Summation of infinite series

Summation of infinite series

We will use residues to calculate sums of infinite series.

Summation of infinite series

We will use residues to calculate sums of infinite series.

We illustrate this by an example.

Summation of infinite series

We will use residues to calculate sums of infinite series.

We illustrate this by an example.

Consider $f(z) = \cot(\pi z)$.

Summation of infinite series

We will use residues to calculate sums of infinite series.

We illustrate this by an example.

Consider $f(z) = \cot(\pi z)$.

We will calculate the residues at its poles seeing it as quotient $\cos(\pi z)/\sin(\pi z)$.

Summation of infinite series

We will use residues to calculate sums of infinite series.

We illustrate this by an example.

Consider $f(z) = \cot(\pi z)$.

We will calculate the residues at its poles seeing it as quotient $\cos(\pi z)/\sin(\pi z)$.

Poles of f=zeros of $\sin(\pi z)$, so poles are the integers.

Summation of infinite series

We will use residues to calculate sums of infinite series.

We illustrate this by an example.

Consider $f(z) = \cot(\pi z)$.

We will calculate the residues at its poles seeing it as quotient $\cos(\pi z)/\sin(\pi z)$.

Poles of f=zeros of $\sin(\pi z)$, so poles are the integers.

(We calculate the zeros of $\sin z$ using $\sin(z) = \frac{e^{iz} - e^{-iz}}{2}$).

$e^{i(x+iy)} = e^{-i(x+iy)} \Rightarrow$ $y = 0$

$e^{2ix} = 1 \Rightarrow x = k\pi$

Summation of infinite series

We will use residues to calculate sums of infinite series.

We illustrate this by an example.

Consider $f(z) = \cot(\pi z)$.

We will calculate the residues at its poles seeing it as quotient $\cos(\pi z)/\sin(\pi z)$.

Poles of f=zeros of $\sin(\pi z)$, so poles are the integers.

(We calculate the zeros of $\sin z$ using $\sin(z) = \frac{e^{iz} - e^{-iz}}{2}$).

Since f is periodic with period 1, it suffices to calculate the principal part of f at $z = 0$.

$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + O(z^7)$ so

$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + O(z^7)$ so

$\sin(z) = z(1 - zh(z))$ where $h(z) = z/3! - z^3/5! + O(z^5)$ is holomorphic at $z = 0$.

$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + O(z^7)$ so

$\sin(z) = z(1 - zh(z))$ where $h(z) = z/3! - z^3/5! + O(z^5)$ is holomorphic at $z = 0$.

$$\frac{1}{\sin(z)} = \frac{1}{z}(1 - zh(z))^{-1} = \frac{1}{z}\Big(1 + \sum_{n \geq 1} z^n h(z)^n\Big) = \frac{1}{z} + h(z) + O(z^2).$$

$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + O(z^7)$ so

$\sin(z) = z(1 - zh(z))$ where $h(z) = z/3! - z^3/5! + O(z^5)$ is holomorphic at $z = 0$.

$$\frac{1}{\sin(z)} = \frac{1}{z}(1 - zh(z))^{-1} = \frac{1}{z}\Big(1 + \sum_{n\geq 1} z^n h(z)^n\Big) = \frac{1}{z} + h(z) + O(z^2).$$

$\cos(z) = 1 + O(z^2)$ so the principal part of $\cot(z)$ is $1/z$. It follows that $\cot(\pi z)$ has a simple pole at each $n \in \mathbb{Z}$ with residue $1/\pi$.

We can also calculate further terms of the Laurent series of $\cot(z)$: As $h(z)$ actually vanishes at $z = 0$, the terms $h(z)^n z^n$ vanish to order $2n$.

We can also calculate further terms of the Laurent series of $\cot(z)$: As $h(z)$ actually vanishes at $z = 0$, the terms $h(z)^n z^n$ vanish to order $2n$.

So, $$\frac{1}{z}\Big(1 + \sum_{n\geq 1} z^n h(z)^n\Big) = \frac{1}{z} + \frac{z}{3!} + O(z^3)$$

$$h(z) = z/3! - z^3/5! + O(z^5)$$

We can also calculate further terms of the Laurent series of $\cot(z)$: As $h(z)$ actually vanishes at $z = 0$, the terms $h(z)^n z^n$ vanish to order $2n$.

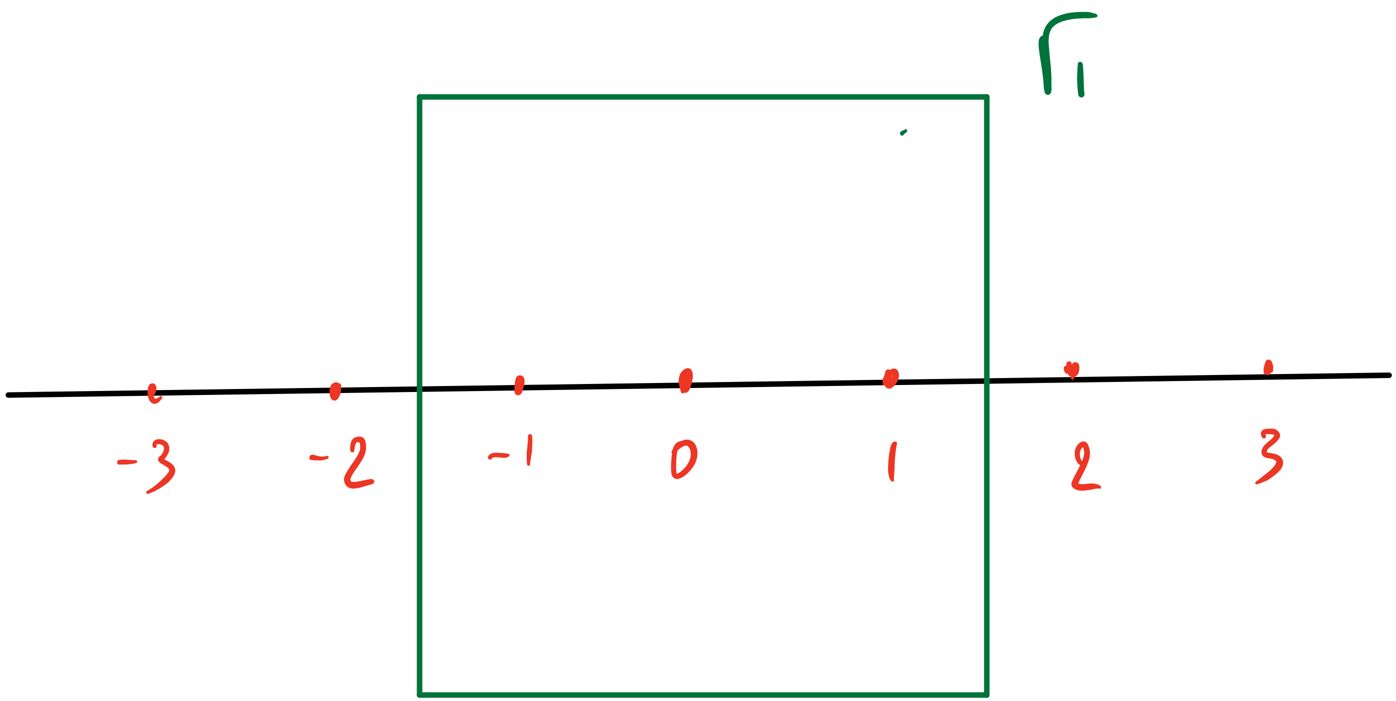
$$\text{So,} \quad \frac{1}{z}\Big(1 + \sum_{n \geq 1} z^n h(z)^n\Big) = \frac{1}{z} + \frac{z}{3!} + O(z^3)$$

Since $\cos(z) = 1 - z^2/2! + O(z^4)$, it follows that $\cot(z)$ has a Laurent series

$$\cot(z) = \left(1 - \frac{z^2}{2!} + O(z^4)\right) \cdot \Big(\frac{1}{z} + \frac{z}{3!} + O(z^3)\Big)$$
$$= \frac{1}{z} - \frac{z}{3} + O(z^3)$$

Lemma

Let $f(z) = \cot(\pi z)$ *and let* Γ_N *denote the* *square path* *with vertices* $(N + 1/2)(\pm 1 \pm i)$ *where* $N \in \mathbb{N}$. *There is a constant* C *independent of* N *such that* $|f(z)| \leq C$ *for all* $z \in \Gamma_N^*$.



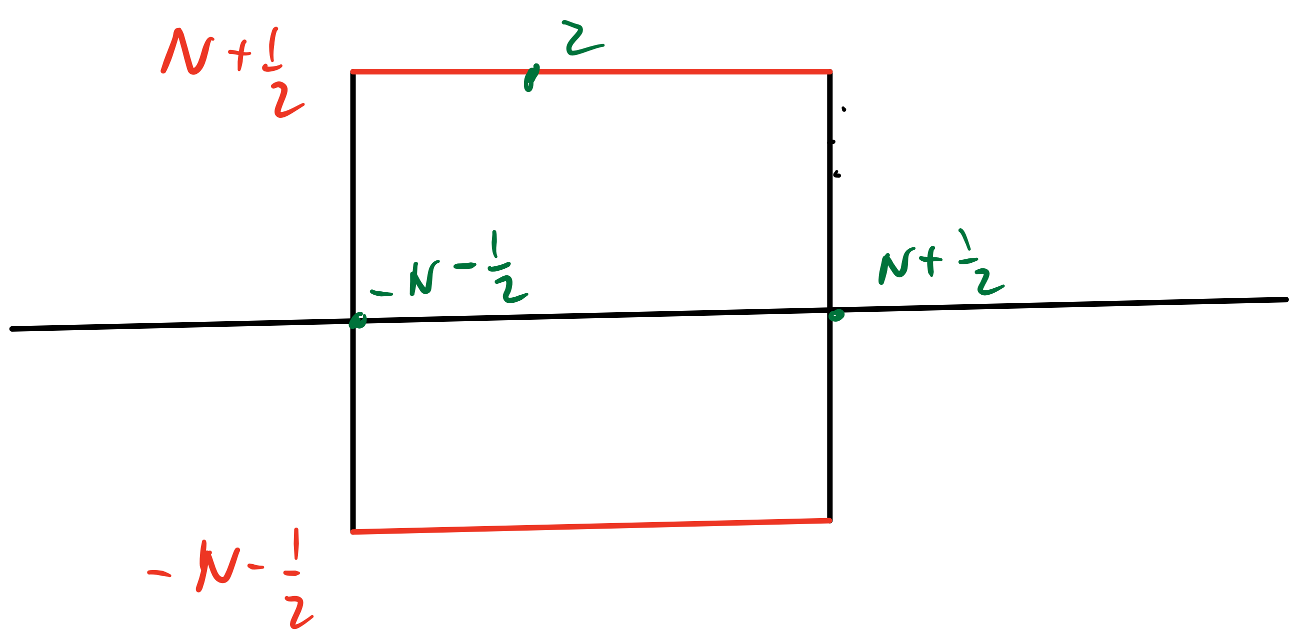
Lemma

Let $f(z) = \cot(\pi z)$ *and let* Γ_N *denote the* *square path* *with vertices* $(N + 1/2)(\pm 1 \pm i)$ *where* $N \in \mathbb{N}$*. There is a constant* C *independent of* N *such that* $|f(z)| \leq C$ *for all* $z \in \Gamma_N^*$.

Proof.

Note that $\cot(\pi z) = (e^{i\pi z} + e^{-i\pi z})/(e^{i\pi z} - e^{-i\pi z})$.

Horizontal sides: $z = x \pm (N + 1/2)i$ and
$-(N + 1/2) \leq x \leq (N + 1/2)$



Lemma

Let $f(z) = \cot(\pi z)$ *and let* Γ_N *denote the* *square path* *with vertices* $(N+1/2)(\pm 1 \pm i)$ *where* $N \in \mathbb{N}$*. There is a constant* C *independent of* N *such that* $|f(z)| \leq C$ *for all* $z \in \Gamma_N^*$*.*

Proof.

Note that $\cot(\pi z) = (e^{i\pi z} + e^{-i\pi z})/(e^{i\pi z} - e^{-i\pi z})$.

Horizontal sides: $z = x \pm (N+1/2)i$ and $-(N+1/2) \leq x \leq (N+1/2)$

$$|\cot(\pi z)| = \left| \frac{e^{i\pi(x\pm(N+1/2)i)} + e^{-i\pi(x\pm(N+1/2)i)}}{e^{i\pi(x\pm(N+1/2)i} - e^{-i\pi(x\pm(N+1/2)i)}} \right|$$

Lemma

Let $f(z) = \cot(\pi z)$ *and let* Γ_N *denote the* *square path* *with vertices* $(N + 1/2)(\pm 1 \pm i)$ *where* $N \in \mathbb{N}$*. There is a constant* C *independent of* N *such that* $|f(z)| \leq C$ *for all* $z \in \Gamma_N^*$*.*

Proof.

Note that $\cot(\pi z) = (e^{i\pi z} + e^{-i\pi z})/(e^{i\pi z} - e^{-i\pi z})$.

Horizontal sides: $z = x \pm (N + 1/2)i$ and
$-(N + 1/2) \leq x \leq (N + 1/2)$

$$|\cot(\pi z)| = \left| \frac{e^{i\pi(x \pm (N+1/2)i)} + e^{-i\pi(x \pm (N+1/2)i)}}{e^{i\pi(x \pm (N+1/2)i} - e^{-i\pi(x \pm (N+1/2)i)}} \right|$$

$$\leq \frac{e^{\pi(N+1/2)} + e^{-\pi(N+1/2)}}{e^{\pi(N+1/2)} - e^{-\pi(N+1/2)}}$$

Lemma

Let $f(z) = \cot(\pi z)$ *and let* Γ_N *denote the* *square path* *with vertices* $(N + 1/2)(\pm 1 \pm i)$ *where* $N \in \mathbb{N}$. *There is a constant* C *independent of* N *such that* $|f(z)| \leq C$ *for all* $z \in \Gamma_N^*$.

Proof.

Note that $\cot(\pi z) = (e^{i\pi z} + e^{-i\pi z})/(e^{i\pi z} - e^{-i\pi z})$.

Horizontal sides: $z = x \pm (N + 1/2)i$ and
$-(N + 1/2) \leq x \leq (N + 1/2)$

$$|\cot(\pi z)| = \left| \frac{e^{i\pi(x \pm (N+1/2)i)} + e^{-i\pi(x \pm (N+1/2)i)}}{e^{i\pi(x \pm (N+1/2)i} - e^{-i\pi(x \pm (N+1/2)i)}} \right|$$
$$\leq \frac{e^{\pi(N+1/2)} + e^{-\pi(N+1/2)}}{e^{\pi(N+1/2)} - e^{-\pi(N+1/2)}}$$

as $|x + e^{i\theta}y| \leq x + y$ for x, y positive reals and $|x - e^{i\theta}y| > x - y$.

so we have

$$|\cot(\pi z)| \leq \frac{e^{\pi(N+1/2)} + e^{-\pi(N+1/2)}}{e^{\pi(N+1/2)} - e^{-\pi(N+1/2)}}$$

so we have

$$
\begin{aligned}
|\cot(\pi z)| &\leq \frac{e^{\pi(N+1/2)} + e^{-\pi(N+1/2)}}{e^{\pi(N+1/2)} - e^{-\pi(N+1/2)}} \\
&= \frac{1 + e^{-2\pi(N+1/2)}}{1 - e^{-2\pi(N+1/2)}}
\end{aligned}
$$

so we have

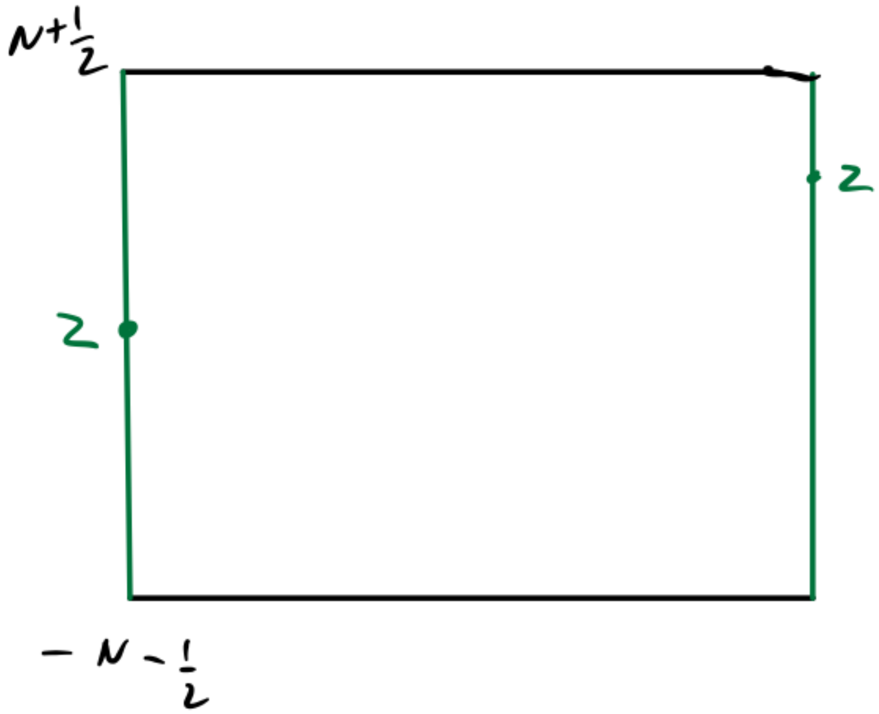
$$
\begin{aligned}
|\cot(\pi z)| &\leq \frac{e^{\pi(N+1/2)} + e^{-\pi(N+1/2)}}{e^{\pi(N+1/2)} - e^{-\pi(N+1/2)}} \\
&= \frac{1 + e^{-2\pi(N+1/2)}}{1 - e^{-2\pi(N+1/2)}} \\
&\leq \frac{2}{1 - e^{-3\pi}}
\end{aligned}
$$

so we have

$$
\begin{aligned}
|\cot(\pi z)| &\leq \frac{e^{\pi(N+1/2)} + e^{-\pi(N+1/2)}}{e^{\pi(N+1/2)} - e^{-\pi(N+1/2)}} \\
&= \frac{1 + e^{-2\pi(N+1/2)}}{1 - e^{-2\pi(N+1/2)}} \\
&\leq \frac{2}{1 - e^{-3\pi}}
\end{aligned}
$$

as e^{-x} is decreasing for $x > 0$.

Vertical sides: $z = \pm(N + 1/2) + iy$, where $-N - 1/2 \leq y \leq N + 1/2$.



Vertical sides: $z = \pm(N + 1/2) + iy$, where $-N - 1/2 \leq y \leq N + 1/2$.

$$|\cot(\pi z)| = \left| \frac{e^{i\pi(\pm(N+1/2)+iy)} + e^{-i\pi(\pm(N+1/2)+iy)}}{e^{i\pi(\pm(N+1/2)+iy)} - e^{-i\pi(\pm(N+1/2)+iy)}} \right|$$

Vertical sides: $z = \pm(N + 1/2) + iy$, where $-N - 1/2 \leq y \leq N + 1/2$.

$$|\cot(\pi z)| = \left| \frac{e^{i\pi(\pm(N+1/2)+iy)} + e^{-i\pi(\pm(N+1/2)+iy)}}{e^{i\pi(\pm(N+1/2)+iy)} - e^{-i\pi(\pm(N+1/2)+iy)}} \right|$$

$$= \left| \frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}} \right| \leq 1$$

Vertical sides: $z = \pm(N + 1/2) + iy$, where $-N - 1/2 \leq y \leq N + 1/2$.

$$|\cot(\pi z)| = \left| \frac{e^{i\pi(\pm(N+1/2)+iy)} + e^{-i\pi(\pm(N+1/2)+iy)}}{e^{i\pi(\pm(N+1/2)+iy)} - e^{-i\pi(\pm(N+1/2)+iy)}} \right|$$

$$= \left| \frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}} \right| \leq 1$$

since $e^{i\pi(\pm(N+1/2))} = \pm i$.

Vertical sides: $z = \pm(N + 1/2) + iy$, where $-N - 1/2 \leq y \leq N + 1/2$.

$$|\cot(\pi z)| = \left| \frac{e^{i\pi(\pm(N+1/2)+iy)} + e^{-i\pi(\pm(N+1/2)+iy)}}{e^{i\pi(\pm(N+1/2)+iy)} - e^{-i\pi(\pm(N+1/2)+iy)}} \right|$$

$$= \left| \frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}} \right| \leq 1$$

since $e^{i\pi(\pm(N+1/2))} = \pm i$.

so we can take $C = \dfrac{2}{1 - e^{-3\pi}}$. □

Example Let $g(z) = \cot(\pi z)/z^2$. By the calculation of Laurent series of $\cot(\pi z)$ at $z = 0$:

$$\frac{\cot(\pi z)}{z^2} = \frac{1}{\pi z^3} - \frac{\pi}{3z} + O(z)$$

Recall

$$\cot(z) = \frac{1}{z} - \frac{z}{3} + O(z^3)$$

Example Let $g(z) = \cot(\pi z)/z^2$. By the calculation of Laurent series of $\cot(\pi z)$ at $z = 0$:

$$\frac{\cot(\pi z)}{z^2} = \frac{1}{\pi z^3} - \frac{\pi}{3z} + O(z)$$

Since $\cot(\pi z) = \cot(\pi(z-n))$ at $z = n$ and $1/z$ is holomorphic near n we have:

$$\frac{\cot(\pi z)}{z^2} = \underbrace{(1/n^2 + O(z-n))}_{\frac{1}{z^2}} \cdot \underbrace{(\tfrac{1}{\pi(z-n)} + O(z-n))}_{\cot(\pi z)} = \frac{1}{\pi n^2(z-n)} + O(1)$$

$$\frac{1}{z^2} = \frac{1}{n^2} + O(z-n)$$

$$\cot(\pi z) = \cot(\pi(z-n)) = \frac{1}{\pi(z-n)} + O(z-n)$$

Example Let $g(z) = \cot(\pi z)/z^2$. By the calculation of Laurent series of $\cot(\pi z)$ at $z = 0$:

$$\frac{\cot(\pi z)}{z^2} = \frac{1}{\pi z^3} - \frac{\pi}{3z} + O(z)$$

Since $\cot(\pi z) = \cot(\pi(z-n))$ at $z = n$ and $1/z$ is holomorphic near n we have:

$$\frac{\cot(\pi z)}{z^2} = (1/n^2 + O(z-n)) \cdot \left(\frac{1}{\pi(z-n)} + O(z-n)\right) = \frac{1}{\pi n^2 (z-n)} + O(1)$$

So $g(z)$ has simple poles with residues $\frac{1}{\pi n^2}$ at each non-zero integer n and residue $-\pi/3$ at $z = 0$.

Example Let $g(z) = \cot(\pi z)/z^2$. By the calculation of Laurent series of $\cot(\pi z)$ at $z = 0$:

$$\frac{\cot(\pi z)}{z^2} = \frac{1}{\pi z^3} - \frac{\pi}{3z} + O(z)$$

Since $\cot(\pi z) = \cot(\pi(z-n))$ at $z = n$ and $1/z$ is holomorphic near n we have:

$$\frac{\cot(\pi z)}{z^2} = (1/n^2 + O(z-n)) \cdot (\frac{1}{\pi(z-n)} + O(z-n)) = \frac{1}{\pi n^2(z-n)} + O(1)$$

So $g(z)$ has simple poles with residues $\frac{1}{\pi n^2}$ at each non-zero integer n and residue $-\pi/3$ at $z = 0$.

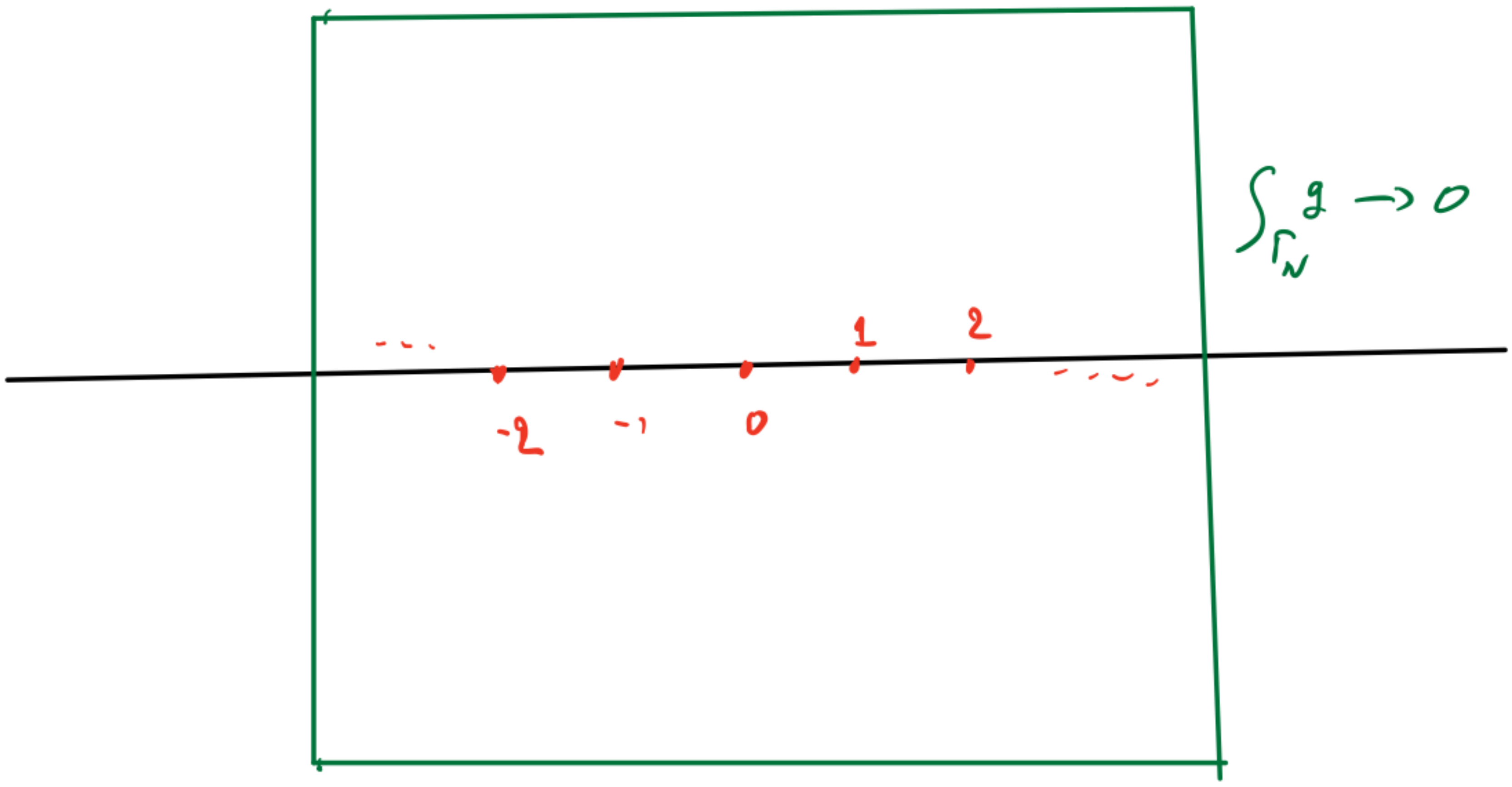
Consider now the integral of $g(z)$ around the paths Γ_N: We know $|g(z)| \leq C/|z|^2$ for $z \in \Gamma_N^*$, and for all $N \geq 1$. Thus by the estimation lemma

$$\left(\int_{\Gamma_N} g(z)dz\right) \leq C \cdot (4N+2)/(N+1/2)^2 \to 0,$$

as $N \to \infty$.

But by the residue theorem we know that

$$\frac{1}{2\pi i}\int_{\Gamma_N} g(z)dz = -\pi/3 + \sum_{\substack{n\neq 0,\\ -N\leq n\leq N}} \frac{1}{\pi n^2}.$$



But by the residue theorem we know that

$$\frac{1}{2\pi i}\int_{\Gamma_N} g(z)dz = -\pi/3 + \sum_{\substack{n\neq 0,\\ -N\leq n\leq N}} \frac{1}{\pi n^2}.$$

It therefore follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$$

But by the residue theorem we know that

$$\frac{1}{2\pi i}\int_{\Gamma_N} g(z)dz = -\pi/3 + \sum_{\substack{n\neq 0,\\ -N\leq n\leq N}} \frac{1}{\pi n^2}.$$

It therefore follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$$

Remark

Notice that the contours Γ_N and the function $\cot(\pi z)$ clearly allows us to sum other infinite series in a similar way – for example if we wished to calculate the sum of the infinite series$\sum_{n\geq 1} \frac{1}{n^2+1}$ then we would consider the integrals of $g(z) = \cot(\pi z)/(1+z^2)$ over the contours Γ_N.

Keyhole contours

Keyhole contours

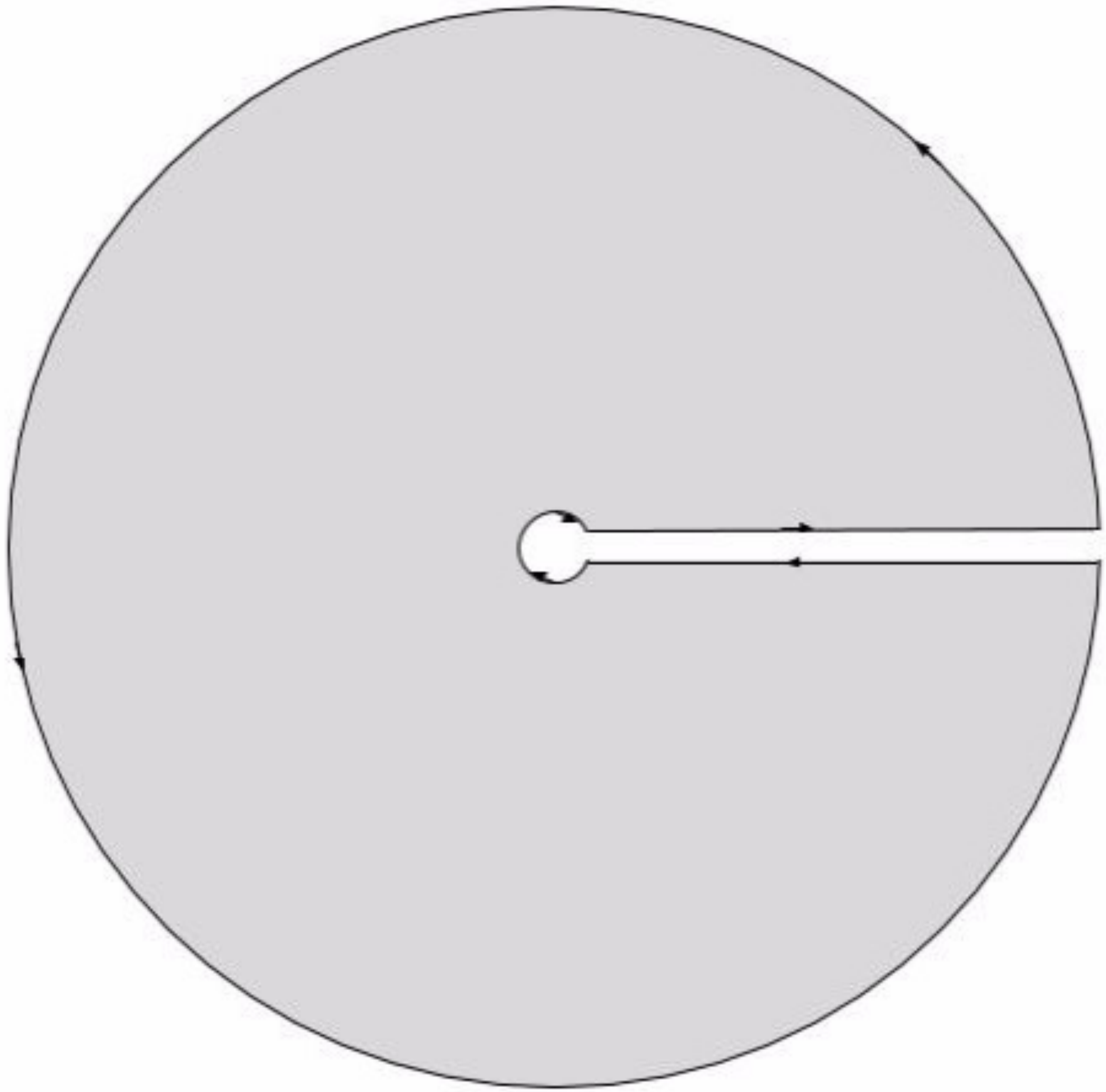


Figure: A keyhole contour.

To take advantage of the residue theorem to calculate integrals of real functions one needs to choose the appropriate contour.

To take advantage of the residue theorem to calculate integrals of real functions one needs to choose the appropriate contour.

The keyhole contour is useful when the integrand is multi-valued as a function on the complex plane. Formally:

To take advantage of the residue theorem to calculate integrals of real functions one needs to choose the appropriate contour.

The keyhole contour is useful when the integrand is multi-valued as a function on the complex plane. Formally:

Given $0 < \epsilon < R$ pick $\delta < \epsilon$ small. Consider two circles C_ϵ, C_R of radius ϵ, R centered at 0.

To take advantage of the residue theorem to calculate integrals of real functions one needs to choose the appropriate contour.

The keyhole contour is useful when the integrand is multi-valued as a function on the complex plane. Formally:

Given $0 < \epsilon < R$ pick $\delta < \epsilon$ small. Consider two circles C_ϵ, C_R of radius ϵ, R centered at 0.

Take two line segments $\eta_+(t) = t + i\delta$, $\eta_-(t) = (R - t) - i\delta$ where $t \in [a, b]$ such that $\eta_+(a), \eta_-(b) \in C_\epsilon$, $\eta_+(b), \eta_-(a) \in C_R$.

To take advantage of the residue theorem to calculate integrals of real functions one needs to choose the appropriate contour.

The keyhole contour is useful when the integrand is multi-valued as a function on the complex plane. Formally:

Given $0 < \epsilon < R$ pick $\delta < \epsilon$ small. Consider two circles C_ϵ, C_R of radius ϵ, R centered at 0.

Take two line segments $\eta_+(t) = t + i\delta$, $\eta_-(t) = (R - t) - i\delta$ where $t \in [a, b]$ such that $\eta_+(a), \eta_-(b) \in C_\epsilon$, $\eta_+(b), \eta_-(a) \in C_R$.

Let γ_R be the positively oriented path on the circle of radius R joining the endpoints of η_+ and η_- on that circle and similarly let γ_ϵ the path on the circle of radius ϵ which is negatively oriented and joins the endpoints of $\gamma_\pm$ on the circle of radius ϵ.

To take advantage of the residue theorem to calculate integrals of real functions one needs to choose the appropriate contour.

The keyhole contour is useful when the integrand is multi-valued as a function on the complex plane. Formally:

Given $0 < \epsilon < R$ pick $\delta < \epsilon$ small. Consider two circles C_ϵ, C_R of radius ϵ, R centered at 0.

Take two line segments $\eta_+(t) = t + i\delta$, $\eta_-(t) = (R - t) - i\delta$ where $t \in [a, b]$ such that $\eta_+(a), \eta_-(b) \in C_\epsilon$, $\eta_+(b), \eta_-(a) \in C_R$.

Let γ_R be the positively oriented path on the circle of radius R joining the endpoints of η_+ and η_- on that circle and similarly let γ_ϵ the path on the circle of radius ϵ which is negatively oriented and joins the endpoints of $\gamma_\pm$ on the circle of radius ϵ.

$\Gamma_{R,\epsilon} = \eta_+ \star \gamma_R \star \eta_- \star \gamma_\epsilon$ is the keyhole contour.

To take advantage of the residue theorem to calculate integrals of real functions one needs to choose the appropriate contour.

The keyhole contour is useful when the integrand is multi-valued as a function on the complex plane. Formally:

Given $0 < \epsilon < R$ pick $\delta < \epsilon$ small. Consider two circles C_ϵ, C_R of radius ϵ, R centered at 0.

Take two line segments $\eta_+(t) = t + i\delta$, $\eta_-(t) = (R - t) - i\delta$ where $t \in [a, b]$ such that $\eta_+(a), \eta_-(b) \in C_\epsilon$, $\eta_+(b), \eta_-(a) \in C_R$.

Let γ_R be the positively oriented path on the circle of radius R joining the endpoints of η_+ and η_- on that circle and similarly let γ_ϵ the path on the circle of radius ϵ which is negatively oriented and joins the endpoints of $\gamma_\pm$ on the circle of radius ϵ.

$\Gamma_{R,\epsilon} = \eta_+ \star \gamma_R \star \eta_- \star \gamma_\epsilon$ is the keyhole contour.

We let $\epsilon \to 0$ and $R \to \infty$.

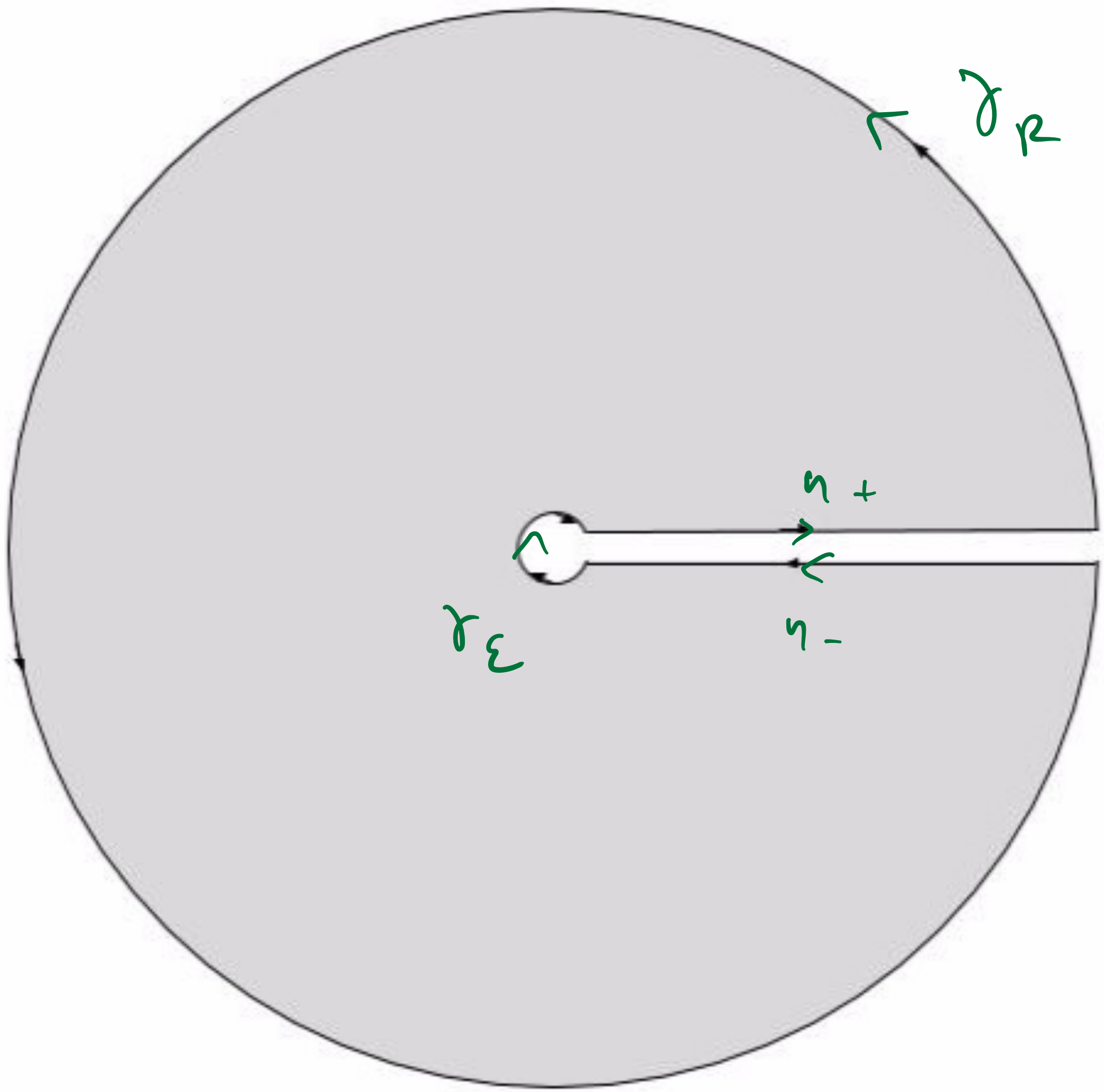


Figure: A keyhole contour.

Example. Calculate the integral

$$\int_0^\infty \frac{x^{1/2}}{1+x^2}dx.$$

Example. Calculate the integral

$$\int_0^\infty \frac{x^{1/2}}{1+x^2} dx.$$

Let $f(z) = z^{1/2}/(1+z^2)$, where we use a continuous branch on $\mathbb{C} \setminus \mathbb{R}_{>0}$, given by $z^{1/2} = r^{1/2} e^{it/2}$ (where $z = re^{it}$ with $t \in [0, 2\pi)$).

Example. Calculate the integral

$$\int_0^\infty \frac{x^{1/2}}{1+x^2}dx.$$

Let $f(z) = z^{1/2}/(1+z^2)$, where we use a continuous branch on $\mathbb{C}\backslash\mathbb{R}_{>0}$, given by $z^{1/2} = r^{1/2}e^{it/2}$ (where $z = re^{it}$ with $t \in [0, 2\pi)$).

We use the keyhole contour $\Gamma_{R,\epsilon}$.

Example. Calculate the integral

$$\int_0^\infty \frac{x^{1/2}}{1+x^2}dx.$$

Let $f(z) = z^{1/2}/(1+z^2)$, where we use a continuous branch on $\mathbb{C}\backslash\mathbb{R}_{>0}$, given by $z^{1/2} = r^{1/2}e^{it/2}$ (where $z = re^{it}$ with $t \in [0, 2\pi)$).

We use the keyhole contour $\Gamma_{R,\epsilon}$.

$$|\int_{\gamma_R} z^{1/2}/(1+z^2)dz| \leq 2\pi R \cdot \frac{R^{1/2}}{R^2-1} \to 0$$

Example. Calculate the integral

$$\int_0^\infty \frac{x^{1/2}}{1+x^2}dx.$$

Let $f(z) = z^{1/2}/(1+z^2)$, where we use a continuous branch on $\mathbb{C}\backslash\mathbb{R}_{>0}$, given by $z^{1/2} = r^{1/2}e^{it/2}$ (where $z = re^{it}$ with $t \in [0, 2\pi)$).

We use the keyhole contour $\Gamma_{R,\epsilon}$.

$$|\int_{\gamma_R} z^{1/2}/(1+z^2)dz| \leq 2\pi R \cdot \frac{R^{1/2}}{R^2-1} \to 0$$

$$|\int_{\gamma_\epsilon} z^{1/2}/(1+z^2)dz| \leq 2\pi\epsilon \cdot \frac{\epsilon^{1/2}}{1-\epsilon^2} \to 0$$

$$\int_{\eta_+} z^{1/2}/(1+z^2)dz \to \int_0^\infty \frac{x^{1/2}}{1+x^2}dx$$

and

$$\int_{\eta_+} z^{1/2}/(1+z^2)dz \to \int_0^\infty \frac{x^{1/2}}{1+x^2}dx$$

and

$$\int_{\eta_-} z^{1/2}/(1+z^2)dz \to \int_0^\infty \frac{x^{1/2}}{1+x^2}dx$$

since

$$\int_{\eta_+} z^{1/2}/(1+z^2)dz \to \int_0^\infty \frac{x^{1/2}}{1+x^2}dx$$

and

$$\int_{\eta_-} z^{1/2}/(1+z^2)dz \to \int_0^\infty \frac{x^{1/2}}{1+x^2}dx$$

since

for $z = re^{i\theta} \in \eta_-$, $z^{1/2} \sim r^{1/2}e^{i\pi} = -r^{1/2}$ and η_- is traversed in the opposite direction from η_+.

$$\int_{\eta_+} z^{1/2}/(1+z^2)dz \to \int_0^\infty \frac{x^{1/2}}{1+x^2}dx$$

and

$$\int_{\eta_-} z^{1/2}/(1+z^2)dz \to \int_0^\infty \frac{x^{1/2}}{1+x^2}dx$$

since

for $z = re^{i\theta} \in \eta_-$, $z^{1/2} \sim r^{1/2}e^{i\pi} = -r^{1/2}$ and η_- is traversed in the opposite direction from η_+.

We use the residue theorem: The function $f(z)$ has simple poles at $z = \pm i$. We calculate the residues:

$$\lim_{z\to i}(z-i)z^{1/2}/(1+z^2) = \frac{1}{2}e^{-\pi i/4},$$

$$\frac{e^{i\frac{\pi}{4}}}{2i} = \frac{e^{i\frac{\pi}{4}}}{2e^{i\frac{\pi}{2}}}$$

$$\lim_{z\to -i}(z+i)z^{1/2}/(1+z^2) = \frac{1}{2}e^{5\pi i/4}.$$

It follows that

$$\int_{\Gamma_{R,\epsilon}} f(z)dz = 2\pi i \left(\frac{1}{2} e^{-\pi i/4} + \frac{1}{2} e^{5\pi i/4} \right) = \pi\sqrt{2}.$$

It follows that

$$\int_{\Gamma_{R,\epsilon}} f(z)dz = 2\pi i \left(\frac{1}{2} e^{-\pi i/4} + \frac{1}{2} e^{5\pi i/4} \right) = \pi\sqrt{2}.$$

Taking the limit as $R \to \infty$ and $\epsilon \to 0$ we see that $2\int_0^\infty \frac{x^{1/2}}{1+x^2} dx = \pi\sqrt{2}$, so that

$$\int_0^\infty \frac{x^{1/2} dx}{1 + x^2} = \frac{\pi}{\sqrt{2}}.$$

Conformal transformations

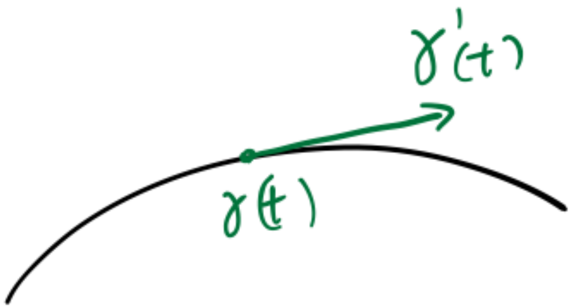
Informally if $U, V \subseteq \mathbb{C}$, $T : U \to V$ is conformal if it preserves the angles at each point.

Conformal transformations

Informally if $U, V \subseteq \mathbb{C}$, $T : U \to V$ is conformal if it preserves the angles at each point.
To make sense of this recall

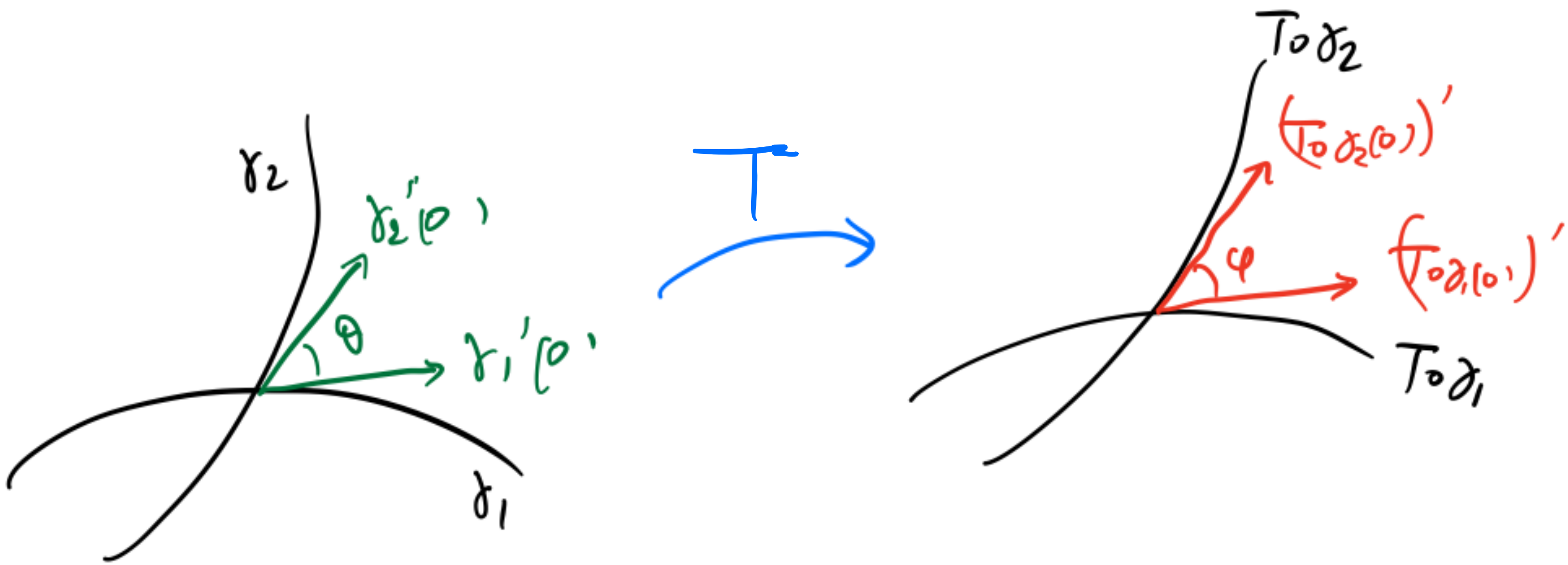
Definition

If $\gamma\colon [-1,1] \to \mathbb{C}$ is a C^1 path which has $\gamma'(t) \neq 0$ for all t, then we say that the line $\{\gamma(t) + s\gamma'(t) : s \in \mathbb{R}\}$ is the *tangent line* to γ at $\gamma(t)$, and the vector $\gamma'(t)$ is a tangent vector at $\gamma(t) \in \mathbb{C}$.



Definition

Let U be an open subset of $\mathbb{C}$ and suppose that $T\colon U \to \mathbb{C}$ is continuously differentiable in the real sense (so all its partial derivatives exist and are continuous). If $\gamma_1, \gamma_2\colon [-1, 1] \to U$ are two C^1 paths with $z_0 = \gamma_1(0) = \gamma_2(0)$ then $\gamma_1'(0)$ and $\gamma_2'(0)$ are two tangent vectors at z_0, and we may consider the (signed) angle between them (formally speaking this is the difference of their arguments).

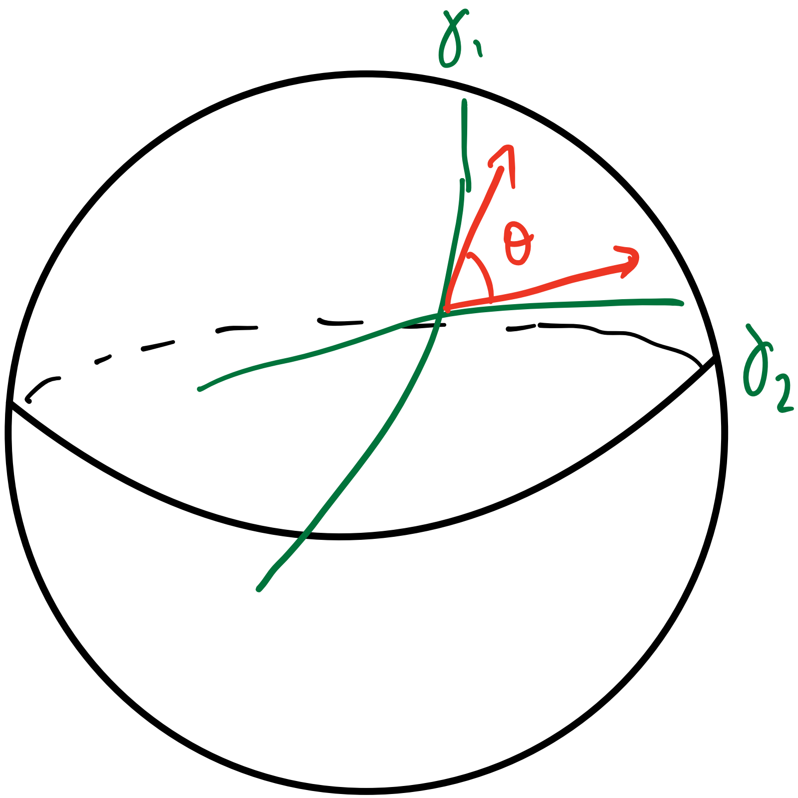


Definition

Let U be an open subset of $\mathbb{C}$ and suppose that $T\colon U \to \mathbb{C}$ is continuously differentiable in the real sense (so all its partial derivatives exist and are continuous). If $\gamma_1, \gamma_2\colon [-1,1] \to U$ are two C^1 paths with $z_0 = \gamma_1(0) = \gamma_2(0)$ then $\gamma_1'(0)$ and $\gamma_2'(0)$ are two tangent vectors at z_0, and we may consider the (signed) angle between them (formally speaking this is the difference of their arguments). By our assumption on T, the compositions $T \circ \gamma_1$ and $T \circ \gamma_2$ are C^1-paths through $T(z_0)$, thus we obtain a pair of tangent vectors at $T(z_0)$. We say that T is *conformal* at z_0 if for every pair of C^1 paths γ_1, γ_2 through z_0, the (signed) angle between their tangent vectors at z_0 is equal to the (signed) angle between the tangent vectors at $T(z_0)$ given by the C^1 paths $T \circ \gamma_1$ and $T \circ \gamma_2$. We say that T is conformal on U if it is conformal at every $z \in U$.

Remark

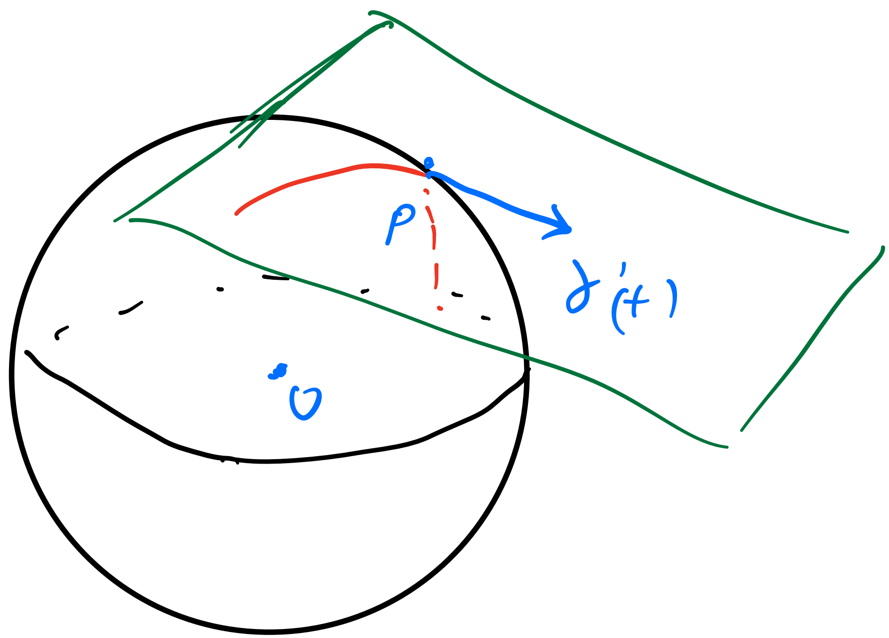
Note that we can define tangent vectors at points on subsets of $\mathbb{R}^n$ using C^1-paths (ie all component functions are C^1).
In particular, if $\mathbb{S}$ is the unit sphere in $\mathbb{R}^3$ we consider C^1 paths on $\mathbb{S}$ ie C^1 paths $\gamma : [a, b] \to \mathbb{R}^3$ whose image lies in $\mathbb{S}$.



Remark

Note that we can define tangent vectors at points on subsets of $\mathbb{R}^n$ *using* C^1*-paths (ie all component functions are* C^1*).*
In particular, if $\mathbb{S}$ *is the unit sphere in* $\mathbb{R}^3$ *we consider* C^1 *paths on* $\mathbb{S}$ *ie* C^1 *paths* $\gamma\colon [a,b] \to \mathbb{R}^3$ *whose image lies in* $\mathbb{S}$.
It is easy to check that the tangent vectors at a point $p \in \mathbb{S}$ *all lie in the plane perpendicular to* p *– simply differentiate the identity* $f(\gamma(t)) = 1$ *where* $f(x, y, z) = x^2 + y^2 + z^2$ *using the chain rule to get*

$$p \cdot \gamma'(t) = 0.$$



Remark

Note that we can define tangent vectors at points on subsets of $\mathbb{R}^n$ using C^1-paths (ie all component functions are C^1).
In particular, if $\mathbb{S}$ is the unit sphere in $\mathbb{R}^3$ we consider C^1 paths on $\mathbb{S}$ ie C^1 paths $\gamma\colon [a,b] \to \mathbb{R}^3$ whose image lies in $\mathbb{S}$.
It is easy to check that the tangent vectors at a point $p \in \mathbb{S}$ all lie in the plane perpendicular to p – simply differentiate the identity $f(\gamma(t)) = 1$ where $f(x,y,z) = x^2 + y^2 + z^2$ using the chain rule to get

$$p \cdot \gamma'(t) = 0.$$

So it makes sense to say that a map $T : \mathbb{S} \to \mathbb{C}$ or $T : \mathbb{S} \to \mathbb{S}$ is conformal.

Remark

Note that we can define tangent vectors at points on subsets of $\mathbb{R}^n$ using C^1-paths (ie all component functions are C^1).
In particular, if $\mathbb{S}$ is the unit sphere in $\mathbb{R}^3$ we consider C^1 paths on $\mathbb{S}$ ie C^1 paths $\gamma\colon [a,b] \to \mathbb{R}^3$ whose image lies in $\mathbb{S}$.
It is easy to check that the tangent vectors at a point $p \in \mathbb{S}$ all lie in the plane perpendicular to p – simply differentiate the identity $f(\gamma(t)) = 1$ where $f(x,y,z) = x^2 + y^2 + z^2$ using the chain rule to get

$$p \cdot \gamma'(t) = 0.$$

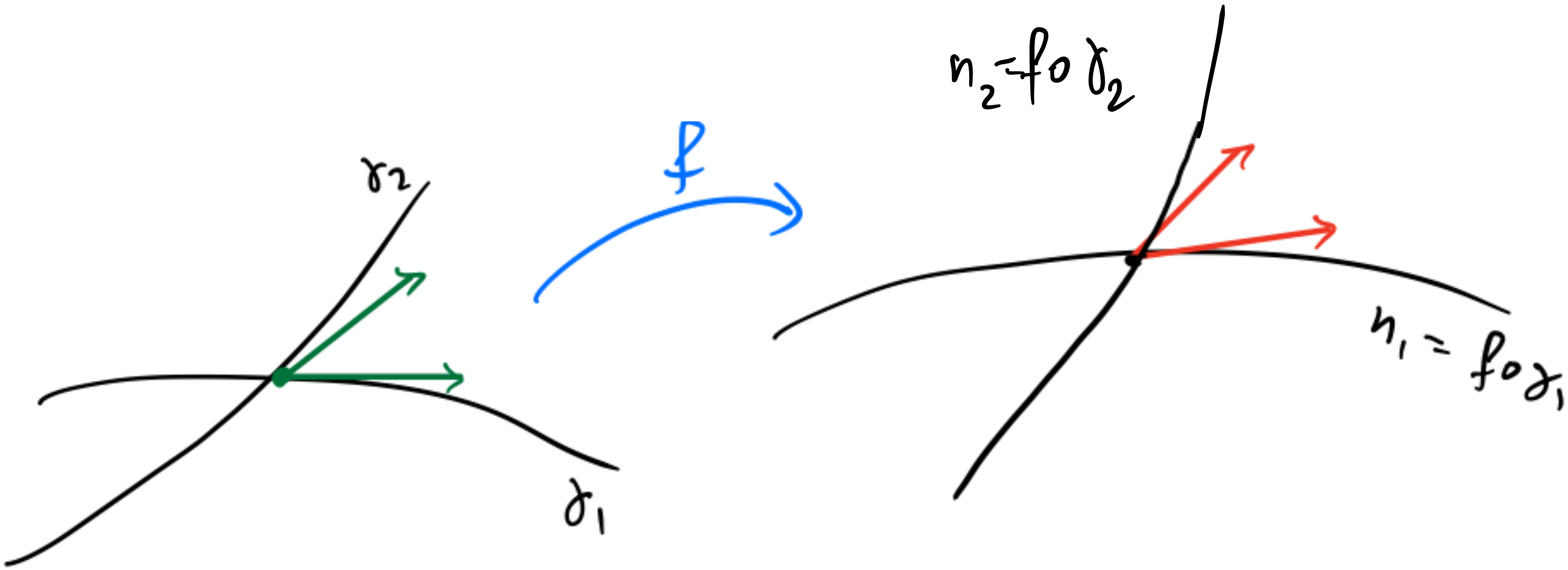
So it makes sense to say that a map $T : \mathbb{S} \to \mathbb{C}$ or $T : \mathbb{S} \to \mathbb{S}$ is conformal.

Proposition

Let $f\colon U \to \mathbb{C}$ be a holomorphic map and let $z_0 \in U$ be such that $f'(z_0) \neq 0$. Then f is conformal at z_0. In particular, if $f\colon U \to \mathbb{C}$ has nonvanishing derivative on all of U, it is conformal on all of U (and locally a biholomorphism).

Proof.

Let γ_1 and γ_2 be C^1-paths with $\gamma_1(0) = \gamma_2(0) = z_0$. Then we obtain paths η_1, η_2 through $f(z_0)$ where $\eta_1(t) = f(\gamma_1(t))$ and $\eta_2(t) = f(\gamma_2(t))$.



Proof.

Let γ_1 and γ_2 be C^1-paths with $\gamma_1(0) = \gamma_2(0) = z_0$. Then we obtain paths η_1, η_2 through $f(z_0)$ where $\eta_1(t) = f(\gamma_1(t))$ and $\eta_2(t) = f(\gamma_2(t))$.

If we set $f'(z_0) = \rho e^{i\theta}$ we have

$$\eta_i'(0) = f'(z_0)\gamma_i'(0) = \rho e^{i\theta}\gamma_i'(0), \quad i = 1, 2.$$

Proof.

Let γ_1 and γ_2 be C^1-paths with $\gamma_1(0) = \gamma_2(0) = z_0$. Then we obtain paths η_1, η_2 through $f(z_0)$ where $\eta_1(t) = f(\gamma_1(t))$ and $\eta_2(t) = f(\gamma_2(t))$.

If we set $f'(z_0) = \rho e^{i\theta}$ we have

$$\eta_i'(0) = f'(z_0)\gamma_i'(0) = \rho e^{i\theta}\gamma_i'(0), \quad i = 1, 2.$$

$$\text{so if } \gamma_1'(0) = r_1 e^{i\phi_1}, \quad \gamma_2'(0) = r_2 e^{i\phi_2}$$

then the angle between $\gamma_1'(0), \gamma_2'(0)$ is $\phi_1 - \phi_2$

Proof.

Let γ_1 and γ_2 be C^1-paths with $\gamma_1(0) = \gamma_2(0) = z_0$. Then we obtain paths η_1, η_2 through $f(z_0)$ where $\eta_1(t) = f(\gamma_1(t))$ and $\eta_2(t) = f(\gamma_2(t))$.

If we set $f'(z_0) = \rho e^{i\theta}$ we have

$$\eta_i'(0) = f'(z_0)\gamma_i'(0) = \rho e^{i\theta}\gamma_i'(0), \quad i = 1, 2.$$

$$\text{so if } \gamma_1'(0) = r_1 e^{i\phi_1}, \quad \gamma_2'(0) = r_2 e^{i\phi_2}$$

then the angle between $\gamma_1'(0), \gamma_2'(0)$ is $\phi_1 - \phi_2$

and the angle between $\eta_1'(0), \eta_2'(0)$ is

$$(\theta + \phi_1) - (\theta + \phi_2) = \phi_1 - \phi_2$$

Proof.

Let γ_1 and γ_2 be C^1-paths with $\gamma_1(0) = \gamma_2(0) = z_0$. Then we obtain paths η_1, η_2 through $f(z_0)$ where $\eta_1(t) = f(\gamma_1(t))$ and $\eta_2(t) = f(\gamma_2(t))$.

If we set $f'(z_0) = \rho e^{i\theta}$ we have

$$\eta_i'(0) = f'(z_0)\gamma_i'(0) = \rho e^{i\theta}\gamma_i'(0), \quad i = 1, 2.$$

$$\text{so if } \gamma_1'(0) = r_1 e^{i\phi_1}, \quad \gamma_2'(0) = r_2 e^{i\phi_2}$$

then the angle between $\gamma_1'(0), \gamma_2'(0)$ is $\phi_1 - \phi_2$

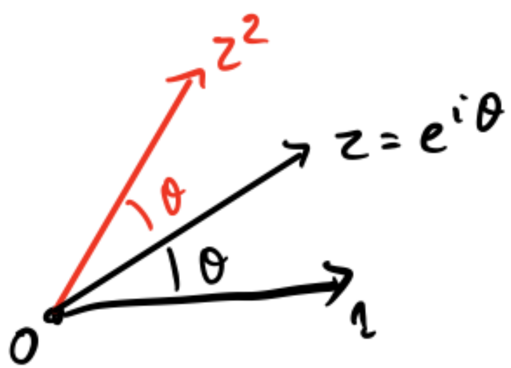
and the angle between $\eta_1'(0), \eta_2'(0)$ is

$$(\theta + \phi_1) - (\theta + \phi_2) = \phi_1 - \phi_2$$

For the final part, note that if $f'(z_0) \neq 0$ then $f(z)$ is locally biholomorphic by the inverse function theorem.

Example

The function $f(z) = z^2$ has $f'(z)$ nonzero everywhere except the origin. It follows f is a conformal map from $\mathbb{C}^\times$ to itself. Note that the condition that $f'(z)$ is non-zero is necessary – if we consider the function $f(z) = z^2$ at $z = 0$, $f'(z) = 2z$ which vanishes precisely at $z = 0$, and it is easy to check that at the origin f in fact doubles the angles between tangent vectors.



The stereographic projection is conformal

The stereographic projection is conformal

Lemma

The stereographic projection map $S\colon \mathbb{C} \to \mathbb{S}$ *is conformal.*

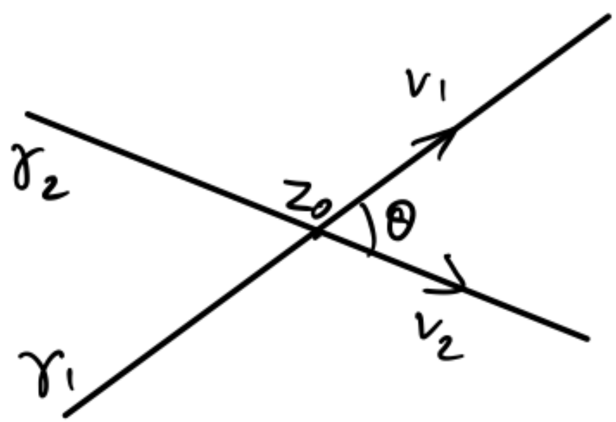
.

The stereographic projection is conformal

Lemma

The stereographic projection map $S\colon \mathbb{C} \to \mathbb{S}$ is conformal.

Proof. Let z_0 be a point in $\mathbb{C}$, and suppose that $\gamma_1(t) = z_0 + tv_1$ and $\gamma_2(t) = z_0 + tv_2$ are two paths having tangents v_1 and v_2 at $z_0 = \gamma_1(0) = \gamma_2(0)$.



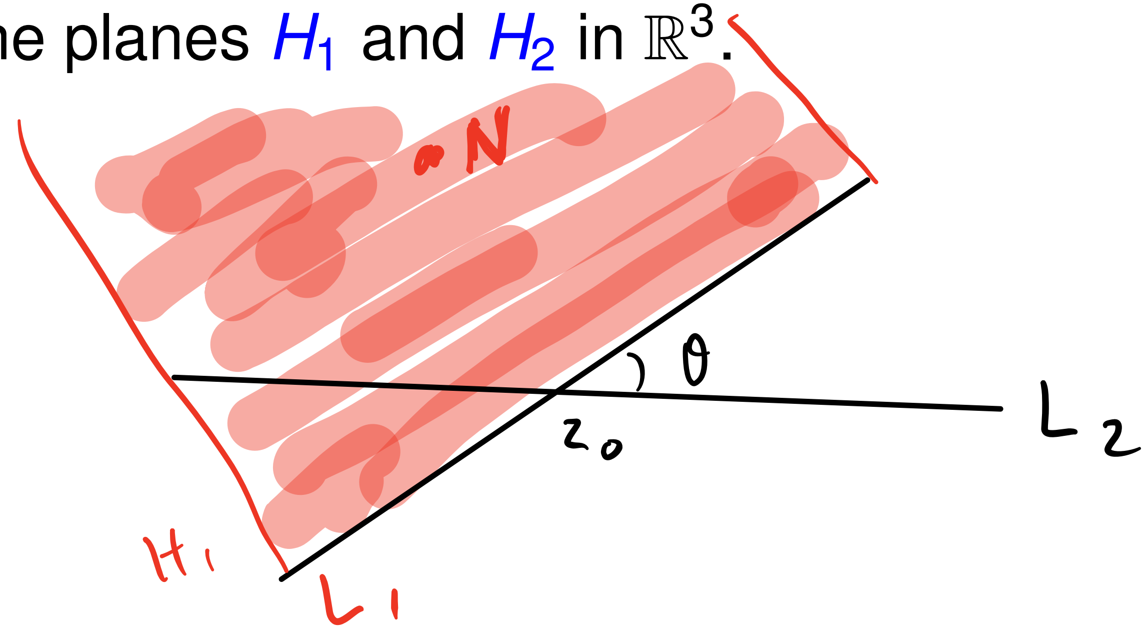
The stereographic projection is conformal

Lemma

The stereographic projection map $S\colon \mathbb{C} \to \mathbb{S}$ is conformal.

Proof. Let z_0 be a point in $\mathbb{C}$, and suppose that $\gamma_1(t) = z_0 + tv_1$ and $\gamma_2(t) = z_0 + tv_2$ are two paths having tangents v_1 and v_2 at $z_0 = \gamma_1(0) = \gamma_2(0)$.

Then the lines L_1 and L_2 they describe, together with north pole of $\mathbb{S}$, N, determine planes H_1 and H_2 in $\mathbb{R}^3$.



The stereographic projection is conformal

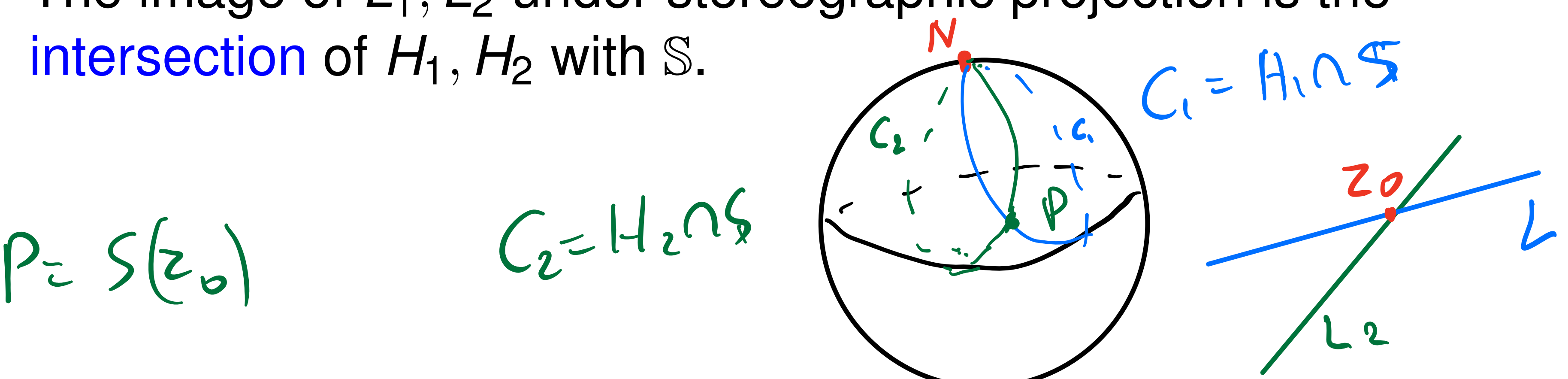
Lemma

The stereographic projection map $S\colon \mathbb{C} \to \mathbb{S}$ *is conformal.*

Proof. Let z_0 be a point in $\mathbb{C}$, and suppose that $\gamma_1(t) = z_0 + tv_1$ and $\gamma_2(t) = z_0 + tv_2$ are two paths having tangents v_1 and v_2 at $z_0 = \gamma_1(0) = \gamma_2(0)$.

Then the lines L_1 and L_2 they describe, together with north pole of $\mathbb{S}$, N, determine planes H_1 and H_2 in $\mathbb{R}^3$.

The image of L_1, L_2 under stereographic projection is the intersection of H_1, H_2 with $\mathbb{S}$.



The stereographic projection is conformal

Lemma

The stereographic projection map $S\colon \mathbb{C} \to \mathbb{S}$ is conformal.

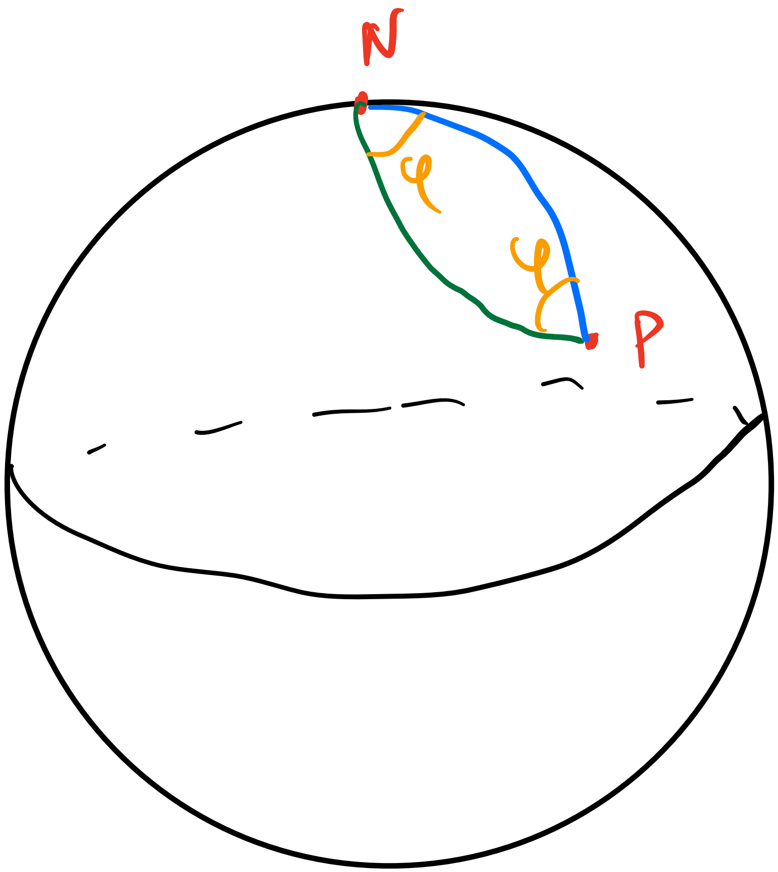
Proof. Let z_0 be a point in $\mathbb{C}$, and suppose that $\gamma_1(t) = z_0 + tv_1$ and $\gamma_2(t) = z_0 + tv_2$ are two paths having tangents v_1 and v_2 at $z_0 = \gamma_1(0) = \gamma_2(0)$.

Then the lines L_1 and L_2 they describe, together with north pole of $\mathbb{S}$, N, determine planes H_1 and H_2 in $\mathbb{R}^3$.

The image of L_1, L_2 under stereographic projection is the intersection of H_1, H_2 with $\mathbb{S}$.

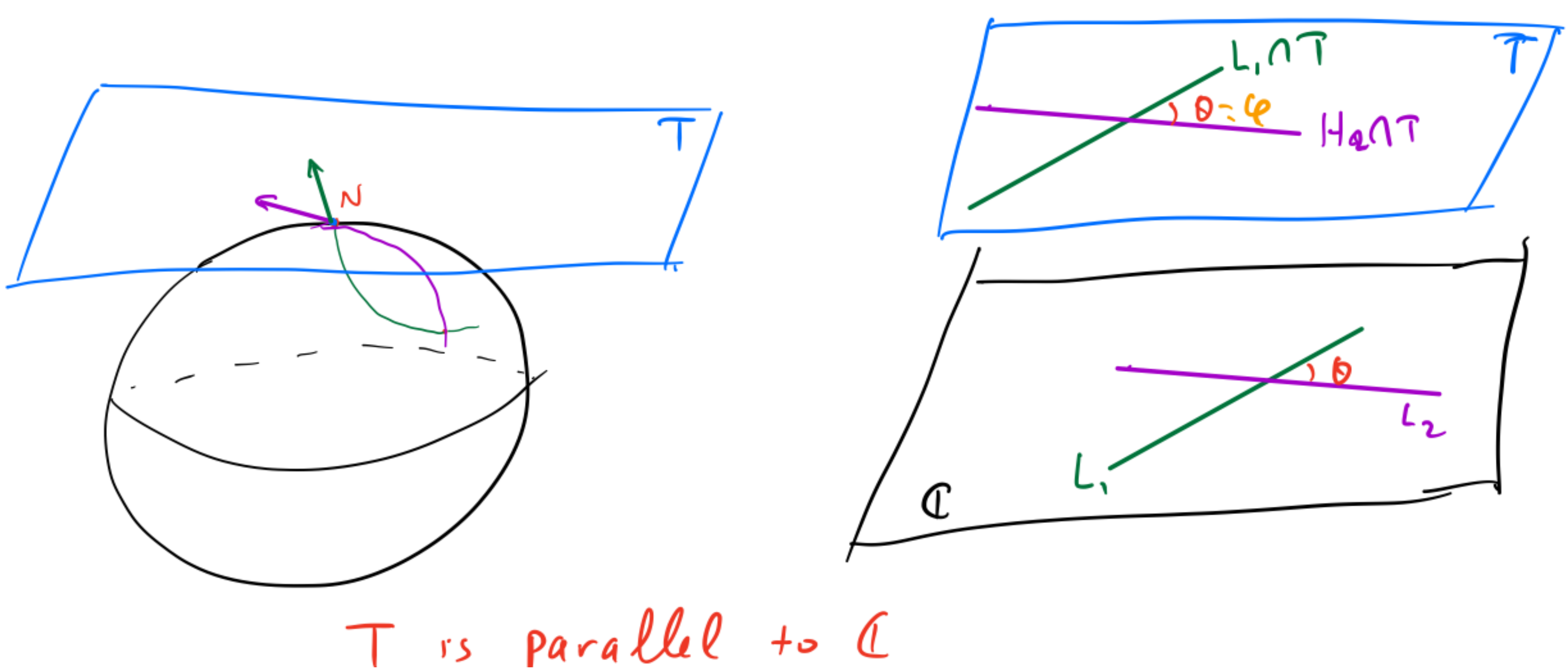
So the paths γ_1 and γ_2 get sent to two circles C_1 and C_2 passing through $P = S(z_0)$ and N.

By symmetry, C_1, C_2 meet at the same angle at N as they do at P.



By symmetry, C_1, C_2 meet at the same angle at N as they do at P.

The tangent lines of C_1 and C_2 at N are just the intersections of H_1 and H_2 with the plane tangent to $\mathbb{S}$ at N.



By symmetry, C_1, C_2 meet at the same angle at N as they do at P.

The tangent lines of C_1 and C_2 at N are just the intersections of H_1 and H_2 with the plane tangent to $\mathbb{S}$ at N.

But this means the angle between them will be the same as that between the intersection of H_1 and H_2 with the complex plane, since it is parallel to the tangent plane of $\mathbb{S}$ at N. Thus the angles between C_1 and C_2 at P and L_1 and L_2 at z_0 coincide as required. $\square$

Lemma
Möbius transformations are conformal.

.

□

Lemma

Möbius transformations are conformal.

Proof.

We note that if $f(z) = \frac{az+b}{cz+d}$ then

$$f'(z) = \frac{ad - bc}{(cz + d)^2} \neq 0,$$

for all $z \neq -d/c$, thus f is conformal at each $z \in \mathbb{C}\backslash\{-d/c\}$. □

Lemma

Möbius transformations are conformal.

Proof.

We note that if $f(z) = \frac{az+b}{cz+d}$ then

$$f'(z) = \frac{ad - bc}{(cz + d)^2} \neq 0,$$

for all $z \neq -d/c$, thus f is conformal at each $z \in \mathbb{C} \backslash \{-d/c\}$. □

We will show further (*off sylabus*) that a Möbius transformation is conformal seen as a map $\mathbb{S} \to \mathbb{S}$ (where $\mathbb{S}$ can be identified with $\mathbb{C} \cup \infty$).

Möbius transformations are conformal

We see now Möbius transformations as maps from the extended complex plane $\mathbb{C} \cup \{\infty\} = \mathbb{S}$ to itself.

Möbius transformations are conformal

We see now Möbius transformations as maps from the extended complex plane $\mathbb{C} \cup \{\infty\} = \mathbb{S}$ to itself.

We note that if f is conformal at z_1 and g is conformal at $f(z_1)$ then $g \circ f$ is conformal at z_1. Since the stereographic projection is conformal a map $f : \mathbb{C} \to \mathbb{C}$ is conformal if and only if the corresponding map $f : \mathbb{S} \to \mathbb{S}$ is conformal.

Möbius transformations are conformal

We see now Möbius transformations as maps from the extended complex plane $\mathbb{C} \cup \{\infty\} = \mathbb{S}$ to itself.

We note that if f is conformal at z_1 and g is conformal at $f(z_1)$ then $g \circ f$ is conformal at z_1. Since the stereographic projection is conformal a map $f : \mathbb{C} \to \mathbb{C}$ is conformal if and only if the corresponding map $f : \mathbb{S} \to \mathbb{S}$ is conformal.

We claim that $1/z$ seen as a map $\mathbb{S} \to \mathbb{S}$ is conformal. Indeed $1/z : \mathbb{S} \to \mathbb{S}$ is the map $(t, u, v) \mapsto (t, -u, -v)$, which is a rotation by π about the x-axis, so clearly it is conformal.

$$(t,u,v) \xrightarrow{S} \left(\frac{t}{1-v} + i\frac{u}{1-v}\right)$$

$$(t,-u,-v) \xrightarrow{S} \left(\frac{t}{1+v} + i\frac{-u}{1+v}\right)$$

$$\left(\frac{t}{1-v} + i\frac{u}{1-v}\right)\cdot\left(\frac{t}{1+v} - i\frac{u}{1+v}\right) = \frac{t^2+u^2}{1-v^2} = 1$$

Möbius transformations are conformal

We see now Möbius transformations as maps from the extended complex plane $\mathbb{C} \cup \{\infty\} = \mathbb{S}$ to itself.

We note that if f is conformal at z_1 and g is conformal at $f(z_1)$ then $g \circ f$ is conformal at z_1. Since the stereographic projection is conformal a map $f : \mathbb{C} \to \mathbb{C}$ is conformal if and only if the corresponding map $f : \mathbb{S} \to \mathbb{S}$ is conformal.

We claim that $1/z$ seen as a map $\mathbb{S} \to \mathbb{S}$ is conformal. Indeed $1/z : \mathbb{S} \to \mathbb{S}$ is the map $(t, u, v) \mapsto (t, -u, -v)$, which is a rotation by π about the x-axis, so clearly it is conformal.

We claim that $z \mapsto z + a$ and $z \mapsto az$ are also conformal maps for $a \in \mathbb{C} \setminus \{0\}$.

The maps $z \mapsto z + a, z \mapsto az\,(a \neq 0)$ are clearly conformal for every $z \in \mathbb{C}$, so they are conformal at every $z \in \mathbb{S} \setminus \{N\}$

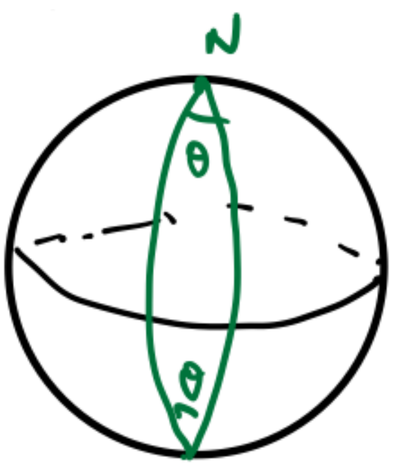
The maps $z \mapsto z + a$, $z \mapsto az$ $(a \neq 0)$ are clearly conformal for every $z \in \mathbb{C}$, so they are conformal at every $z \in \mathbb{S} \setminus \{N\}$

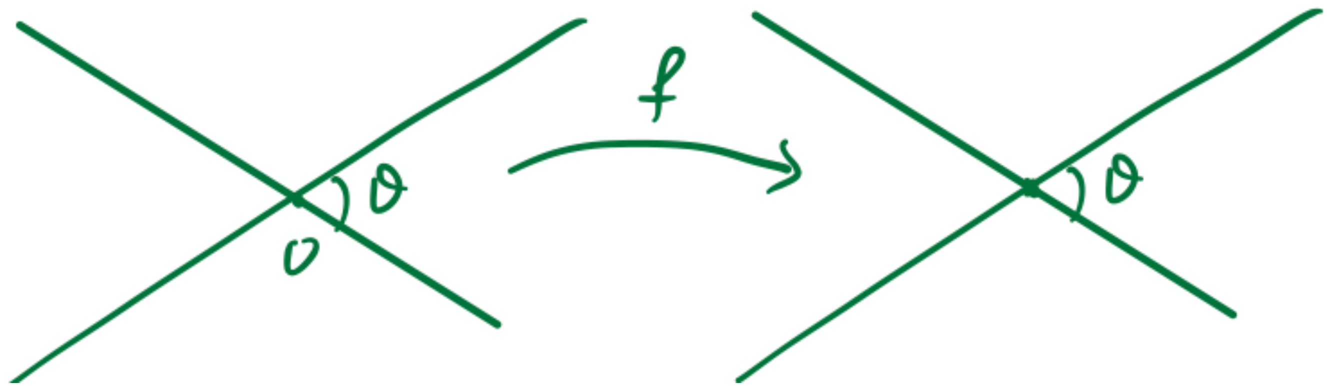
We claim that if f is $z \mapsto z + a$ or $z \mapsto az$ then f is conformal at N as well.

The maps $z \mapsto z + a$, $z \mapsto az$ $(a \neq 0)$ are clearly conformal for every $z \in \mathbb{C}$, so they are conformal at every $z \in \mathbb{S} \setminus \{N\}$

We claim that if f is $z \mapsto z + a$ or $z \mapsto az$ then f is conformal at N as well.

To see this we consider the images of great circles through N.

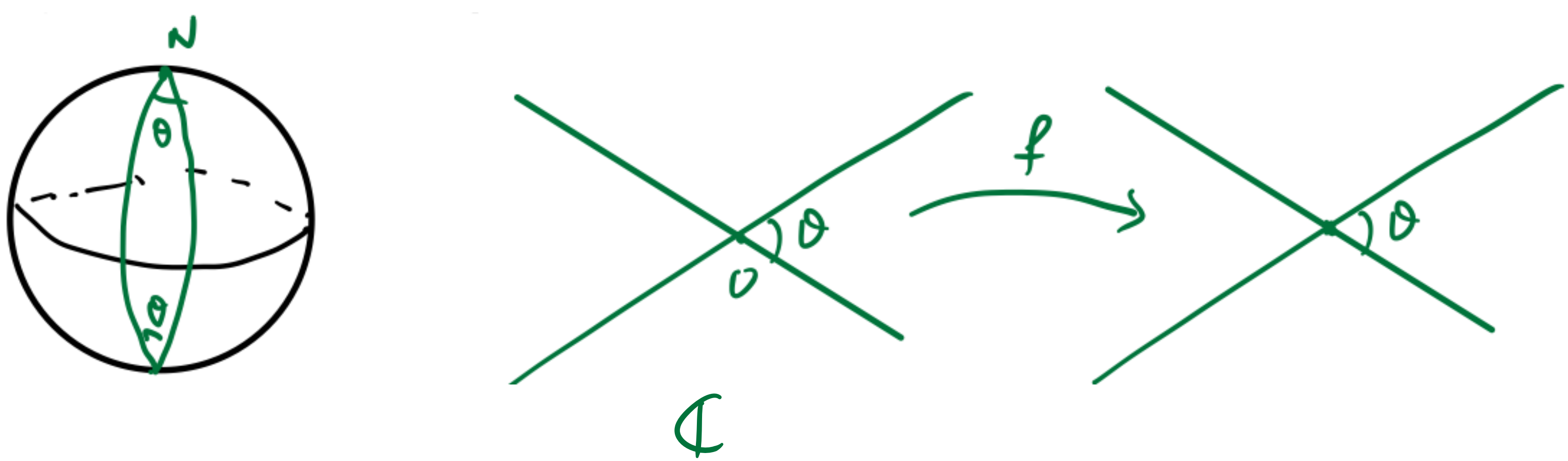




The maps $z \mapsto z + a$, $z \mapsto az$ $(a \neq 0)$ are clearly conformal for every $z \in \mathbb{C}$, so they are conformal at every $z \in \mathbb{S} \setminus \{N\}$

We claim that if f is $z \mapsto z + a$ or $z \mapsto az$ then f is conformal at N as well.

To see this we consider the images of great circles through N. These circles correspond to lines through 0 under S and as in the previous lemma we note that the angles of two such circles at N is equal to the angle of the lines at 0.



The maps $z \mapsto z + a, z \mapsto az\,(a \neq 0)$ are clearly conformal for every $z \in \mathbb{C}$, so they are conformal at every $z \in \mathbb{S} \setminus \{N\}$

We claim that if f is $z \mapsto z + a$ or $z \mapsto az$ then f is conformal at N as well.

To see this we consider the images of great circles through N. These circles correspond to lines through 0 under S and as in the previous lemma we note that the angles of two such circles at N is equal to the angle of the lines at 0. But, since f is conformal as a map $\mathbb{C} \to \mathbb{C}$ the angles at 0 are preserved by f, so the angles at N are preserved as well.

The maps $z \mapsto z + a$, $z \mapsto az$ $(a \neq 0)$ are clearly conformal for every $z \in \mathbb{C}$, so they are conformal at every $z \in \mathbb{S} \setminus \{N\}$

We claim that if f is $z \mapsto z + a$ or $z \mapsto az$ then f is conformal at N as well.

To see this we consider the images of great circles through N. These circles correspond to lines through 0 under S and as in the previous lemma we note that the angles of two such circles at N is equal to the angle of the lines at 0. But, since f is conformal as a map $\mathbb{C} \to \mathbb{C}$ the angles at 0 are preserved by f, so the angles at N are preserved as well.

We have seen that any Möbius transformation can be written as a composition of dilations, translations and an inversion. Since all these are conformal maps $\mathbb{S} \to \mathbb{S}$ their compositions are conformal as well. So Möbius tranformations are conformal.

Proposition

If z_1, z_2, z_3 and w_1, w_2, w_3 are triples of pairwise distinct complex numbers, then there is a unique Möbius transformation f such that $f(z_i) = w_i$ for each $i = 1, 2, 3$.

Proposition

If z_1, z_2, z_3 and w_1, w_2, w_3 are triples of pairwise distinct complex numbers, then there is a unique Möbius transformation f such that $f(z_i) = w_i$ for each $i = 1, 2, 3$.

Proof. It is enough to show that, given any triple (z_1, z_2, z_3) of complex numbers, we can find a Möbius transformations which takes z_1, z_2, z_3 to $0, 1, \infty$ respectively.

Proposition

If z_1, z_2, z_3 and w_1, w_2, w_3 are triples of pairwise distinct complex numbers, then there is a unique Möbius transformation f such that $f(z_i) = w_i$ for each $i = 1, 2, 3$.

Proof. It is enough to show that, given any triple (z_1, z_2, z_3) of complex numbers, we can find a Möbius transformations which takes z_1, z_2, z_3 to $0, 1, \infty$ respectively.
Indeed if f_1 is such a transformation, and f_2 takes $0, 1, \infty$ to w_1, w_2, w_3 respectively, then clearly $f_2 \circ f_1^{-1}$ is a Möbius transformation which takes z_i to w_i for each i.

Proposition

If z_1, z_2, z_3 and w_1, w_2, w_3 are triples of pairwise distinct complex numbers, then there is a unique Möbius transformation f such that $f(z_i) = w_i$ for each $i = 1, 2, 3$.

Proof. It is enough to show that, given any triple (z_1, z_2, z_3) of complex numbers, we can find a Möbius transformations which takes z_1, z_2, z_3 to $0, 1, \infty$ respectively.
Indeed if f_1 is such a transformation, and f_2 takes $0, 1, \infty$ to w_1, w_2, w_3 respectively, then clearly $f_2 \circ f_1^{-1}$ is a Möbius transformation which takes z_i to w_i for each i.

Now consider

$$f(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

It is easy to check that $f(z_1) = 0, f(z_2) = 1, f(z_3) = \infty$, and clearly f is a Möbius transformation as required.

If $z_1 = \infty$ then we set $f(z) = \frac{z_2 - z_3}{z - z_3}$; if $z_2 = \infty$, we take $f(z) = \frac{z - z_1}{z - z_3}$; and finally if $z_3 = \infty$ take $f(z) = \frac{z - z_1}{z_2 - z_1}$.

If $z_1 = \infty$ then we set $f(z) = \frac{z_2 - z_3}{z - z_3}$; if $z_2 = \infty$, we take $f(z) = \frac{z - z_1}{z - z_3}$; and finally if $z_3 = \infty$ take $f(z) = \frac{z - z_1}{z_2 - z_1}$.

Uniqueness: Suppose f_1 and f_2 both take z_1, z_2, z_3 to w_1, w_2, w_3.

If $z_1 = \infty$ then we set $f(z) = \frac{z_2 - z_3}{z - z_3}$; if $z_2 = \infty$, we take $f(z) = \frac{z - z_1}{z - z_3}$; and finally if $z_3 = \infty$ take $f(z) = \frac{z - z_1}{z_2 - z_1}$.

Uniqueness: Suppose f_1 and f_2 both take z_1, z_2, z_3 to w_1, w_2, w_3.

If g, h are Möbius maps sending z_1, z_2, z_3 and w_1, w_2, w_3 to $0, 1, \infty$ then hf_1g^{-1} and hf_2g^{-1} both take $(0, 1, \infty)$ to $(0, 1, \infty)$.

If $z_1 = \infty$ then we set $f(z) = \frac{z_2 - z_3}{z - z_3}$; if $z_2 = \infty$, we take $f(z) = \frac{z - z_1}{z - z_3}$; and finally if $z_3 = \infty$ take $f(z) = \frac{z - z_1}{z_2 - z_1}$.

Uniqueness: Suppose f_1 and f_2 both take z_1, z_2, z_3 to w_1, w_2, w_3.

If g, h are Möbius maps sending z_1, z_2, z_3 and w_1, w_2, w_3 to $0, 1, \infty$ then hf_1g^{-1} and hf_2g^{-1} both take $(0, 1, \infty)$ to $(0, 1, \infty)$.

But suppose $T(z) = \dfrac{az + b}{cz + d}$ is Möbius with $T(0) = 0$, $T(1) = 1$ and $T(\infty) = \infty$. Since T fixes ∞ it follows $c = 0$. Since $T(0) = 0$ it follows that $b/d = 0$ hence $b = 0$, thus $T(z) = a/d \cdot z$, and since $T(1) = 1$ it follows $a/d = 1$ and hence $T(z) = z$.

If $z_1 = \infty$ then we set $f(z) = \frac{z_2 - z_3}{z - z_3}$; if $z_2 = \infty$, we take $f(z) = \frac{z - z_1}{z - z_3}$; and finally if $z_3 = \infty$ take $f(z) = \frac{z - z_1}{z_2 - z_1}$.

Uniqueness: Suppose f_1 and f_2 both take z_1, z_2, z_3 to w_1, w_2, w_3.

If g, h are Möbius maps sending z_1, z_2, z_3 and w_1, w_2, w_3 to $0, 1, \infty$ then hf_1g^{-1} and hf_2g^{-1} both take $(0, 1, \infty)$ to $(0, 1, \infty)$.

But suppose $T(z) = \dfrac{az + b}{cz + d}$ is Möbius with $T(0) = 0$, $T(1) = 1$ and $T(\infty) = \infty$. Since T fixes ∞ it follows $c = 0$. Since $T(0) = 0$ it follows that $b/d = 0$ hence $b = 0$, thus $T(z) = a/d \cdot z$, and since $T(1) = 1$ it follows $a/d = 1$ and hence $T(z) = z$.

Hence

$$hf_1g^{-1} = hf_2g^{-1} = \mathrm{id},$$

and so $f_1 = f_2$. □

Examples of conformal maps

Examples of conformal maps

Möbius tranformations give us a source of conformal maps. They have some useful geometric properties as they map circles/lines to circles/lines, they are bijective, and are determined by their value in 3 points.

Examples of conformal maps

Möbius tranformations give us a source of conformal maps. They have some useful geometric properties as they map circles/lines to circles/lines, they are bijective, and are determined by their value in 3 points.

Example Find a conformal map that takes the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ to the unit disk $B(0, 1)$.

Examples of conformal maps

Möbius tranformations give us a source of conformal maps. They have some useful geometric properties as they map circles/lines to circles/lines, they are bijective, and are determined by their value in 3 points.

Example Find a conformal map that takes the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ to the unit disk $B(0, 1)$.

The boundary of the half plane is a line, so by a Möbius map we can map it to the boundary of the disc:

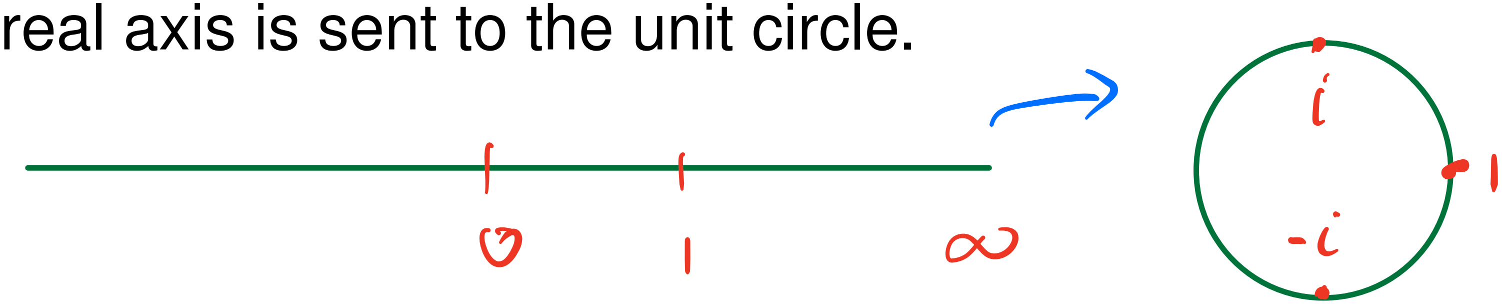
Examples of conformal maps

Möbius tranformations give us a source of conformal maps. They have some useful geometric properties as they map circles/lines to circles/lines, they are bijective, and are determined by their value in 3 points.

Example Find a conformal map that takes the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ to the unit disk $B(0, 1)$.

The boundary of the half plane is a line, so by a Möbius map we can map it to the boundary of the disc:

Take f the Möbius defined by $0 \mapsto -i$, $1 \mapsto 1$, $\infty \mapsto i$. Then the real axis is sent to the unit circle.



We calculate:

$$f(z) = \frac{iz+1}{z+i}$$

$$f(z) = \frac{az+b}{cz+d} \qquad f(0) = \frac{b}{d} = -i \qquad f(\infty) = \frac{a}{c} = i \qquad f(1) = \frac{a+b}{c+d} = 1$$

set $\boxed{c=1}$ then $\boxed{a=i}$ $\quad b = -id \qquad i - id = 1 + d$

$$\boxed{d = \frac{i-1}{1+i} = i}$$

We calculate:

$$f(z) = \frac{iz + 1}{z + i}$$

f is continuous so it maps connected sets to connected sets.

We calculate:

$$f(z) = \frac{iz+1}{z+i}$$

f is continuous so it maps connected sets to connected sets.

We note that $\mathbb{C}\backslash\mathbb{R}$ has two connected components, the upper and lower half planes, $\mathbb{H}$ and $-\mathbb{H}$, and similarly $\mathbb{C}\backslash\mathbb{S}^1$ has two connected components, $B(0,1)$ and $\mathbb{C}\backslash\bar{B}(0,1)$.

We calculate:

$$f(z) = \frac{iz+1}{z+i}$$

f is continuous so it maps connected sets to connected sets.

We note that $\mathbb{C}\backslash\mathbb{R}$ has two connected components, the upper and lower half planes, $\mathbb{H}$ and $-\mathbb{H}$, and similarly $\mathbb{C}\backslash\mathbb{S}^1$ has two connected components, $B(0,1)$ and $\mathbb{C}\backslash\bar{B}(0,1)$.

As f is 1-1 one of the two open half planes maps to the disc and the other to the complement of $\bar{B}(0,1)$.

We calculate:

$$f(z) = \frac{iz+1}{z+i}$$

f is continuous so it maps connected sets to connected sets.

We note that $\mathbb{C}\backslash\mathbb{R}$ has two connected components, the upper and lower half planes, $\mathbb{H}$ and $-\mathbb{H}$, and similarly $\mathbb{C}\backslash\mathbb{S}^1$ has two connected components, $B(0,1)$ and $\mathbb{C}\backslash\bar{B}(0,1)$.

As f is 1-1 one of the two open half planes maps to the disc and the other to the complement of $\bar{B}(0,1)$.

We calculate $f(i) = 0 \in B(0,1)$, so $f(\mathbb{H}) = B(0,1)$.

Note that if we had taken $g(z) = (z + i)/(iz + 1)$, then g also maps $\mathbb{R}$ to the unit circle $\mathbb{S}^1$, but $g(-i) = 0$

Note that if we had taken $g(z) = (z + i)/(iz + 1)$, then g also maps $\mathbb{R}$ to the unit circle $\mathbb{S}^1$, but $g(-i) = 0$

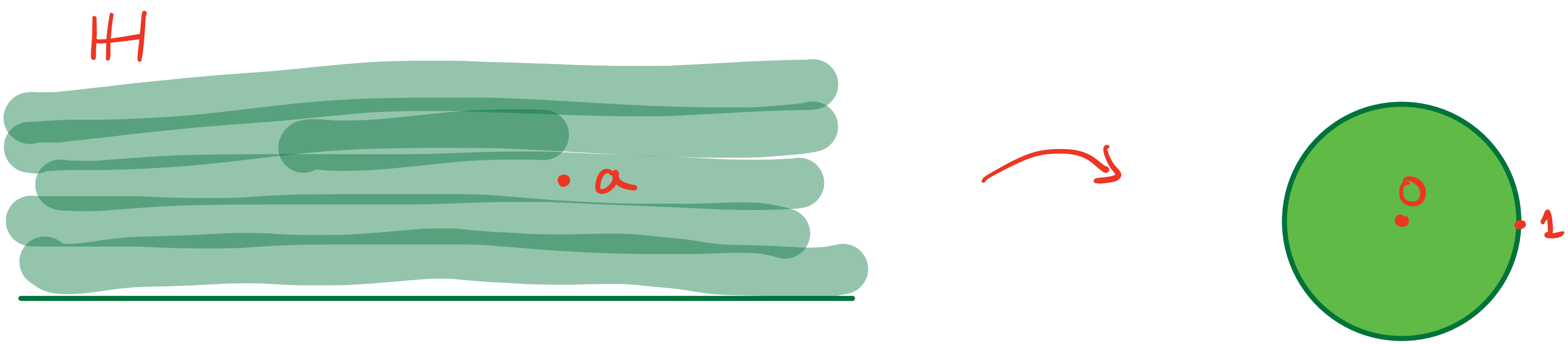
However it is easy to correct this as $R(z) = -z$ maps $\mathbb{H}$ to $-\mathbb{H}$ so we may take $g(-z)$ as our map instead.

Note that if we had taken $g(z) = (z + i)/(iz + 1)$, then g also maps $\mathbb{R}$ to the unit circle $\mathbb{S}^1$, but $g(-i) = 0$

However it is easy to correct this as $R(z) = -z$ maps $\mathbb{H}$ to $-\mathbb{H}$ so we may take $g(-z)$ as our map instead.

In particular the conformal map taking $\mathbb{H}$ to $B(0, 1)$ is far from unique. Any Möbius map that preserves $B(0, 1)$ will give another such map. Thus for example $e^{i\theta} \cdot f$ is another such map.

Example Find a conformal map that takes the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ to the unit disk $B(0,1)$ and sends $a \in \mathbb{H}$ to 0.

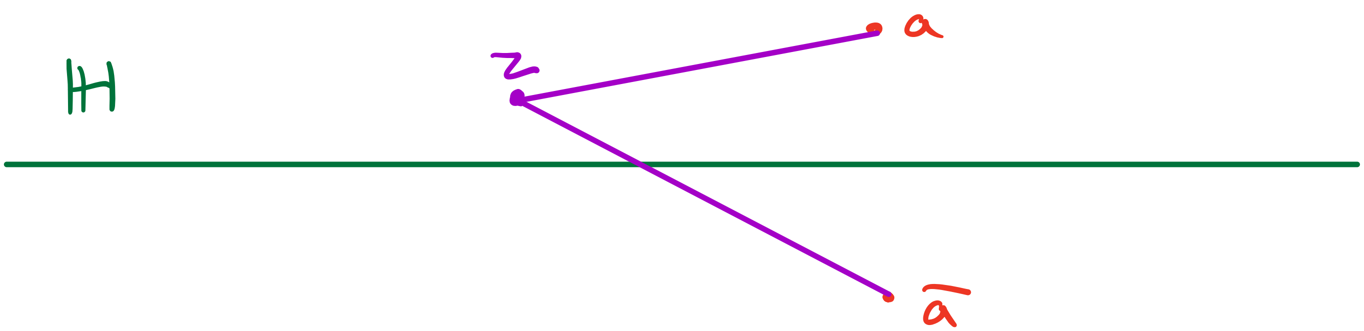


Example Find a conformal map that takes the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ to the unit disk $B(0, 1)$ and sends $a \in \mathbb{H}$ to 0.

Note that

$$\left| \frac{z - a}{z - \bar{a}} \right| < 1$$

if and only if $z \in \mathbb{H}$.



Example Find a conformal map that takes the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ to the unit disk $B(0,1)$ and sends $a \in \mathbb{H}$ to 0.

Note that

$$\left|\frac{z-a}{z-\bar{a}}\right| < 1$$

if and only if $z \in \mathbb{H}$.

So if

$$f(z) = \frac{z-a}{z-\bar{a}}$$

then $f(\mathbb{H}) = B(0,1)$ and $f(a) = 0$.

The exponential map

Consider the exponential map $z \mapsto e^z$. Then the vertical line $x = a$ maps to the set $\{e^a e^{iy} : y \in \mathbb{R}\}$ ie a circle of radius $r = e^a$.

The exponential map

Consider the exponential map $z \mapsto e^z$. Then the vertical line $x = a$ maps to the set $\{e^a e^{iy} : y \in \mathbb{R}\}$ ie a circle of radius $r = e^a$.

The horizontal line $y = b$ maps to the set $\{e^x e^{ib} : x \in \mathbb{R}\}$ ie an (open) ray at angle b from the x-axis.

The exponential map

Consider the exponential map $z \mapsto e^z$. Then the vertical line $x = a$ maps to the set $\{e^a e^{iy} : y \in \mathbb{R}\}$ ie a circle of radius $r = e^a$.

The horizontal line $y = b$ maps to the set $\{e^x e^{ib} : x \in \mathbb{R}\}$ ie an (open) ray at angle b from the x-axis.

So the exponential map is a conformal map mapping an open strip between two vertical lines onto an annulus.

The exponential map

Consider the exponential map $z \mapsto e^z$. Then the vertical line $x = a$ maps to the set $\{e^a e^{iy} : y \in \mathbb{R}\}$ ie a circle of radius $r = e^a$.

The horizontal line $y = b$ maps to the set $\{e^x e^{ib} : x \in \mathbb{R}\}$ ie an (open) ray at angle b from the x-axis.

So the exponential map is a conformal map mapping an open strip between two vertical lines onto an annulus.

Similarly an open strip between two horizontal lines maps by $\exp$ to a cyclic sector at 0.

The exponential map

Consider the exponential map $z \mapsto e^z$. Then the vertical line $x = a$ maps to the set $\{e^a e^{iy} : y \in \mathbb{R}\}$ ie a circle of radius $r = e^a$.

The horizontal line $y = b$ maps to the set $\{e^x e^{ib} : x \in \mathbb{R}\}$ ie an (open) ray at angle b from the x-axis.

So the exponential map is a conformal map mapping an open strip between two vertical lines onto an annulus.

Similarly an open strip between two horizontal lines maps by $\exp$ to a cyclic sector at 0.

Note that any two such cyclic sectors are conformally equivalent using power maps z^c. The logarithm Log maps these same domains in the reverse direction.

Riemann mapping theorem

Definition

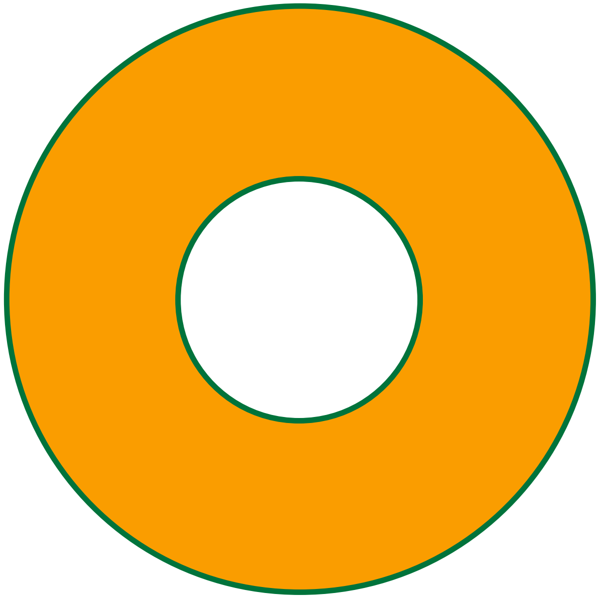
If there is a bijective conformal transformation between two domains U and V in the complex plane then we say that they are conformally equivalent.

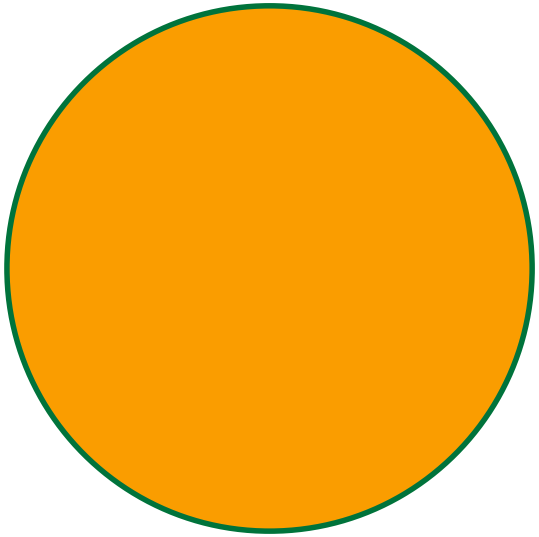
Riemann mapping theorem

Definition

If there is a bijective conformal transformation between two domains U and V in the complex plane then we say that they are conformally equivalent.

Since two conformally equivalent domains are in particular homeomorphic, one can not expect that any two domains are conformally equivalent.





Riemann mapping theorem

Definition

If there is a bijective conformal transformation between two domains U and V in the complex plane then we say that they are conformally equivalent.

Since two conformally equivalent domains are in particular homeomorphic, one can not expect that any two domains are conformally equivalent.

Denote by $\mathbb{D}$ the unit disc $B(0, 1)$.

Riemann mapping theorem

Definition

If there is a bijective conformal transformation between two domains U and V in the complex plane then we say that they are conformally equivalent.

Since two conformally equivalent domains are in particular homeomorphic, one can not expect that any two domains are conformally equivalent.

Denote by $\mathbb{D}$ the unit disc $B(0, 1)$.

Theorem

(Riemann's mapping theorem): Let U be an open connected and simply-connected proper subset of $\mathbb{C}$. Then for any $z_0 \in U$ there is a unique bijective conformal transformation $f : U \to \mathbb{D}$ such that $f(z_0) = 0$, $f'(z_0) > 0$.

Riemann mapping theorem

Definition

If there is a bijective conformal transformation between two domains U and V in the complex plane then we say that they are conformally equivalent.

Since two conformally equivalent domains are in particular homeomorphic, one can not expect that any two domains are conformally equivalent.

Denote by $\mathbb{D}$ the unit disc $B(0,1)$.

Theorem

(Riemann's mapping theorem): Let U be an open connected and simply-connected proper subset of $\mathbb{C}$. Then for any $z_0 \in U$ there is a unique bijective conformal transformation $f : U \to \mathbb{D}$ such that $f(z_0) = 0$, $f'(z_0) > 0$.

For the proof see eg Shakarchi and Stein's Complex Analysis book.

Liouville's theorem implies that there can be no bijective conformal transformation taking $\mathbb{C}$ to $B(0, 1)$, so the whole complex plane is an exception.

Liouville's theorem implies that there can be no bijective conformal transformation taking $\mathbb{C}$ to $B(0, 1)$, so the whole complex plane is an exception.

Say D_1, D_2 are open proper simply connected subsets. How do we construct $f : D_1 \to D_2$ conformal?

Liouville's theorem implies that there can be no bijective conformal transformation taking $\mathbb{C}$ to $B(0, 1)$, so the whole complex plane is an exception.

Say D_1, D_2 are open proper simply connected subsets. How do we construct $f : D_1 \to D_2$ conformal?

Some useful maps: Möbius transformations, the exponential function, branches of the multifunction $[z^\alpha]$ (away from the origin)

Liouville's theorem implies that there can be no bijective conformal transformation taking $\mathbb{C}$ to $B(0, 1)$, so the whole complex plane is an exception.

Say D_1, D_2 are open proper simply connected subsets. How do we construct $f : D_1 \to D_2$ conformal?

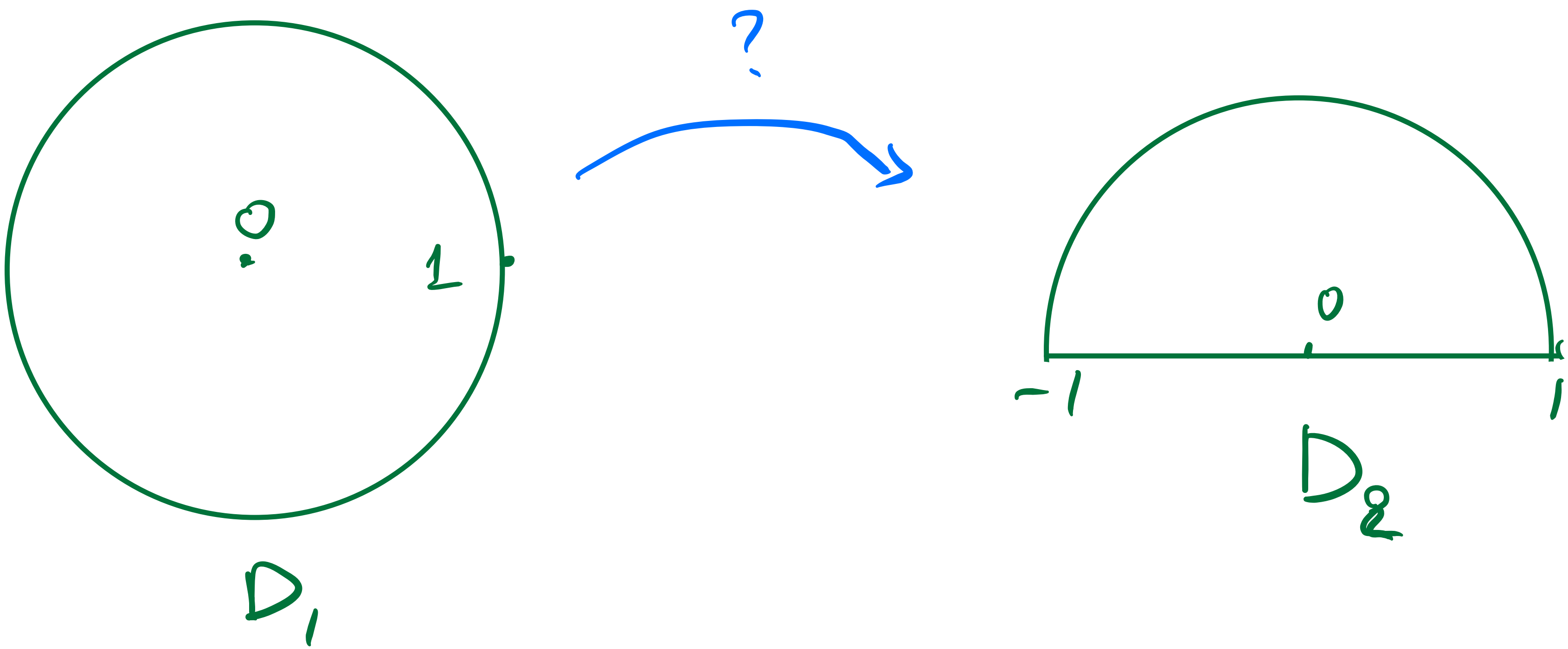
Some useful maps: Möbius transformations, the exponential function, branches of the multifunction $[z^\alpha]$ (away from the origin)

Note also that conformal maps preserve angles, sometimes this helps determine the image of a conformal map.

More examples of conformal maps

More examples of conformal maps

Example.
Let $D_1 = B(0, 1)$ and $D_2 = \{z \in \mathbb{C} : |z| < 1, \Im(z) > 0\}$. Since these domains are both convex, they are simply-connected, so by Riemann's mapping theorem there is a conformal map sending D_2 to D_1.

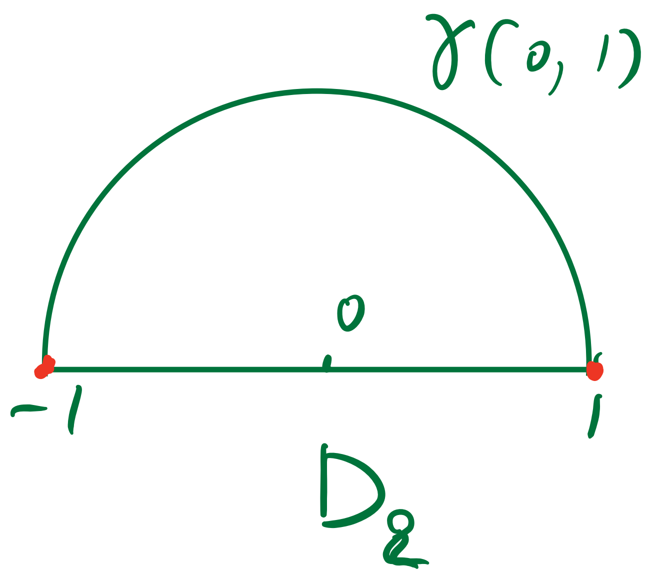


More examples of conformal maps

Example.

Let $D_1 = B(0,1)$ and $D_2 = \{z \in \mathbb{C} : |z| < 1, \Im(z) > 0\}$. Since these domains are both convex, they are simply-connected, so by Riemann's mapping theorem there is a conformal map sending D_2 to D_1.

The boundary of D_2 consists of two curves $\gamma(0,1)$ and $[-1,1]$ which intersect on $-1, 1$.



More examples of conformal maps

Example.
Let $D_1 = B(0,1)$ and $D_2 = \{z \in \mathbb{C} : |z| < 1, \Im(z) > 0\}$. Since these domains are both convex, they are simply-connected, so by Riemann's mapping theorem there is a conformal map sending D_2 to D_1.

The boundary of D_2 consists of two curves $\gamma(0,1)$ and $[-1,1]$ which intersect on $-1, 1$.

We map ± 1 to 0 and ∞ by a Möbius transformation:

$$f(z) = \frac{z-1}{z+1},$$

More examples of conformal maps

Example.
Let $D_1 = B(0,1)$ and $D_2 = \{z \in \mathbb{C} : |z| < 1, \Im(z) > 0\}$. Since these domains are both convex, they are simply-connected, so by Riemann's mapping theorem there is a conformal map sending D_2 to D_1.

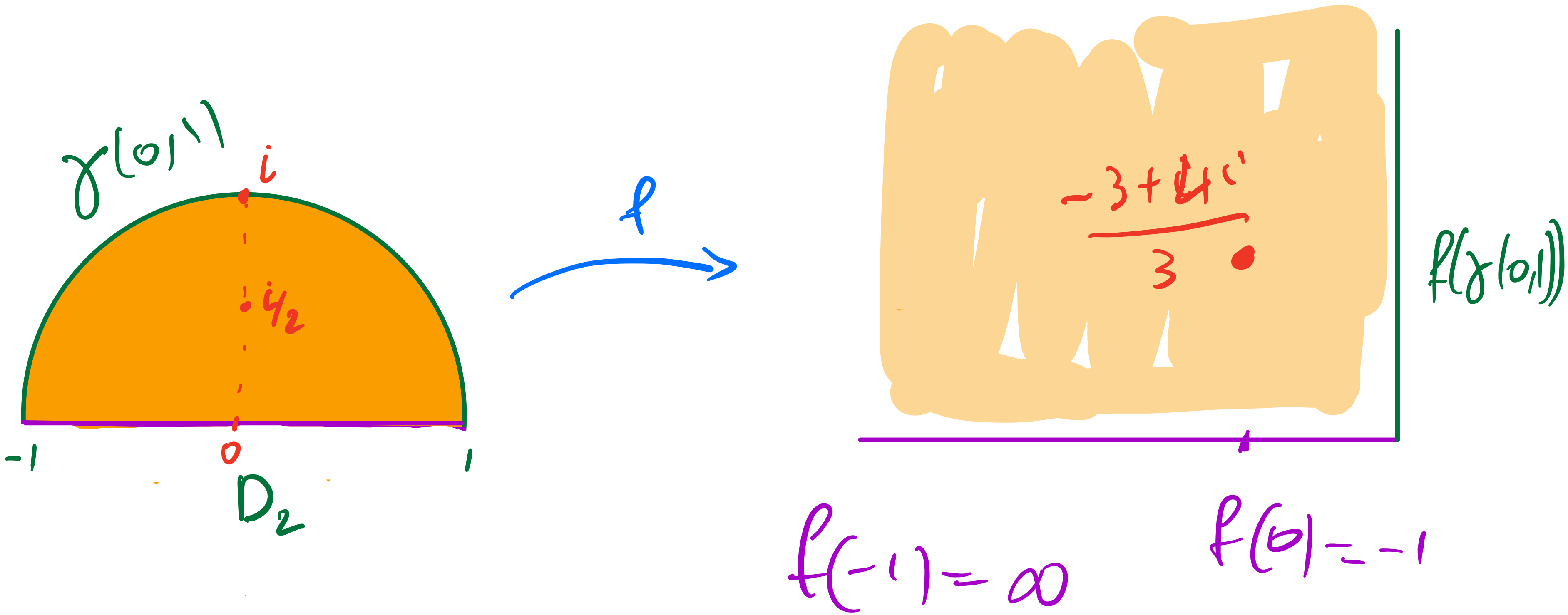
The boundary of D_2 consists of two curves $\gamma(0,1)$ and $[-1,1]$ which intersect on $-1, 1$.

We map ± 1 to 0 and ∞ by a Möbius transformation:

$$f(z) = \frac{z-1}{z+1},$$

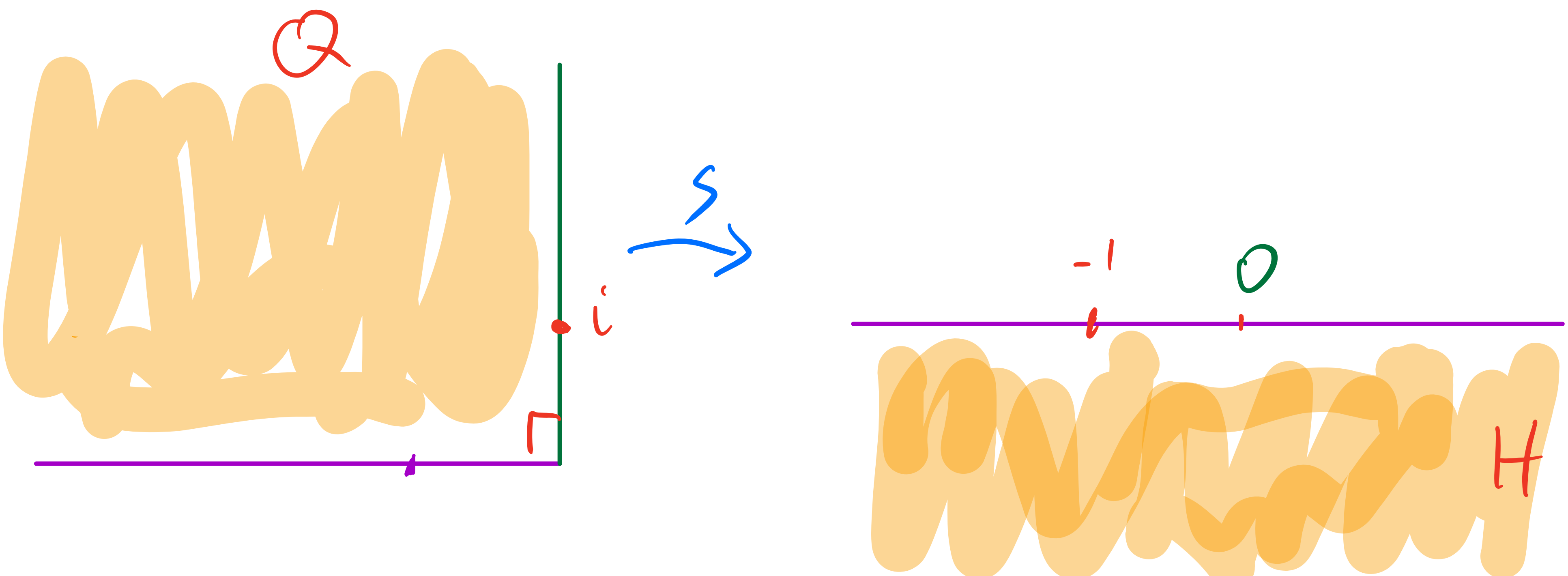
Since f is Möbius and $f(-1) = \infty$, $f(1) = 0$ both $\gamma(0,1)$, $[-1,1]$ map to half lines from 0.

$f(0) = -1$ so $[-1, 1]$ maps to the negative real axis.
$f(i) = \frac{i-1}{i+1} = i$ so $\gamma(0, 1)$ maps to the imaginary axis. Since $f(i/2) = (-3 + 4i)/5$ it follows by connectedness that $f(D_1)$ is the second quadrant $Q = \{w \in \mathbb{C} : \Re(z) < 0, \Im(z) > 0\}$.



$f(0) = -1$ so $[-1, 1]$ maps to the negative real axis. $f(i) = \frac{i-1}{i+1} = i$ so $\gamma(0, 1)$ maps to the imaginary axis. Since $f(i/2) = (-3 + 4i)/5$ it follows by connectedness that $f(D_1)$ is the second quadrant $Q = \{w \in \mathbb{C} : \Re(z) < 0, \Im(z) > 0\}$.

Now the squaring map $s\colon \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$ maps Q bijectively to the lower half-plane $H = \{w \in \mathbb{C} : \Im(w) < 0\}$, and is conformal except at $z = 0$ (0 does not lie in Q).



$f(0) = -1$ so $[-1, 1]$ maps to the negative real axis. $f(i) = \frac{i-1}{i+1} = i$ so $\gamma(0, 1)$ maps to the imaginary axis. Since $f(i/2) = (-3 + 4i)/5$ it follows by connectedness that $f(D_1)$ is the second quadrant $Q = \{w \in \mathbb{C} : \Re(z) < 0, \Im(z) > 0\}$.

Now the squaring map $s\colon \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$ maps Q bijectively to the lower half-plane $H = \{w \in \mathbb{C} : \Im(w) < 0\}$, and is conformal except at $z = 0$ (0 does not lie in Q).

We may then use a Möbius map to take this half-plane to the unit disc: as in a previous example we see that $g(z) = \frac{z+i}{iz+1}$ takes H to the disk.

$f(0) = -1$ so $[-1, 1]$ maps to the negative real axis. $f(i) = \frac{i-1}{i+1} = i$ so $\gamma(0, 1)$ maps to the imaginary axis. Since $f(i/2) = (-3 + 4i)/5$ it follows by connectedness that $f(D_1)$ is the second quadrant $Q = \{w \in \mathbb{C} : \Re(z) < 0, \Im(z) > 0\}$.

Now the squaring map $s \colon \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$ maps Q bijectively to the lower half-plane $H = \{w \in \mathbb{C} : \Im(w) < 0\}$, and is conformal except at $z = 0$ (0 does not lie in Q).

We may then use a Möbius map to take this half-plane to the unit disc: as in a previous example we see that $g(z) = \frac{z+i}{iz+1}$ takes H to the disk.

So $F = g \circ s \circ f$ is a conformal transformation taking D_1 to D_2.

$f(0) = -1$ so $[-1, 1]$ maps to the negative real axis.
$f(i) = \frac{i-1}{i+1} = i$ so $\gamma(0, 1)$ maps to the imaginary axis. Since $f(i/2) = (-3 + 4i)/5$ it follows by connectedness that $f(D_1)$ is the second quadrant $Q = \{w \in \mathbb{C} : \Re(z) < 0, \Im(z) > 0\}$.

Now the squaring map $s\colon \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$ maps Q bijectively to the lower half-plane $H = \{w \in \mathbb{C} : \Im(w) < 0\}$, and is conformal except at $z = 0$ (0 does not lie in Q).

We may then use a Möbius map to take this half-plane to the unit disc: as in a previous example we see that $g(z) = \frac{z+i}{iz+1}$ takes H to the disk.

So $F = g \circ s \circ f$ is a conformal transformation taking D_1 to D_2.
We calculate:

$$F(z) = i\left(\frac{z^2 + 2iz + 1}{z^2 - 2iz + 1}\right)$$

General principles: If we have circular arcs on the boundary we may transform them to half-lines by Möbius transformations that map one of the endpoints to ∞.

General principles: If we have circular arcs on the boundary we may transform them to half-lines by Möbius transformations that map one of the endpoints to ∞.

Branches of fractional power maps $[z^\alpha]$ allow us to change the angle at the points of intersection of arcs of the boundary.

General principles: If we have circular arcs on the boundary we may transform them to half-lines by Möbius transformations that map one of the endpoints to ∞.

Branches of fractional power maps $[z^\alpha]$ allow us to change the angle at the points of intersection of arcs of the boundary.

Möbius transformations allow us to map half planes to discs.

The Laplace equation

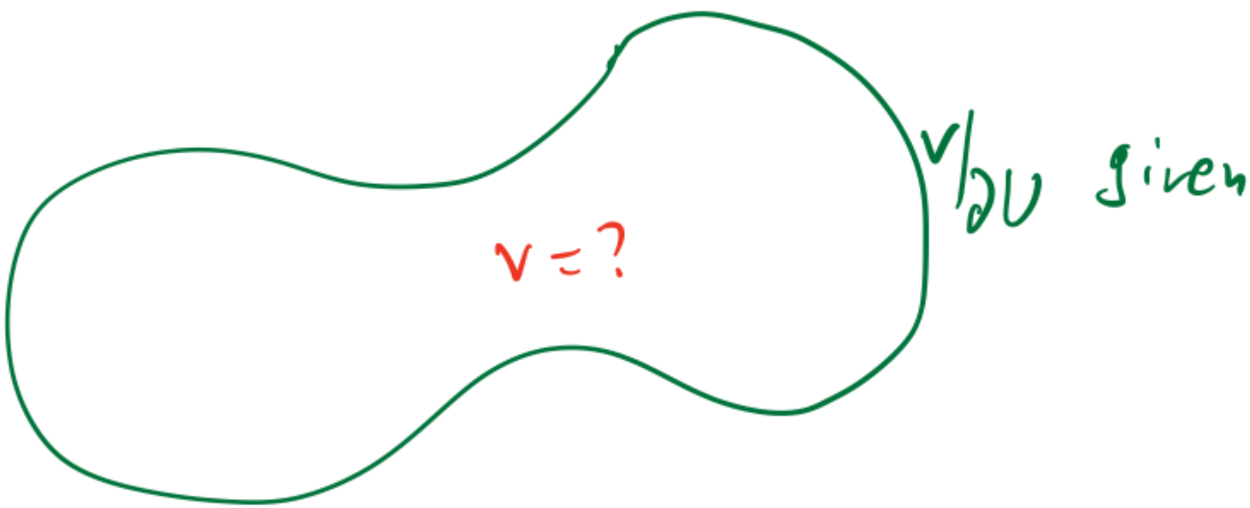
The Laplace equation

We say that a C^2 function $v : \mathbb{R}^2 \to \mathbb{R}$ sarisfies the Laplace equation if $\partial_x^2 v + \partial_y^2 v = 0$.

The Laplace equation

We say that a C^2 function $v : \mathbb{R}^2 \to \mathbb{R}$ sarisfies the Laplace equation if $\partial_x^2 v + \partial_y^2 v = 0$.

A function $v \colon \mathbb{R}^2 \to \mathbb{R}$ is said to be harmonic if it is twice differentiable and $\partial_x^2 v + \partial_y^2 v = 0$. Often one seeks to find solutions to this equation on a domain $U \subset \mathbb{R}^2$ where we specify the values of v on the boundary ∂U of U. This problem is known as the Dirichlet problem.



The Laplace equation

We say that a C^2 function $v : \mathbb{R}^2 \to \mathbb{R}$ sarisfies the Laplace equation if $\partial_x^2 v + \partial_y^2 v = 0$.

A function $v \colon \mathbb{R}^2 \to \mathbb{R}$ is said to be harmonic if it is twice differentiable and $\partial_x^2 v + \partial_y^2 v = 0$. Often one seeks to find solutions to this equation on a domain $U \subset \mathbb{R}^2$ where we specify the values of v on the boundary ∂U of U. This problem is known as the Dirichlet problem.

Lemma

Suppose that $U \subset \mathbb{C}$ is a simply-connected open subset of $\mathbb{C}$ and $v \colon U \to \mathbb{R}$ is twice continuously differentiable and harmonic. Then there is a holomorphic function $f \colon U \to \mathbb{C}$ such that $\Re(f) = v$. In particular, any such function v is analytic.

Proof.

(sketch)Consider the function $g(z) = \partial_x v - i\partial_y v$. Then since v is twice continuously differentiable, the partial derivatives of g are continuous and

$$\partial_x^2 v = -\partial_y^2 v; \quad \partial_y \partial_x v = \partial_x \partial_y v,$$

Proof.

(sketch)Consider the function $g(z) = \partial_x v - i\partial_y v$. Then since v is twice continuously differentiable, the partial derivatives of g are continuous and

$$\partial_x^2 v = -\partial_y^2 v; \quad \partial_y \partial_x v = \partial_x \partial_y v,$$

ie g satisfies the Cauchy-Riemann equations, hence g is holomorphic.

Recall

$f = u + iw$

and $\partial_x u = \partial_y w$

$\partial_x w = -\partial_y u$

$u, w \; C^2$

$\Rightarrow f$ holomorphic

Proof.

(sketch)Consider the function $g(z) = \partial_x v - i\partial_y v$. Then since v is twice continuously differentiable, the partial derivatives of g are continuous and

$$\partial_x^2 v = -\partial_y^2 v; \quad \partial_y \partial_x v = \partial_x \partial_y v,$$

ie g satisfies the Cauchy-Riemann equations, hence g is holomorphic.

Since U is simply-connected, g has a primitive $G\colon U \to \mathbb{C}$, $G = u + iw$.

Proof.

(sketch)Consider the function $g(z) = \partial_x v - i\partial_y v$. Then since v is twice continuously differentiable, the partial derivatives of g are continuous and

$$\partial_x^2 v = -\partial_y^2 v; \quad \partial_y \partial_x v = \partial_x \partial_y v,$$

ie g satisfies the Cauchy-Riemann equations, hence g is holomorphic.

Since U is simply-connected, g has a primitive $G\colon U \to \mathbb{C}$, $G = u + iw$.

$G' = \partial_x u + i\partial_x w = -i\partial_y u + \partial_y w$ so $\partial_x u = \partial_x v$, $\partial_y u = \partial_y v$.

Proof.

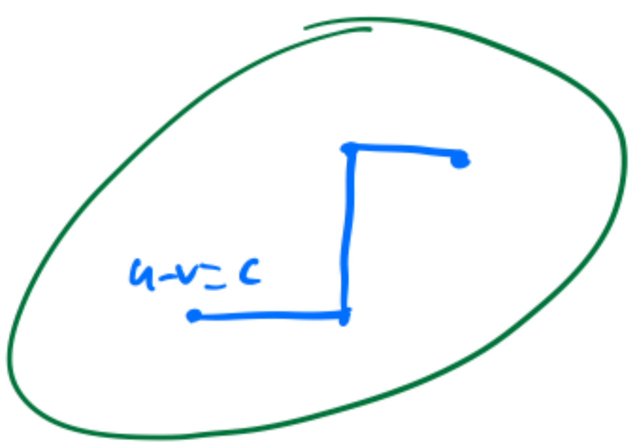
(sketch)Consider the function $g(z) = \partial_x v - i\partial_y v$. Then since v is twice continuously differentiable, the partial derivatives of g are continuous and

$$\partial_x^2 v = -\partial_y^2 v; \quad \partial_y \partial_x v = \partial_x \partial_y v,$$

ie g satisfies the Cauchy-Riemann equations, hence g is holomorphic.

Since U is simply-connected, g has a primitive $G\colon U \to \mathbb{C}$, $G = u + iw$.

$G' = \partial_x u + i\partial_x w = -i\partial_y u + \partial_y w$ so $\partial_x u = \partial_x v$, $\partial_y u = \partial_y v$. It follows that u, v differ by a constant on each vertical and on each horizontal path.



Proof.

(sketch)Consider the function $g(z) = \partial_x v - i\partial_y v$. Then since v is twice continuously differentiable, the partial derivatives of g are continuous and

$$\partial_x^2 v = -\partial_y^2 v; \quad \partial_y \partial_x v = \partial_x \partial_y v,$$

ie g satisfies the Cauchy-Riemann equations, hence g is holomorphic.

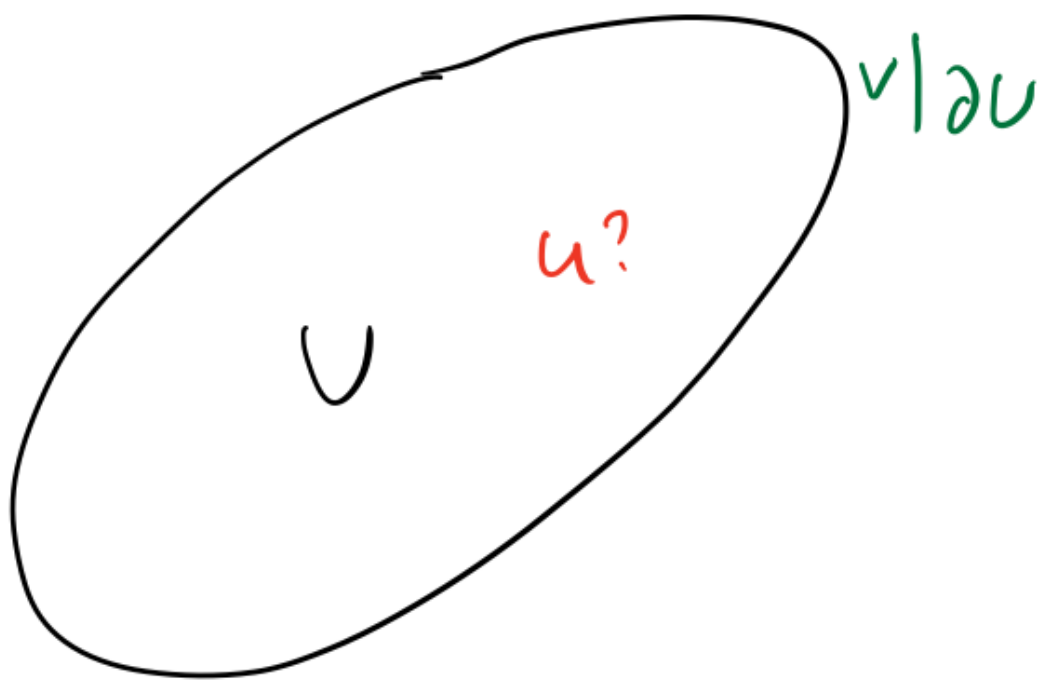
Since U is simply-connected, g has a primitive $G\colon U \to \mathbb{C}$, $G = u + iw$.

$G' = \partial_x u + i\partial_x w = -i\partial_y u + \partial_y w$ so $\partial_x u = \partial_x v$, $\partial_y u = \partial_y v$.
It follows that u, v differ by a constant on each vertical and on each horizontal path.
However since U is open connected there is a path consisting of vertical and horizontal segments joining any two points of U.
It follows that $u - v = c$ a constant and v is the real part of $f = G - c$.

Dirichlet problem and holomorphic maps

Dirichlet problem and holomorphic maps

Recall the Dirichlet Problem: Given a continuous function v on ∂U for some domain U find a harmonic function u extending v to U. So u is continuous on $\overline{U}$ and equal to v on ∂U.



Dirichlet problem and holomorphic maps

Recall the Dirichlet Problem: Given a continuous function v on ∂U for some domain U find a harmonic function u extending v to U. So u is continuous on $\overline{U}$ and equal to v on ∂U.

We have shown that if u is a harmonic function on a simply connected domain U then u is the real part of a holomorphic function. Conversely given a holomorphic function f we obtain a harmonic function by taking its real part.

Dirichlet problem and holomorphic maps

Recall the Dirichlet Problem: Given a continuous function v on ∂U for some domain U find a harmonic function u extending v to U. So u is continuous on $\overline{U}$ and equal to v on ∂U.

We have shown that if u is a harmonic function on a simply connected domain U then u is the real part of a holomorphic function. Conversely given a holomorphic function f we obtain a harmonic function by taking its real part.

So to solve the Dirichlet problem for a simply connected domain U for a given function g on ∂U, it suffices to find a holomorphic function f on U such that $\Re(f) = g$ on the boundary ∂U.

Relationship with conformal mappings

Relationship with conformal mappings

If we have a solution u to the Dirichlet problem for a domain V and $G\colon U \to V$ is a conformal mapping then we can 'transport' our solution to U.

Relationship with conformal mappings

If we have a solution u to the Dirichlet problem for a domain V and $G\colon U \to V$ is a conformal mapping then we can 'transport' our solution to U.

This is because (locally) u is the real part of a holomorphic function f and $f \circ G$ is holomorphic.
Precisely we have:

Relationship with conformal mappings

If we have a solution u to the Dirichlet problem for a domain V and $G\colon U \to V$ is a conformal mapping then we can 'transport' our solution to U.

This is because (locally) u is the real part of a holomorphic function f and $f \circ G$ is holomorphic.
Precisely we have:

Lemma
If U and V are domains and $G\colon U \to V$ is a conformal transformation, then if $u\colon V \to \mathbb{R}$ is a harmonic function on V, the composition $u \circ G$ is harmonic on U.

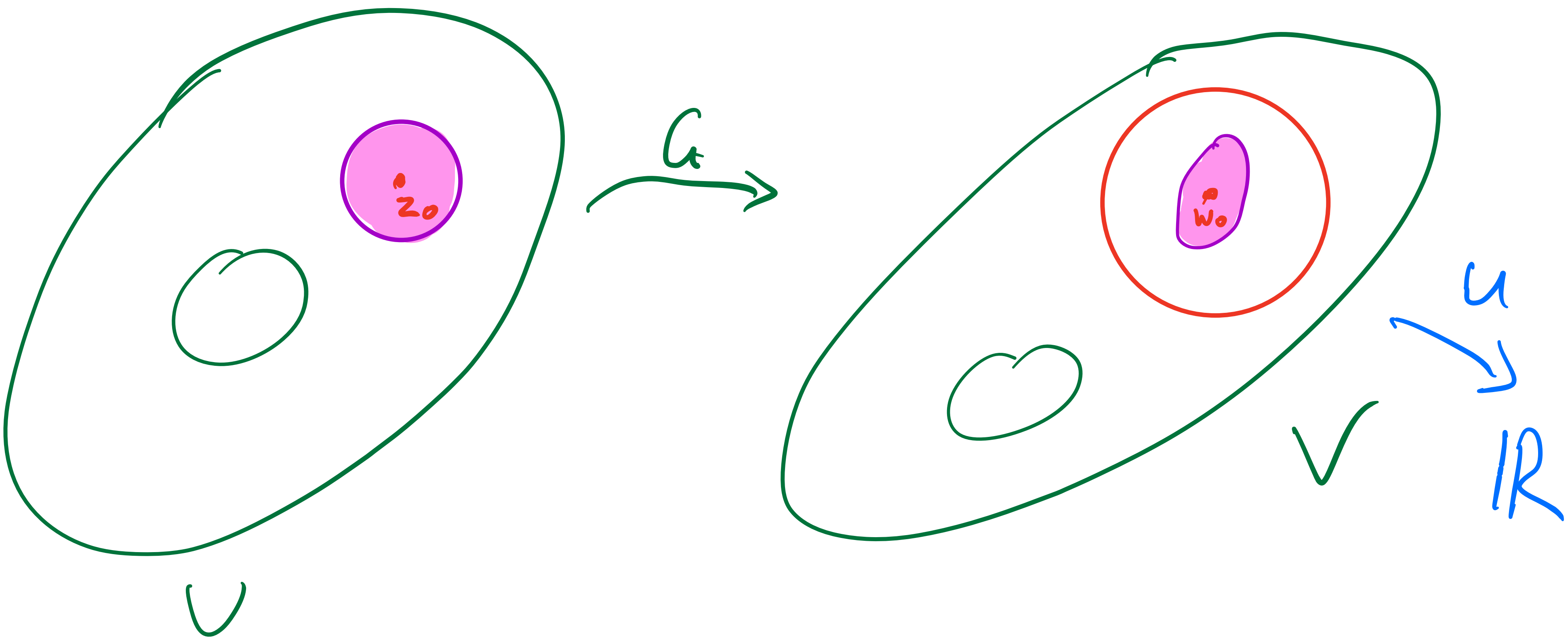
Proof.

To see that $u \circ G$ is harmonic we need only check this in a disk $B(z_0, r) \subseteq U$ about any point $z_0 \in U$.

Proof.

To see that $u \circ G$ is harmonic we need only check this in a disk $B(z_0, r) \subseteq U$ about any point $z_0 \in U$.

There are $\delta, \epsilon > 0$ such that $G(B(z_0, \delta)) \subseteq B(w_0, \epsilon) \subseteq V$.



Proof.

To see that $u \circ G$ is harmonic we need only check this in a disk $B(z_0, r) \subseteq U$ about any point $z_0 \in U$.

There are $\delta, \epsilon > 0$ such that $G(B(z_0, \delta)) \subseteq B(w_0, \epsilon) \subseteq V$.

But now since $B(w_0, \epsilon)$ is simply-connected we can find a holomorphic function $f(z)$ with $u = \Re(f)$.

Proof.

To see that $u \circ G$ is harmonic we need only check this in a disk $B(z_0, r) \subseteq U$ about any point $z_0 \in U$.

There are $\delta, \epsilon > 0$ such that $G(B(z_0, \delta)) \subseteq B(w_0, \epsilon) \subseteq V$.

But now since $B(w_0, \epsilon)$ is simply-connected we can find a holomorphic function $f(z)$ with $u = \Re(f)$.

But then on $B(z_0, \delta)$ we have $u \circ G = \Re(f \circ G)$, and by the chain rule $f \circ G$ is holomorphic, so its real part is harmonic. □

By the Riemann mapping theorem there is a conformal map from any proper simply connected open subset of $\mathbb{C}$ to the disk.

By the Riemann mapping theorem there is a conformal map from any proper simply connected open subset of $\mathbb{C}$ to the disk.

Strategy in two steps for solving the Dirichlet problem on a simply connected domain U.
We are given a continuous function $h : \partial U \to \mathbb{R}$ and we would like to extend this to a harmonic function defined on U.

By the Riemann mapping theorem there is a conformal map from any proper simply connected open subset of $\mathbb{C}$ to the disk.

Strategy in two steps for solving the Dirichlet problem on a simply connected domain U.
We are given a continuous function $h : \partial U \to \mathbb{R}$ and we would like to extend this to a harmonic function defined on U.

Step 1: Find a conformal map $G : U \to \mathbb{D}$ where $\mathbb{D} = B(0, 1)$.
We need to check then that G extends continuously to the boundary ∂U.

By the Riemann mapping theorem there is a conformal map from any proper simply connected open subset of $\mathbb{C}$ to the disk.

Strategy in two steps for solving the Dirichlet problem on a simply connected domain U.
We are given a continuous function $h : \partial U \to \mathbb{R}$ and we would like to extend this to a harmonic function defined on U.

Step 1: Find a conformal map $G : U \to \mathbb{D}$ where $\mathbb{D} = B(0, 1)$.
We need to check then that G extends continuously to the boundary ∂U.

Then $h_1 = h \circ G^{-1}$ is a continuous function on $\partial \mathbb{D}$.

By the Riemann mapping theorem there is a conformal map from any proper simply connected open subset of $\mathbb{C}$ to the disk.

Strategy in two steps for solving the Dirichlet problem on a simply connected domain U.
We are given a continuous function $h : \partial U \to \mathbb{R}$ and we would like to extend this to a harmonic function defined on U.

Step 1: Find a conformal map $G : U \to \mathbb{D}$ where $\mathbb{D} = B(0, 1)$.
We need to check then that G extends continuously to the boundary ∂U.

Then $h_1 = h \circ G^{-1}$ is a continuous function on $\partial \mathbb{D}$.

Step 2: Solve the Diriclet problem on the disk $\mathbb{D}$, i.e. find a harmonic function u_1 extending h_1 to the whole of $\mathbb{D}$. Then $u = G \circ u_1$ is harmonic on U and equal to h on ∂U.

Step 1: The Riemann mapping theorem states that *every* domain which is simply connected, other than the whole complex plane itself, is in fact conformally equivalent to $B(0, 1)$.

Step 1: The Riemann mapping theorem states that *every* domain which is simply connected, other than the whole complex plane itself, is in fact conformally equivalent to $B(0, 1)$.

For the solution of Dirichlet's problem one needs something slightly stronger:

Theorem

Let U, V be bounded domains in $\mathbb{C}$ and let $f : U \to V$ be a conformal map. If $\partial U, \partial V$ are piecewise C^1 simple closed curves the conformal map $f : U \to V$ can be extended to a homeomorphism $\bar{f} : \bar{U} \to \bar{V}$.

(for a proof see the book Introduction to Complex Analysis by K. Kodaira, p. 215)

Step 2: Suppose that u is a harmonic function defined on $B(0, r)$ for some $r > 1$. Then there is a holomorphic function $f\colon B(0, r) \to \mathbb{C}$ such that $u = \Re(f)$.

Step 2: Suppose that u is a harmonic function defined on $B(0, r)$ for some $r > 1$. Then there is a holomorphic function $f: B(0, r) \to \mathbb{C}$ such that $u = \Re(f)$.

We sketch this argument now (off syllabus). By Cauchy's integral formula, if γ is a parametrization of the positively oriented unit circle, then for all $w \in B(0, 1)$ we have $f(w) = \frac{1}{2\pi i} \int_\gamma f(z)/(z - w) dz$, and so

$$u(z) = \Re\left(\frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{z - w}\right).$$

Step 2: Suppose that u is a harmonic function defined on $B(0, r)$ for some $r > 1$. Then there is a holomorphic function $f: B(0, r) \to \mathbb{C}$ such that $u = \Re(f)$.

We sketch this argument now (off syllabus). By Cauchy's integral formula, if γ is a parametrization of the positively oriented unit circle, then for all $w \in B(0, 1)$ we have $f(w) = \frac{1}{2\pi i} \int_\gamma f(z)/(z - w)dz$, and so

$$u(z) = \Re\left(\frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{z - w}\right).$$

Since the integrand uses only the values of f on the boundary circle, we have almost recovered the function u from its values on the boundary. But we need the values of f rather than u on the boundary. The next lemma gives an expression that only depends on u.

Lemma

If u is harmonic on $B(0,r)$ *for* $r > 1$ *then for all* $w \in B(0,1)$ *we have*

$$u(w) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\frac{1-|w|^2}{|e^{i\theta}-w|^2}d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\Re\left(\frac{e^{i\theta}+w}{e^{i\theta}-w}\right)d\theta.$$

Proof (*Sketch*.) Let $f(z)$ be holomorphic with $\Re(f) = u$ on $B(0, r)$. Then letting γ be a parametrization of the positively oriented unit circle we have

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-w} - \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-\bar{w}^{-1}}$$

where the first term is $f(w)$ by the integral formula and the second term is zero because $f(z)/(z-\bar{w}^{-1})$ is holomorphic inside all of $B(0, 1)$. So

$|\bar{w}^{-1}| > 1$

Proof (*Sketch.*) Let $f(z)$ be holomorphic with $\Re(f) = u$ on $B(0, r)$. Then letting γ be a parametrization of the positively oriented unit circle we have

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-w} - \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-\bar{w}^{-1}}$$

where the first term is $f(w)$ by the integral formula and the second term is zero because $f(z)/(z-\bar{w}^{-1})$ is holomorphic inside all of $B(0, 1)$. So

$$f(w) = \frac{1}{2\pi}\int_\gamma f(z)\frac{1-|w|^2}{|z-w|^2}\frac{dz}{iz} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{1-|w|^2}{|e^{i\theta}-w|^2}d\theta.$$

$\boxed{\bar{z} = \frac{1}{z}}$

$$\frac{1}{z-w} - \frac{1}{z-\bar{w}^{-1}} = \frac{z - \frac{1}{\bar{w}} - z + w}{z(1-w\bar{z})\frac{(z\bar{w}-1)}{\bar{w}}} = \frac{1}{z}\cdot\frac{1-|w|^2}{|1-w\bar{z}|^2}$$

$$|1-w\bar{z}|^2 = |z\bar{z}-w\bar{z}|^2 = |z-w|^2$$

Proof (*Sketch.*) Let $f(z)$ be holomorphic with $\Re(f) = u$ on $B(0, r)$. Then letting γ be a parametrization of the positively oriented unit circle we have

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z - w} - \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z - \bar{w}^{-1}}$$

where the first term is $f(w)$ by the integral formula and the second term is zero because $f(z)/(z - \bar{w}^{-1})$ is holomorphic inside all of $B(0, 1)$. So

$$f(w) = \frac{1}{2\pi}\int_\gamma f(z)\frac{1 - |w|^2}{|z - w|^2}\frac{dz}{iz} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{1 - |w|^2}{|e^{i\theta} - w|^2}d\theta.$$

The real part is

$$u(w) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\frac{1 - |w|^2}{|e^{i\theta} - w|^2}d\theta.$$

Finally for the second integral expression note that if $|z| = 1$ then

$$\frac{z+w}{z-w} = \frac{(z+w)(\bar{z}-\bar{w})}{|z-w|^2} = \frac{1-|w|^2+(\bar{z}w-z\bar{w})}{|z-w|^2}.$$

from which one readily sees the real part agrees with the corresponding factor in our first expression. □

Finally for the second integral expression note that if $|z| = 1$ then

$$\frac{z+w}{z-w} = \frac{(z+w)(\bar{z}-\bar{w})}{|z-w|^2} = \frac{1-|w|^2+(\bar{z}w - z\bar{w})}{|z-w|^2}.$$

from which one readily sees the real part agrees with the corresponding factor in our first expression. □

Given now a function $h : \partial\mathbb{D} \to \mathbb{R}$ we define

$$u(w) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{i\theta})\frac{1-|w|^2}{|e^{i\theta}-w|^2}d\theta. \quad (*)$$

Finally for the second integral expression note that if $|z| = 1$ then

$$\frac{z+w}{z-w} = \frac{(z+w)(\bar{z}-\bar{w})}{|z-w|^2} = \frac{1-|w|^2+(\bar{z}w-z\bar{w})}{|z-w|^2}.$$

from which one readily sees the real part agrees with the corresponding factor in our first expression. □

Given now a function $h : \partial\mathbb{D} \to \mathbb{R}$ we define

$$u(w) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{i\theta})\frac{1-|w|^2}{|e^{i\theta}-w|^2}d\theta. \quad (*)$$

As we have seen in the proof of the lemma this is the real part of

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{h(z)dz}{z-w}$$

which is clearly holomorphic. So its real part u is harmonic.

Finally for the second integral expression note that if $|z| = 1$ then

$$\frac{z+w}{z-w} = \frac{(z+w)(\bar{z}-\bar{w})}{|z-w|^2} = \frac{1-|w|^2+(\bar{z}w - z\bar{w})}{|z-w|^2}.$$

from which one readily sees the real part agrees with the corresponding factor in our first expression. □

Given now a function $h : \partial\mathbb{D} \to \mathbb{R}$ we define

$$u(w) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{i\theta})\frac{1-|w|^2}{|e^{i\theta}-w|^2}d\theta. \quad (*)$$

As we have seen in the proof of the lemma this is the real part of

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{h(z)dz}{z-w}$$

which is clearly holomorphic. So its real part u is harmonic. It remains to show that as $z \to z_0 \in \partial\mathbb{D}$, $u(z) \to h(z_0)$ for all $z_0 \in \partial\mathbb{D}$.

To see this applying $(*)$ to the constant function 1 we get

$$1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |w|^2}{|e^{i\theta} - w|^2} d\theta,$$

To see this applying $(*)$ to the constant function 1 we get

$$1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |w|^2}{|e^{i\theta} - w|^2} d\theta,$$

if $w_0 = e^{i\theta_0}$

$$u(w) - h(w_0) = \frac{1}{2\pi} \int_0^{2\pi} (h(e^{i\theta}) - h(e^{i\theta_0})) \frac{1 - |w|^2}{|e^{i\theta} - w|^2} d\theta.$$

To see this applying $(*)$ to the constant function 1 we get

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|w|^2}{|e^{i\theta}-w|^2} d\theta,$$

if $w_0 = e^{i\theta_0}$

$$u(w) - h(w_0) = \frac{1}{2\pi}\int_0^{2\pi} (h(e^{i\theta}) - h(e^{i\theta_0})) \frac{1-|w|^2}{|e^{i\theta}-w|^2} d\theta.$$

We split the integral in two parts. If $J = [\theta_0 - \delta, \theta_0 + \delta]$ for some 'small' δ and $I = [0, 2\pi] - J$ we have that

To see this applying $(*)$ to the constant function 1 we get

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|w|^2}{|e^{i\theta}-w|^2}\,d\theta,$$

if $w_0 = e^{i\theta_0}$

$$u(w) - h(w_0) = \frac{1}{2\pi}\int_0^{2\pi} (h(e^{i\theta}) - h(e^{i\theta_0}))\frac{1-|w|^2}{|e^{i\theta}-w|^2}\,d\theta.$$

We split the integral in two parts. If $J = [\theta_0 - \delta, \theta_0 + \delta]$ for some 'small' δ and $I = [0, 2\pi] - J$ we have that

$$\frac{1}{2\pi}\int_J (h(e^{i\theta}) - 1)\frac{1-|w|^2}{|e^{i\theta}-w|^2}\,d\theta$$

is 'small'.

On the other hand if we assume that $|w - w_0| < \epsilon$ for some ϵ 'much smaller' than δ we have that

$$\frac{1 - |w|^2}{|e^{i\theta} - w|^2} \leq \frac{2\epsilon}{\delta}$$

for $\theta \in I$.

On the other hand if we assume that $|w - w_0| < \epsilon$ for some ϵ 'much smaller' than δ we have that

$$\frac{1 - |w|^2}{|e^{i\theta} - w|^2} \leq \frac{2\epsilon}{\delta}$$

for $\theta \in I$.
So the integral

$$\frac{1}{2\pi} \int_I (h(e^{i\theta}) - 1) \frac{1 - |w|^2}{|e^{i\theta} - w|^2} d\theta$$

is 'small' as well,

On the other hand if we assume that $|w - w_0| < \epsilon$ for some ϵ 'much smaller' than δ we have that

$$\frac{1 - |w|^2}{|e^{i\theta} - w|^2} \leq \frac{2\epsilon}{\delta}$$

for $\theta \in I$.
So the integral

$$\frac{1}{2\pi} \int_I (h(e^{i\theta}) - 1) \frac{1 - |w|^2}{|e^{i\theta} - w|^2} d\theta$$

is 'small' as well,
which proves the continuity of $u(w)$ at w_0. $\square$