B8.5 Graph Theory

Sheet 0 — MT23

Solutions for students

Suggested use: having tried the questions, read a solution or two and compare with yours. If there are major differences (in level of detail), try to modify your solutions to the next questions before looking at mine, to see if you can end up with a close match by the end of the sheet. The solutions are one to a page for this reason!

(Of course, for some questions there are a number of different approaches possible; this is more about the level of rigour.)

As always, if you find an error please check the website, and if it has not already been corrected, e-mail: `Paul.Balister@maths.ox.ac.uk`.

1. Let x and y be vertices of a graph G. Show that G contains an (i.e., at least one) x-y walk if and only if G contains an x-y path.

The definition of an x-y path is exactly that it's an x-y walk $v_0v_1 \cdots v_t$ in which $v_0, \ldots, v_t$ are distinct, so any x-y path is an x-y walk. Thus one direction is trivial.

For the converse, suppose that G contains an x-y walk $W = v_0 \cdots v_t$. If W is a path then we are done. Otherwise, by definition, there are $0 \leq i < j \leq t$ such that $v_i = v_j$. (There may be several such pairs.) Let $W' = v_0 \cdots v_i v_{j+1} \cdots v_t$.[1] This is an x-y walk: the first vertex is x, the last is y, and each is joined to the next since W is a walk and (if $j < t$) $v_i v_{j+1} = v_j v_{j+1} \in E(G)$. Also W' is strictly shorter than W: it has length $t - (j - i) < t$. If W' is a path, we are done. Otherwise repeat – this cannot continue indefinitely, so there exists an x-y path.

[This is the argument as one might first think of it; the slick proof is to start by saying that among all x-y walks there is (at least) one with minimum length, and start from that. Be careful that a walk may intersect itself in many ways; if you try to remove all the 'loops' in one go things may well go wrong.]

[1] As usual in maths, if $j = t$ then we interpret $v_{j+1} \cdots v_t$ as being an empty list, so in this case $W' = v_0 \cdots v_i$

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2. Let $G = (V, E)$ be a graph, and define a relation $\sim$ on V by $x \sim y$ if x and y are connected in G, i.e., if there is an x-y path/walk in G (possibly of length 0). Show, giving full details, that $\sim$ is an equivalence relation.

The relation is reflexive since for any vertex x, $v_0 = x$ is an x-x path/walk of length 0.

If $x = v_0v_1 \cdots v_t = y$ is an x-y path/walk of length $t \geq 0$, then $v_tv_{t-1} \cdots v_0$ is a y-x path/walk, so the relation is symmetric.

For transitivity, suppose $x \sim y$ and $y \sim z$. Then there exist paths/walks $x = v_0 \cdots v_t = y$ and $y = w_0 \cdots w_s = z$. Then $v_0 \cdots v_tw_1 \cdots w_s$ is an x-z *walk*: the first vertex is x, the last is z, and each is adjacent to the next, from the walks we started with plus the fact that $v_tw_1 = w_0w_1$. (To be really complete, $s = 0$ needs a trivial special case. Or it's ok to write $v_0 \cdots v_t = w_0 \cdots w_s$; the middle entry only appears once.)

[For the first two parts you can work with paths or walks; it makes no difference. For the last, walks are better – even if we start with paths, the combined walk may not be a path. But this is fine; we know that an x-z walk guarantees the existence of an x-z path by question 1.]

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3. [A little tedious; omit if you like.] Check that any graph G is the disjoint union of its components (maximal connected subgraphs). It may help to first show that the components correspond to equivalence classes of the relation $\sim$ in the previous question.

Let $G = (V, E)$. Define the relation $\sim$ as in question 2; this is an equivalence relation on the set V, so it partitions V into equivalence classes $V_1, \dots, V_k$. These will be the vertex sets of the components: let $G_i = G[V_i]$, the graph with vertex set V_i containing all edges of G with both ends in V_i. Then we expect that the G_i are the components.

Firstly, lets check that G is the disjoint union of $G_1, \dots, G_k$. They are (vertex)[2] disjoint since equivalence classes are disjoint. Also, $V(G_1 \cup \dots \cup G_k) = V_1 \cup \dots \cup V_k = V$. Certainly $E(G_1 \cup \dots \cup G_k) \subseteq E(G)$, so we just need to check that every edge of G is an edge of some G_i. But if $xy \in E(G)$ then there is an x-y path (xy) of length 1, so $x \sim y$ and x, y are in the same equivalence class V_i. Then $xy \in E(G_i)$.

Secondly, we need to show that the G_i are the components (maximal connected subgraphs) of G.

(i) each G_i is connected: if $x, y \in V(G_i)$, then $x, y \in V_i$, so $x \sim y$. Hence there is an x-y path $P = v_0 v_1 \cdots v_t$ in G. For each vertex v_s on the path there is an x-v_s path, namely $v_0 v_1 \cdots v_s$; so $x \sim v_s$ and also $v_s \in V_i$.[3] It follows that P is a path in G_i (all the vertices are in G_i and, from the way we defined G_i, so are all the edges). Since x and y were any two vertices of G_i, then G_i is connected.

(ii) we need to show maximality. So suppose G_i is a strict subgraph of H, itself a connected subgraph of G. We cannot have $V(H) = V(G_i) = V_i$ since G_i already contains all edges of G within the set V_i. So H contains a vertex $x \notin V_i$. Let $y \in V_i$. Then y is a vertex of G_i and so of H. Since H is connected, there is an x-y path in H, and hence in G. Thus $x \sim y$. But this contradicts $y \in V_i$, $x \notin V_i$.

[2] Vertex disjointness is the strongest notion of disjointness for graphs; if two graphs have no common vertices, they can't have any common edges. So we often just say 'disjoint' to mean vertex disjoint. If we want to allow common vertices but not common edges, we say 'edge disjoint'.

[3] Or, say that $v_{s-1}v_s \in E(G)$ for each s, so $v_{s-1} \sim v_s$, so (formally by induction on s) all the v_s are in the same equivalence class.

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4. Show that TFAE (The Following Are Equivalent): (a) T is a tree, (b) T is a minimal (w.r.t. edges) connected graph, (c) T is a maximal (w.r.t. edges) acyclic graph.

(a)$\Rightarrow$(b): Let $T = (V, E)$ be a tree. Then T is connected by definition. If $T - e$ were connected for some edge $e = xy$ of T then there would exist an x-y path in $T - e$, say $x = v_0v_1 \cdots v_t = y$. Now $t \geq 2$, since $xy = e$ is not an edge in $T - e$. Thus $v_0, v_1, \dots, v_t$ are ≥ 3 distinct vertices of T. We have $v_0v_1, \dots, v_{t-1}v_t \in E(T - e) \subseteq E(T)$ (from the path) and $v_tv_0 = xy \in E(T)$. Thus $v_0 \cdots v_t$ defines a cycle in T, contradicting T being a tree.[4]

(b)$\Rightarrow$(a): Conversely, suppose that T is a minimal connected graph. If T contains a cycle $v_1v_2 \cdots v_k$, then removing the edge $e = v_1v_k$ from T leaves $T - e$ connected: for any two vertices u and v, there is a path from u to v in T, and if that path uses e, we can replace e by with $v_1v_2 \cdots v_k$ or $v_k \cdots v_2v_1$ to obtain a *walk* (there may be some repeated vertices) from u to v in $T - e$. Thus $T - e$ would be connected, a contradiction. Hence T is acyclic and hence a tree.

(a)$\Rightarrow$(c): Let $T = (V, E)$ be a tree. Then T is acyclic by definition. Suppose $T' = (V, E')$ with $E \subsetneq E'$ and $e = xy \in E' \setminus E$. Then as T is connected, there exists an x-y path $x = v_0 \cdots v_t = y$ in T. It is of length at least 2 as $xy \notin E(T)$. Thus $v_0v_1 \cdots v_tv_0$ is a cycle in T'.

(c)$\Rightarrow$(a): Conversely, suppose T is a maximal acyclic graph. It T were not connected, there would exist $x, y \in V$ with no x-y walk in T. In particular $xy \notin E(T)$. Consider $T' = T + xy$. If there were a cycle in T' then the cycle would have to use the edge xy, as otherwise it would be a cycle in T. Let the cycle be $xyv_1 \cdots v_tx$. As $x, y, v_1, \dots, v_t$ are distinct, none of the edges $yv_1, v_1v_2, \dots, v_tx$ are equal to xy, and so all lie in T. But then $xv_t \cdots v_1y$ is an x-y path in T, a contradiction. Hence T is connected and hence a tree.

[4] Actually, to be really complete we should consider deleting more than one edge too: if $T' = (V, E')$ with $E' \subsetneq E$, then pick any $e \in E \setminus E'$. Then $T - e$ is not connected and T' is a spanning subgraph of $T - e$. It follows that T' is not connected (any path in T' is a path in $T - e$).

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5. Modify the argument in lectures to show that any tree with at least two vertices has at least two leaves.

Let T be a tree with at least two vertices. If any vertex v has degree 0 then T is not connected, a contradiction. Indeed, pick $u \neq v$ and consider a v-u path $vv_1 \dots u$. Then v_1 is a neighbour of v and so $d(v) \geq 1$. If at least two vertices have degree 1 we are done. Otherwise, there is at most 1 vertex with degree 1, and all others have degree at least 2.[5] In particular there is a vertex v_0 (with degree ≥ 1) such that every other vertex has degree at least 2. We now obtain a contradiction by (carefully!) building a cycle.

Let v_1 be a neighbour of v_0 (which exists as $d(v_0) \geq 1$). Having chosen v_i, if $v_i = v_0$ then we stop; otherwise $d(v_i) \geq 2$ so v_i has a neighbour v_{i+1} other than v_{i-1}. This way we construct a finite or infinite sequence $v_0, v_1, \dots$ such that (as in lectures) any three consecutive vertices are distinct. There must exist $i < j$ such that $v_i = v_j$ (either because we stop with $v_i = v_0$, or because the sequence is infinite). The rest of the argument is exactly as in lectures: $j - i \geq 3$ since any three consecutive vertices are distinct, and then $v_i \cdots v_{j-1}$ forms a cycle.

[An incorrect proof would be as follows: start at v_0, which has $d(v_0) \geq 1$ so we can choose a neighbour v_i. For $i \geq 1$, v_i has degree ≥ 2, so we can find a neighbour v_{i+1} of v_i other than v_{i-1}, continue as in lectures. The problem is that we don't know that $v_i \neq v_0$ in general. We fix this by stopping if we repeat – in this case we've already built a cycle.]

[5] Or say if all ≥ 2 then done in lectures, so we can assume exactly one has degree 1, the rest ≥ 2.

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6. Let T be a tree with $|T| \geq 2$, and let P be a longest path in T. Prove, giving full details, that the ends of P are leaves. Deduce that T has at least two leaves.

Let $P = v_0 v_1 \cdots v_t$ be a longest path in T. It's crucial (and trivial!) to note that $t \geq 1$: T certainly has at last one edge (a graph with ≥ 2 vertices and no edges is not connected), so there is at least one path in G of length at least 1. So the two ends of P are distinct.

Also, v_0 has one neighbour v_1 (using $t \geq 1$). Suppose for a contradiction that it has another neighbour w. Then either w is not on the path; but then $wv_0 \cdots v_t$ is a longer path in T, a contradiction. Or w is on the path, so $w = v_i$ for some $2 \leq i \leq t$. But then $v_0 v_1 \cdots v_i$ defines a cycle in T, a contradiction. So v_0 has exactly one neighbour in T and is a leaf. Similarly for v_t.

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7. Show that any two vertices of a tree T are joined by a *unique* path in T.

By definition (connectedness) any two vertices of a tree T are joined by at least one path, so we have to rule out two vertices joined by two or more paths. I'll give two proofs: a pedestrian one, and a 'slick' one.

Pedestrian: (You really need to draw a picture as you read this!)

Let v and w be vertices of T, and suppose for a contradiction that there are at least two v-w paths in T. Let $P_1 = x_0x_1 \cdots x_r$ and $P_2 = y_0y_1 \cdots y_s$ be two different v-w paths, with $x_0 = y_0 = v$ and $x_r = y_s = w$.

Since $P_1 \neq P_2$ there is some $i \leq \min\{r, s\}$ such that $x_i \neq y_i$. (Otherwise, one of the paths would have to extend the other. If, wlog, P_2 were longer, we would have $y_r = x_r = w$ and $y_s = w$, contradicting $y_0, y_1, \ldots, y_s$ being distinct.) Pick the least such i, so $x_k = y_k$ for $0 \leq k \leq i - 1$.

There is some $j \geq i$ such that y_j is a vertex of P_1; indeed, this holds for $j = s$ since $y_s = w = x_r$. Pick the least $j \geq i$ with this property. Since y_j is on P_1 we have $y_j = x_k$ for some k. Since $y_j \notin \{y_1, \ldots, y_{i-1}\} = \{x_1, \ldots, x_{i-1}\}$, we have $k \geq i$.

Since (by definition of j) the vertices $y_i, y_{i+1}, \ldots, y_{j-1}$ are not on P_1, the vertices $x_{i-1} = y_{i-1}, y_i, y_{i+1}, \ldots, y_j = x_k, x_{k-1}, \ldots, x_i$ are distinct. Also, each vertex in this list is adjacent to the next, and the last to the first. (The relevant edges are all edges of P_1 or of P_2.) Finally, there are $k - i + 1 + j - i + 1$ vertices in this list. This number is at least 3, since otherwise $k = i$ and $j = i$, contradicting $x_k = y_j$ and $x_i \neq y_i$. Hence $x_{i-1} = y_{i-1}, y_i, y_{i+1}, \ldots, y_j = x_k, x_{k-1}, \ldots, x_i$ is a cycle in T, contradicting our assumption that T is a tree.

Slick(ish; you can say it shorter, but with details it's not that short):

Suppose that, somewhere in T, there exist two distinct paths P_1 and P_2 with the same start and endpoints. Pick such a pair with the sum of the lengths of P_1 and P_2 minimal. Say $P_1 = x_0x_1 \cdots x_r$ and $P_2 = y_0y_1 \cdots y_s$, with $x_0 = y_0 = v$ and $x_r = y_s = w$, say. We can't have $v = w$ (the only v-w path has length 0). So $r, s > 0$. We can't have $r = s = 1$, otherwise the paths are the same. If the paths share no vertices other than v and w then, since $r + s \geq 3$, $x_0 \cdots x_r = y_sy_{s-1} \cdots y_0$ forms a cycle, a contradiction. So the paths meet in some other vertex u. But then $u = x_i = y_j$ for some $0 < i < r$ and $0 < j < s$. Then $P_1' = x_0 \cdots x_i$ and $P_2' = y_0 \cdots y_j$ is a pair of (not necessarily distinct) paths with the same start and end, shorter than the pair we started with. Similarly, $P_1'' = x_i \cdots x_r$ and $P_2'' = y_j \cdots y_s$ is a such a pair. By minimality of the original pair we thus have $P_1' = P_2'$ and $P_1'' = P_2''$. But then $P_1 = P_2$, a contradiction.

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8. Let $(d_1, \dots, d_n)$ be a sequence of integers with $n \geq 2$. Show that there is a tree on $[n]$ with $d(i) = d_i$ for each i if and only if $d_i \geq 1$ for all i and $\sum_{i=1}^{n} d_i = 2n - 2$.

Suppose first that there is a tree on $[n]$ with $d(i) = d_i$ for every i. Then (as in lectures) T is a tree with $|T| \geq 2$, so every vertex has degree at least 1, i.e., $d_i \geq 1$ for every i. Also, $\sum d_i = \sum_{i \in V(T)} d(i) = 2e(T) = 2(n-1) = 2n - 2$ by the handshaking lemma.

For the reverse direction we use induction on n. The case $n = 2$ is easy. (The only sequence is $(1, 1)$.)

Suppose that $n \geq 3$ and the result holds for $n - 1$. We prove it for n. Crucially, this means that we start with a *sequence*, not a tree.[6] More precisely, let $(d_1, \dots, d_n)$ be any sequence satisfying the given conditions. We must build a corresponding n-vertex tree.

[Informally: our plan is to use induction, which allows us to get a tree from a sequence of length $n - 1$. So we try to shorten our sequence appropriately, get an $n - 1$ vertex tree, and extend that.]

To do this, first note that there is some i (wlog $i = n$) such that $d_i = 1$. Otherwise $d_i \geq 2$ for every i and so $\sum d_i \geq 2n > 2n - 2$. Also, there is some j (wlog $j = 1$) such that $d_j \geq 2$: otherwise $d_j = 1$ for all j and $\sum d_j = n < 2n - 2$. Consider the sequence $(d'_1, \dots, d'_{n-1}) = (d_1 - 1, d_2, \dots, d_{n-1})$. This has length $n - 1 \geq 2$. Every entry is at least 1 (since $d_1 \geq 2$). Also, since we decreased d_1 and deleted $d_n = 1$, we have $\sum d'_k = \sum d_k - 2 = 2n - 4 = 2(n-1) - 2$. Thus by induction there is a tree T' on $[n-1]$ in which vertex i has degree d'_i. Let $T = T' + 1n$ (i.e., add the vertex n and the edge $1n$). This is a tree (by Lemma 2.4) and each vertex i has degree d_i.

[6] Of course, the way to come up with this proof is to think about deleting a leaf from an n-vertex tree; but that is not how the logic of the proof works.

9. Show that deleting any edge from a tree T leaves a graph with exactly two components. Show that deleting a vertex v leaves $d(v)$ components. [Hint: you could do this directly, or try a short cut using what we know about numbers of edges in trees.]

A pedestrian proof goes by considering what happens when we *add* an edge to a graph: if it does not create a cycle, then there was no path between its ends, so they were in different components, and these two components become united. So deleting an edge not in a cycle will split one component into two. Use this repeatedly for the vertex case.

Or: we know that in a tree $e(T) = |T| - 1$. So summing over components, in a forest with k components, $e(F) = |F| - k$. If we delete an edge from a tree T we certainly get a forest as the graph is still acyclic; we've deleted no vertices and one edge, so $e(F) = |F| - 2$ and there must be exactly two components. If we delete a vertex, the resulting forest has $|F| = |T| - 1$ and $e(F) = e(T) - d(v) = |T| - 1 - d(v)$, so $e(F) = |F| - d(v)$ and there are $d(v)$ components.

[Of course, one can also say that every vertex $u \neq v$ was joined to v by a path, and split according to the last vertex on the path before v. This will (with some checking) give the $d(v)$ components in a more intuitive way.]

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