Analysis II: Continuity and Differentiability   Sheet 6   HT 2022

1. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable, with $f''(x) \geq 0$ for all $x \in \mathbb{R}$. Prove that
$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\left(f(x)+f(y)\right) \text{ for all } x, y \in \mathbb{R}.$$

2. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and let $a \in \mathbb{R}$. Suppose that $f''(a)$ exists. Prove that
$$\lim_{h\to 0} \frac{f(a+h)+f(a-h)-2f(a)}{h^2} = f''(a).$$

(b) Assume that $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathbb{R}$. Suppose that f satisfies the following convex inequality
$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\left(f(x)+f(y)\right) \text{ for all } x, y \in \mathbb{R}.$$
Using (a) to show that $f''(a) \geq 0$ for all $a \in \mathbb{R}$.

3. (a) Prove that $\cos x$ and $\sin x$, given by their power series:
$$\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$
and
$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$
are differentiable on $\mathbb{R}$. Hence prove that
$$\cos(x+y) = \cos x \cos y - \sin x \sin y \text{ for all } x, y \in \mathbb{R}.$$
Deduce from the addition formula for $\cos x$ the corresponding addition formula for $\sin x$, and prove that
$$|\cos x| \leq 1 \text{ and } |\sin x| \leq 1 \text{ for all } x \in \mathbb{R}.$$

(b) Prove that $\sin x \geq x - \frac{x^3}{3!}$ and that $\cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$ for $x \geq 0$, and deduce that $\cos 2 < 0$, $\sin x > 0$ for $x \in (0,2)$ and cos is strictly decreasing on $[0,2]$. Hence, by using IVT, prove that there exists a unique $p \in [0,2]$ such that $\cos p = 0$ and $\sin p = 1$.

Define $\pi = 2p$. Show that $\cos(x+2\pi) = \cos x$, and $\sin(x+2\pi) = \sin x$ for all $x \in \mathbb{R}$, and 2π is the smallest strictly positive period of sin (or the same cos) function.

(c) Let $q \in \mathbb{R}$. Prove that $\sin q = 0$ only if q is of the form $q = k\pi$ for some $k \in \mathbb{Z}$.

(d) Describe the continuous functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy $\sin(f(x)) = \sin x$ for all $x \in \mathbb{R}$.

[*Graphical presentation of answer acceptable.*]

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable everywhere.

(a) Prove that if $f'(x) = af(x)$ for all x, then $f(x) = A\exp(ax)$ for some constant A.

(b) Prove that if $f''(x) - 5f'(x) + 6f(x) = 0$ then $f(x) = A\exp(2x) + B\exp(3x)$ for some constants A, B.

[*Consider* $g(x) = f'(x) - 2f(x)$ *and* $h(x) = f'(x) - 3f(x)$.]

(c) Prove that if $f''(x) + 25f(x) = 0$ then $f(x) = A\cos(5x) + B\sin(5x)$ for some constants A, B.

[*Put* $A := f(0)$ and $B := f'(0)/5$; *look at* $g(x) := f(x) - A\cos(5x) - B\sin(5x)$; *now take a hint from the applied mathematicians and consider* $\frac{25}{2}g(x)^2 + \frac{1}{2}g'(x)^2$.]

(d) What can be said about the solutions of the differential equation $f''(x) - 4f'(x) + 4f(x) = 0$?