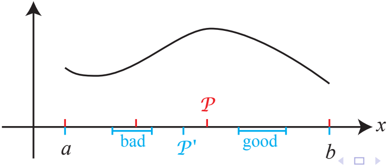
$$\psi(t) \leq \phi_+^{\mathcal{P}}(t) + 2M.$$

Now if j is bad then we have $x_i \in [x'_{j-1}, x'_j]$ for some i. No x_i can belong to more than two intervals $[x'_{j-1}, x'_j]$, so there cannot be more than $2n$ bad values of j. Therefore the total length of the corresponding intervals (x'_{j-1}, x'_j) is at most $2\delta n = 2\varepsilon/M$.

Considering both the good and bad intervals,

$$\Sigma(f, \mathcal{P}', \vec{\xi'}) = I(\psi) \leq I(\phi_+^{\mathcal{P}}) + 2M \cdot \frac{2\varepsilon}{M} = I(\phi_+^{\mathcal{P}}) + 4\varepsilon \leq \int_a^b f + 5\varepsilon.$$

We also have a similar lower bound. $\square$



Chapter 4A: The fundamental theorem of calculus

The fundamental theorem of calculus

The fundamental theorem of calculus

There are two theorems:

The fundamental theorem of calculus

There are two theorems:

1. first integrate, then differentiate;

The fundamental theorem of calculus

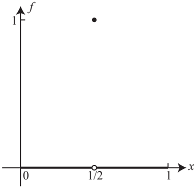
There are two theorems:

1. first integrate, then differentiate;
2. first differentiate, then integrate.

An example

Let $f : [0, 1] \to \mathbb{R}$ be

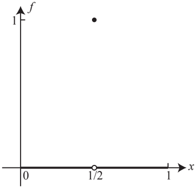
$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{2}; \\ 0 & \text{otherwise.} \end{cases}$$



An example

Let $f : [0, 1] \to \mathbb{R}$ be

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{2}; \\ 0 & \text{otherwise.} \end{cases}$$



Define

$$F(x) = \int_0^x f = \int_0^x f(t)\ dt.$$

An example

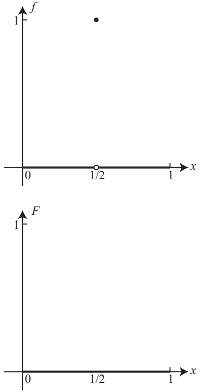
Let $f : [0,1] \to \mathbb{R}$ be

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$F(x) = \int_0^x f = \int_0^x f(t)\, dt.$$

Then F is identically zero.



An example

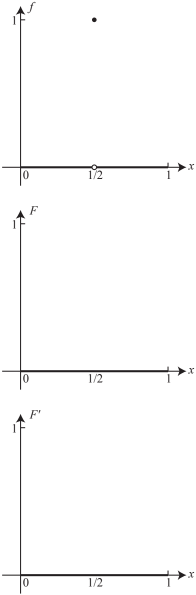
Let $f : [0,1] \to \mathbb{R}$ be

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$F(x) = \int_0^x f = \int_0^x f(t)\ dt.$$

Then F is identically zero.
So, F' is also identically zero.



An example

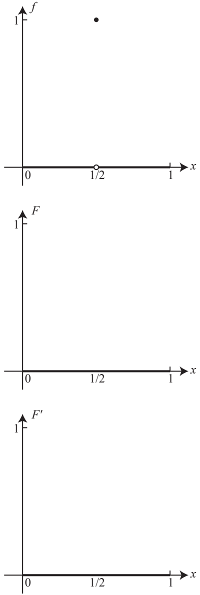
Let $f : [0,1] \to \mathbb{R}$ be

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$F(x) = \int_0^x f = \int_0^x f(t)\, dt.$$

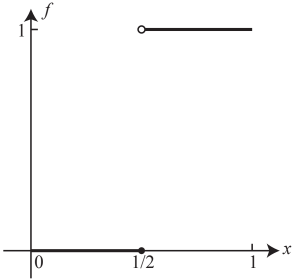
Then F is identically zero.
So, F' is also identically zero.
So, $F' \neq f$.



Another example

Let $f : [0,1] \to \mathbb{R}$ be

$$f(x) = \begin{cases} 0 & \text{if } x \le \frac{1}{2}; \\ 1 & \text{if } x > \frac{1}{2}. \end{cases}$$



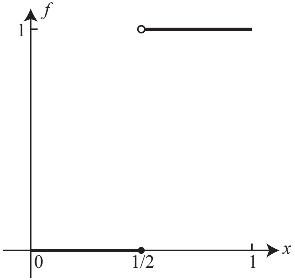
Another example

Let $f : [0,1] \to \mathbb{R}$ be

$$f(x) = \begin{cases} 0 & \text{if } x \le \frac{1}{2}; \\ 1 & \text{if } x > \frac{1}{2}. \end{cases}$$

Define

$$F(x) = \int_0^x f = \int_0^x f(t)\, dt.$$



Another example

Let $f : [0,1] \to \mathbb{R}$ be

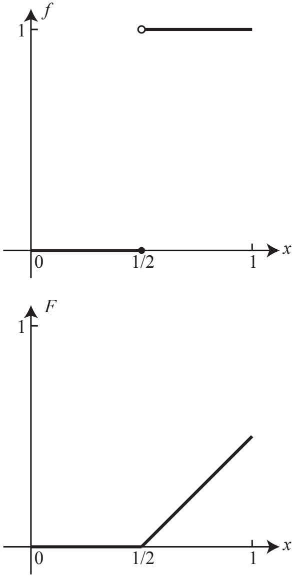
$$f(x) = \begin{cases} 0 & \text{if } x \le \frac{1}{2}; \\ 1 & \text{if } x > \frac{1}{2}. \end{cases}$$

Define

$$F(x) = \int_0^x f = \int_0^x f(t)\, dt.$$

Then

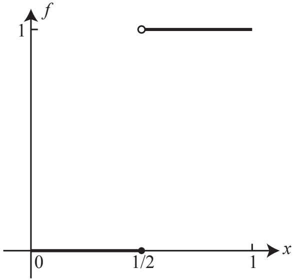
$$F(x) = \begin{cases} 0 & \text{if } x \le \frac{1}{2}; \\ x - \frac{1}{2} & \text{if } x > \frac{1}{2}. \end{cases}$$



Another example

Let $f : [0,1] \to \mathbb{R}$ be

$$f(x) = \begin{cases} 0 & \text{if } x \le \frac{1}{2}; \\ 1 & \text{if } x > \frac{1}{2}. \end{cases}$$

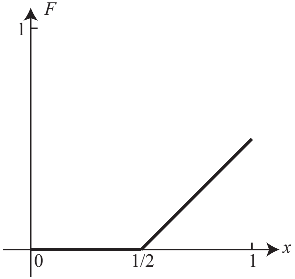


Define

$$F(x) = \int_0^x f = \int_0^x f(t)\ dt.$$

Then

$$F(x) = \begin{cases} 0 & \text{if } x \le \frac{1}{2}; \\ x - \frac{1}{2} & \text{if } x > \frac{1}{2}. \end{cases}$$



So, F is not differentable at $x = \frac{1}{2}$.

The first fundamental theorem of calculus

The first fundamental theorem of calculus

Theorem 4.1. Suppose that f is integrable on (a, b). Define a new function $F : [a, b] \to \mathbb{R}$ by

$$F(x) := \int_a^x f.$$

Then F is continuous.

The first fundamental theorem of calculus

Theorem 4.1. Suppose that f is integrable on (a, b). Define a new function $F : [a, b] \to \mathbb{R}$ by

$$F(x) := \int_a^x f.$$

Then F is continuous. Moreover, if f is continuous at $c \in (a, b)$ then F is differentiable at c and $F'(c) = f(c)$.

The first fundamental theorem of calculus

Theorem 4.1. Suppose that f is integrable on (a, b). Define a new function $F : [a, b] \to \mathbb{R}$ by

$$F(x) := \int_a^x f.$$

Then F is continuous. Moreover, if f is continuous at $c \in (a, b)$ then F is differentiable at c and $F'(c) = f(c)$.

Proof.

The first fundamental theorem of calculus

Theorem 4.1. Suppose that f is integrable on (a, b). Define a new function $F : [a, b] \to \mathbb{R}$ by

$$F(x) := \int_a^x f.$$

Then F is continuous. Moreover, if f is continuous at $c \in (a, b)$ then F is differentiable at c and $F'(c) = f(c)$.

Proof. As f is integrable, it is bounded ie $|f| \leq M$.

The first fundamental theorem of calculus

Theorem 4.1. Suppose that f is integrable on (a, b). Define a new function $F : [a, b] \to \mathbb{R}$ by

$$F(x) := \int_a^x f.$$

Then F is continuous. Moreover, if f is continuous at $c \in (a, b)$ then F is differentiable at c and $F'(c) = f(c)$.

Proof. As f is integrable, it is bounded ie $|f| \leq M$.
So for any $c \in [a, b]$,

$$|F(c+h) - F(c)| = \left| \int_c^{c+h} f \right| \leq \int_c^{c+h} |f| \leq Mh.$$

The first fundamental theorem of calculus

Theorem 4.1. Suppose that f is integrable on (a, b). Define a new function $F : [a, b] \to \mathbb{R}$ by

$$F(x) := \int_a^x f.$$

Then F is continuous. Moreover, if f is continuous at $c \in (a, b)$ then F is differentiable at c and $F'(c) = f(c)$.

Proof. As f is integrable, it is bounded ie $|f| \leq M$.
So for any $c \in [a, b]$,

$$|F(c+h) - F(c)| = \left|\int_c^{c+h} f\right| \leq \int_c^{c+h} |f| \leq Mh.$$

Hence, F is Lipschitz, hence uniformly continuous, hence continuous.

Proof (second part)

We will show that if f is continuous at any $c \in (a, b)$, then F is differentiable at c.

Proof (second part)

We will show that if f is continuous at any $c \in (a, b)$, then F is differentiable at c. If $h > 0$ is sufficiently small that $c + h < b$, then

$$F(c+h) - F(c) = \int_c^{c+h} f.$$

Proof (second part)

We will show that if f is continuous at any $c \in (a, b)$, then F is differentiable at c. If $h > 0$ is sufficiently small that $c + h < b$, then

$$F(c+h) - F(c) = \int_c^{c+h} f.$$

Let $\epsilon > 0$.

Proof (second part)

We will show that if f is continuous at any $c \in (a, b)$, then F is differentiable at c. If $h > 0$ is sufficiently small that $c + h < b$, then

$$F(c+h) - F(c) = \int_c^{c+h} f.$$

Let $\epsilon > 0$.
Since f is continuous at c, there is a $\delta > 0$ such that for all $t \in [c - \delta, c + \delta]$, we have $|f(t) - f(c)| \leq \varepsilon$.

Proof (second part)

We will show that if f is continuous at any $c \in (a, b)$, then F is differentiable at c. If $h > 0$ is sufficiently small that $c + h < b$, then

$$F(c+h) - F(c) = \int_c^{c+h} f.$$

Let $\epsilon > 0$.
Since f is continuous at c, there is a $\delta > 0$ such that for all $t \in [c - \delta, c + \delta]$, we have $|f(t) - f(c)| \leq \varepsilon$. Therefore, for any $h \in (0, \delta)$,

$$|F(c+h) - F(c) - hf(c)| = \left| \int_c^{c+h} (f(t) - f(c)) dt \right| \leq \varepsilon h.$$

Proof (second part)

We will show that if f is continuous at any $c \in (a, b)$, then F is differentiable at c. If $h > 0$ is sufficiently small that $c + h < b$, then

$$F(c+h) - F(c) = \int_c^{c+h} f.$$

Let $\epsilon > 0$.
Since f is continuous at c, there is a $\delta > 0$ such that for all $t \in [c - \delta, c + \delta]$, we have $|f(t) - f(c)| \leq \varepsilon$. Therefore, for any $h \in (0, \delta)$,

$$|F(c+h) - F(c) - hf(c)| = \left| \int_c^{c+h} (f(t) - f(c))dt \right| \leq \varepsilon h.$$

Divide through by h:

$$\left| \frac{F(c+h) - F(c)}{h} - f(c) \right| \leq \varepsilon.$$

Proof (second part)

We will show that if f is continuous at any $c \in (a, b)$, then F is differentiable at c. If $h > 0$ is sufficiently small that $c + h < b$, then

$$F(c+h) - F(c) = \int_c^{c+h} f.$$

Let $\epsilon > 0$.
Since f is continuous at c, there is a $\delta > 0$ such that for all $t \in [c - \delta, c + \delta]$, we have $|f(t) - f(c)| \leq \varepsilon$. Therefore, for any $h \in (0, \delta)$,

$$|F(c+h) - F(c) - hf(c)| = \left| \int_c^{c+h} (f(t) - f(c))dt \right| \leq \varepsilon h.$$

Divide through by h:

$$\left| \frac{F(c+h) - F(c)}{h} - f(c) \right| \leq \varepsilon.$$

Essentially the same argument works for $h < 0$.

Proof (second part)

We will show that if f is continuous at any $c \in (a, b)$, then F is differentiable at c. If $h > 0$ is sufficiently small that $c + h < b$, then

$$F(c+h) - F(c) = \int_c^{c+h} f.$$

Let $\epsilon > 0$.
Since f is continuous at c, there is a $\delta > 0$ such that for all $t \in [c - \delta, c + \delta]$, we have $|f(t) - f(c)| \leq \varepsilon$. Therefore, for any $h \in (0, \delta)$,

$$|F(c+h) - F(c) - hf(c)| = \left| \int_c^{c+h} (f(t) - f(c))dt \right| \leq \varepsilon h.$$

Divide through by h:

$$\left| \frac{F(c+h) - F(c)}{h} - f(c) \right| \leq \varepsilon.$$

Essentially the same argument works for $h < 0$. Hence, F is differentiable at c with derivative $f(c)$.

Chapter 4B: The second fundamental theorem of calculus

The second fundamental theorem of calculus

Here, we differentiate, then integrate.

The second fundamental theorem of calculus

Here, we differentiate, then integrate.

Example. Let $F : [-1, 1] \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

The second fundamental theorem of calculus

Here, we differentiate, then integrate.

Example. Let $F : [-1, 1] \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Then F is differentable everywhere, with $f = F'$ given by

$$f(x) = \begin{cases} 2x \sin(1/x^2) - \frac{2}{x} \cos(1/x^2) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

The second fundamental theorem of calculus

Here, we differentiate, then integrate.

Example. Let $F : [-1,1] \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Then F is differentable everywhere, with $f = F'$ given by

$$f(x) = \begin{cases} 2x \sin(1/x^2) - \frac{2}{x} \cos(1/x^2) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

In particular, f is unbounded on any interval containing 0, and so it has no majorants and is not integrable according to our definition.

The Volterra function

This is a function $F : [0, 1] \to \mathbb{R}$ such that

The Volterra function

This is a function $F : [0, 1] \to \mathbb{R}$ such that

- F is differentiable,

The Volterra function

This is a function $F : [0, 1] \to \mathbb{R}$ such that

- F is differentiable,
- F' is bounded,

The Volterra function

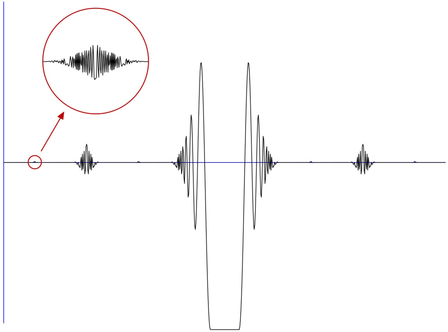
This is a function $F : [0, 1] \to \mathbb{R}$ such that

- F is differentiable,
- F' is bounded,
- but F' is not integrable.

The Volterra function

This is a function $F : [0, 1] \to \mathbb{R}$ such that

- F is differentiable,
- F' is bounded,
- but F' is not integrable.



The second fundamental theorem of calculus, applications

The second fundamental theorem of calculus, applications

Theorem 4.2. Suppose that $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b).

The second fundamental theorem of calculus, applications

Theorem 4.2. Suppose that $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Suppose furthermore that its derivative F' is integrable on (a, b).

The second fundamental theorem of calculus, applications

Theorem 4.2. Suppose that $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Suppose furthermore that its derivative F' is integrable on (a, b). Then

$$\int_a^b F' = F(b) - F(a).$$

The second fundamental theorem of calculus, applications

Theorem 4.2. Suppose that $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Suppose furthermore that its derivative F' is integrable on (a, b). Then

$$\int_a^b F' = F(b) - F(a).$$

Proof.

The second fundamental theorem of calculus, applications

Theorem 4.2. Suppose that $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Suppose furthermore that its derivative F' is integrable on (a, b). Then

$$\int_a^b F' = F(b) - F(a).$$

Proof. Let $\mathcal{P}$ be a partition, $a = x_0 < x_1 < \cdots < x_n = b$.

The second fundamental theorem of calculus, applications

Theorem 4.2. Suppose that $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Suppose furthermore that its derivative F' is integrable on (a, b). Then

$$\int_a^b F' = F(b) - F(a).$$

Proof. Let $\mathcal{P}$ be a partition, $a = x_0 < x_1 < \cdots < x_n = b$. We claim that some Riemann sum $\Sigma(F'; \mathcal{P}, \xi)$ is equal to $F(b) - F(a)$.

The second fundamental theorem of calculus, applications

Theorem 4.2. Suppose that $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Suppose furthermore that its derivative F' is integrable on (a, b). Then

$$\int_a^b F' = F(b) - F(a).$$

Proof. Let $\mathcal{P}$ be a partition, $a = x_0 < x_1 < \cdots < x_n = b$. We claim that some Riemann sum $\Sigma(F'; \mathcal{P}, \xi)$ is equal to $F(b) - F(a)$. By Proposition 3.2 (the harder direction of the equivalence between integrability and limits of Riemann sums), the second fundamental theorem follows immediately from this.

The second fundamental theorem of calculus, applications

Theorem 4.2. Suppose that $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Suppose furthermore that its derivative F' is integrable on (a, b). Then

$$\int_a^b F' = F(b) - F(a).$$

Proof. Let $\mathcal{P}$ be a partition, $a = x_0 < x_1 < \cdots < x_n = b$.
We claim that some Riemann sum $\Sigma(F'; \mathcal{P}, \xi)$ is equal to $F(b) - F(a)$. By Proposition 3.2 (the harder direction of the equivalence between integrability and limits of Riemann sums), the second fundamental theorem follows immediately from this.
By the mean value theorem, we may choose $\xi_i \in (x_{i-1}, x_i)$ so that $F'(\xi_i)(x_i - x_{i-1}) = F(x_i) - F(x_{i-1})$.

The second fundamental theorem of calculus, applications

Theorem 4.2. Suppose that $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Suppose furthermore that its derivative F' is integrable on (a, b). Then

$$\int_a^b F' = F(b) - F(a).$$

Proof. Let $\mathcal{P}$ be a partition, $a = x_0 < x_1 < \cdots < x_n = b$.
We claim that some Riemann sum $\Sigma(F'; \mathcal{P}, \xi)$ is equal to $F(b) - F(a)$. By Proposition 3.2 (the harder direction of the equivalence between integrability and limits of Riemann sums), the second fundamental theorem follows immediately from this.
By the mean value theorem, we may choose $\xi_i \in (x_{i-1}, x_i)$ so that $F'(\xi_i)(x_i - x_{i-1}) = F(x_i) - F(x_{i-1})$.
Summing from $i = 1$ to n gives

$$\Sigma(F'; \mathcal{P}, \xi) = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = F(b) - F(a).$$

Integration by parts

Integration by parts

Proposition 4.5. Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous functions, differentiable on (a, b).

Integration by parts

Proposition 4.5. Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous functions, differentiable on (a, b). Suppose that the derivatives f', g' are integrable on (a, b).

Integration by parts

Proposition 4.5. Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous functions, differentiable on (a, b). Suppose that the derivatives f', g' are integrable on (a, b). Then fg' and $f'g$ are integrable on (a, b), and

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g.$$

Integration by parts

Proposition 4.5. Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous functions, differentiable on (a, b). Suppose that the derivatives f', g' are integrable on (a, b). Then fg' and $f'g$ are integrable on (a, b), and

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g.$$

Proof. We use the second form of the fundamental theorem of calculus, applied to the function $F = fg$.

Integration by parts

Proposition 4.5. Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous functions, differentiable on (a, b). Suppose that the derivatives f', g' are integrable on (a, b). Then fg' and $f'g$ are integrable on (a, b), and

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g.$$

Proof. We use the second form of the fundamental theorem of calculus, applied to the function $F = fg$.
We know that F is differentiable and $F' = f'g + fg'$.

Integration by parts

Proposition 4.5. Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous functions, differentiable on (a, b). Suppose that the derivatives f', g' are integrable on (a, b). Then fg' and $f'g$ are integrable on (a, b), and

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g.$$

Proof. We use the second form of the fundamental theorem of calculus, applied to the function $F = fg$.
We know that F is differentiable and $F' = f'g + fg'$.
By Proposition 1.19 and the assumption that f', g' are integrable, F' is integrable on (a, b).

Integration by parts

Proposition 4.5. Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous functions, differentiable on (a, b). Suppose that the derivatives f', g' are integrable on (a, b). Then fg' and $f'g$ are integrable on (a, b), and

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g.$$

Proof. We use the second form of the fundamental theorem of calculus, applied to the function $F = fg$.
We know that F is differentiable and $F' = f'g + fg'$.
By Proposition 1.19 and the assumption that f', g' are integrable, F' is integrable on (a, b).
Applying the fundamental theorem gives

$$\int_a^b F' = F(b) - F(a).$$

Substitution

Substitution

Proposition 4.6.

Substitution

Proposition 4.6. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $\phi : [c, d] \to [a, b]$ is continuous on $[c, d]$, has $\phi(c) = a$ and $\phi(d) = b$, and maps (c, d) to (a, b).

Substitution

Proposition 4.6. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $\phi : [c, d] \to [a, b]$ is continuous on $[c, d]$, has $\phi(c) = a$ and $\phi(d) = b$, and maps (c, d) to (a, b). Suppose moreover that ϕ is differentiable on (c, d)

Substitution

Proposition 4.6. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $\phi : [c, d] \to [a, b]$ is continuous on $[c, d]$, has $\phi(c) = a$ and $\phi(d) = b$, and maps (c, d) to (a, b). Suppose moreover that ϕ is differentiable on (c, d) and that its derivative ϕ' is integrable on this interval.

Substitution

Proposition 4.6. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $\phi : [c, d] \to [a, b]$ is continuous on $[c, d]$, has $\phi(c) = a$ and $\phi(d) = b$, and maps (c, d) to (a, b). Suppose moreover that ϕ is differentiable on (c, d) and that its derivative ϕ' is integrable on this interval. Then

$$\int_a^b f = \int_c^d (f \circ \phi)\phi'.$$

Substitution

Proposition 4.6. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $\phi : [c, d] \to [a, b]$ is continuous on $[c, d]$, has $\phi(c) = a$ and $\phi(d) = b$, and maps (c, d) to (a, b). Suppose moreover that ϕ is differentiable on (c, d) and that its derivative ϕ' is integrable on this interval. Then

$$\int_a^b f = \int_c^d (f \circ \phi)\phi'.$$

Written out in full:

$$\int_a^b f(x)dx = \int_c^d f(\phi(t))\frac{d\phi}{dt}dt.$$

Substitution (proof).

$$\int_a^b f = \int_c^d (f \circ \phi)\phi'.$$

Substitution (proof).

$$\int_a^b f = \int_c^d (f \circ \phi)\phi'.$$

Note that $f \circ \phi$ is continuous and hence integrable on $[c, d]$.

Substitution (proof).

$$\int_a^b f = \int_c^d (f \circ \phi)\phi'.$$

Note that $f \circ \phi$ is continuous and hence integrable on $[c, d]$.

It therefore follows from Proposition 1.19 that $(f \circ \phi)\phi'$ is integrable on $[c, d]$, so the statement does at least make sense.

Substitution (proof).

$$\int_a^b f = \int_c^d (f \circ \phi)\phi'.$$

Note that $f \circ \phi$ is continuous and hence integrable on $[c, d]$.

It therefore follows from Proposition 1.19 that $(f \circ \phi)\phi'$ is integrable on $[c, d]$, so the statement does at least make sense.

Since f is continuous on $[a, b]$, it is integrable. The first fundamental theorem of calculus implies that its antiderivative

$$F(x) := \int_a^x f$$

is continuous on $[a, b]$, differentiable on (a, b) and that $F' = f$.

Substitution (proof).

$$\int_a^b f = \int_c^d (f \circ \phi)\phi'.$$

Substitution (proof).

$$\int_a^b f = \int_c^d (f \circ \phi)\phi'.$$

By the chain rule and the fact that $\phi((c, d)) \subset (a, b)$, $F \circ \phi$ is differentiable on (c, d), and

$$(F \circ \phi)' = (F' \circ \phi)\phi' = (f \circ \phi)\phi',$$

Substitution (proof).

$$\int_a^b f = \int_c^d (f \circ \phi)\phi'.$$

By the chain rule and the fact that $\phi((c,d)) \subset (a,b)$, $F \circ \phi$ is differentiable on (c,d), and

$$(F \circ \phi)' = (F' \circ \phi)\phi' = (f \circ \phi)\phi',$$

which we have checked is an integrable function.

Substitution (proof).

$$\int_a^b f = \int_c^d (f \circ \phi)\phi'.$$

By the chain rule and the fact that $\phi((c,d)) \subset (a,b)$, $F \circ \phi$ is differentiable on (c,d), and

$$(F \circ \phi)' = (F' \circ \phi)\phi' = (f \circ \phi)\phi',$$

which we have checked is an integrable function.

By the second form of the fundamental theorem,

$$\begin{aligned} \int_c^d (f \circ \phi)\phi' &= \int_c^d (F \circ \phi)' \\ &= (F \circ \phi)(d) - (F \circ \phi)(c) \\ &= F(b) - F(a) \\ &= F(b) = \int_a^b f. \end{aligned}$$

Chapter 5A: Interchanging limits and integration

Interchanging limits and integration

$$\lim_{n\to\infty} \int_a^b f_n \stackrel{?}{=} \int_a^b \lim_{n\to\infty} f_n$$

Interchanging limits and integration

$$\lim_{n\to\infty}\int_a^b f_n \stackrel{?}{=} \int_a^b \lim_{n\to\infty} f_n$$

Example. This is not necessarily true if f_n just converges pointwise.

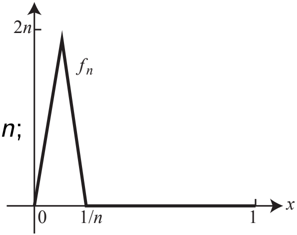
Interchanging limits and integration

$$\lim_{n\to\infty} \int_a^b f_n \stackrel{?}{=} \int_a^b \lim_{n\to\infty} f_n$$

Example. This is not necessarily true if f_n just converges pointwise.

Let $f_n : [0,1] \to \mathbb{R}$ be

$$f_n(x) = \begin{cases} 4n^2 x & \text{if } x \le 1/(2n); \\ 4n - 4n^2 x & \text{if } 1/(2n) < x < 1/n; \\ 0 & \text{otherwise} \end{cases}$$



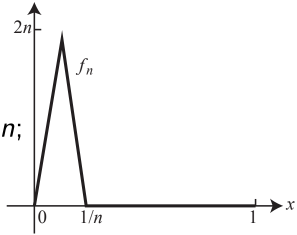
Interchanging limits and integration

$$\lim_{n\to\infty} \int_a^b f_n \stackrel{?}{=} \int_a^b \lim_{n\to\infty} f_n$$

Example. This is not necessarily true if f_n just converges pointwise.

Let $f_n : [0,1] \to \mathbb{R}$ be

$$f_n(x) = \begin{cases} 4n^2 x & \text{if } x \le 1/(2n); \\ 4n - 4n^2 x & \text{if } 1/(2n) < x < 1/n; \\ 0 & \text{otherwise} \end{cases}$$



Then f_n converges pointwise to the zero function.

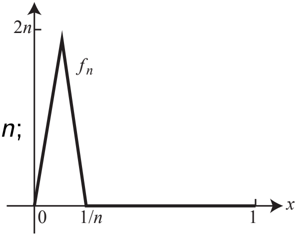
Interchanging limits and integration

$$\lim_{n\to\infty} \int_a^b f_n \stackrel{?}{=} \int_a^b \lim_{n\to\infty} f_n$$

Example. This is not necessarily true if f_n just converges pointwise.

Let $f_n : [0,1] \to \mathbb{R}$ be

$$f_n(x) = \begin{cases} 4n^2 x & \text{if } x \le 1/(2n); \\ 4n - 4n^2 x & \text{if } 1/(2n) < x < 1/n; \\ 0 & \text{otherwise} \end{cases}$$



Then f_n converges pointwise to the zero function.
But $\int_0^1 f_n = 1$.

Uniform convergence

Uniform convergence

Theorem 5.2.

Uniform convergence

Theorem 5.2. Suppose that $f_n : [a, b] \to \mathbb{R}$ are integrable, and that $f_n \to f$ uniformly on $[a, b]$.

Uniform convergence

Theorem 5.2. Suppose that $f_n : [a, b] \to \mathbb{R}$ are integrable, and that $f_n \to f$ uniformly on $[a, b]$. Then f is also integrable, and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f = \int_a^b \lim_{n\to\infty} f_n.$$

Uniform convergence

Theorem 5.2. Suppose that $f_n : [a, b] \to \mathbb{R}$ are integrable, and that $f_n \to f$ uniformly on $[a, b]$. Then f is also integrable, and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f = \int_a^b \lim_{n\to\infty} f_n.$$

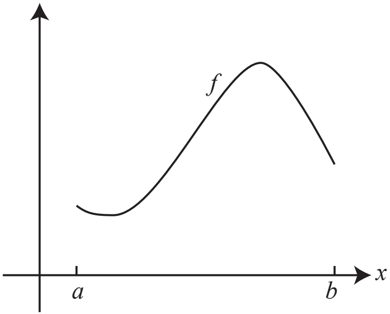
Proof.

Uniform convergence

Theorem 5.2. Suppose that $f_n : [a, b] \to \mathbb{R}$ are integrable, and that $f_n \to f$ uniformly on $[a, b]$. Then f is also integrable, and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f = \int_a^b \lim_{n\to\infty} f_n.$$

Proof. Let $\varepsilon > 0$.

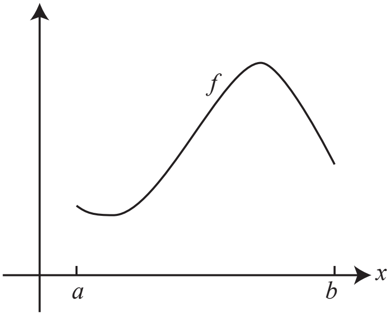


Uniform convergence

Theorem 5.2. Suppose that $f_n : [a, b] \to \mathbb{R}$ are integrable, and that $f_n \to f$ uniformly on $[a, b]$. Then f is also integrable, and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f = \int_a^b \lim_{n\to\infty} f_n.$$

Proof. Let $\varepsilon > 0$.
Since $f_n \to f$ uniformly, there is some choice of n such that we have $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in [a, b]$.

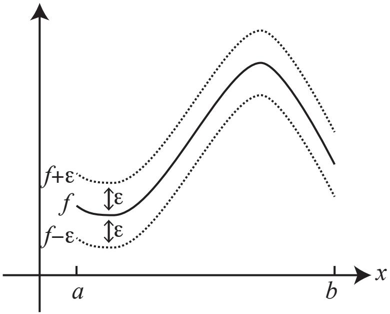


Uniform convergence

Theorem 5.2. Suppose that $f_n : [a, b] \to \mathbb{R}$ are integrable, and that $f_n \to f$ uniformly on $[a, b]$. Then f is also integrable, and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f = \int_a^b \lim_{n\to\infty} f_n.$$

Proof. Let $\varepsilon > 0$.
Since $f_n \to f$ uniformly, there is some choice of n such that we have $|f_n(x) - f(x)| \le \varepsilon$ for all $x \in [a, b]$.

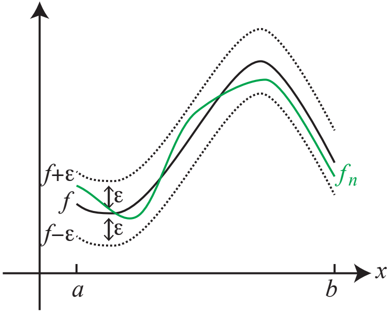


Uniform convergence

Theorem 5.2. Suppose that $f_n : [a, b] \to \mathbb{R}$ are integrable, and that $f_n \to f$ uniformly on $[a, b]$. Then f is also integrable, and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f = \int_a^b \lim_{n\to\infty} f_n.$$

Proof. Let $\varepsilon > 0$.
Since $f_n \to f$ uniformly, there is some choice of n such that we have $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in [a, b]$.



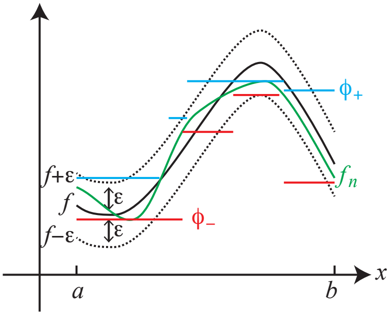
Uniform convergence

Theorem 5.2. Suppose that $f_n : [a, b] \to \mathbb{R}$ are integrable, and that $f_n \to f$ uniformly on $[a, b]$. Then f is also integrable, and

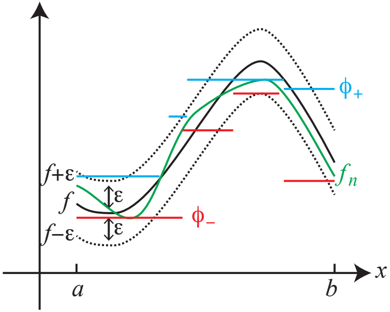
$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f = \int_a^b \lim_{n\to\infty} f_n.$$

Proof. Let $\varepsilon > 0$.
Since $f_n \to f$ uniformly, there is some choice of n such that we have $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in [a, b]$.

Now f_n is integrable, and so there is a majorant ϕ_+ and a minorant ϕ_- for f_n with $I(\phi_+) - I(\phi_-) \leq \varepsilon$.

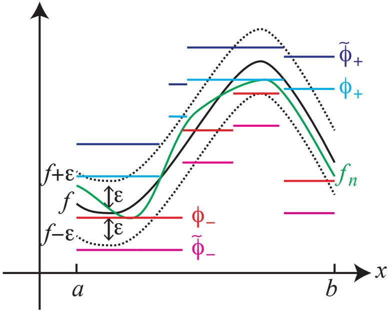


Proof (continued)



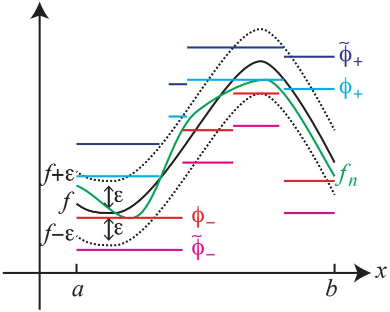
Proof (continued)

Define $\tilde{\phi}_+ := \phi_+ + \varepsilon$ and $\tilde{\phi}_- := \phi_- - \varepsilon$.



Proof (continued)

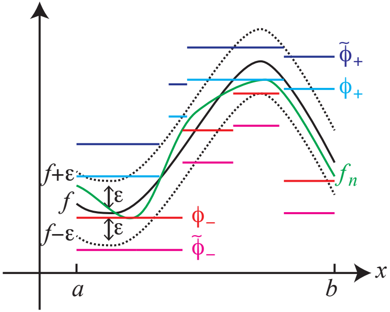
Define $\tilde{\phi}_+ := \phi_+ + \varepsilon$ and $\tilde{\phi}_- := \phi_- - \varepsilon$.
Then $\tilde{\phi}_-, \tilde{\phi}_+$ are minorant/majorant for f,



Proof (continued)

Define $\tilde{\phi}_+ := \phi_+ + \varepsilon$ and $\tilde{\phi}_- := \phi_- - \varepsilon$.
Then $\tilde{\phi}_-, \tilde{\phi}_+$ are minorant/majorant for f, and

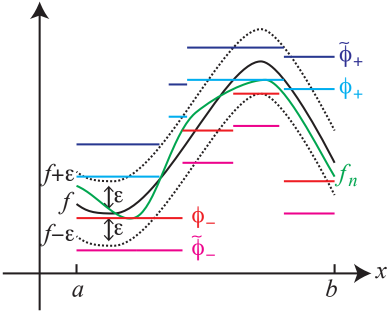
$$\begin{aligned} &I(\tilde{\phi}_+) - I(\tilde{\phi}_-) \\ &\leq 2\varepsilon(b-a) + I(\phi_+) - I(\phi_-) \\ &\leq 2\varepsilon(b-a) + \varepsilon. \end{aligned}$$



Proof (continued)

Define $\tilde{\phi}_+ := \phi_+ + \varepsilon$ and $\tilde{\phi}_- := \phi_- - \varepsilon$.
Then $\tilde{\phi}_-, \tilde{\phi}_+$ are minorant/majorant for f, and

$$I(\tilde{\phi}_+) - I(\tilde{\phi}_-) \le 2\varepsilon(b-a) + I(\phi_+) - I(\phi_-) \le 2\varepsilon(b-a) + \varepsilon.$$

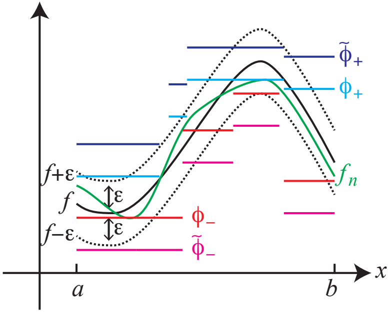


Since ε was arbitrary, this shows that f is integrable.

Proof (continued)

Define $\tilde{\phi}_+ := \phi_+ + \varepsilon$ and $\tilde{\phi}_- := \phi_- - \varepsilon$.
Then $\tilde{\phi}_-, \tilde{\phi}_+$ are minorant/majorant for f, and

$$I(\tilde{\phi}_+) - I(\tilde{\phi}_-) \\ \leq 2\varepsilon(b-a) + I(\phi_+) - I(\phi_-) \\ \leq 2\varepsilon(b-a) + \varepsilon.$$



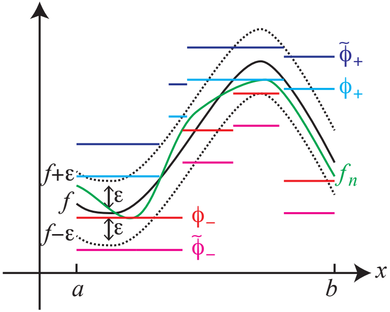
Since ε was arbitrary, this shows that f is integrable. Now

$$\left|\int_a^b f_n - \int_a^b f\right| \leq \int_a^b |f_n - f| \leq (b-a) \sup_{x\in[a,b]} |f_n(x) - f(x)|.$$

Proof (continued)

Define $\tilde{\phi}_+ := \phi_+ + \varepsilon$ and $\tilde{\phi}_- := \phi_- - \varepsilon$.
Then $\tilde{\phi}_-, \tilde{\phi}_+$ are minorant/majorant for f, and

$$\begin{aligned} &I(\tilde{\phi}_+) - I(\tilde{\phi}_-) \\ &\le 2\varepsilon(b-a) + I(\phi_+) - I(\phi_-) \\ &\le 2\varepsilon(b-a) + \varepsilon. \end{aligned}$$



Since ε was arbitrary, this shows that f is integrable. Now

$$|\int_a^b f_n - \int_a^b f| \le \int_a^b |f_n - f| \le (b-a) \sup_{x\in[a,b]} |f_n(x) - f(x)|.$$

Since $f_n \to f$ uniformly, it follows that

$$\lim_{n\to\infty} |\int_a^b f_n - \int_a^b f| = 0.$$

.

Integration and sums

Integration and sums

Corollary 5.3.

Integration and sums

Corollary 5.3. Suppose that $\phi_i : [a, b] \to \mathbb{R}$, $i = 1, 2, \ldots$ are integrable functions

Integration and sums

Corollary 5.3. Suppose that $\phi_i : [a, b] \to \mathbb{R}$, $i = 1, 2, \ldots$ are integrable functions and that $|\phi_i(x)| \le M_i$ for all $x \in [a, b]$, where $\sum_{i=1}^{\infty} M_i < \infty$.

Integration and sums

Corollary 5.3. Suppose that $\phi_i : [a, b] \to \mathbb{R}$, $i = 1, 2, \ldots$ are integrable functions and that $|\phi_i(x)| \leq M_i$ for all $x \in [a, b]$, where $\sum_{i=1}^{\infty} M_i < \infty$. Then the sum $\sum_i \phi_i$ is integrable and

$$\int_a^b \sum_i \phi_i = \sum_i \int_a^b \phi_i.$$

Integration and sums

Corollary 5.3. Suppose that $\phi_i : [a, b] \to \mathbb{R}$, $i = 1, 2, \ldots$ are integrable functions and that $|\phi_i(x)| \leq M_i$ for all $x \in [a, b]$, where $\sum_{i=1}^{\infty} M_i < \infty$. Then the sum $\sum_i \phi_i$ is integrable and

$$\int_a^b \sum_i \phi_i = \sum_i \int_a^b \phi_i.$$

Proof.

Integration and sums

Corollary 5.3. Suppose that $\phi_i : [a, b] \to \mathbb{R}$, $i = 1, 2, \ldots$ are integrable functions and that $|\phi_i(x)| \leq M_i$ for all $x \in [a, b]$, where $\sum_{i=1}^{\infty} M_i < \infty$. Then the sum $\sum_i \phi_i$ is integrable and

$$\int_a^b \sum_i \phi_i = \sum_i \int_a^b \phi_i.$$

Proof. This is immediate from the Weierstrass M-test and Theorem 5.2, applied with $f_n = \sum_{i=1}^{n} \phi_i$. □

Chapter 5B: Interchanging limits and differentiation

An example

Take $f_n(x) = \frac{1}{n}\sin(n^2 x)$.

An example

Take $f_n(x) = \frac{1}{n}\sin(n^2 x)$.

Then $f_n \to 0$ uniformly on $[0, 1]$.

An example

Take $f_n(x) = \frac{1}{n}\sin(n^2 x)$.

Then $f_n \to 0$ uniformly on $[0, 1]$.

We have $f_n'(x) = -n\cos(n^2 x)$.

An example

Take $f_n(x) = \frac{1}{n}\sin(n^2x)$.

Then $f_n \to 0$ uniformly on $[0, 1]$.

We have $f_n'(x) = -n\cos(n^2x)$.

If n is a multiple of 4 then $f_n'(\pi/4) = -n$.

An example

Take $f_n(x) = \frac{1}{n}\sin(n^2 x)$.

Then $f_n \to 0$ uniformly on $[0, 1]$.

We have $f_n'(x) = -n\cos(n^2 x)$.

If n is a multiple of 4 then $f_n'(\pi/4) = -n$.

So, $f_n'(\pi/4)$ does not converge as $n \to \infty$.

Uniform convergence of derivatives

Proposition 5.5. Suppose that $f_n : [a, b] \to \mathbb{R}$, $n = 1, 2, \ldots$ is a sequence of functions such that

Uniform convergence of derivatives

Proposition 5.5. Suppose that $f_n : [a, b] \to \mathbb{R}$, $n = 1, 2, \dots$ is a sequence of functions such that

- f_n is continuously differentiable on (a, b),

Uniform convergence of derivatives

Proposition 5.5. Suppose that $f_n : [a, b] \to \mathbb{R}$, $n = 1, 2, \ldots$ is a sequence of functions such that

- f_n is continuously differentiable on (a, b),
- f_n converges pointwise to some function f on $[a, b]$, and

Uniform convergence of derivatives

Proposition 5.5. Suppose that $f_n : [a, b] \to \mathbb{R}$, $n = 1, 2, \ldots$ is a sequence of functions such that

- f_n is continuously differentiable on (a, b),
- f_n converges pointwise to some function f on $[a, b]$, and
- f_n' converges uniformly to some bounded function g on (a, b).

Uniform convergence of derivatives

Proposition 5.5. Suppose that $f_n : [a, b] \to \mathbb{R}$, $n = 1, 2, \ldots$ is a sequence of functions such that

- f_n is continuously differentiable on (a, b),
- f_n converges pointwise to some function f on $[a, b]$, and
- f_n' converges uniformly to some bounded function g on (a, b).

Then f is differentiable and $f' = g$. In particular, $\lim_{n\to\infty} f_n' = (\lim_{n\to\infty} f_n)'$.

Proof

The f_n' are continuous and $f_n' \to g$ uniformly, and so g is continuous.

Proof

The f_n' are continuous and $f_n' \to g$ uniformly, and so g is continuous.

Since we are also assuming g is bounded, it follows from Theorem 2.2 that g is integrable.

Proof

The f_n' are continuous and $f_n' \to g$ uniformly, and so g is continuous.
Since we are also assuming g is bounded, it follows from Theorem 2.2 that g is integrable.
Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x g(t)dt$.

Proof

The f_n' are continuous and $f_n' \to g$ uniformly, and so g is continuous.
Since we are also assuming g is bounded, it follows from Theorem 2.2 that g is integrable.
Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x g(t)dt$. Then (by the first fundamental theorem) F is differentiable with $F' = g$.

Proof

The f_n' are continuous and $f_n' \to g$ uniformly, and so g is continuous.
Since we are also assuming g is bounded, it follows from Theorem 2.2 that g is integrable.
Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x g(t)dt$. Then (by the first fundamental theorem) F is differentiable with $F' = g$.
By the second fundamental theorem applied to f_n, we have

$$\int_a^x f_n'(t)dt = f_n(x) - f_n(a).$$

Proof

The f_n' are continuous and $f_n' \to g$ uniformly, and so g is continuous.

Since we are also assuming g is bounded, it follows from Theorem 2.2 that g is integrable.

Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x g(t)dt$. Then (by the first fundamental theorem) F is differentiable with $F' = g$.

By the second fundamental theorem applied to f_n, we have

$$\int_a^x f_n'(t)dt = f_n(x) - f_n(a).$$

Since $f_n \to f$ pointwise, $\lim_{n\to\infty} \int_a^x f_n'(t)dt = f(x) - f(a)$.

Proof

The f_n' are continuous and $f_n' \to g$ uniformly, and so g is continuous.
Since we are also assuming g is bounded, it follows from Theorem 2.2 that g is integrable.
Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x g(t)dt$. Then (by the first fundamental theorem) F is differentiable with $F' = g$.
By the second fundamental theorem applied to f_n, we have

$$\int_a^x f_n'(t)dt = f_n(x) - f_n(a).$$

Since $f_n \to f$ pointwise, $\lim_{n\to\infty} \int_a^x f_n'(t)dt = f(x) - f(a)$.
Since $f_n' \to g$ uniformly, $\lim_{n\to\infty} \int_a^x f_n'(t)dt = \int_a^x g(t)dt$ by Theorem 5.1.

Proof

The f_n' are continuous and $f_n' \to g$ uniformly, and so g is continuous.
Since we are also assuming g is bounded, it follows from Theorem 2.2 that g is integrable.
Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x g(t)dt$. Then (by the first fundamental theorem) F is differentiable with $F' = g$.
By the second fundamental theorem applied to f_n, we have

$$\int_a^x f_n'(t)dt = f_n(x) - f_n(a).$$

Since $f_n \to f$ pointwise, $\lim_{n\to\infty} \int_a^x f_n'(t)dt = f(x) - f(a)$.
Since $f_n' \to g$ uniformly, $\lim_{n\to\infty} \int_a^x f_n'(t)dt = \int_a^x g(t)dt$ by Theorem 5.1. Thus

$$F(x) = \int_a^x g(t)dt = f(x) - f(a).$$

Proof

The f_n' are continuous and $f_n' \to g$ uniformly, and so g is continuous.
Since we are also assuming g is bounded, it follows from Theorem 2.2 that g is integrable.
Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x g(t)dt$. Then (by the first fundamental theorem) F is differentiable with $F' = g$.
By the second fundamental theorem applied to f_n, we have

$$\int_a^x f_n'(t)dt = f_n(x) - f_n(a).$$

Since $f_n \to f$ pointwise, $\lim_{n\to\infty} \int_a^x f_n'(t)dt = f(x) - f(a)$.
Since $f_n' \to g$ uniformly, $\lim_{n\to\infty} \int_a^x f_n'(t)dt = \int_a^x g(t)dt$ by Theorem 5.1. Thus

$$F(x) = \int_a^x g(t)dt = f(x) - f(a).$$

It follows immediately that f is differentiable and that its derivative is the same as that of F, namely g. □

Term-by-term differentation of series

Corollary 5.6. Suppose we have a sequence of continuous functions $\phi_i : [a, b] \to \mathbb{R}$, continuously differentiable on (a, b), with $\sum_i \phi_i$ converging pointwise. Suppose that $|\phi_i'(x)| \leq M_i$ for all $x \in (a, b)$, where $\sum_i M_i < \infty$. Then $\sum \phi_i$ is differentiable and

$$(\sum_i \phi_i)' = \sum_i \phi_i'.$$

Term-by-term differentation of series

Corollary 5.6. Suppose we have a sequence of continuous functions $\phi_i : [a, b] \to \mathbb{R}$, continuously differentiable on (a, b), with $\sum_i \phi_i$ converging pointwise. Suppose that $|\phi_i'(x)| \leq M_i$ for all $x \in (a, b)$, where $\sum_i M_i < \infty$. Then $\sum \phi_i$ is differentiable and

$$(\sum_i \phi_i)' = \sum_i \phi_i'.$$

Proof. Apply Proposition 5.5 with $f_n := \sum_{i=1}^{n} \phi_i$.

Term-by-term differentation of series

Corollary 5.6. Suppose we have a sequence of continuous functions $\phi_i : [a, b] \to \mathbb{R}$, continuously differentiable on (a, b), with $\sum_i \phi_i$ converging pointwise. Suppose that $|\phi_i'(x)| \leq M_i$ for all $x \in (a, b)$, where $\sum_i M_i < \infty$. Then $\sum \phi_i$ is differentiable and

$$(\sum_i \phi_i)' = \sum_i \phi_i'.$$

Proof. Apply Proposition 5.5 with $f_n := \sum_{i=1}^n \phi_i$. By the Weierstrass M-test, $f_n' = \sum_{i=1}^n \phi_i'$ converges uniformly to some bounded function, which we may call g. □

Chapter 5C: Radius of convergence

Power series and radius of convergence

Definition. Now suppose we have a sequence $(a_i)_{i=0}^{\infty}$ of real numbers. Then the expression $\sum_{i=0}^{\infty} a_i x^i$ is called a (formal) power series.

Power series and radius of convergence

Definition. Now suppose we have a sequence $(a_i)_{i=0}^{\infty}$ of real numbers. Then the expression $\sum_{i=0}^{\infty} a_i x^i$ is called a (formal) power series.

Definition. Given a formal power series $\sum_i a_i x^i$, we define its radius of convergence R to be the supremum of all $|x|$ for which the sum $\sum_{i=0}^{\infty} |a_i x^i|$ converges. If this sum converges for all x, we write $R = \infty$.

Main theorem

Theorem 5.9. Suppose a formal power series $\sum_{i=0}^{\infty} a_i x^i$ has radius of convergence R.

Main theorem

Theorem 5.9. Suppose a formal power series $\sum_{i=0}^{\infty} a_i x^i$ has radius of convergence R. Then the series converges for $|x| < R$, giving a well-defined function $f(x) = \sum_{i=0}^{\infty} a_i x^i$.

Main theorem

Theorem 5.9. Suppose a formal power series $\sum_{i=0}^{\infty} a_i x^i$ has radius of convergence R. Then the series converges for $|x| < R$, giving a well-defined function $f(x) = \sum_{i=0}^{\infty} a_i x^i$. Moreover, f is differentiable on $(-R, R)$, and its derivative is given by term-by-term differentiation, that is to say $f'(x) = \sum_{i=1}^{\infty} i a_i x^{i-1}$.

Main theorem

Theorem 5.9. Suppose a formal power series $\sum_{i=0}^{\infty} a_i x^i$ has radius of convergence R. Then the series converges for $|x| < R$, giving a well-defined function $f(x) = \sum_{i=0}^{\infty} a_i x^i$. Moreover, f is differentiable on $(-R, R)$, and its derivative is given by term-by-term differentiation, that is to say $f'(x) = \sum_{i=1}^{\infty} i a_i x^{i-1}$. Moreover, the radius of convergence for this power series for f' is at least R.

Geometric series

Lemma. Suppose that $0 \le \lambda < 1$. Then $\sum_{i=0}^{\infty} \lambda^i$ and $\sum_{i=1}^{\infty} i\lambda^{i-1}$ both converge.

Geometric series

Lemma. Suppose that $0 \leq \lambda < 1$. Then $\sum_{i=0}^{\infty} \lambda^i$ and $\sum_{i=1}^{\infty} i\lambda^{i-1}$ both converge.

Proof. By the geometric series formula we have

$$\sum_{i=0}^{n-1} \lambda^i = \frac{1-\lambda^n}{1-\lambda}.$$

Geometric series

Lemma. Suppose that $0 \leq \lambda < 1$. Then $\sum_{i=0}^{\infty} \lambda^i$ and $\sum_{i=1}^{\infty} i\lambda^{i-1}$ both converge.

Proof. By the geometric series formula we have

$$\sum_{i=0}^{n-1} \lambda^i = \frac{1-\lambda^n}{1-\lambda}.$$

Letting $n \to \infty$ gives $\sum_{i=0}^{\infty} \lambda^i = \frac{1}{1-\lambda}$.

Geometric series

Lemma. Suppose that $0 \leq \lambda < 1$. Then $\sum_{i=0}^{\infty} \lambda^i$ and $\sum_{i=1}^{\infty} i\lambda^{i-1}$ both converge.

Proof. By the geometric series formula we have

$$\sum_{i=0}^{n-1} \lambda^i = \frac{1-\lambda^n}{1-\lambda}.$$

Letting $n \to \infty$ gives $\sum_{i=0}^{\infty} \lambda^i = \frac{1}{1-\lambda}$.

For the second statement, we differentiate the geometric series formula. This gives

$$\sum_{i=1}^{n-1} i\lambda^{i-1} = \frac{1+(n-1)\lambda^n - n\lambda^{n-1}}{(1-\lambda)^2},$$

Geometric series

Lemma. Suppose that $0 \le \lambda < 1$. Then $\sum_{i=0}^{\infty} \lambda^i$ and $\sum_{i=1}^{\infty} i\lambda^{i-1}$ both converge.

Proof. By the geometric series formula we have

$$\sum_{i=0}^{n-1} \lambda^i = \frac{1-\lambda^n}{1-\lambda}.$$

Letting $n \to \infty$ gives $\sum_{i=0}^{\infty} \lambda^i = \frac{1}{1-\lambda}$.

For the second statement, we differentiate the geometric series formula. This gives

$$\sum_{i=1}^{n-1} i\lambda^{i-1} = \frac{1+(n-1)\lambda^n - n\lambda^{n-1}}{(1-\lambda)^2},$$

which tends to $\frac{1}{(1-\lambda)^2}$ as $n \to \infty$. $\square$

Main theorem

Theorem 5.9. Suppose a formal power series $\sum_{i=0}^{\infty} a_i x^i$ has radius of convergence R.

Main theorem

Theorem 5.9. Suppose a formal power series $\sum_{i=0}^{\infty} a_i x^i$ has radius of convergence R. Then the series converges for $|x| < R$, giving a well-defined function $f(x) = \sum_{i=0}^{\infty} a_i x^i$.

Main theorem

Theorem 5.9. Suppose a formal power series $\sum_{i=0}^{\infty} a_i x^i$ has radius of convergence R. Then the series converges for $|x| < R$, giving a well-defined function $f(x) = \sum_{i=0}^{\infty} a_i x^i$. Moreover, f is differentiable on $(-R, R)$, and its derivative is given by term-by-term differentiation, that is to say $f'(x) = \sum_{i=1}^{\infty} i a_i x^{i-1}$.

Main theorem

Theorem 5.9. Suppose a formal power series $\sum_{i=0}^{\infty} a_i x^i$ has radius of convergence R. Then the series converges for $|x| < R$, giving a well-defined function $f(x) = \sum_{i=0}^{\infty} a_i x^i$. Moreover, f is differentiable on $(-R, R)$, and its derivative is given by term-by-term differentiation, that is to say $f'(x) = \sum_{i=1}^{\infty} i a_i x^{i-1}$. Moreover, the radius of convergence for this power series for f' is at least R.

Proof

If $R = 0$, there is nothing to prove. Suppose that $R > 0$. Let R_1 satisfy $0 < R_1 < R$.

Proof

If $R = 0$, there is nothing to prove. Suppose that $R > 0$. Let R_1 satisfy $0 < R_1 < R$. We will apply Corollary 5.6 with $\phi_i(x) = a_i x^i$ and $[a, b] = [-R_1, R_1]$.

Proof

If $R = 0$, there is nothing to prove. Suppose that $R > 0$. Let R_1 satisfy $0 < R_1 < R$. We will apply Corollary 5.6 with $\phi_i(x) = a_i x^i$ and $[a, b] = [-R_1, R_1]$.

Conditions of Corollary 5.6:

1. ϕ_i continuous of $[a, b]$ and continuously differentiable on (a, b);
2. $\sum_i \phi_i$ converging pointwise;
3. $|\phi_i'(x)| \leq M_i$ for all $x \in (a, b)$, where $\sum_i M_i < \infty$.

Proof

If $R = 0$, there is nothing to prove. Suppose that $R > 0$. Let R_1 satisfy $0 < R_1 < R$. We will apply Corollary 5.6 with $\phi_i(x) = a_i x^i$ and $[a, b] = [-R_1, R_1]$.

Conditions of Corollary 5.6:

1. ϕ_i continuous of $[a, b]$ and continuously differentiable on (a, b);
2. $\sum_i \phi_i$ converging pointwise;
3. $|\phi_i'(x)| \leq M_i$ for all $x \in (a, b)$, where $\sum_i M_i < \infty$.

(1) is immediate.

Proof

If $R = 0$, there is nothing to prove. Suppose that $R > 0$. Let R_1 satisfy $0 < R_1 < R$. We will apply Corollary 5.6 with $\phi_i(x) = a_i x^i$ and $[a, b] = [-R_1, R_1]$.

Conditions of Corollary 5.6:

1. ϕ_i continuous of $[a, b]$ and continuously differentiable on (a, b);
2. $\sum_i \phi_i$ converging pointwise;
3. $|\phi_i'(x)| \leq M_i$ for all $x \in (a, b)$, where $\sum_i M_i < \infty$.

(1) is immediate.

(2) Let R_0 satisfy $R_1 < R_0 < R$. By assumption, $\sum_i |a_i R_0^i|$ converges, and so $|a_i R_0^i| \leq K$ uniformly in i. Then if $x \in [a, b]$ we have

$$|\phi_i(x)| \leq K(\frac{R_1}{R_0})^i$$

and so by the geometric series lemma (first part), $\sum_i \phi_i(x)$ converges pointwise.

Proof (continued)

Proof (continued)

(3) If $x \in [a, b]$, then

$$|\phi_i'(x)| \le \frac{K}{R_0} i (\frac{R_1}{R_0})^{i-1}.$$

Apply the geometric series lemma (second part).

Proof (continued)

(3) If $x \in [a, b]$, then

$$|\phi_i'(x)| \leq \frac{K}{R_0} i (\frac{R_1}{R_0})^{i-1}.$$

Apply the geometric series lemma (second part).

It now follows from Corollary 5.6 that f is differentiable on $(-R_1, R_1)$, and that is derivative is given by term-by-term differentiation of the power series for f.

Proof (continued)

(3) If $x \in [a, b]$, then

$$|\phi_i'(x)| \leq \frac{K}{R_0} i (\frac{R_1}{R_0})^{i-1}.$$

Apply the geometric series lemma (second part).

It now follows from Corollary 5.6 that f is differentiable on $(-R_1, R_1)$, and that is derivative is given by term-by-term differentiation of the power series for f. Since $R_1 < R$ was arbitrary, we may assert the same on $(-R, R)$.

Proof (continued)

(3) If $x \in [a, b]$, then

$$|\phi_i'(x)| \le \frac{K}{R_0} i (\frac{R_1}{R_0})^{i-1}.$$

Apply the geometric series lemma (second part).

It now follows from Corollary 5.6 that f is differentiable on $(-R_1, R_1)$, and that is derivative is given by term-by-term differentiation of the power series for f. Since $R_1 < R$ was arbitrary, we may assert the same on $(-R, R)$.

By the geometric series lemma, the radius of convergence of the power series for f' is at least R_1.

Proof (continued)

(3) If $x \in [a, b]$, then

$$|\phi_i'(x)| \leq \frac{K}{R_0} i (\frac{R_1}{R_0})^{i-1}.$$

Apply the geometric series lemma (second part).

It now follows from Corollary 5.6 that f is differentiable on $(-R_1, R_1)$, and that is derivative is given by term-by-term differentiation of the power series for f. Since $R_1 < R$ was arbitrary, we may assert the same on $(-R, R)$.

By the geometric series lemma, the radius of convergence of the power series for f' is at least R_1. Since $R_1 < R$ was arbitrary, the radius of convergence of this power series is at least R. □

Chapter 6A: The exponential function

A simple differential equation

A simple differential equation

Lemma 6.1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f' = f$ identically and $f(0) = 0$.

A simple differential equation

Lemma 6.1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f' = f$ identically and $f(0) = 0$. Then f is identically zero.

A simple differential equation

Lemma 6.1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f' = f$ identically and $f(0) = 0$. Then f is identically zero.

Proof.

A simple differential equation

Lemma 6.1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f' = f$ identically and $f(0) = 0$. Then f is identically zero.

Proof. Since f is continuous, it attains its maximum value on $[0, \frac{1}{2}]$ at some point x.

A simple differential equation

Lemma 6.1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f' = f$ identically and $f(0) = 0$. Then f is identically zero.

Proof. Since f is continuous, it attains its maximum value on $[0, \frac{1}{2}]$ at some point x. Suppose that $x > 0$.

A simple differential equation

Lemma 6.1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f' = f$ identically and $f(0) = 0$. Then f is identically zero.

Proof. Since f is continuous, it attains its maximum value on $[0, \frac{1}{2}]$ at some point x. Suppose that $x > 0$. By the MVT,
$f(x) = f(x) - f(0) = xf'(\xi) = xf(\xi)$ for some point $\xi \in (0, x)$.

A simple differential equation

Lemma 6.1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f' = f$ identically and $f(0) = 0$. Then f is identically zero.

Proof. Since f is continuous, it attains its maximum value on $[0, \frac{1}{2}]$ at some point x. Suppose that $x > 0$. By the MVT, $f(x) = f(x) - f(0) = xf'(\xi) = xf(\xi)$ for some point $\xi \in (0, x)$. Therefore $f(x) \leq xf(x) \leq \frac{1}{2}f(x)$, which implies that $f(x) \leq 0$.

A simple differential equation

Lemma 6.1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f' = f$ identically and $f(0) = 0$. Then f is identically zero.

Proof. Since f is continuous, it attains its maximum value on $[0, \frac{1}{2}]$ at some point x. Suppose that $x > 0$. By the MVT,
$f(x) = f(x) - f(0) = xf'(\xi) = xf(\xi)$ for some point $\xi \in (0, x)$.
Therefore $f(x) \leq xf(x) \leq \frac{1}{2}f(x)$, which implies that $f(x) \leq 0$.
That is, $f \leq 0$ on $[0, \frac{1}{2}]$.

A simple differential equation

Lemma 6.1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f' = f$ identically and $f(0) = 0$. Then f is identically zero.

Proof. Since f is continuous, it attains its maximum value on $[0, \frac{1}{2}]$ at some point x. Suppose that $x > 0$. By the MVT,
$f(x) = f(x) - f(0) = xf'(\xi) = xf(\xi)$ for some point $\xi \in (0, x)$.
Therefore $f(x) \le xf(x) \le \frac{1}{2}f(x)$, which implies that $f(x) \le 0$.
That is, $f \le 0$ on $[0, \frac{1}{2}]$. Applying the same argument to $-f$ gives $f \ge 0$ on $[0, \frac{1}{2}]$, and so $f = 0$ identically on $[0, \frac{1}{2}]$.

A simple differential equation

Lemma 6.1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f' = f$ identically and $f(0) = 0$. Then f is identically zero.

Proof. Since f is continuous, it attains its maximum value on $[0, \frac{1}{2}]$ at some point x. Suppose that $x > 0$. By the MVT, $f(x) = f(x) - f(0) = xf'(\xi) = xf(\xi)$ for some point $\xi \in (0, x)$. Therefore $f(x) \leq xf(x) \leq \frac{1}{2}f(x)$, which implies that $f(x) \leq 0$. That is, $f \leq 0$ on $[0, \frac{1}{2}]$. Applying the same argument to $-f$ gives $f \geq 0$ on $[0, \frac{1}{2}]$, and so $f = 0$ identically on $[0, \frac{1}{2}]$.

We may now apply the same argument to $g(x) = f(x + \frac{1}{2})$, which satisfies $g' = g$ and $g(0) = 0$.

A simple differential equation

Lemma 6.1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f' = f$ identically and $f(0) = 0$. Then f is identically zero.

Proof. Since f is continuous, it attains its maximum value on $[0, \frac{1}{2}]$ at some point x. Suppose that $x > 0$. By the MVT, $f(x) = f(x) - f(0) = xf'(\xi) = xf(\xi)$ for some point $\xi \in (0, x)$. Therefore $f(x) \leq xf(x) \leq \frac{1}{2}f(x)$, which implies that $f(x) \leq 0$. That is, $f \leq 0$ on $[0, \frac{1}{2}]$. Applying the same argument to $-f$ gives $f \geq 0$ on $[0, \frac{1}{2}]$, and so $f = 0$ identically on $[0, \frac{1}{2}]$.

We may now apply the same argument to $g(x) = f(x + \frac{1}{2})$, which satisfies $g' = g$ and $g(0) = 0$. We conclude that g is identically zero on $[0, \frac{1}{2}]$, and hence that f is identically zero on $[\frac{1}{2}, 1]$ and hence on $[0, 1]$.

A simple differential equation

Lemma 6.1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f' = f$ identically and $f(0) = 0$. Then f is identically zero.

Proof. Since f is continuous, it attains its maximum value on $[0, \frac{1}{2}]$ at some point x. Suppose that $x > 0$. By the MVT, $f(x) = f(x) - f(0) = xf'(\xi) = xf(\xi)$ for some point $\xi \in (0, x)$. Therefore $f(x) \leq xf(x) \leq \frac{1}{2}f(x)$, which implies that $f(x) \leq 0$. That is, $f \leq 0$ on $[0, \frac{1}{2}]$. Applying the same argument to $-f$ gives $f \geq 0$ on $[0, \frac{1}{2}]$, and so $f = 0$ identically on $[0, \frac{1}{2}]$.

We may now apply the same argument to $g(x) = f(x + \frac{1}{2})$, which satisfies $g' = g$ and $g(0) = 0$. We conclude that g is identically zero on $[0, \frac{1}{2}]$, and hence that f is identically zero on $[\frac{1}{2}, 1]$ and hence on $[0, 1]$. Continuing in this manner eventually shows that f is identically zero on the whole of $\mathbb{R}$. $\square$

Simple properties of the exponential function

Simple properties of the exponential function

Theorem 6.2. For $x \in \mathbb{R}$, define

$$e(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Simple properties of the exponential function

Theorem 6.2. For $x \in \mathbb{R}$, define

$$e(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Then

1. The series converges for all x, and $e(x)$ is a differentiable function satisfying $e' = e$.

Simple properties of the exponential function

Theorem 6.2. For $x \in \mathbb{R}$, define

$$e(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Then

1. The series converges for all x, and $e(x)$ is a differentiable function satisfying $e' = e$.
2. We have $e(x) > 0$ for all $x \in \mathbb{R}$.

Simple properties of the exponential function

Theorem 6.2. For $x \in \mathbb{R}$, define

$$e(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Then

1. The series converges for all x, and $e(x)$ is a differentiable function satisfying $e' = e$.
2. We have $e(x) > 0$ for all $x \in \mathbb{R}$.
3. We have $e(x + y) = e(x)e(y)$ for all $x, y \in \mathbb{R}$.

Proof of 1

$$e(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

The series converges for all x, and $e(x)$ is a differentiable function satisfying $e' = e$.

Proof of 1

$$e(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

The series converges for all x, and $e(x)$ is a differentiable function satisfying $e' = e$.

Term-by-term differentiation gives the same series back again.

Proof of 1

$$e(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

The series converges for all x, and $e(x)$ is a differentiable function satisfying $e' = e$.

Term-by-term differentiation gives the same series back again. So by Theorem 5.9, it is enough to show that the radius of convergence is infinite ie that $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges for all x.

Proof of 1

$$e(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

The series converges for all x, and $e(x)$ is a differentiable function satisfying $e' = e$.

Term-by-term differentiation gives the same series back again. So by Theorem 5.9, it is enough to show that the radius of convergence is infinite ie that $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges for all x.
This is a simple consequence of the ratio test (limit form):

$$\frac{x^{k+1}}{(k+1)!} \Big/ \frac{x^k}{k!} = \frac{x}{k+1} \to 0 \quad \text{as } k \to \infty.$$

Proof of 2

We have $e(x) > 0$ for all $x \in \mathbb{R}$.

Proof of 2

We have $e(x) > 0$ for all $x \in \mathbb{R}$.

Suppose that $e(a) = 0$ for some $a \in \mathbb{R}$.

Proof of 2

We have $e(x) > 0$ for all $x \in \mathbb{R}$.

Suppose that $e(a) = 0$ for some $a \in \mathbb{R}$. Consider the function $f(x) = e(x + a)$; then $f(0) = 0$ and $f' = f$.

Proof of 2

We have $e(x) > 0$ for all $x \in \mathbb{R}$.

Suppose that $e(a) = 0$ for some $a \in \mathbb{R}$. Consider the function $f(x) = e(x + a)$; then $f(0) = 0$ and $f' = f$.
By Lemma 6.1, f is identically zero and hence so is e. But this is a contradiction, as e is clearly not identically zero (for example $e(0) = 1$).

Proof of 2

We have $e(x) > 0$ for all $x \in \mathbb{R}$.

Suppose that $e(a) = 0$ for some $a \in \mathbb{R}$. Consider the function $f(x) = e(x + a)$; then $f(0) = 0$ and $f' = f$.
By Lemma 6.1, f is identically zero and hence so is e. But this is a contradiction, as e is clearly not identically zero (for example $e(0) = 1$).
Thus e never vanishes. Since it is continuous, and positive somewhere, the intermediate value theorem implies that it is positive everywhere.

Proof of 3

We have $e(x+y) = e(x)e(y)$ for all $x, y \in \mathbb{R}$.

Proof of 3

We have $e(x+y) = e(x)e(y)$ for all $x, y \in \mathbb{R}$.

Consider the function $\tilde{e}(x) = \frac{e(x+y)}{e(y)}$.

Proof of 3

We have $e(x+y) = e(x)e(y)$ for all $x, y \in \mathbb{R}$.

Consider the function $\tilde{e}(x) = \frac{e(x+y)}{e(y)}$. As just established, $e(y) \neq 0$ and so for every fixed y this is a continuous function of x.

Proof of 3

We have $e(x+y) = e(x)e(y)$ for all $x, y \in \mathbb{R}$.

Consider the function $\tilde{e}(x) = \frac{e(x+y)}{e(y)}$. As just established, $e(y) \neq 0$ and so for every fixed y this is a continuous function of x. Moreover by the chain rule we have $\tilde{e}'(x) = \tilde{e}(x)$,

Proof of 3

We have $e(x+y) = e(x)e(y)$ for all $x, y \in \mathbb{R}$.

Consider the function $\tilde{e}(x) = \frac{e(x+y)}{e(y)}$. As just established, $e(y) \neq 0$ and so for every fixed y this is a continuous function of x. Moreover by the chain rule we have $\tilde{e}'(x) = \tilde{e}(x)$, and by direct substitution we have $\tilde{e}(0) = e(0) = 1$.

Proof of 3

We have $e(x+y) = e(x)e(y)$ for all $x, y \in \mathbb{R}$.

Consider the function $\tilde{e}(x) = \frac{e(x+y)}{e(y)}$. As just established, $e(y) \neq 0$ and so for every fixed y this is a continuous function of x. Moreover by the chain rule we have $\tilde{e}'(x) = \tilde{e}(x)$, and by direct substitution we have $\tilde{e}(0) = e(0) = 1$.

Therefore the function $f := e - \tilde{e}$ satisfies the hypotheses of Lemma 6.1.

Proof of 3

We have $e(x+y) = e(x)e(y)$ for all $x, y \in \mathbb{R}$.

Consider the function $\tilde{e}(x) = \frac{e(x+y)}{e(y)}$. As just established, $e(y) \neq 0$ and so for every fixed y this is a continuous function of x. Moreover by the chain rule we have $\tilde{e}'(x) = \tilde{e}(x)$, and by direct substitution we have $\tilde{e}(0) = e(0) = 1$.

Therefore the function $f := e - \tilde{e}$ satisfies the hypotheses of Lemma 6.1. It follows that $\tilde{e}(x) = e(x)$. $\square$

Chapter 6B: The logarithm function

The logarithm function

The logarithm function

Theorem 6.3. For $x > 0$, define

$$L(x) = \int_1^x \frac{dy}{y}.$$

The logarithm function

Theorem 6.3. For $x > 0$, define

$$L(x) = \int_1^x \frac{dy}{y}.$$

Then

1. L is differentiable with derivative $\frac{1}{x}$ at each $x > 0$;

The logarithm function

Theorem 6.3. For $x > 0$, define

$$L(x) = \int_1^x \frac{dy}{y}.$$

Then

1. L is differentiable with derivative $\frac{1}{x}$ at each $x > 0$;
2. $L(e^t) = t$ for all $t \in \mathbb{R}$.

The logarithm function

Theorem 6.3. For $x > 0$, define

$$L(x) = \int_1^x \frac{dy}{y}.$$

Then

1. L is differentiable with derivative $\frac{1}{x}$ at each $x > 0$;
2. $L(e^t) = t$ for all $t \in \mathbb{R}$.

(When $x < 1$, we define $\int_b^a f$ to be $-\int_a^b f$ when $a < b$.)

Proof of 1

$$L(x) = \int_1^x \frac{dy}{y}.$$

Then L is differentiable with derivative $\frac{1}{x}$ at each $x > 0$.

Proof of 1

$$L(x) = \int_1^x \frac{dy}{y}.$$

Then L is differentiable with derivative $\frac{1}{x}$ at each $x > 0$.

This is *almost* immediate from the first fundamental theorem of calculus except that we need to convince ourselves that it still applies when $x \leq 1$. This may be done as follows.

Proof of 1

$$L(x) = \int_1^x \frac{dy}{y}.$$

Then L is differentiable with derivative $\frac{1}{x}$ at each $x > 0$.

This is *almost* immediate from the first fundamental theorem of calculus except that we need to convince ourselves that it still applies when $x \leq 1$. This may be done as follows.
Let $c > 0$ and write

$$\int_1^x \frac{dy}{y} = \int_c^x \frac{dy}{y} - \int_c^1 \frac{dy}{y}.$$

It is easy to check that this holds for any $c > 0$.

Proof of 1

$$L(x) = \int_1^x \frac{dy}{y}.$$

Then L is differentiable with derivative $\frac{1}{x}$ at each $x > 0$.

This is *almost* immediate from the first fundamental theorem of calculus except that we need to convince ourselves that it still applies when $x \leq 1$. This may be done as follows.
Let $c > 0$ and write

$$\int_1^x \frac{dy}{y} = \int_c^x \frac{dy}{y} - \int_c^1 \frac{dy}{y}.$$

It is easy to check that this holds for any $c > 0$.
Then we may apply the fundamental theorem of calculus to get that $L'(x) = \frac{1}{x}$ for any $x > c$. Since c was arbitrary, the result follows.

Proof of 2

$L(e^t) = t$ for all $t \in \mathbb{R}$.

Proof of 2

$L(e^t) = t$ for all $t \in \mathbb{R}$.

We use the substitution rule, Proposition 4.6:
Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $\phi : [c, d] \to [a, b]$ is continuous on $[c, d]$, has $\phi(c) = a$ and $\phi(d) = b$, and maps (c, d) to (a, b). Suppose moreover that ϕ is differentiable on (c, d) and that its derivative ϕ' is integrable on this interval. Then

$$\int_a^b f = \int_c^d (f \circ \phi)\phi'.$$

Proof of 2

$L(e^t) = t$ for all $t \in \mathbb{R}$.

We use the substitution rule, Proposition 4.6:
Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $\phi : [c, d] \to [a, b]$ is continuous on $[c, d]$, has $\phi(c) = a$ and $\phi(d) = b$, and maps (c, d) to (a, b). Suppose moreover that ϕ is differentiable on (c, d) and that its derivative ϕ' is integrable on this interval. Then

$$\int_a^b f = \int_c^d (f \circ \phi)\phi'.$$

Set $f(y) = \frac{1}{y}$ and $\phi(t) = e^t$.

Proof of 2

$L(e^t) = t$ for all $t \in \mathbb{R}$.

We use the substitution rule, Proposition 4.6:
Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $\phi : [c, d] \to [a, b]$ is continuous on $[c, d]$, has $\phi(c) = a$ and $\phi(d) = b$, and maps (c, d) to (a, b). Suppose moreover that ϕ is differentiable on (c, d) and that its derivative ϕ' is integrable on this interval. Then

$$\int_a^b f = \int_c^d (f \circ \phi)\phi'.$$

Set $f(y) = \frac{1}{y}$ and $\phi(t) = e^t$.
Note that $f(\phi(t))\phi'(t) = 1$, since $\phi' = \phi$.

Proof of 2

$L(e^t) = t$ for all $t \in \mathbb{R}$.

We use the substitution rule, Proposition 4.6:
Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $\phi : [c, d] \to [a, b]$ is continuous on $[c, d]$, has $\phi(c) = a$ and $\phi(d) = b$, and maps (c, d) to (a, b). Suppose moreover that ϕ is differentiable on (c, d) and that its derivative ϕ' is integrable on this interval. Then

$$\int_a^b f = \int_c^d (f \circ \phi)\phi'.$$

Set $f(y) = \frac{1}{y}$ and $\phi(t) = e^t$.
Note that $f(\phi(t))\phi'(t) = 1$, since $\phi' = \phi$. We therefore have

$$\int_1^{e^x} \frac{dt}{t} = \int_0^x (f \circ \phi)\phi' = x.$$

Chapter 7: Improper integrals

Example 7.1.

Consider the function $f(x) = \log x$.

Example 7.1.

Consider the function $f(x) = \log x$. This is continuous on $(0, 1]$ but it is not integrable there since it is not bounded (it tends to $-\infty$ as $x \to 0$).

Example 7.1.

Consider the function $f(x) = \log x$. This is continuous on $(0, 1]$ but it is not integrable there since it is not bounded (it tends to $-\infty$ as $x \to 0$). However, it *is* integrable on any interval $[\varepsilon, 1]$, $\varepsilon > 0$.

Example 7.1.

Consider the function $f(x) = \log x$. This is continuous on $(0, 1]$ but it is not integrable there since it is not bounded (it tends to $-\infty$ as $x \to 0$). However, it *is* integrable on any interval $[\varepsilon, 1]$, $\varepsilon > 0$. Set $F(x) = x \log x - x$ then $F'(x) = \log x$,

Example 7.1.

Consider the function $f(x) = \log x$. This is continuous on $(0, 1]$ but it is not integrable there since it is not bounded (it tends to $-\infty$ as $x \to 0$). However, it *is* integrable on any interval $[\varepsilon, 1]$, $\varepsilon > 0$. Set $F(x) = x \log x - x$ then $F'(x) = \log x$, and so by the second fundamental theorem of calculus we have

$$\int_{\varepsilon}^{1} \log x \; dx = [x \log x - x]_{\varepsilon}^{1} = -1 - \varepsilon \log \varepsilon + \varepsilon.$$

We claim that $\lim_{\varepsilon \to 0^+} \varepsilon \log \varepsilon = 0$.

Example 7.1.

Consider the function $f(x) = \log x$. This is continuous on $(0, 1]$ but it is not integrable there since it is not bounded (it tends to $-\infty$ as $x \to 0$). However, it *is* integrable on any interval $[\varepsilon, 1]$, $\varepsilon > 0$. Set $F(x) = x \log x - x$ then $F'(x) = \log x$, and so by the second fundamental theorem of calculus we have

$$\int_{\varepsilon}^{1} \log x \; dx = [x \log x - x]_{\varepsilon}^{1} = -1 - \varepsilon \log \varepsilon + \varepsilon.$$

We claim that $\lim_{\varepsilon \to 0^+} \varepsilon \log \varepsilon = 0$. Once this is shown, it follows that

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \log x \; dx = -1.$$

Example 7.1.

Consider the function $f(x) = \log x$. This is continuous on $(0, 1]$ but it is not integrable there since it is not bounded (it tends to $-\infty$ as $x \to 0$). However, it *is* integrable on any interval $[\varepsilon, 1]$, $\varepsilon > 0$. Set $F(x) = x \log x - x$ then $F'(x) = \log x$, and so by the second fundamental theorem of calculus we have

$$\int_{\varepsilon}^{1} \log x \; dx = [x \log x - x]_{\varepsilon}^{1} = -1 - \varepsilon \log \varepsilon + \varepsilon.$$

We claim that $\lim_{\varepsilon \to 0^+} \varepsilon \log \varepsilon = 0$. Once this is shown, it follows that

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \log x \; dx = -1.$$

This will often be written as

$$\int_{0}^{1} \log x \; dx = -1,$$

but strictly speaking, as remarked above, this is not an integral as discussed in this course.

Proof of claim

$\lim_{\varepsilon \to 0^+} \varepsilon \log \varepsilon = 0.$

Proof of claim

$\lim_{\varepsilon \to 0^+} \varepsilon \log \varepsilon = 0$.

$$\log \varepsilon = - \int_{\varepsilon}^{1} \frac{dx}{x}$$

for $\varepsilon < 1$.

Proof of claim

$\lim_{\varepsilon \to 0^+} \varepsilon \log \varepsilon = 0$.

$$\log \varepsilon = -\int_{\varepsilon}^{1} \frac{dx}{x}$$

for $\varepsilon < 1$. We divide the range of integration into the ranges $[\varepsilon, \sqrt{\varepsilon}]$ and $[\sqrt{\varepsilon}, 1]$.

Proof of claim

$\lim_{\varepsilon \to 0^+} \varepsilon \log \varepsilon = 0$.

$$\log \varepsilon = -\int_{\varepsilon}^{1} \frac{dx}{x}$$

for $\varepsilon < 1$. We divide the range of integration into the ranges $[\varepsilon, \sqrt{\varepsilon}]$ and $[\sqrt{\varepsilon}, 1]$. On the first range we have $1/x \leq 1/\varepsilon$ and so

$$|\int_{\varepsilon}^{\sqrt{\varepsilon}} \frac{dx}{x}| \leq \frac{1}{\sqrt{\varepsilon}}.$$

Proof of claim

$\lim_{\varepsilon \to 0^+} \varepsilon \log \varepsilon = 0$.

$$\log \varepsilon = -\int_{\varepsilon}^{1} \frac{dx}{x}$$

for $\varepsilon < 1$. We divide the range of integration into the ranges $[\varepsilon, \sqrt{\varepsilon}]$ and $[\sqrt{\varepsilon}, 1]$. On the first range we have $1/x \leq 1/\varepsilon$ and so

$$|\int_{\varepsilon}^{\sqrt{\varepsilon}} \frac{dx}{x}| \leq \frac{1}{\sqrt{\varepsilon}}.$$

On the second range we have $1/x \leq 1/\sqrt{\varepsilon}$ and so

$$|\int_{\sqrt{\varepsilon}}^{1} \frac{dx}{x}| \leq \frac{1}{\sqrt{\varepsilon}}.$$

Proof of claim

$\lim_{\varepsilon \to 0^+} \varepsilon \log \varepsilon = 0$.

$$\log \varepsilon = -\int_\varepsilon^1 \frac{dx}{x}$$

for $\varepsilon < 1$. We divide the range of integration into the ranges $[\varepsilon, \sqrt{\varepsilon}]$ and $[\sqrt{\varepsilon}, 1]$. On the first range we have $1/x \leq 1/\varepsilon$ and so

$$|\int_\varepsilon^{\sqrt{\varepsilon}} \frac{dx}{x}| \leq \frac{1}{\sqrt{\varepsilon}}.$$

On the second range we have $1/x \leq 1/\sqrt{\varepsilon}$ and so

$$|\int_{\sqrt{\varepsilon}}^1 \frac{dx}{x}| \leq \frac{1}{\sqrt{\varepsilon}}.$$

It follows that

$$|\log \varepsilon| \leq \frac{2}{\sqrt{\varepsilon}},$$

from which the claim follows immediately. □

Example 7.2.

Consider the function $f(x) = 1/x^2$ for $x \in [1, \infty)$.

Example 7.2.

Consider the function $f(x) = 1/x^2$ for $x \in [1, \infty)$.

This is not permitted by the way we have defined the integral, which requires a bounded interval.

Example 7.2.

Consider the function $f(x) = 1/x^2$ for $x \in [1, \infty)$.

This is not permitted by the way we have defined the integral, which requires a bounded interval. However, on any bounded interval $[1, K]$ we have

$$\int_1^K \frac{1}{x^2} dx = \big[-\frac{1}{x} \big]_1^K = 1 - \frac{1}{K}.$$

Example 7.2.

Consider the function $f(x) = 1/x^2$ for $x \in [1, \infty)$.

This is not permitted by the way we have defined the integral, which requires a bounded interval. However, on any bounded interval $[1, K]$ we have

$$\int_1^K \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^K = 1 - \frac{1}{K}.$$

Therefore

$$\lim_{K \to \infty} \int_1^K \frac{1}{x^2} dx = 1.$$

Example 7.2.

Consider the function $f(x) = 1/x^2$ for $x \in [1, \infty)$.

This is not permitted by the way we have defined the integral, which requires a bounded interval. However, on any bounded interval $[1, K]$ we have

$$\int_1^K \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^K = 1 - \frac{1}{K}.$$

Therefore

$$\lim_{K \to \infty} \int_1^K \frac{1}{x^2} dx = 1.$$

This is invariably written

$$\int_1^\infty \frac{1}{x^2} dx = 1.$$

Example 7.3.

Define $f(x)$ to be $\log x$ if $0 < x \leq 1$, and $f(x) = \frac{1}{x^2}$ for $x \geq 1$.

Example 7.3.

Define $f(x)$ to be $\log x$ if $0 < x \leq 1$, and $f(x) = \frac{1}{x^2}$ for $x \geq 1$. Then it makes sense to write

$$\int_0^\infty f(x)dx = 0,$$

Example 7.3.

Define $f(x)$ to be $\log x$ if $0 < x \leq 1$, and $f(x) = \frac{1}{x^2}$ for $x \geq 1$. Then it makes sense to write

$$\int_0^\infty f(x)dx = 0,$$

by which we mean

$$\lim_{K\to\infty,\varepsilon\to 0} \int_\varepsilon^K f(x)dx = 0.$$

Example 7.3.

Define $f(x)$ to be $\log x$ if $0 < x \leq 1$, and $f(x) = \frac{1}{x^2}$ for $x \geq 1$. Then it makes sense to write

$$\int_0^\infty f(x)dx = 0,$$

by which we mean

$$\lim_{K \to \infty, \varepsilon \to 0} \int_\varepsilon^K f(x)dx = 0.$$

By this 'double limit', we formally mean the following:

Example 7.3.

Define $f(x)$ to be $\log x$ if $0 < x \leq 1$, and $f(x) = \frac{1}{x^2}$ for $x \geq 1$. Then it makes sense to write

$$\int_0^\infty f(x)dx = 0,$$

by which we mean

$$\lim_{K\to\infty, \varepsilon\to 0} \int_\varepsilon^K f(x)dx = 0.$$

By this 'double limit', we formally mean the following: For all $\varepsilon' > 0$, there are $N \in (0, \infty)$ and $\delta > 0$ such that for all $K > N$ and all $\varepsilon \in (0, \delta)$,

$$\left| \int_\varepsilon^K f(x)dx - 0 \right| < \varepsilon'.$$

Example 7.4.

Define $f(x)$ to be $1/x$ for $0 < |x| \leq 1$, and $f(0) = 0$.

Example 7.4.

Define $f(x)$ to be $1/x$ for $0 < |x| \leq 1$, and $f(0) = 0$. Then f is unbounded as $x \to 0$, and so we cannot define the integral $\int_{-1}^{1} f$.

Example 7.4.

Define $f(x)$ to be $1/x$ for $0 < |x| \leq 1$, and $f(0) = 0$. Then f is unbounded as $x \to 0$, and so we cannot define the integral $\int_{-1}^{1} f$. Excising the problematic region around 0, one can look at

$$I_{\varepsilon,\varepsilon'} := \int_{\varepsilon}^{1} f(x)dx + \int_{-1}^{-\varepsilon'} f(x)dx = \log \frac{\varepsilon'}{\varepsilon}.$$

Example 7.4.

Define $f(x)$ to be $1/x$ for $0 < |x| \leq 1$, and $f(0) = 0$. Then f is unbounded as $x \to 0$, and so we cannot define the integral $\int_{-1}^{1} f$. Excising the problematic region around 0, one can look at

$$I_{\varepsilon,\varepsilon'} := \int_{\varepsilon}^{1} f(x)dx + \int_{-1}^{-\varepsilon'} f(x)dx = \log \frac{\varepsilon'}{\varepsilon}.$$

This does not necessarily tend to a limit as $\varepsilon, \varepsilon' \to 0$ (for example, if $\varepsilon' = \varepsilon^2$ it does not tend to a limit).

Example 7.4.

Define $f(x)$ to be $1/x$ for $0 < |x| \leq 1$, and $f(0) = 0$. Then f is unbounded as $x \to 0$, and so we cannot define the integral $\int_{-1}^{1} f$. Excising the problematic region around 0, one can look at

$$I_{\varepsilon,\varepsilon'} := \int_{\varepsilon}^{1} f(x)dx + \int_{-1}^{-\varepsilon'} f(x)dx = \log \frac{\varepsilon'}{\varepsilon}.$$

This does not necessarily tend to a limit as $\varepsilon, \varepsilon' \to 0$ (for example, if $\varepsilon' = \varepsilon^2$ it does not tend to a limit).

The *Cauchy principal value* (PV) is the limit $\lim_{\varepsilon \to 0} I_{\varepsilon,\varepsilon} = 0$.

Example 7.4.

Define $f(x)$ to be $1/x$ for $0 < |x| \leq 1$, and $f(0) = 0$. Then f is unbounded as $x \to 0$, and so we cannot define the integral $\int_{-1}^{1} f$. Excising the problematic region around 0, one can look at

$$I_{\varepsilon,\varepsilon'} := \int_{\varepsilon}^{1} f(x)dx + \int_{-1}^{-\varepsilon'} f(x)dx = \log \frac{\varepsilon'}{\varepsilon}.$$

This does not necessarily tend to a limit as $\varepsilon, \varepsilon' \to 0$ (for example, if $\varepsilon' = \varepsilon^2$ it does not tend to a limit).

The *Cauchy principal value* (PV) is the limit $\lim_{\varepsilon \to 0} I_{\varepsilon,\varepsilon} = 0$.

It is *not* appropriate to write $\int_{-1}^{1} \frac{1}{x} dx = 0$; one could possibly write PV $\int_{-1}^{1} \frac{1}{x} dx = 0$.

Example 7.5.

Similarly to the last example, one should not write $\int_{-\infty}^{\infty} \sin x \; dx = 0$, even though $\lim_{K\to\infty} \int_{-K}^{K} \sin x \; dx = 0$ (because sin is an odd function). In this case, $\lim_{K,K'\to\infty} \int_{-K'}^{K} \sin x \; dx$ does not exist.

Example 7.5.

Similarly to the last example, one should not write $\int_{-\infty}^{\infty} \sin x \; dx = 0$, even though $\lim_{K\to\infty} \int_{-K}^{K} \sin x \; dx = 0$ (because sin is an odd function). In this case, $\lim_{K,K'\to\infty} \int_{-K'}^{K} \sin x \; dx$ does not exist.

One could maybe write

$$\mathrm{PV} \int_{-\infty}^{\infty} \sin x \; dx = 0,$$

but I would not be tempted to do so.