B7.1 Classical Mechanics

Lectured by Lionel Mason, for Michaelmas term 2023, notes by James Sparks, 2017

About these notes

These are lecture notes for the B7.1 Classical Mechanics course, which is a third year option in the mathematics syllabus at the University of Oxford. In putting together the notes I have drawn freely from the large literature on the subject; notably from the reading list below, but also from many other books and lecture notes (notably those of Paul Tod and David Tong).

Familiarity with the Oxford first year Dynamics course and second year Calculus of Variations option will be assumed, although we will essentially develop the theory *ab initio*. Starred paragraphs are not examinable: this material is often included to make contact with other courses (and may be safely ignored by students not taking those courses), and/or is more advanced. There are four problem sheets. Please send any questions/corrections/comments to `sparks@maths.ox.ac.uk`.

Reading

- H. Goldstein, C. P. Poole, J. L. Safko, *Classical Mechanics*.
- L. D. Landau, E. M. Lifshitz, *Mechanics* (*Course of Theoretical Physics, Vol. 1*).
- N. M. J. Woodhouse, *Introduction to Analytical Mechanics*.
- V. I. Arnold, *Mathematical Methods of Classical Mechanics*.

Introduction

This course is about the Lagrangian and Hamiltonian formulations of classical mechanics. These were introduced and developed in the late 18th and 19th centuries, and recast Newton's laws in different mathematical frameworks. The reader might immediately ask what the point of this is: what mileage do you get out of rewriting Newtonian mechanics in a different language? Moreover, why is it important to study this subject?

In order to answer these questions, let's begin with the Newtonian theory itself. Newtonian mechanics is an extremely accurate theory, valid over a vast range of scales, and is applicable to many classes of dynamical problems. The axioms are also clear and simple to state. Although the later developments of special relativity, general relativity and quantum mechanics undoubtedly provide a more accurate description of the real world, these are all much more complex descriptions

of Nature. Scientists, even theoretical physicists, will often try to revert to using the techniques of classical mechanics where possible. Indeed, measuring the differences from Newtonian theory usually requires very well-designed and delicate experiments.

From a computational point of view Newtonian theory, as summarized for point particles in section 1, quickly becomes cumbersome and inefficient as the systems become more complicated. In particular Newton's laws require one to work in an inertial frame in Cartesian coordinates, which is often inconvenient. The Lagrangian and Hamiltonian formalisms, on the other hand, are coordinate independent, and provide a more elegant and computationally efficient framework in which to work. For example, it much easier to solve constrained systems, as there is no need to introduce constraint forces (such as the tension in a simple pendulum, or the constraint forces for a bead moving on a wire); rigid body motion is also easier to implement. The Lagrangian formulation also introduces a new fundamental principle: the *principle of least action*, also known as *Hamilton's principle*. This gives a surprising amount of insight into classical mechanics, for example making clear the relation between symmetries and conservation laws (via Noether's theorem).

The ideas and techniques developed in the Lagrangian and Hamiltonian formulations of classical mechanics also generalize to other areas of theoretical physics. For example, there are Lagrangian and Hamiltonian descriptions of electromagnetism and general relativity, which play an important role in formulating those theories. The ideas and principles we shall encounter were also key to the development of quantum mechanics in the 20th century. Those who have studied quantum mechanics may have noticed that the theory looks nothing like Newtonian mechanics. In fact the Hamiltonian formulation of the latter is closely related to the quantum description of a non-relativistic particle you may have already seen. The principle of least action also led to Feynman's formulation of quantum mechanics as a path integral/sum over histories, which in turn has been central to the development of particle physics in the second half of the 20th century. Finally, Lagrangian and Hamiltonian mechanics have also had a significant impact on the development of various branches of pure mathematics, particularly geometry.

Hopefully this has at least partly addressed the questions posed at the beginning of the introduction.

Contents

1 Newtonian mechanics

We begin with an overview of Newtonian mechanics for point particles. Although much of this material may already be quite familiar, it will be important to have a good grasp of these basic notions when we come to generalize the results later in the course.

1.1 Reference frames

In classical mechanics space is modelled by $\mathbb{R}^3$, equipped with the usual Euclidean metric. More precisely, a *reference frame* $\mathcal{S}$ is specified by a choice of origin O, together with a choice of Cartesian coordinate axes. With respect to this frame one then writes a position vector as $\mathbf{r} = (x, y, z)$. The trajectory of a point particle is represented by a curve $\mathbf{r} = \mathbf{r}(t)$, parametrized by time t. The *velocity* of the particle in the frame $\mathcal{S}$ is then $\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z})$, where a dot will denote derivative with respect to time. Similarly, its *acceleration* is $\ddot{\mathbf{r}} = (\ddot{x}, \ddot{y}, \ddot{z})$.

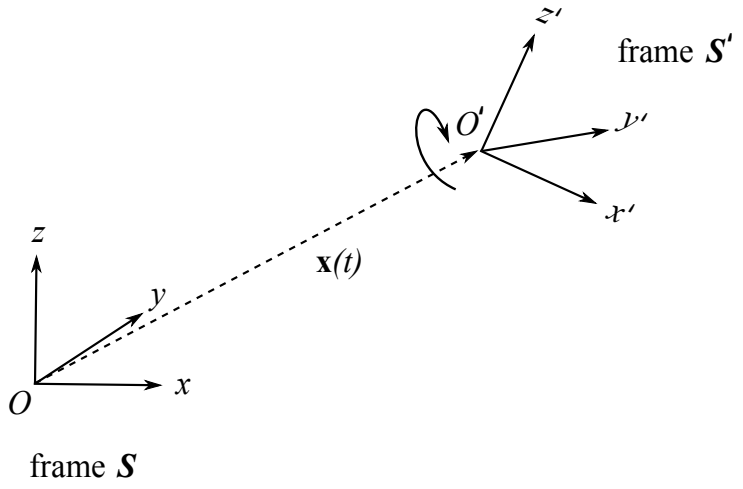

Figure 1: Relative to a choice of reference frame $\mathcal{S}$, the origin O' of another frame $\mathcal{S}'$ is at $\mathbf{x}(t)$, and the coordinate axes of $\mathcal{S}'$ may also rotate with respect to the axes of $\mathcal{S}$.

Of course the choice of frame is far from unique, and the motion of a particle viewed in two different frames can look very different. If we fix an initial choice of $\mathcal{S}$, then the origin O' of any other frame $\mathcal{S}'$ will be at some position $\mathbf{x}(t)$, measured with respect to the origin O of $\mathcal{S}$. Moreover, the coordinate axes of $\mathcal{S}'$ may rotate relative to the axes of $\mathcal{S}$ – see Figure 1. In particular if $\mathcal{S}$ and $\mathcal{S}'$ have the same origin (so $\mathbf{x}(t) = \mathbf{0}$ for all t), then the frames differ only by a rotation of the axes. A position vector $\mathbf{r}$ measured in $\mathcal{S}$ is then $\mathbf{r}' = (x', y', z') = \mathcal{R}\,\mathbf{r}$ as measured in $\mathcal{S}'$. Here $\mathcal{R} = \mathcal{R}(t)$ is a 3×3 orthogonal matrix, which in general can depend on time t. Mathematically the three-dimensional rotation group is $O(3) \subset GL(3, \mathbb{R})$, the set of real 3×3 matrices satisfying $\mathcal{R}^T\mathcal{R} = \mathcal{R}\mathcal{R}^T = \mathbb{1}$, where $\mathbb{1}$ denotes the 3×3 identity matrix. This ensures the transformation $\mathcal{R} \in O(3)$ between frames preserves Euclidean distances – all observers in Newtonian mechanics are postulated to measure the same distance between any two points. We shall describe the rotation

group and rotating frames in much greater detail in section 4 when we come to discuss rigid body motion.

1.2 Newton's laws

Newton's laws of motion apply to *point particles*. These are objects whose dimensions may be neglected, to a good approximation, in describing their motion. For example, this is the case if the size of the object is small compared to the distances involved in the dynamics, *e.g.* the motion of a planet around the Sun. Of course, it is no good treating the Earth as a point particle if you want to understand the effects of its rotation. On the other hand, as we shall see in section 4 the centre of mass of an extended rigid body does behave like a point particle. Point particles have a *mass* m, and an associated *(linear) momentum* $\mathbf{p} = m\dot{\mathbf{r}}$.

The first of Newton's laws singles out a special equivalence class of reference frames, called *inertial frames*. These are defined by

> **N1**: Inertial frames exist. In such a frame an object either remains at rest or moves with constant momentum (uniform motion in a straight line), unless acted on by an external force.

Here the force that is acting is understood to have an identifiable source, *e.g.* gravity, electromagnetism, friction. A non-inertial frame $\mathcal{S}'$ is accelerating with respect to an inertial frame $\mathcal{S}$. That is, the origin O' of $\mathcal{S}'$ is accelerating with respect to O, or the axes of $\mathcal{S}'$ are rotating relative to the axes of $\mathcal{S}$. In the non-inertial frame a particle will appear to be acted on by "ficticious forces", for example the Coriolis force, or centrifugal force (more about this in section 4.2). A frame at rest on the surface of the Earth is a very good approximation to an inertial frame, ignoring the rotation of the Earth around its axis and its acceleration about the Sun.[1] Compare such a frame $\mathcal{S}$ to someone standing on a roundabout, whose frame $\mathcal{S}'$ rotates around a fixed vertical axis relative to $\mathcal{S}$: a stationary object in $\mathcal{S}$ will appear to be accelerating to the person on the roundabout.

In an inertial frame the dynamics of a particle is governed by

> **N2**: $$\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t) \;=\; \dot{\mathbf{p}}.$$

Assuming the mass m is constant the right hand side of Newton's second law is $\dot{\mathbf{p}} = m\ddot{\mathbf{r}}$, although one could also consider variable mass bodies, *e.g.* a rocket ship that expels the spent fuel. In this course we'll assume m is constant. The external force $\mathbf{F}$ can in general depend on the particle's position $\mathbf{r}$, its velocity $\dot{\mathbf{r}}$ (*e.g.* the drag force due to motion through a fluid), and on time t (*e.g.* a charged particle moving in a time-dependent electromagnetic field). Newton's second law is then a second order ODE for $\mathbf{r}(t)$. General theorems from the theory of differential equations guarantee

[1] The former does give rise to a very small measurable effect. For example, *Foucault's pendulum* detects the rotation of the Earth.

that under suitable conditions on the function $\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$, specifying the position $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$ at some initial time $t = t_0$ gives a unique solution for the particle trajectory $\mathbf{r}(t)$.

Finally, if we have more than one particle, then

> **N3**: If particle 1 exerts a force $\mathbf{F} = \mathbf{F}_{21}$ on particle 2, then particle 2 also exerts a force $\mathbf{F}_{12} = -\mathbf{F}$ on particle 1.

In other words, $\mathbf{F}_{12} = -\mathbf{F}_{21}$. This is often paraphrased by saying that every action has an equal and opposite reaction. There is also a *strong form* of Newton's third law:

> **N3′**: The force in **N3** above acts along the vector connecting particle 1 and particle 2.

That is, $\mathbf{F} = \mathbf{F}_{21} \propto (\mathbf{r}_1 - \mathbf{r}_2)$, where $\mathbf{r}_I$ denotes the position vector of particle I. Such forces are called *central forces*. The strong form of Newton's third law is true for the gravitational and electrostatic forces between particles:

Example (gravitational force between two point masses): According to Newton (and Hooke), the gravitational force on particle 1 due to particle 2 is given by

$$\mathbf{F}_{12} \;=\; -G_N \frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2} \frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} \;. \tag{1.1}$$

Here m_I is the mass of particle I, and $G_N \simeq 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$ is Newton's gravitational constant. The Coulomb electrostatic force between two point charges e_1, e_2 takes a similar central inverse square law form, except unlike masses charges can be both positive and negative, leading to repulsive as well as attractive forces.

There are examples of forces that do not satisfy the strong form of Newton's third law, notably the magnetic component of the Lorentz force acting on a charged particle moving in the electromagnetic field generated by another charged particle. However, a proper discussion of this would take us too far into electromagnetic theory, and in any case requires special relativity (rather than Galilean relativity) to understand properly, so we will not discuss this further here.

1.3 Galilean transformations

Inertial frames are not unique. Rather there is an equivalence class of inertial frames, related to each other by *Galilean transformations*. Specifically, we have the following transformations of an inertial frame $\mathcal{S}$:

$$\begin{cases} \text{temporal translations, } t' = t - s, \text{ where } s \text{ is a constant,} \\ \text{spatial translations, } \mathbf{r}' = \mathbf{r} - \mathbf{x}, \text{ where } \mathbf{x} \text{ is a constant vector,} \\ \text{constant rotations, } \mathbf{r}' = \mathcal{R}\,\mathbf{r}, \text{ where } \mathcal{R} \in O(3) \text{ is a constant } 3 \times 3 \text{ orthogonal matrix,} \\ \text{Galilean boosts, } \mathbf{r}' = \mathbf{r} - \mathbf{v}t, \text{ where } \mathbf{v} \text{ is a constant velocity.} \end{cases}$$

These map uniform motion in $\mathcal{S}$ to uniform motion in $\mathcal{S}'$. Altogether they generate the 1+3+3+3 = 10-dimensional *Galilean group*.

Notice that in the first transformation we have been tacitly assuming there is a notion of *absolute time*, unique up to what we call time $t = 0$. In particular in Newtonian mechanics all observers measure the same time interval between any two events. Of course this is *not* true in special relativity (essentially it took until Einstein to realize that assuming there is an absolute notion of time *is* an assumption). Galilean boost transformations mean there is no absolute rest frame in Newtonian mechanics.

The first Galilean transformation is a statement of *homogeneity of time* (Newtonian physics is invariant under $t \to t - s$), while the second and third Galilean transformations say that space is respectively *homogeneous* and *isotropic*. These are symmetries of Galilean spacetime.

1.4 Closed systems and Galilean relativity

A *closed system* is one in which all forces are internal, acting between the constituents of the system. To be concrete let us consider a closed system of N point particles. We suppose these have constant masses m_I, $I = 1, \ldots, N$, and are at positions $\mathbf{r}_I(t)$ as measured in an inertial frame. Newton's second law **N2** for the system may be written

$$\mathbf{F}_I(\mathbf{r}_1, \ldots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N, t) \;=\; \dot{\mathbf{p}}_I \;=\; m_I \ddot{\mathbf{r}}_I \;, \tag{1.2}$$

where $\mathbf{F}_I$ denotes the total force on the Ith particle. For such a closed system we have

> *Galileo's principle of relativity*: Newtonian dynamics is invariant under Galilean transformations.

More precisely, if we apply the same Galilean transformation to each of the particle trajectories $\{\mathbf{r}_I(t)\}$ solving (1.2), then the resulting trajectories solve the *same* system of equations. For example, if $\{\mathbf{r}_I(t)\}$ solves (1.2) then $\{\mathbf{r}_I(t - s)\}$ must solve the same system, for all $s \in \mathbb{R}$. This invariance under temporal translations immediately implies that $\mathbf{F}_I$ must be independent of time t, so that we may write $\mathbf{F}_I = \mathbf{F}_I(\mathbf{r}_1, \ldots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N)$. Similarly, invariance under spatial translations and boosts means that the forces depend only on the relative positions $(\mathbf{r}_J - \mathbf{r}_K)$ and relative velocities $(\dot{\mathbf{r}}_J - \dot{\mathbf{r}}_K)$ of the particles, respectively. Finally, invariance under a rotation $\mathcal{R} \in O(3)$ means that $\mathbf{F}_I$ is further constrained to obey

$$\mathcal{R}\,\mathbf{F}_I(\mathbf{r}_1, \ldots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N) \;=\; \mathbf{F}_I(\mathcal{R}\,\mathbf{r}_1, \ldots, \mathcal{R}\,\mathbf{r}_N, \mathcal{R}\,\dot{\mathbf{r}}_1, \ldots, \mathcal{R}\,\dot{\mathbf{r}}_N) \;. \tag{1.3}$$

A closed system consisting of a *single* point particle is not very interesting: a little thought shows that the above constraints imply $\mathbf{F} = \mathbf{0}$ (a single particle cannot act on itself). Such a particle moves at constant momentum/velocity by **N1**. When treating a single point particle subject to an *external* force, we have in mind that (a) something else is responsible for producing that force,

and (b) we are entirely ignoring the effect the particle has on whatever that something else is (*i.e.* we are ignoring its *back-reaction*). One can imagine such a model arising from a closed system, in which the particle has been singled out and the "external force" $\mathbf{F}$ is simply the sum of the forces on the particle from the rest of the system. Such effective descriptions of the dynamics typically won't be Galilean invariant in the way we have described.

Since this is quite a subtle point, it is worth pausing to discuss an explicit example. Consider a small body moving through a fluid with a linear drag force $\mathbf{F} = -b\,\dot{\mathbf{r}}$, where $b > 0$ is a constant. Newton's second law for our body is clearly not invariant under Galilean boosts, which take $\dot{\mathbf{r}} \to \dot{\mathbf{r}} - \mathbf{v}$. However, this is because the force law is only valid in the *rest frame* of the fluid, where the average velocity of the fluid particles is zero. The individual fluid particles will certainly be moving relative to each other, and colliding, but in the rest frame there is no net movement of the fluid as a whole. If the fluid has a net velocity $\mathbf{u}$ in our inertial frame (think of a steady flowing river), then the force law reads $\mathbf{F} = -b\,(\dot{\mathbf{r}} - \mathbf{u})$. Notice that here $\mathbf{F} = \mathbf{F}(\dot{\mathbf{r}}\,;\mathbf{u})$ also depends on the "external" parameter $\mathbf{u}$. Of course the Galilean boost also acts as $\mathbf{u} \to \mathbf{u} - \mathbf{v}$ on the fluid velocity, and with this understanding Newton's second law for the body is now Galilean invariant. The linear drag force is an *effective* force, which at a microscopic level arises due to many collisions of the body with the fluid particles. A more accurate (but completely impractical) description of the system would treat the fluid as a large number of point particles. These would all have initial velocity $\mathbf{u}$. The collisions between the fluid particles and our body, which can be treated as another particle, can then be described in a fully Galilean invariant way. The collisions will change the velocities of the fluid particles, and the linear drag force with a constant $\mathbf{u}$ is entirely ignoring this.

Of course, whether or not it is reasonable to neglect the effects of a particle on the rest of the system depends on the circumstances. It is perhaps worth pointing out though that closed systems can sometimes be rewritten/reinterpreted as non-closed systems. A good example is the *two-body problem*, of two point masses m_1, m_2 interacting via the gravitational force (1.1). This closed system may be rewritten as a problem for a *single* point particle (with mass given by the *reduced mass* $\mu = \frac{m_1 m_2}{m_1+m_2}$) moving in an external central potential. We shall examine this example in more detail in section 2.5.

1.5 Energy and angular momentum

Consider a single point particle of mass m, acted on by an external force $\mathbf{F}$. The *work done* W by the force $\mathbf{F}$ along a path connecting position vectors $\mathbf{r}^{(1)}$, $\mathbf{r}^{(2)}$ is the line integral

$$W \;\equiv\; \int_{\mathbf{r}^{(1)}}^{\mathbf{r}^{(2)}} \mathbf{F}\cdot \mathrm{d}\mathbf{r}\ . \tag{1.4}$$

If we now consider a trajectory of the particle $\mathbf{r}(t)$ satisfying Newton's second law $m\ddot{\mathbf{r}} = \mathbf{F}$, we may compute

$$\begin{aligned} W &= \int_{\mathbf{r}^{(1)}}^{\mathbf{r}^{(2)}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{t_1}^{t_2} \mathbf{F}\cdot\dot{\mathbf{r}}\,\mathrm{d}t = m\int_{t_1}^{t_2} \ddot{\mathbf{r}}\cdot\dot{\mathbf{r}}\,\mathrm{d}t = \frac{1}{2}m\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})\,\mathrm{d}t \\ &= T(t_2) - T(t_1)\ . \end{aligned} \tag{1.5}$$

Here the particle trajectory begins at position $\mathbf{r}^{(1)} = \mathbf{r}(t_1)$ at time t_1, and ends at $\mathbf{r}^{(2)} = \mathbf{r}(t_2)$ at time t_2, and we have defined the *kinetic energy* of the particle as

$$T \equiv \frac{1}{2}m|\dot{\mathbf{r}}|^2\ . \tag{1.6}$$

Notice that T is not invariant under Galilean boosts, and so in general depends on the choice of inertial frame $\mathcal{S}$ we choose to measure it in. The equality in (1.5) is called the *work-energy theorem*: the work done by the force $\mathbf{F}$ along the particle trajectory is equal to the change in kinetic energy.

In the remainder of this course we will almost entirely focus on *conservative forces*. Recall that these arise from the gradient of a *potential function* $V = V(\mathbf{r})$ via

$$\mathbf{F} = -\nabla V(\mathbf{r})\ . \tag{1.7}$$

In Cartesian coordinates $\mathbf{r} = (x, y, z)$ this reads $\mathbf{F} = (-\partial_x V, -\partial_y V, -\partial_z V)$. Notice in particular that V, and hence also $\mathbf{F}$, depends *only* on the position vector $\mathbf{r}$. Note also that V is defined only up to an additive constant. It is natural to fix this freedom, where it makes sense to do so, by requiring that the potential is zero when the particle (or more generally distances between particles) is at infinity, *c.f.* (1.11) below. Conservative forces have the property that the work done is independent of the path from $\mathbf{r}^{(1)}$ to $\mathbf{r}^{(2)}$. This follows from the fundamental theorem of calculus:

$$\int_{\mathbf{r}^{(1)}}^{\mathbf{r}^{(2)}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = -\int_{\mathbf{r}^{(1)}}^{\mathbf{r}^{(2)}} \nabla V\cdot\mathrm{d}\mathbf{r} = -V(\mathbf{r}^{(2)}) + V(\mathbf{r}^{(1)})\ . \tag{1.8}$$

Combining with the work-energy theorem (1.5) and rearranging we thus deduce the *conservation of energy*

$$T(t_1) + V(\mathbf{r}(t_1)) = T(t_2) + V(\mathbf{r}(t_2)) \equiv E\ . \tag{1.9}$$

Thus the total energy

$$E \equiv T + V\ , \tag{1.10}$$

is constant along the trajectory.

The inverse square law force (1.1), that governs the gravitational or Coulomb interaction between point masses/charges, is conservative:

$$V(\mathbf{r}) = -\frac{\kappa}{r} \qquad \Longrightarrow \qquad \mathbf{F} = -\nabla V = -\frac{\kappa}{r^3}\mathbf{r}\ , \tag{1.11}$$

where $r = |\mathbf{r}|$ and κ is a constant. Non-conservative forces include friction/drag forces that depend on $\dot{\mathbf{r}}$. In particular, the effective linear drag force discussed in the last subsection is not conservative. However, the *fundamental* forces in Nature seem to be conservative. For example, a body that experiences a frictional force will typically lose energy, but in practice we know what happens to this energy: it is converted into heat (average kinetic energy). Modelling this takes us outside the realm of classical mechanics and into thermodynamics. But at a fundamental level, we expect energy to be conserved.

Still focusing on the single point particle, recall that its *angular momentum* (about the origin O of $\mathcal{S}$) is defined as

$$\mathbf{L} \;=\; \mathbf{r} \wedge \mathbf{p} \;. \tag{1.12}$$

We will sometimes write the cross product of two vectors using the Levi-Civita alternating symbol, discussed in the appendix at the end of these lecture notes. The angular momentum (1.12) is the *moment* of the momentum $\mathbf{p}$ about O. Similarly, the moment of the force about O is called the *torque*

$$\boldsymbol{\tau} \;=\; \mathbf{r} \wedge \mathbf{F} \;. \tag{1.13}$$

Since $\dot{\mathbf{L}} = \dot{\mathbf{r}} \wedge \mathbf{p} + \mathbf{r} \wedge \dot{\mathbf{p}} = \mathbf{r} \wedge \dot{\mathbf{p}}$ (as $\mathbf{p} = m\dot{\mathbf{r}}$ is parallel to the velocity $\dot{\mathbf{r}}$), the moment of **N2** about the origin gives the equation

$$\boldsymbol{\tau} \;=\; \dot{\mathbf{L}} \;. \tag{1.14}$$

In particular *central forces* have $\mathbf{F} \propto \mathbf{r}$, so that the torque $\boldsymbol{\tau} = \mathbf{0}$. It then follows from (1.14) that central forces lead to *conservation of angular momentum*, $\dot{\mathbf{L}} = \mathbf{0}$.

It is straightforward to extend these concepts to our closed system of N point particles in section 1.4. If particle J exerts a force $\mathbf{F}_{IJ}$ on particle I for $J \neq I$, then by **N3** therefore $\mathbf{F}_{JI} = -\mathbf{F}_{IJ}$. Newton's second law for particle I (1.2) then reads

$$\sum_{J \neq I} \mathbf{F}_{IJ} \;=\; \mathbf{F}_I \;=\; \dot{\mathbf{p}}_I \;. \tag{1.15}$$

We could also add an external force $\mathbf{F}_I^{\text{external}}$ to the left hand side, but the system would then not be closed. The *total momentum* of the system of particles is defined to be $\mathbf{P} = \sum_{I=1}^{N} \mathbf{p}_I$, and by summing (1.15) over all particles we deduce that

$$\dot{\mathbf{P}} \;=\; \sum_{I=1}^{N} \sum_{J \neq I} \mathbf{F}_{IJ} \;=\; \mathbf{0} \;. \tag{1.16}$$

Here the $N(N-1)$ terms in the sum of forces cancel pairwise due to **N3**: $\mathbf{F}_{IJ} = -\mathbf{F}_{JI}$. Thus for this closed system of point particles we see that the total momentum is conserved.

We may similarly define the *total angular momentum* about the origin O as

$$\mathbf{L} \;=\; \sum_{I=1}^{N} \mathbf{r}_I \wedge \mathbf{p}_I \ . \tag{1.17}$$

As for the single point particle we compute

$$\dot{\mathbf{L}} \;=\; \sum_{I=1}^{N} \mathbf{r}_I \wedge \dot{\mathbf{p}}_I \;=\; \sum_{I=1}^{N} \mathbf{r}_I \wedge \sum_{J\neq I} \mathbf{F}_{IJ} \ . \tag{1.18}$$

In particular we have $\frac{1}{2}N(N-1)$ pairs of terms on the right hand side

$$\mathbf{r}_I \wedge \mathbf{F}_{IJ} + \mathbf{r}_J \wedge \mathbf{F}_{JI} \;=\; (\mathbf{r}_I - \mathbf{r}_J) \wedge \mathbf{F}_{IJ} \ , \tag{1.19}$$

where we have used **N3**. Imposing the *strong form* of Newton's third law **N3′** means that $\mathbf{F}_{IJ} \propto (\mathbf{r}_I - \mathbf{r}_J)$, and we see that the total angular momentum is conserved, $\dot{\mathbf{L}} = \mathbf{0}$.

One of the general results we shall derive in Lagrangian mechanics is that conservation of energy, momentum, and angular momentum for any closed system follow from the symmetries of Galilean spacetime described in section 1.3.

2 Lagrangian mechanics

In this section we introduce the Lagrangian formulation of classical mechanics. The general theory is developed in sections 2.1 – 2.4, with a number of detailed examples presented in section 2.5. The reader should feel free to dip into the examples while digesting the general theory, especially on a first reading.

2.1 Generalized coordinates

In order to study the dynamics of a physical system, one first needs to describe its possible configurations. Any set of independent quantities that specify uniquely the position of the system at a given time are called *generalized coordinates*. These label the points of the *configuration space* $\mathcal{Q}$. The number of generalized coordinates is called the number of *degrees of freedom*.

For example, for the point particle discussed in section 1 we used Cartesian coordinates $\mathbf{r} = (x, y, z)$ with respect to an inertial frame $\mathcal{S}$. These coordinates obviously specify uniquely the position of the particle in $\mathbb{R}^3$ at a given time t, so $\mathcal{Q} = \mathbb{R}^3$ and the number of degrees of freedom is 3. For N point particles we similarly used N sets of Cartesian coordinates $\mathbf{r}_I = (x_I, y_I, z_I)$, $I = 1, \ldots, N$, giving a configuration space $\mathcal{Q} = \mathbb{R}^{3N}$ with $3N$ degrees of freedom.

For the point particle we may also consider changing to non-Cartesian coordinates. Consider the general coordinate change

$$q_i \;=\; q_i(x, y, z, t) \;, \qquad i \;=\; 1, 2, 3 \;, \tag{2.1}$$

which might depend explicitly on time t (that is, the new coordinate system is moving relative to the original coordinate system). We thus replace the Cartesian coordinates $(x, y, z) \to (q_1, q_2, q_3)$. Introducing $x_1 = x$, $x_2 = y$, $x_3 = z$ we will also have the inverse coordinate transformation

$$x_i \;=\; x_i(q_1, q_2, q_3, t) \;, \qquad i \;=\; 1, 2, 3 \;. \tag{2.2}$$

This latter change of coordinates is said to be *non-singular* at a point (q_1, q_2, q_3) if the *Jacobian determinant* $\det\left(\partial x_i/\partial q_j\right)$ is non-zero at that point. This condition plays a role when writing differential equations in different coordinate systems, as the Jacobian $\partial x_i/\partial q_j$ and its inverse $\partial q_i/\partial x_j$ enter the chain rule.[2]

A familiar example is *cylindrical polars*

$$x \;=\; \varrho\cos\phi \;, \qquad y \;=\; \varrho\sin\phi \;, \qquad z \;=\; z \;, \tag{2.3}$$

where $\varrho \geq 0$ and $\phi \in [0, 2\pi)$. Here $(q_1, q_2, q_3) = (\varrho, \phi, z)$, and the coordinates are adapted to rotation about the z-axis. Notice that $\varrho = 0$ (the z-axis) is a coordinate singularity, as $\det\left(\partial x_i/\partial q_j\right) = \varrho$

[2] * We may regard a coordinate transformation $(x_1, x_2, x_3) \to (q_1, q_2, q_3)$ at fixed time t as a map $x : \mathbb{R}^3 \to \mathbb{R}^3$, with components $x_i = x_i(q_1, q_2, q_3)$. By the *inverse function theorem*, if the Jacobian determinant $\det\left(\partial x_i/\partial q_j\right)$ is non-zero at a point $P = (q_1, q_2, q_3)$, then there is an open set containing P on which this map is invertible with differentiable inverse.

is zero at $\varrho = 0$. In practice this causes no problems provided one remembers it is there (for example it explains why the Laplacian $\partial_x^2 + \partial_y^2 = \partial_\varrho^2 + \frac{1}{\varrho}\partial_\varrho + \frac{1}{\varrho^2}\partial_\phi^2$ looks singular at $\varrho = 0$).

Another natural coordinate system is *spherical polars*

$$x \;=\; r\sin\theta\cos\varphi\ , \qquad y \;=\; r\sin\theta\sin\varphi\ , \qquad z \;=\; r\cos\theta\ , \tag{2.4}$$

where $r \geq 0$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$. Here $(q_1, q_2, q_3) = (r, \theta, \varphi)$, with the coordinates adapted to rotation about the origin $r = 0$. Again, there are coordinate singularities at $r = 0$ and at $\theta = 0, \pi$, which are the zeros of the Jacobian determinant $\det(\partial x_i/\partial q_j) = r^2 \sin\theta$.

The notion of generalized coordinates is particularly useful for *constrained* problems. You have already met many such problems in Dynamics. Perhaps the simplest example is the *simple pendulum*. Here a point mass m swings on a rigid (but very light) rod of length l, freely pivoted at one end to the origin O, and constrained to move in a vertical plane containing the z-axis – see Figure 2. Taking the plane of motion to be the (x, z) plane at $y = 0$, the rod constrains the point mass coordinates to satisfy $x^2 + z^2 = l^2$. It is then convenient to describe the position of the system by the angle θ given by $x = l\sin\theta$, $z = -l\cos\theta$. Thus $\theta \in [-\pi, \pi)$ is a generalized coordinate for this problem, $q = \theta$, which has one degree of freedom. Since $\theta = \pi$ is identified with $\theta = -\pi$, notice that the configuration space is a circle, $\mathcal{Q} = S^1$.

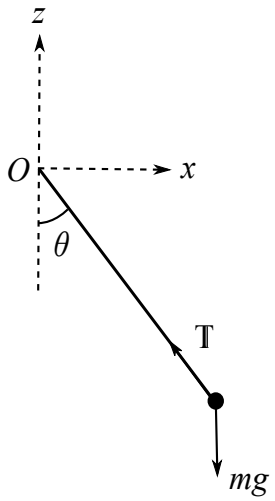

Figure 2: A simple pendulum with generalized coordinate $q = \theta$.

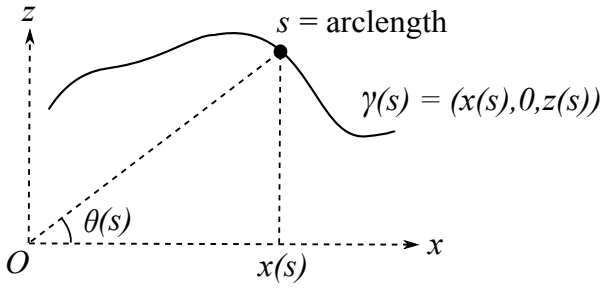

Figure 3: A bead moving on a fixed wire. We might use arclength s, the x-coordinate, or the angle θ as a generalized coordinate.

As another example consider a bead moving on a fixed wire. Here there are several natural choices of generalized coordinate. For simplicity suppose that the wire lies in the (x, z) plane at $y = 0$, as in Figure 3. We can specify the wire as the image of a parametrized curve $\gamma : [0, 1] \to \mathbb{R}^3$ given by $\gamma(s) = (x(s), 0, z(s))$. A natural generalized coordinate is then simply the arclength s. But there are other choices. Provided the projection onto the x coordinate is injective (so that distinct points on the curve correspond to distinct values of x), we could also use the x coordinate

of the bead as a generalized coordinate. Another choice is to use the angle θ subtended from the origin, again provided the lines θ = constant through the origin intersect the curve in at most one point. For example, the latter would be a natural choice if the curve was a segment of a circle centred at the origin. This problem again has one degree of freedom, and the configuration space is an interval (assuming something stops the bead from falling off either end of the wire). Constrained motion on a surface is similar, with two degrees of freedom.

In section 4 we will consider rigid body motion. A rigid body has six degrees of freedom: three describe the position of the centre of mass, two determine the orientation in space of some axis fixed in the body, and there is one more describing rotations about that axis. For generalized coordinates it is then natural to use a mixture of Cartesian coordinates, describing the centre of mass motion, and angles, describing the orientation of the rigid body.

In general then a physical system will have n degrees of freedom, described by n real generalized coordinates $q_a = q_a(t)$, $a = 1, \ldots, n$. Their time derivatives $\dot{q}_a$ are called *generalized velocities*, $\ddot{q}_a$ are *generalized accelerations*, *etc.* We will use the boldface notation $\mathbf{q} = (q_1, \ldots, q_n)$, although as the examples above illustrate this is not to be confused (in general) with a vector in physical space $\mathbb{R}^3$. The dynamics of the system will trace out a curve $\mathbf{q}(t)$ in the configuration space $\mathcal{Q}$, which one could think of as a *quasi-particle* moving on $\mathcal{Q}$, whose position at any time determines the configuration of the whole system.[3] However, to avoid any possible confusion we shall generally refer to $\mathbf{q}(t)$ as the *system trajectory*, or *path*.

***** The intuitive ideas of configuration space $\mathcal{Q}$ and generalized coordinates we have described can be made much more mathematically precise. A more rigorous account of a general framework for classical mechanics would take $\mathcal{Q}$ to be a *differentiable manifold*. Roughly, these are topological spaces that look locally like $\mathbb{R}^n$ (the open sets are homeomorphic to open sets in $\mathbb{R}^n$), but globally they can have much more non-trivial topology. Familiar examples are smooth curves and surfaces in $\mathbb{R}^3$ (with $n = 1$, $n = 2$ respectively). The study of geometry and calculus on these spaces is called *differential geometry*. However, expanding on these ideas would take us too far from our main subject, and in any case one doesn't need the full machinery of differential geometry to understand the fairly simple systems we shall study in these lectures. Those wishing to see more details might consult chapter 8 of *Introduction to Analytical Dynamics* by N. M. J. Woodhouse, or the book *Mathematical Methods of Classical Mechanics* by V. I. Arnold.

[3] For example, the configuration space for two point particles moving in $\mathbb{R}^3$ is $\mathcal{Q} = \mathbb{R}^6$, which may be coordinatized by Cartesian coordinates $\mathbf{q} = (x_1, y_1, z_1, x_2, y_2, z_2)$ for the two particles. The position of the quasi-particle in $\mathcal{Q}$ then determines both physical positions of the two actual partices in $\mathbb{R}^3$.

2.2 The principle of least action

We are now in a position to introduce a new fundamental principle that governs the dynamics of physical systems. The *principle of least action*, or *Hamilton's principle*, asserts that systems are characterized by a function $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$, called the *Lagrangian*. We write this more briefly as $L(\mathbf{q}, \dot{\mathbf{q}}, t)$. The Lagrangian function in turn determines the dynamics of the system (the *equations of motion*) as a calculus of variations problem.

To describe this, consider all smooth paths $\mathbf{q}(t)$ in $\mathcal{Q}$ with fixed boundary conditions at the endpoints. That is, we consider all smooth $\mathbf{q}(t)$ satisfying

$$\mathbf{q}(t_1) \;=\; \mathbf{q}^{(1)} \;, \qquad \mathbf{q}(t_2) \;=\; \mathbf{q}^{(2)} \;, \tag{2.5}$$

for some fixed time interval $[t_1, t_2]$, and with $\mathbf{q}^{(i)}$ fixed for $i = 1, 2$. Then the path followed by the system between these two positions is such that the *action*

$$S[\mathbf{q}(t)] \;\equiv\; \int_{t_1}^{t_2} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\, \mathrm{d}t \;, \tag{2.6}$$

is *extremized*. The action S is a *functional* – that is, a function whose argument is itself a function – and the above extremal problem is of the same type encountered in the second year Calculus of Variations course.

That the Lagrangian depends only on $\mathbf{q}$ and $\dot{\mathbf{q}}$, but not higher derivatives, is related to the fact that one expects the dynamics of a system to be determined uniquely once one specifies $\mathbf{q}$ and $\dot{\mathbf{q}}$ at some initial time $t = t_0$. Compare to our discussion of Newton's second law in section 1.2. This expectation is itself sometimes elevated to a principle: the *Newton-Laplace determinacy principle*. It also means we expect the equations of motion to be second order differential equations for $\mathbf{q}(t)$.

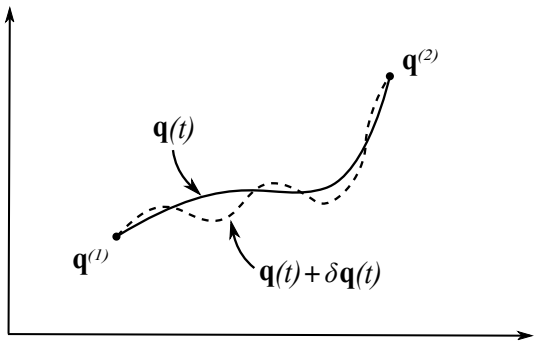

Figure 4: A trajectory $\mathbf{q}(t)$ in configuration space $\mathcal{Q}$, with fixed endpoints $\mathbf{q}(t_1) = \mathbf{q}^{(1)}$ and $\mathbf{q}(t_2) = \mathbf{q}^{(2)}$, together with a variation $\mathbf{q}(t) + \delta\mathbf{q}(t)$.

Let us determine the differential equations that result from extremizing the action S in (2.6). We suppose the extremum occurs at a critical function $\mathbf{q}(t)$, such that the change $\delta S = 0$ when $\mathbf{q}(t)$ is replaced by any variation $\mathbf{q}(t) \to \mathbf{q}(t) + \delta\mathbf{q}(t)$. Here $\delta\mathbf{q}(t)$ is a small change in $\mathbf{q}(t)$, where the fixed endpoints of the path require the boundary conditions $\delta\mathbf{q}(t_1) = \delta\mathbf{q}(t_2) = \mathbf{0}$. We may

make this more precise by writing

$$\delta\mathbf{q}(t) \;=\; \epsilon\,\mathbf{u}(t)\ , \tag{2.7}$$

$\mathbf{u}(t_1)=\mathbf{u}(t_2)=\mathbf{0}$, and ϵ is a real number. We then compute

$$\begin{aligned} S[\mathbf{q}(t)+\delta\mathbf{q}(t)] &= \int_{t_1}^{t_2} L(\mathbf{q}+\epsilon\,\mathbf{u},\dot{\mathbf{q}}+\epsilon\,\dot{\mathbf{u}},t)\,\mathrm{d}t \\ &= S[\mathbf{q}(t)]+\epsilon\sum_{a=1}^{n}\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_a}u_a+\frac{\partial L}{\partial\dot{q}_a}\dot{u}_a\right)\mathrm{d}t+O(\epsilon^2)\ . \end{aligned} \tag{2.8}$$

Here we have simply Taylor expanded around $\epsilon=0$ using the chain rule. Thus the first order variation in S is

$$\begin{aligned} \delta S &\equiv \epsilon\sum_{a=1}^{n}\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_a}u_a+\frac{\partial L}{\partial\dot{q}_a}\dot{u}_a\right)\mathrm{d}t \\ &= \epsilon\sum_{a=1}^{n}\left\{\int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q_a}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_a}\right)\right]u_a\,\mathrm{d}t+\left[\frac{\partial L}{\partial\dot{q}_a}u_a\right]_{t_1}^{t_2}\right\} \\ &= \epsilon\sum_{a=1}^{n}\int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q_a}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_a}\right)\right]u_a\,\mathrm{d}t\ , \end{aligned} \tag{2.9}$$

the last equality holding since $\mathbf{u}(t_1)=\mathbf{u}(t_2)=\mathbf{0}$. The requirement that $\mathbf{q}(t)$ is an extremum of S, $\delta S=0$, for *all* such variations $\delta\mathbf{q}(t)=\epsilon\,\mathbf{u}(t)$ then means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\mathbf{q}}}\right)-\frac{\partial L}{\partial\mathbf{q}} \;=\; \mathbf{0}\ , \tag{2.10}$$

or in components

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_a}\right)-\frac{\partial L}{\partial q_a} \;=\; 0\ ,\qquad a=1,\ldots,n\ , \tag{2.11}$$

The equations (2.10) are known in classical mechanics as *Lagrange's equations*, or the *Euler-Lagrange equations*. They are a set of n second-order differential equations for $\mathbf{q}(t)$. Notice that nothing we have said guarantees that the solution $\mathbf{q}(t)$ to (2.10) actually *minimizes* the action S. In fact there are examples where it does not *i.e.* the stationary point solution to $\delta S=0$ is not a minimum of the action (see Problem Sheet 1). It should thus more properly be called the *principle of stationary action*.

N.B. In deriving the Lagrange equations (2.10) one should be wary of the *first fundamental confusion of calculus* (N. M. J. Woodhouse, *Introduction to Analytical Dynamics*). The partial derivatives of L appearing in the second line of (2.8) involve regarding $L=L(\mathbf{q},\mathbf{v},t)$, where $\mathbf{v}=\dot{\mathbf{q}}$ is treated as an independent variable from $\mathbf{q}$. However, in computing the derivative $\mathrm{d}/\mathrm{d}t$ in the Lagrange equation (2.10) we mean the derivative of $\frac{\partial L}{\partial\mathbf{v}}(\mathbf{q}(t),\dot{\mathbf{q}}(t),t)$, which is a function only of time t. If you follow the derivation carefully these comments are self-evident, but it is also easy

to get confused! The potential confusion arises because we treat $\mathbf{q}$ and $\dot{\mathbf{q}}$ as independent variables on which L depend, but also $\mathbf{q} = \mathbf{q}(t)$ for the system trajectory.

As mentioned in the introduction, Lagrange's equations (2.10) hold in any coordinate system. This actually follows from the formulation of the principle of least action: it is *paths* that extremize S. If we change coordinates then the form of the path will also change, but only due to the coordinate transformation itself – it is the *same path* in $\mathcal{Q}$. One can check this statement very directly: if we make a coordinate transformation

$$\mathbf{q} \;=\; \mathbf{q}(\tilde{\mathbf{q}}, t) \;, \tag{2.12}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}_a}\right) - \frac{\partial \tilde{L}}{\partial \tilde{q}_a} \;=\; \sum_{b=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_b}\right) - \frac{\partial L}{\partial q_b}\right]\frac{\partial q_b}{\partial \tilde{q}_a} \;, \qquad a = 1, \ldots, n \;. \tag{2.13}$$

Here we use the chain rule to compute

$$\dot{q}_a \;=\; \sum_{b=1}^{n}\frac{\partial q_a}{\partial \tilde{q}_b}\dot{\tilde{q}}_b + \frac{\partial q_a}{\partial t} \;=\; \dot{q}_a(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t) \;, \tag{2.14}$$

and define

$$\tilde{L}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t) \;\equiv\; L(\mathbf{q}(\tilde{\mathbf{q}}, t), \dot{\mathbf{q}}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t), t) \;. \tag{2.15}$$

That is, the transformed Lagrangian $\tilde{L}$ is obtained simply by substituting the coordinate transformation (2.12), (2.14) into the original Lagrangian. The Lagrangian is said to *transform as a scalar*. It is then common to drop the tilde on L, and simply remember that the notation $\partial L/\partial \tilde{\mathbf{q}}$ means one first substitutes $\mathbf{q} = \mathbf{q}(\tilde{\mathbf{q}}, t)$ to regard L as a function of $(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t)$, and then takes the partial derivative with respect to $\tilde{\mathbf{q}}$, with $\dot{\tilde{\mathbf{q}}}$ and t held fixed.

Going back to (2.13), since for a (non-singular) coordinate transformation the Jacobian matrix $\partial q_b/\partial \tilde{q}_a$ is invertible, we see that Lagrange equations (2.10) hold if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{\tilde{\mathbf{q}}}}\right) - \frac{\partial \tilde{L}}{\partial \tilde{\mathbf{q}}} \;=\; \mathbf{0} \tag{2.16}$$

hold. Verifying (2.13) amounts to a short computation using the chain rule, and is left as an exercise on Problem Sheet 1.

Notice that if we have two Lagrangians L_1, L_2 related by

$$L_2(\mathbf{q}, \dot{\mathbf{q}}, t) \;=\; L_1(\mathbf{q}, \dot{\mathbf{q}}, t) + \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{q}, t) \;, \tag{2.17}$$

then these lead to the *same* Lagrange equations. This follows since the corresponding actions are related by

$$S_2 \;=\; S_1 + f(\mathbf{q}^{(2)}, t_2) - f(\mathbf{q}^{(1)}, t_1) \;, \tag{2.18}$$

where we have used the fundamental theorem of calculus. The actions thus differ by a term whose variation is zero. In classical mechanics (2.17) should be regarded as an equivalence relation on Lagrangians. Another comment is that if we have two Lagrangians L_1, L_2 describing *decoupled* systems, then the Lagrangian for the combined system is simply $L = L_1 + L_2$. This is reasonable, as then the equations of motion for both systems follow from that for L.

So far our comments have been very general. As mentioned in the introduction, the principle of least action (appropriately generalized) actually applies to many physical theories, including for example electromagnetism and general relativity. However, in classical mechanics the Lagrangian for a system takes the simple form

$$L \;=\; T - V \;, \tag{2.19}$$

where T is the kinetic energy of the system, and V is its potential energy. We shall see more precisely which systems this applies to in the next subsection, and why the Lagrange equations are equivalent to Newton's equations. Our application of Lagrangian mechanics to such systems will then always begin by identifying T and V, in whichever generalized coordinates we have chosen to use. Let us first see that this correctly reproduces Newton's second law as the Lagrange equations for a point particle:

Example (point particle in Cartesian coordinates): As in section 1.5 we consider a single point particle of mass m, in Cartesian coordinates $\mathbf{r} = (x_1, x_2, x_3)$ for an inertial frame $\mathcal{S}$. The particle is acted on by an external force $\mathbf{F} = -\nabla V$, where $V = V(\mathbf{r})$ is the potential. The Lagrangian (2.19) is thus

$$L \;=\; \frac{1}{2}m|\dot{\mathbf{r}}|^2 - V(\mathbf{r}) \;. \tag{2.20}$$

Here the generalized coordinates are $q_a = x_a$, $a = 1, 2, 3$. Since $\partial L/\partial \mathbf{q} = -\partial V/\partial \mathbf{r} = \mathbf{F}$ and $\partial L/\partial \dot{\mathbf{q}} = m\dot{\mathbf{r}} = \mathbf{p}$, we immediately deduce that the Lagrange equations (2.10) are simply Newton's second law. It is straightforward to extend this computation to a system of N particles with masses m_I at positions $\mathbf{r}_I(t)$, interacting through a potential $V = V(\mathbf{r}_1, \ldots, \mathbf{r}_N)$.

We thus see that Hamilton's principle (equivalently the Lagrange equations of motion) is equivalent to Newton's second law for point particles. Let's look at another simple example:

Example (simple pendulum): Let us return to the simple pendulum discussed in the previous subsection, and shown in Figure 2. Resolving the tension force $\mathbb{T}$ in the x and z directions, Newton's second law gives

$$m\ddot{x} \;=\; -\mathbb{T}\sin\theta \;, \qquad m\ddot{z} \;=\; -mg + \mathbb{T}\cos\theta \;. \tag{2.21}$$

Here recall $x = l\sin\theta$, $z = -l\cos\theta$. Taking $\cos\theta$ times the x equation of motion plus $\sin\theta$ times the z equation of motion we easily derive

$$\ddot{\theta} \;=\; -\frac{g}{l}\sin\theta \;, \tag{2.22}$$

which is the equation of motion for the coordinate θ. The other linear combination determines the tension

$$\mathbb{T} \;=\; mg\cos\theta + ml\dot{\theta}^2 \;, \tag{2.23}$$

which can be understood as balancing the component of the weight along the rod and the centripetal force for circular motion about the origin O, respectively.

On the other hand the Lagrangian description with generalized coordinate $q = \theta$ leads immediately to (2.22). The kinetic energy is $T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) = \frac{1}{2}ml^2\dot{\theta}^2$, while the potential energy is $V = mgz = -mgl\cos\theta$. Thus the Lagrangian is

$$L \;=\; T - V \;=\; \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \;, \tag{2.24}$$

and the Lagrange equation (2.10) with $q = \theta$ indeed gives (2.22). This short calculation might look like a sleight of hand: we have (apparently) entirely ignored the tension force, but still found the correct dynamics. In fact this is one of the advantages of the Lagrangian formalism, and we'll explain why (2.24) correctly describes the dynamics in the next subsection.

2.3 Constraints

In the simple pendulum example just studied the motion of the mass m in the (x, z) plane was constrained to satisfy $x^2 + z^2 = l^2$, due to the rod pivoted at O. This is an example of a *holonomic constraint*. To describe this more generally, we need some more notation. The constrained motion is by definition embedded inside a larger *unconstrained configuration space*. Let $x_1, \ldots, x_d$ denote (generalized) coordinates on this larger space, denoting $\mathbf{x} = (x_1, \ldots, x_d)$. For example, for the constrained motion of a single particle moving in $\mathbb{R}^3$ we may take $\mathbf{x} = (x, y, z)$ to be Cartesian coordinates. Then *holonomic constraints* on the motion constrain the coordinates to satisfy

$$f_A(\mathbf{x}, t) \;=\; 0 \;, \qquad A \;=\; 1, \ldots, d-n \;. \tag{2.25}$$

Here n is the number degrees of freedom satisfying the constraints. For example, for the simple pendulum the constraint on motion in the (x, z) plane is $f(x, z) = 0$ where $f(x, z) = x^2 + z^2 - l^2$. This is an example of a *scleronomous* constraint, meaning it doesn't depend on time t. More generally these are given by $f_A(\mathbf{x}) = 0$. The general time-dependent case in (2.25) are called *rheonomous* constraints, *e.g.* compare the bead moving on the fixed wire in Figure 3 to the case where the wire is rotating about the x-axis.

Let us fix the time t, and consider $f_A(\mathbf{x}, t) = 0$, $A = 1, \ldots, d-n$, which we assume are $d-n$ independent equations. As such, we expect to be able to eliminate $d-n$ of the d variables x_i, $i = 1, \ldots, d$, in terms of the remaining n variables. Equivalently we can say that there are functions

$$\mathbf{x} \;=\; \mathbf{x}(\mathbf{q}, t) \;, \tag{2.26}$$

which parametrize the general solution to (2.25). The variables q_a, $a = 1, \ldots, n$, are then generalized coordinates for the constrained motion. For example, for the simple pendulum we have $x = l\sin\theta$, $z = -l\cos\theta$, which solve the constraint $f(x, z) = 0$ where $f(x, z) = x^2 + z^2 - l^2$.

A precise definition of the constraints being *independent* is that the $(d-n) \times d$ matrix $\partial f_A / \partial x_i$ has maximal rank $d-n$ at every point. It then follows from the *implicit function theorem* that we can solve the constraints as in (2.26).[4] Moreover, there is always a system of coordinates $y_1, \ldots, y_d$ such that the constraints are simply

$$y_{n+A} \;=\; 0 \;, \qquad A \;=\; 1, \ldots, d-n \;. \tag{2.27}$$

We may then take $y_a = q_a$ for $a = 1, \ldots, n$ to be our generalized coordinates. Such a coordinate system is said to be *adapted* to the constraints $\{f_A = 0\}$. We will not give a proof of this result, but it follows from methods similar to those seen in the Part A course Introduction to Manifolds. As an example, the coordinates $y_1 = q_1 = \theta$, $y_2 = q_2 = \varphi$, $y_3 = r - a$ are adapted to the constraint $f(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0$ for motion on a sphere of radius a.

Going back to our dynamical problem, there will be a Lagrangian $L_0(\mathbf{x}, \dot{\mathbf{x}}, t)$ that describes the motion without the constraints imposed. For example, for a single particle of mass m moving in $\mathbb{R}^3$ under the influence of a potential V we have Cartesian coordinates $\mathbf{x} = (x_1, x_2, x_3) = (x, y, z)$ and this Lagrangian is $L_0 = L_0(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - V(x_1, x_2, x_3)$. Provided we can find generalized coordinates $\mathbf{q}$ that solve the constraints via $\mathbf{x} = \mathbf{x}(\mathbf{q}, t)$, as in (2.26), the Lagrangian for the constrained problem is simply

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) \;\equiv\; L_0(\mathbf{x}(\mathbf{q}, t), \dot{\mathbf{x}}(\mathbf{q}, \dot{\mathbf{q}}, t), t) \;. \tag{2.28}$$

In particular here

$$\dot{x}_i \;=\; \sum_{a=1}^{n} \frac{\partial x_i}{\partial q_a}\dot{q}_a + \frac{\partial x_i}{\partial t} \;, \qquad i = 1, \ldots, d \;, \tag{2.29}$$

by the chain rule. This is precisely what we did for the simple pendulum, for example computing the kinetic energy to be $T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) = \frac{1}{2}ml^2\dot{\theta}^2$, where $x = l\sin\theta$, $z = -l\cos\theta$ solve the constraint in terms of the single generalized coordinate $q = \theta$. Solving the constrained problem in this way, the Lagrange equations for (2.28) are called *Lagrange equations of the second kind*.

[4] Those who have taken the Part A course Introduction to Manifolds may recognize the conditions we just stated.

Proof of Hamilton's principle for workless, holonomic constraints

In writing (2.28) we are stating that the Lagrangian for the constrained motion is obtained by simply substituting the solution to the constraints into the Lagrangian for the unconstrained motion. While this might seem reasonable, there is also an explanation. This is related to the *Lagrange equations of the first kind* where the constraints are not solved first, but are rather imposed via Lagrange multipliers. Specificially, let us introduce the Lagrangian

$$\hat{L} \;=\; \hat{L}(\mathbf{x},\dot{\mathbf{x}},\boldsymbol{\lambda},t) \;=\; L_0(\mathbf{x},\dot{\mathbf{x}},t) + \sum_{A=1}^{d-n} \lambda_A f_A(\mathbf{x},t)\ , \tag{2.30}$$

where the $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_{d-n})$ are called *Lagrange multipliers*, which are treated as additional generalized coordinates. The Lagrange equations for an extremum of the action $\hat{S} = \int_{t_1}^{t_2} \hat{L}\,\mathrm{d}t$ then read

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_0}{\partial \dot{x}_i}\right) - \frac{\partial L_0}{\partial x_i} \;=\; \sum_{A=1}^{d-n} \lambda_A \frac{\partial f_A}{\partial x_i}\ , \qquad i \;=\; 1,\ldots,d\ , \tag{2.31}$$

$$f_A(\mathbf{x},t) \;=\; 0\ , \qquad\qquad A \;=\; 1,\ldots,d-n\ . \tag{2.32}$$

Equation (2.31) is the Lagrange equation for x_i, while (2.32) follows from the Lagrange equation for λ_A: notice $\partial\hat{L}/\partial\lambda_A = f_A$, while $\hat{L}$ is independent of $\dot{\lambda}_A$.

We may now compare the equations (2.31), (2.32) to what we would obtain by applying Newton's laws. Of course (2.32) simply imposes the constraints. The left hand side of (2.31) is the unconstrained equation of motion for x_i. We have already shown that for an unconstrained system of point particles interacting via a general potential V, this is the same as Newton's second law. Concretely, for a single particle of mass m moving in $\mathbb{R}^3$ in a potential $V = V(\mathbf{x})$ (2.31) gives

$$m\ddot{x}_i + \frac{\partial V}{\partial x_i} \;=\; \sum_{A=1}^{d-n} \lambda_A \frac{\partial f_A}{\partial x_i} \;\equiv\; R_i\ , \qquad i = 1,\ldots,d\ . \tag{2.33}$$

The right hand side of (2.31)/(2.33) may then be interpreted as the (generalized) *constraint force* $\mathbf{R} = (R_1,\ldots,R_d)$. An example is the tension for the simple pendulum. By saying that a constraint force $\mathbf{R}$ is *workless* we mean that the work $\mathbf{R}\cdot\delta\mathbf{x} = 0$, where $\delta\mathbf{x}$ is any displacement tangent to the constraint space.[5] Geometrically this means that the vector $\mathbf{R}$ is orthogonal/normal to the constraint space. Since $\partial f_A/\partial x_i$ has maximal rank $d-n$, and for fixed A the vector field $\partial f_A/\partial\mathbf{x}$ is orthogonal to the constraint space $\{f_A = 0, A = 1,\ldots,d-n\}$, it follows that the set of $d-n$ vectors $\{\partial f_A/\partial\mathbf{x}, A = 1,\ldots,d-n\}$ form a basis for the normal space to the constraint space. Hence *any* constraint force $\mathbf{R}$ can be written as a linear combination $\mathbf{R} = \sum_{A=1}^{d-n}\lambda_A\frac{\partial f_A}{\partial\mathbf{x}}$. The Lagrangian (2.30) then determines the constrained dynamics, with the Lagrange multipliers determining the constraint forces.

[5] Some textbooks refer to such constraint forces as *ideal*, with $\mathbf{R}\cdot\delta\mathbf{x}$ the *virtual work* under the displacement $\delta\mathbf{x}$. It is "virtual" in the sense that $\delta\mathbf{x}$ isn't necessarily an actual displacement of the system obeying the equations of motion, but rather *any* potential displacement obeying the constraints.

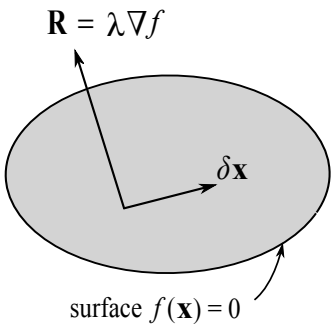

Figure 5: The figure shows a constraint surface defined by the single equation $f(\mathbf{x}) = 0$ in $\mathbb{R}^3$. A constraint force $\mathbf{R}$ is workless if $\mathbf{R} \cdot \delta\mathbf{x} = 0$ for any displacement vector $\delta\mathbf{x}$ tangent to the constraint space. Since $\nabla f = \partial f/\partial \mathbf{x}$ is everywhere normal to the constraint space we may hence write $\mathbf{R} = \lambda \nabla f$, where λ is the Lagrange multiplier.

That the Lagrange equations of the first and second kind are equivalent is then straightforward to see when we use coordinates $x_i = y_i$ that are adapted to the constraints. Recall that in this coordinate system the constraints are simply (2.27), with $y_a = q_a$, $a = 1, \ldots, n$, being generalized coordinates for the constrained motion. The constraints (2.32) then simply set $y_{n+A} = 0$ for each $A = 1, \ldots, d-n$, while the Lagrange equations (2.31) for $i = 1, \ldots, n$ read

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} \;=\; \sum_{A=1}^{d-n} \lambda_A \frac{\partial f_A}{\partial \mathbf{q}} \;=\; \mathbf{0}\ . \tag{2.34}$$

Here $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$ is defined by (2.28), and the last equality $\partial f_A/\partial \mathbf{q} = \mathbf{0}$ holds since in this coordinate system the constraint functions $f_A = y_{n+A}$ are independent of $\mathbf{q}$. Notice that in this argument we have implicitly used the fact that the Lagrange equations hold in one coordinate system if and only if they hold in any other coordinate system. The remaining $d-n$ equations in (2.31) determine the Lagrange multipliers λ_A.

Example: Since the above discussion is a little abstract, let's return to our example of the simple pendulum. The unconstrained coordinates are $x_1 = x$, $x_2 = z$, while the single constraint function may be taken to be $f(x, z) = x^2 + z^2 - l^2$. The Lagrangian for the unconstrained motion is $L_0 = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) - mgz$, which is the Lagrangian for a particle of mass m moving in (x, z) space under gravity. The Lagrange equations of the first kind (2.31), (2.32) hence read

$$\begin{aligned} m\ddot{x} &= 2\lambda x\ , \\ m\ddot{z} + mg &= 2\lambda z\ , \\ x^2 + z^2 - l^2 &= 0\ . \end{aligned} \tag{2.35}$$

Here λ is the Lagrange multiplier. Polar coodinates (ϱ, θ), defined by $x = \varrho \sin\theta$, $z = -\varrho\cos\theta$, are essentially adapted to the constraint. More precisely, adapted coordinates may be taken to be

$y_1 = q = \theta$, $y_2 = \varrho^2 - l^2$, so the constraint is simply $y_2 = 0$. Imposing this of course sets $\varrho = l$, and the first two equations in (2.35) become

$$\begin{aligned} ml(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2) &= 2l\lambda\sin\theta\ , \\ ml(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2) + mg &= -2l\lambda\cos\theta\ . \end{aligned} \tag{2.36}$$

Taking $\cos\theta$ times the first equation plus $\sin\theta$ times the second equation then gives the equation of motion (2.22). We've already seen that this indeed arises as the Lagrange equation of the second kind for the system. On the other hand $\sin\theta$ times the first equation in (2.36) minus $\cos\theta$ times the second equation instead gives

$$-ml\dot{\theta}^2 - mg\cos\theta = 2l\lambda\ . \tag{2.37}$$

Comparing to (2.23) we thus see that the Lagrange multipler λ is indeed proportional to the constraint tension $\mathbb{T}$

$$\lambda = -\frac{1}{2l}\mathbb{T}\ . \tag{2.38}$$

The constraint force vector $\mathbf{R} = (2\lambda x, 2\lambda z) = (2l\lambda\sin\theta, -2l\lambda\cos\theta) = (-\mathbb{T}\sin\theta, \mathbb{T}\cos\theta)$ appears on the right hand side of (2.35), which is a vector of magnitude $\mathbb{T}$ that is normal to the constraint space.

It is also possible to have constrained motion where the constraints cannot be written in the form (2.25). These are called *non-holonomic constraints*. Examples include velocity dependent constraints $f(\mathbf{x}, \dot{\mathbf{x}}, t) = 0$, and constraints defined by inequalities, say $f(\mathbf{x}, t) \geq 0$. Velocity dependent constraints arise naturally in certain problems involving the rolling of rigid bodies, since the velocity of the point of contact is instantaneously zero. A particle confined to a box may be defined by inequalities on the coordinates. One can solve both types of problem, but the point is that our above treatment of holonomic constraints is not directly applicable. There is no general theory – the constraints are implemented on a case-by-case basis. However, it is perhaps worth noting that some velocity dependent constraints can be "integrated" to give equivalent holonomic constraints. For example consider the constraint $A(x,t)\dot{x} + B(x,t) = 0$. In general this would not be holonomic, but if there is a function $f(x,t)$ such that $A = \partial f/\partial x$ and $B = \partial f/\partial t$ then by the chain rule the constraint may be rewritten as $\mathrm{d}f/\mathrm{d}t = 0$, so that $f(x,t) =$ constant is an equivalent holonomic constraint (where we then need to also vary the constant in $f(x,t) =$ constant).

2.4 Noether's theorem

The Lagrangian formulation of classical mechanics leads to a very clear relationship between *symmetries* and *conserved quantities*, via *Noether's theorem*. In this section we give a general account of Noether's theorem, and then discuss how certain symmetries lead to conservation of energy, momentum and angular momentum.

A *conserved quantity* is a function $F(\mathbf{q}, \dot{\mathbf{q}}, t)$ that is constant when evaluated on a solution to the Lagrange equations of motion, *i.e.*

$$0 \;=\; \frac{\mathrm{d}}{\mathrm{d}t} F(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) \;=\; \frac{\partial F}{\partial t} + \sum_{a=1}^{n} \left(\frac{\partial F}{\partial q_a} \dot{q}_a + \frac{\partial F}{\partial \dot{q}_a} \ddot{q}_a \right) \;, \tag{2.39}$$

where $\mathbf{q}(t)$ solves the Lagrange equations (2.10).

We next need to make precise what we mean by a *symmetry* of a Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$. There are various ways to approach this. We shall define a *generator of an infinitesimal deformation* to be a function $\boldsymbol{\rho} = \boldsymbol{\rho}(\mathbf{q}, \dot{\mathbf{q}}, t)$, which leads to a first order variation in any path $\mathbf{q}(t)$ in configuration space $\mathcal{Q}$ via

$$\delta\mathbf{q}(t) \;=\; \epsilon\, \boldsymbol{\rho}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) \;\equiv\; \epsilon\, \mathbf{u}(t) \;. \tag{2.40}$$

Notice here that $\mathbf{u}(t)$ depends on the path $\mathbf{q}(t)$, although for fixed path it is a function only of time t. Unlike the variations of the action in section 2.2 we do not require $\mathbf{u}(t)$ to satisfy any boundary conditions. Then we say that $\boldsymbol{\rho}$ generates a (infinitesimal) *symmetry* of L if there exists a function $f(\mathbf{q}, \dot{\mathbf{q}}, t)$ such that for all paths $\mathbf{q}(t)$ we have

$$\frac{\partial}{\partial \epsilon} L(\mathbf{q}(t) + \epsilon\, \mathbf{u}(t), \dot{\mathbf{q}}(t) + \epsilon\, \dot{\mathbf{u}}(t), t) \bigg|_{\epsilon=0} \;=\; \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) \;. \tag{2.41}$$

If one multiplies the left hand side of (2.41) by ϵ, this is the first order variation $\delta L \equiv \epsilon \, \left(\frac{\partial L}{\partial \epsilon} \big|_{\epsilon=0} \right)$ of L under $\mathbf{q} \to \mathbf{q} + \delta\mathbf{q}$. Thus (2.41) simply says that the first order variation in the Lagrangian is a total time derivative.

Noether's theorem: Suppose we have a symmetry of a Lagrangian L in the above sense. Then

$$F \;=\; F(\mathbf{q}, \dot{\mathbf{q}}, t) \;\equiv\; \sum_{a=1}^{n} \frac{\partial L}{\partial \dot{q}_a} \rho_a - f \tag{2.42}$$

is a *conserved quantity*.

The proof of Noether's theorem is by direct calculation. Evaluating on a solution $\mathbf{q}(t)$ to the Lagrange equations we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left[\sum_{a=1}^{n} \frac{\partial L}{\partial \dot{q}_a} \rho_a - f \right] \;&=\; \sum_{a=1}^{n} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_a} \right) u_a + \frac{\partial L}{\partial \dot{q}_a} \dot{u}_a \right] - \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) \\ &=\; \sum_{a=1}^{n} \left[\frac{\partial L}{\partial q_a} u_a + \frac{\partial L}{\partial \dot{q}_a} \dot{u}_a \right] - \frac{\partial}{\partial \epsilon} L(\mathbf{q}(t) + \epsilon\, \mathbf{u}(t), \dot{\mathbf{q}}(t) + \epsilon\, \dot{\mathbf{u}}(t), t) \bigg|_{\epsilon=0} \\ &=\; 0 \;. \end{aligned} \tag{2.43}$$

Here in the second equality we have used the Lagrange equation on the first term, and (2.41) for the last term. The final equality simply follows from the chain rule.

The simplest case is when $f = 0$, so that the Lagrangian is actually invariant under the deformation to first order, $\delta L = 0$. In fact this is the version of Noether's theorem usually given in textbooks. The more general form we have presented will allow us to treat conservation of energy, momentum and angular momentum on the same footing. Notice that in deriving the conservation law (2.43) we only used the definition of the symmetry (2.41) for a solution $\mathbf{q}(t)$ to the equations of motion (rather than a completely general path).

Conservation of energy

Suppose we have a Lagrangian that is invariant under time translations, meaning that $L(\mathbf{q}, \dot{\mathbf{q}}, t + \epsilon) = L(\mathbf{q}, \dot{\mathbf{q}}, t)$ is independent of ϵ. Hence $\partial L/\partial t = 0$ and $\boldsymbol{\rho} = \dot{\mathbf{q}}$ generates a symmetry of L since

$$\frac{\partial}{\partial \epsilon} L(\mathbf{q}(t) + \epsilon\, \dot{\mathbf{q}}(t), \dot{\mathbf{q}} + \epsilon\, \ddot{\mathbf{q}}(t), t)\Big|_{\epsilon=0} \;=\; \frac{\mathrm{d}}{\mathrm{d}t} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\ . \tag{2.44}$$

The reasoning behind this is that the corresponding first order variation in a path arises from $\mathbf{q}(t+\epsilon) = \mathbf{q}(t) + \epsilon\, \dot{\mathbf{q}}(t) + O(\epsilon^2)$, so that $\delta \mathbf{q}(t) = \epsilon\, \dot{\mathbf{q}}(t)$. Thus from (2.44) we may simply take $f = L$ in Noether's theorem, and the conserved quantity is

$$H \;=\; \sum_{a=1}^{n} \frac{\partial L}{\partial \dot{q}_a} \dot{q}_a - L\ . \tag{2.45}$$

This is indeed what we would usually call the *energy*. To see this, let us suppose that $L = T - V$ with $V = V(\mathbf{q})$ and the kinetic energy being quadratic in the generalized velocities $\dot{\mathbf{q}}$

$$T \;=\; \frac{1}{2} \sum_{a,b=1}^{n} T_{ab}(\mathbf{q})\, \dot{q}_a \dot{q}_b\ . \tag{2.46}$$

For example this is the case if one is describing a point particle where the Cartesian coordinates x_a are related to the generalized coordinates q_a via the scleronomic change of variable $\mathbf{x} = \mathbf{x}(\mathbf{q})$. Then $T_{ab} = m \sum_{c=1}^{n} \frac{\partial x_c}{\partial q_a} \frac{\partial x_c}{\partial q_b}$. A simple computation then gives

$$H \;=\; 2T - L \;=\; T + V\ . \tag{2.47}$$

It is worth stressing that although H defined in (2.45) is always conserved when $\partial L/\partial t = 0$, it is not *always* the case that H is simply the sum of kinetic and potential energies $T + V$, which one would perhaps think of as the total energy. We might then have called this section "conservation of H", but what one defines as the "energy" is in any case a matter of convention.

We may also now define the *generalized momentum* $\mathbf{p}$ *conjugate* to $\mathbf{q}$ as

$$\mathbf{p} \;\equiv\; \frac{\partial L}{\partial \dot{\mathbf{q}}}\ . \tag{2.48}$$

For example, for a point particle in $\mathbb{R}^3$ this is simply the momentum $\mathbf{p} = m\dot{\mathbf{r}}$, where $\mathbf{r} = (x_1, x_2, x_3)$ are Cartesian coordinates. When $H = \sum_a p_a \dot{q}_a - L$ is regarded as a function of $(\mathbf{q}, \mathbf{p}, t)$, rather

than $(\mathbf{q}, \dot{\mathbf{q}}, t)$, it is called the *Hamiltonian* of the system, and will be the subject of section 5. Notice that for a *closed* system the Lagrangian should indeed be time translation invariant, as this is one of the Galilean symmetries discussed in sections 1.3 and 1.4. Conservation of energy follows directly from this symmetry.

Conservation of momentum

Suppose now that we have a *translational symmetry* of the configuation space, meaning that $\boldsymbol{\rho} = \mathbf{v}$ is a symmetry of the Lagrangian L with $f = 0$ and $\mathbf{v}$ a fixed vector. The corresponding first order variation in a path is $\delta\mathbf{q}(t) = \epsilon\mathbf{v}$ and $\mathbf{u}(t) = \mathbf{v}$. Since $\dot{\mathbf{u}} = \mathbf{0}$ the symmetry of the Lagrangian is equivalent to

$$\sum_{a=1}^{n} \frac{\partial L}{\partial q_a} v_a \quad = \quad 0 \; . \tag{2.49}$$

For example, for $\mathbf{v} = (1, 0, \ldots, 0)$ this means that $\partial L/\partial q_1 = 0$ and the coordinate q_1 is an *ignorable coordinate* (also sometimes called a *cyclic coordinate*). Noether's theorem immediately tells us that the quantity

$$\sum_{a=1}^{n} \frac{\partial L}{\partial \dot{q}_a} v_a \; = \; \sum_{a=1}^{n} p_a v_a \tag{2.50}$$

is conserved. Of course, this result may also be very simply derived by multiplying the Lagrange equations (2.10) by v_a and summing over a. Going back to our example with $\mathbf{v} = (1, 0, \ldots, 0)$ we see that it is the conjugate momentum p_1 that is conserved. Thus translational symmetry leads to conservation of momentum.

As an example consider the free motion of a particle in spherical polar coordinates $(q_1, q_2, q_3) = (r, \theta, \varphi)$, related to Cartesian coordinates by (2.4). Here the Lagrangian is $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\varphi}^2)$. Since $\partial L/\partial\varphi = 0$ we see that $\mathbf{v} = (0, 0, 1)$ generates a symmetry of L, and the conjugate momentum $p_3 = p_\varphi = \partial L/\partial\dot{\varphi} = mr^2 \sin^2\theta\, \dot{\varphi}$ is conserved.

More generally for a closed system in an inertial frame we have the Galilean spatial translational symmetry discussed in sections 1.3, 1.4. Consider a system of N particles with masses m_I, position vectors $\mathbf{r}_I$, and interacting through a potential $V = V(\{|\mathbf{r}_I - \mathbf{r}_J|\})$. Here translational symmetry implies that the potential depends only on the distances between the particles $\{|\mathbf{r}_I - \mathbf{r}_J|\}$, and the Lagrangian

$$L \quad = \quad \frac{1}{2}\sum_{I=1}^{N} m_I |\dot{\mathbf{r}}_I|^2 - V(\{|\mathbf{r}_I - \mathbf{r}_J|\}) \tag{2.51}$$

is invariant under the spatial translations $\mathbf{r}_I \to \mathbf{r}_I + \epsilon\,\mathbf{k}$, for any ϵ and fixed vector $\mathbf{k}$. Here the configuration space has dimension $3N$ with generalized coordinates $(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ and in the above notation the $3N$-vector is $\mathbf{v} = (\mathbf{k}, \ldots, \mathbf{k})$. Thus Noether's theorem gives us that

$$\sum_{I=1}^{N} \frac{\partial L}{\partial \dot{\mathbf{r}}_I} \cdot \mathbf{k} \; = \; \sum_{I=1}^{N} \mathbf{p}_I \cdot \mathbf{k} \tag{2.52}$$

is conserved. Since this is true for all $\mathbf{k}$ we deduce that the *total momentum*

$$\mathbf{P} \;=\; \sum_{I=1}^{N} \mathbf{p}_I \tag{2.53}$$

is conserved. Compare this to our discussion in section 1.5.

Conservation of angular momentum

The Lagrangian (2.51) also has *rotational symmetry*. By this we mean that L is invariant under $\mathbf{r}_I \to \mathcal{R}\,\mathbf{r}_I$ for all $I = 1, \ldots, N$, where $\mathcal{R} \in O(3)$ is any rotation matrix. This is simply because L depends on $\mathbf{r}_I - \mathbf{r}_J$ and $\dot{\mathbf{r}}_I$ only via their lengths, which are (by definition) invariant under orthogonal transformations. We shall study rotations in more detail in section 4.1. Consider then a one-parameter family of rotations $\mathcal{R}(\epsilon) \in O(3)$, where $\mathcal{R}(0) = \mathbb{1}$ is the identity matrix. We may Taylor expand this in ϵ to write

$$\mathcal{R}(\epsilon) \;=\; \mathbb{1} + \epsilon \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} + O(\epsilon^2)\ , \tag{2.54}$$

where $\mathbf{n}$ is a fixed vector describing the first order axis of rotation. Here we have simply used the fact that $\mathrm{d}\mathcal{R}/\mathrm{d}\epsilon \;|_{\epsilon=0}$ is an anti-symmetric matrix, which in turn follows from $\mathcal{R}(\epsilon)$ being orthogonal (see the discussion around equation (4.5)). Then $\mathbf{r}_I \to \mathcal{R}(\epsilon)\,\mathbf{r}_I$ is to first order in ϵ given by

$$\mathbf{r}_I \;\;\to\;\; \mathbf{r}_I + \epsilon\,\mathbf{n} \wedge \mathbf{r}_I\ . \tag{2.55}$$

Invariance of the Lagrangian L in (2.51) gives, by Noether's theorem, that

$$\sum_{I=1}^{N} \frac{\partial L}{\partial \dot{\mathbf{r}}_I} \cdot (\mathbf{n} \wedge \mathbf{r}_I) \;=\; \sum_{I=1}^{N} \mathbf{n} \cdot (\mathbf{r}_I \wedge \mathbf{p}_I) \;=\; \mathbf{n} \cdot \mathbf{L} \tag{2.56}$$

is conserved. Here we have used the scalar triple product identity in the second equality, and the definition of the *total angular momentum* $\mathbf{L}$ in (4.24) in the last equality. Since $\mathbf{n}$ is arbitrary we thus deduce that systems with rotational symmetry have conserved angular momentum. In comparing this to the more direct analysis in section 1.5 notice that the strong form of Newton's third law is obeyed by the Lagrangian (2.51), in the sense that for fixed I the force term $\partial L/\partial \mathbf{r}_I$ on the Ith particle is a sum of terms $\mathbf{F}_{IJ}$ for $J \neq I$, where $\mathbf{F}_{IJ} = -\mathbf{F}_{JI} \propto (\mathbf{r}_I - \mathbf{r}_J)$ follows since V depends only on $|\mathbf{r}_I - \mathbf{r}_J|$.

These are the most important symmetries and conservation laws in classical mechanics, but there are others. For example the Galilean boost symmetry of section 1.3 also leads to a conserved quantity, which again is not often mentioned in textbooks. The details are left to Problem Sheet 2.

There are also conserved quantities that, while they can be derived from the form of Noether's theorem we have presented, are not obviously related to any geometric symmetry of the system. These are sometimes called *hidden symmetries*. The most famous example arises in the *Kepler problem* you studied in first year Dynamics. The Lagrangian is $L = \frac{1}{2}m|\dot{\mathbf{r}}|^2 + \frac{\kappa}{r}$, with κ a constant. This is the same potential as (1.11), and for example describes the motion of a planet around the Sun (more on this two-body problem in the next subsection). Then one can check explicitly that the *Laplace-Runge-Lenz vector*

$$\mathbf{A} \equiv \mathbf{p} \wedge \mathbf{L} - m\kappa \frac{\mathbf{r}}{|\mathbf{r}|} \tag{2.57}$$

is conserved, where $\mathbf{p} = m\dot{\mathbf{r}}$ and $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ are the momentum and angular momentum about $\mathbf{r} = \mathbf{0}$, respectively. This is a fascinating conserved quantity, with a very interesting history. Writing $\mathbf{r} = (x_1, x_2, x_3)$, one can show that

$$\rho_a(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}m\left[2\dot{x}_a x_b - x_a \dot{x}_b - \delta_{ab}\,\mathbf{r}\cdot\dot{\mathbf{r}}\right] , \tag{2.58}$$

where $b = 1, 2, 3$ is fixed generates a symmetry of the Lagrangian via $x_a(t) \to x_a(t) + \epsilon\rho_a(\mathbf{r}(t), \dot{\mathbf{r}}(t))$. The corresponding Noether conserved quantity in (2.42) (with $q_a = x_a$) is precisely the component A_b of the vector $\mathbf{A} = (A_1, A_2, A_3)$.

***** Pauli used this vector to correctly derive the energy levels of the hydrogen atom *before* Schrödinger discovered his equation for quantum mechanics! Essentially the energy levels and their degeneracies follow from symmetry principles alone – one doesn't need to solve any differential equations. We'll come back to this topic again briefly in section 5. In the meantime there is more on the classical Laplace-Runge-Lenz vector (2.57) on Problem Sheet 1.

2.5 Examples

So far most of our discussion has been fairly abstract, sprinkled with a few very simple examples to hopefully clarify the basic ideas. However, one really gets to grips with Lagrangian mechanics by looking at various examples in detail. In this section we'll focus on the new ideas: finding generalized coordinates, deriving the Lagrangians, identifying symmetries and conserved quantities. Typically the Lagrange equations in any case cannot be solved in closed form (there are *chaotic* examples, such as the double pendulum). We will study systems analytically near to equilibrium in section 3, but more generally one will have to settle for a qualitative understanding and/or numerical simulations of the equations of motion.

Pendulum on a horizontally moving support

Consider a simple pendulum of mass m and length l, attached to a pivot of mass M that is free to move along the x-axis. As usual we denote by θ the angle the pendulum makes with the vertical z direction. If X denotes the position of the pivot mass M, then the pendulum mass m coordinates in the (x, z) plane are $(x, z) = (X + l\sin\theta, -l\cos\theta)$ – see Figure 6.

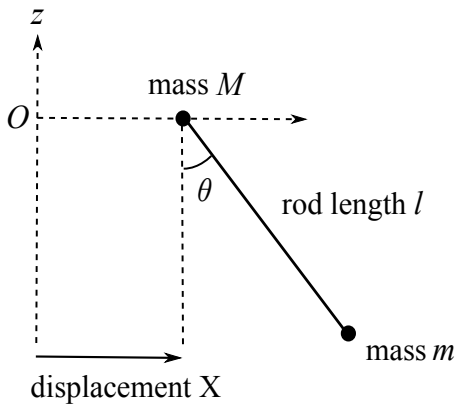

Figure 6: A pendulum on a horizontally moving support. The simple pendulum has mass m and length l, and the support pivot has mass M and moves freely along the x-axis with displacement X from the origin. The coordinates of the mass m are $(x, z) = (X + l\sin\theta, -l\cos\theta)$.

The total kinetic energy of the system is the sum of the kinetic energies of the two masses:

$$T \;=\; \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{x}^2+\dot{z}^2) \;=\; \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left[(\dot{X}+l\cos\theta\,\dot{\theta})^2 + (l\sin\theta\,\dot{\theta})^2\right] \;. \tag{2.59}$$

The potential energy is $V = mgz = -mgl\cos\theta$, precisely as it is for the usual simple pendulum. Gravity also acts on the pivot mass M, but since it is constrained to $z = 0$ this force is cancelled by a reaction force that we don't need to determine in the Lagrangian approach. Thus the Lagrangian is

$$L \;=\; T - V \;=\; \frac{1}{2}(M+m)\dot{X}^2 + ml\cos\theta\,\dot{X}\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \;. \tag{2.60}$$

Notice here that X is an *ignorable coordinate*, $\partial L/\partial X = 0$, so its conjugate momentum

$$p_X \;=\; \frac{\partial L}{\partial \dot{X}} \;=\; (M+m)\dot{X} + ml\cos\theta\,\dot{\theta} \tag{2.61}$$

is conserved (by Noether's theorem or by the Lagrange equation for X). The Lagrange equation of motion for θ is

$$0 \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\left[ml^2\dot{\theta} + ml\cos\theta\,\dot{X}\right] + ml\sin\theta\dot{X}\dot{\theta} + mgl\sin\theta \;. \tag{2.62}$$

The cross term in $\dot{X}\dot{\theta}$ cancels and this reduces to

$$\ddot{\theta} + \frac{1}{l}\cos\theta\,\ddot{X} + \frac{g}{l}\sin\theta \;=\; 0 \;. \tag{2.63}$$

One can then eliminate the $\ddot{X}$ term in favour of terms depending only on θ, $\dot{\theta}$, $\ddot{\theta}$ using $\dot{p}_X = 0$, thus obtaining a single second order ODE for $\theta(t)$:

$$(M + m\sin^2\theta)\,\ddot{\theta} + m\sin\theta\cos\theta\,\dot{\theta}^2 + \frac{g}{l}(M+m)\sin\theta \;=\; 0 \;. \tag{2.64}$$

However, since $\partial L/\partial t = 0$ we have another conserved quantity, namely the energy E. Since T is quadratic in the generalized velocities $\dot{X}$, $\dot{\theta}$ the argument after equation (2.46) applies and the conserved energy is hence

$$E \;=\; T+V \;=\; \frac{1}{2}(M+m)\dot{X}^2 + ml\cos\theta\,\dot{X}\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta\ . \tag{2.65}$$

We may now substitute for the conserved generalized momentum p_X in (2.61) to obtain

$$E - \frac{1}{2(M+m)}p_X^2 \;=\; \frac{ml^2}{2(M+m)}(M+m\sin^2\theta)\dot{\theta}^2 - mgl\cos\theta\ . \tag{2.66}$$

We have thus reduced the problem to a *first order* ODE for $\theta(t)$. This was possible because of the existence of the two conserved quantities E and p_X. Equation (2.66) may be written as $\dot{\theta}^2 = f(\theta)$, which integrates to $t = \int \mathrm{d}\theta/\sqrt{f(\theta)}$. The solution $\theta(t)$ may then be substituted into the conserved momentum (2.61), which is a first order ODE for $X(t)$ which may similarly be directly integrated. We have thus *reduced the problem to quadratures*. Generally speaking this means writing the solution to n differential equations as a complete set of n independent definite integrals (generally using $n-1$ conservation laws, plus conservation of energy, with a final integration of the remaining variable after substituting for these conserved quantities). Such systems are also called *completely integrable*.

Bead on a circular wire/spherical pendulum

Next we discuss two closely related problems. Consider a small bead of mass m that slides freely on a circular wire of radius a which rotates about a vertical diameter. Here it is natural to introduce spherical polar coordinates (2.4) as these are adapted to the constraints. In particular the angle θ measures the position of the bead with respect to the vertical axis of rotation, while φ is the angle between the plane of the wire and some fixed fiducial (initial) choice of this plane – see Figure 7.

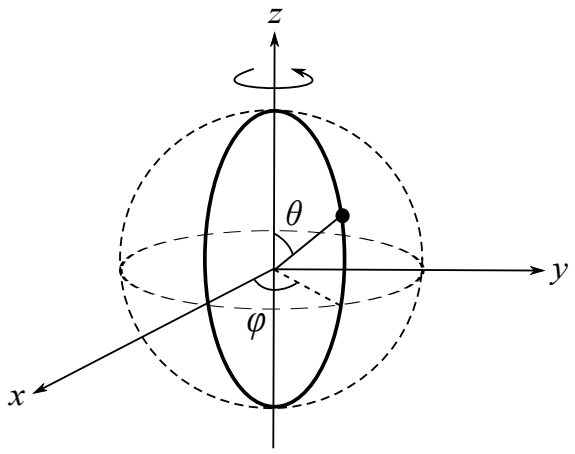

Figure 7: A bead of mass m sliding freely on a circular wire of radius a that rotates about a vertical diameter. One can use the spherical polar angles θ and φ as generalized coordinates.

The position of the bead in Cartesian coordinates is

$$\mathbf{r} \;=\; (x,y,z) \;=\; (a\sin\theta\cos\varphi, a\sin\theta\sin\varphi, a\cos\theta)\ . \tag{2.67}$$

One then calculates the kinetic energy

$$T \;=\; \frac{1}{2}m|\dot{\mathbf{r}}|^2 \;=\; \frac{1}{2}ma^2(\dot\theta^2+\sin^2\theta\,\dot\varphi^2)\ . \tag{2.68}$$

The potential energy is (notice that with respect to the simple pendulum we have $\theta\to\pi-\theta$)

$$V \;=\; mgz \;=\; mga\cos\theta\ . \tag{2.69}$$

We may then consider the following two dynamical situations:

1. The wire is forced to rotate at a constant angular velocity $\dot\varphi=\omega$.

2. The wire rotates freely about the vertical diameter.

Case 1 is an example of a time-dependent (rheonomous) holonomic constraint, since $\varphi(t)$ is fixed to be $\varphi(t)=\varphi_0+\omega t$, which leaves just one degree of freedom in the angle θ. Case 2 on the other hand has two degrees of freedom θ, φ, and is equivalent to a *spherical pendulum* since the only constraint is that the bead is distance a from the origin. The two Lagrangians are hence

$$L_1 \;=\; L_1(\theta,\dot\theta) \;=\; \frac{1}{2}ma^2\dot\theta^2+\frac{1}{2}ma^2\omega^2\sin^2\theta-mga\cos\theta\ , \tag{2.70}$$

$$L_2 \;=\; L_2(\theta,\varphi,\dot\theta,\dot\varphi) \;=\; \frac{1}{2}ma^2(\dot\theta^2+\sin^2\theta\,\dot\varphi^2)-mga\cos\theta\ . \tag{2.71}$$

Focusing first on case 1, since $\partial L_1/\partial t=0$ we may derive a first order equation from

$$E_1 \;=\; \frac{\partial L_1}{\partial\dot\theta}\dot\theta-L_1 \;=\; \frac{1}{2}ma^2(\dot\theta^2-\omega^2\sin^2\theta)+mga\cos\theta \tag{2.72}$$

being conserved. Notice this is *not* the same as taking $T+V$ given by (2.68), (2.69) and putting $\dot\varphi=\omega$. This gives something very similar to (2.72), but with a minus sign difference (and is not conserved). Notice also that in (2.70) the kinetic term arising from $\frac{1}{2}ma^2\sin^2\theta\,\dot\varphi^2$ with $\dot\varphi=\omega$ is absorbed into an effective potential

$$V_{\rm eff}(\theta) \;=\; mga\cos\theta-\frac{1}{2}ma^2\omega^2\sin^2\theta \tag{2.73}$$

for the dynamics of $\theta(t)$. We may again in principle integrate the equation $E_1=\frac{1}{2}ma^2\dot\theta^2+V_{\rm eff}(\theta)$, thus reducing to quadratures. The equation of motion for θ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_1}{\partial\dot\theta}\right) \;=\; \frac{\partial L_1}{\partial\theta} \qquad\Longleftrightarrow\qquad ma^2\ddot\theta \;=\; -\frac{\partial V_{\rm eff}}{\partial\theta} \;=\; ma\sin\theta(g+a\omega^2\cos\theta)\ , \tag{2.74}$$

and there are hence equilibrium points (see section 3) at $\theta=0$ and $\theta=\pi$, so that the bead is on the axis of rotation, and provided $\omega\geq\sqrt{g/a}$ also at

$$\theta_0 \;=\; \cos^{-1}\left(-\frac{g}{\omega^2a}\right)\ . \tag{2.75}$$

The stability of these equilibria is examined in Problem Sheet 2 (using the methods of section 3).

Going back to case 2, which is the spherical pendulum, the coordinate φ is ignorable, so its conjugate momentum

$$p_\varphi = \frac{\partial L_2}{\partial \dot\varphi} = ma^2 \sin^2\theta\, \dot\varphi \tag{2.76}$$

is conserved. This is simply the component of angular momentum about the vertical axis. Again $\partial L_2/\partial t = 0$ so

$$\begin{aligned} E_2 &= \frac{\partial L_2}{\partial \dot\theta}\dot\theta + \frac{\partial L_2}{\partial \dot\varphi}\dot\varphi - L_2 \\ &= \frac{1}{2}ma^2(\dot\theta^2 + \sin^2\theta\dot\varphi^2) + mga\cos\theta \end{aligned} \tag{2.77}$$

is conserved. This is simply $T + V$ for the bead. One can eliminate $\dot\varphi$ using (2.76), thus again reducing to quadratures.

The double Atwood machine

Next we consider the *double Atwood machine*, shown in Figure 8. This consists of a cable of length l passing over a *fixed* pulley, with a mass m_1 tied to one end of the cable and another *moveable* pulley tied to the other end. Over this second pulley passes a cable of length l', with masses m_2, m_3 tied to each end. We treat the pulleys and cables as massless, there is no friction, and moreover we take the radii of the pulleys to be negligible compared to l, l'.

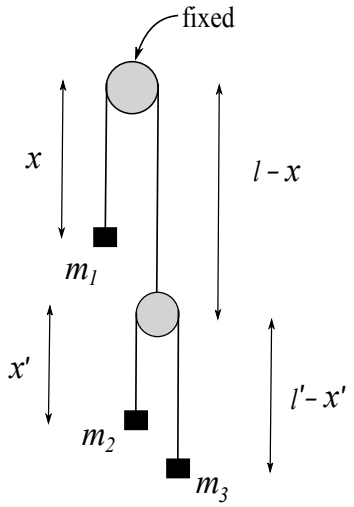

Figure 8: A double Atwood machine. A cable of length l passes over a fixed pulley. Tied to one end of the cable is a mass m_1, while at the other end is a moveable pulley. A cable of length l' passes over this second pulley, with masses m_2, m_3 attached to its two ends. The pulleys and cables are treated as massless, there is no friction, and we assume the radii of the pulleys are negligible compared to l, l'. We may use the coordinates x, x' shown as generalized coordinates.

This system has two degrees of freedom: the vertical displacement x of mass m_1 from the fixed pulley, and the vertical displacement x' of mass m_2 from the moveable pulley. We take these as

our generalized coordinates. If we denote the vertical displacements of the three masses from the fixed pulley by x_i, $i = 1, 2, 3$, and similarly the vertical displacement of the moveable pulley from the fixed pulley by x_p, then

$$x_1 \;=\; x \;, \qquad x_2 \;=\; (l - x) + x' \;, \qquad x_3 \;=\; (l - x) + (l' - x') \;, \qquad x_p \;=\; l - x \;. \tag{2.78}$$

The kinetic energy is hence

$$T \;=\; \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x}' - \dot{x})^2 + \frac{1}{2} m_3 (\dot{x}' + \dot{x})^2 \;, \tag{2.79}$$

while the potential energy is

$$V \;=\; -m_1 g x - m_2 g (l - x + x') - m_3 g (l + l' - x - x') \;. \tag{2.80}$$

In general neither coordinate is ignorable, and Lagrange's equations for x, x' read

$$\begin{aligned} m_1 \ddot{x} + m_2 (\ddot{x} - \ddot{x}') + m_3 (\ddot{x} + \ddot{x}') &= (m_1 - m_2 - m_3) g \;, \\ -m_2 (\ddot{x} - \ddot{x}') + m_3 (\ddot{x} + \ddot{x}') &= (m_2 - m_3) g \;. \end{aligned} \tag{2.81}$$

A little algebra allows us to solve for the accelerations

$$\begin{aligned} \ddot{x} &= \frac{m_1 (m_2 + m_3) - 4 m_2 m_3}{m_1 (m_2 + m_3) + 4 m_2 m_3} g \;, \\ \ddot{x}' &= \frac{2 m_1 (m_2 - m_3)}{m_1 (m_2 + m_3) + 4 m_2 m_3} g \;. \end{aligned} \tag{2.82}$$

In particular notice that if $m_2 = m_3$ then $\ddot{x}' = 0$ and x' moves with constant speed. This also follows since in this case x' is an ignorable coordinate, and this statement is conservation of the conjugate momentum $p_{x'}$. Also notice that if $m_1 = m$, $m_2 = m_3 = \frac{m}{2}$ then $\ddot{x} = \ddot{x}' = 0$, and both p_x and $p_{x'}$ are conserved.

Charged particle in an electromagnetic field

Consider a particle of mass m and charge e moving in an electromagnetic field. For our purposes here we don't need to know anything about the theory of electromagnetism, except that the electric field $\mathbf{E}(\mathbf{r}, t)$ and magnetic field $\mathbf{B}(\mathbf{r}, t)$ can be written in terms of a scalar potential $\phi(\mathbf{r}, t)$ and vector potential $\mathbf{A}(\mathbf{r}, t)$ via

$$\mathbf{E} \;=\; -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \;, \qquad \mathbf{B} \;=\; \nabla \wedge \mathbf{A} \;. \tag{2.83}$$

The Lagrangian of the charged particle in such a background electromagnetic field is simply

$$L \;=\; \frac{1}{2} m |\dot{\mathbf{r}}|^2 - e \left(\phi - \dot{\mathbf{r}} \cdot \mathbf{A} \right) \;. \tag{2.84}$$

The first term is of course the kinetic energy of the particle, while the second term gives rise to the *Lorentz force law*. Let us see this by working out Lagrange's equations. These read

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{\mathbf{r}}} \right) \;=\; \frac{\partial L}{\partial \mathbf{r}} \;, \tag{2.85}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{\mathbf{r}} + e\mathbf{A}\right) \;=\; -e\left(\nabla\phi - \nabla(\dot{\mathbf{r}}\cdot\mathbf{A})\right)\ . \tag{2.86}$$

Recalling the vector calculus identity

$$\nabla(\mathbf{a}\cdot\mathbf{b}) \;=\; (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\wedge(\nabla\wedge\mathbf{b}) + \mathbf{b}\wedge(\nabla\wedge\mathbf{a})\ , \tag{2.87}$$

we may rewrite the last term in (2.86) as

$$\nabla(\dot{\mathbf{r}}\cdot\mathbf{A}) \;=\; (\dot{\mathbf{r}}\cdot\nabla)\mathbf{A} + \dot{\mathbf{r}}\wedge(\nabla\wedge\mathbf{A})\ . \tag{2.88}$$

On the other hand on the left hand side of (2.86), using the chain rule we have the term

$$\dot{\mathbf{A}} \;=\; \frac{\partial\mathbf{A}}{\partial t} + (\dot{\mathbf{r}}\cdot\nabla)\mathbf{A}\ . \tag{2.89}$$

The terms involving $(\dot{\mathbf{r}}\cdot\nabla)\mathbf{A}$ cancel in (2.86), which rearranges to give

$$m\ddot{\mathbf{r}} \;=\; e\left(-\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}\right) + e\,\dot{\mathbf{r}}\wedge(\nabla\wedge\mathbf{A}) \;=\; e\left(\mathbf{E} + \dot{\mathbf{r}}\wedge\mathbf{B}\right)\ , \tag{2.90}$$

and in the last step we have used (2.83). The force $\mathbf{F} = e(\mathbf{E} + \dot{\mathbf{r}}\wedge\mathbf{B})$ on the right hand side of (2.90) is called the *Lorentz force*. Notice that the corresponding potential term $e(\phi - \dot{\mathbf{r}}\cdot\mathbf{A})$ in the Lagrangian (2.84) depends on the particle's velocity $\dot{\mathbf{r}}$. Note also in this example that the momentum $\mathbf{p}$ conjugate to $\mathbf{r}$ is

$$\mathbf{p} \;=\; \frac{\partial L}{\partial\dot{\mathbf{r}}} \;=\; m\dot{\mathbf{r}} + e\,\mathbf{A}\ . \tag{2.91}$$

Thus the canonical momentum is the usual linear momentum $m\dot{\mathbf{r}}$, shifted by the charge e times the vector potential $\mathbf{A}$. We thus need to be careful what we mean by "momentum" in this case.

The two-body problem

Let us return to the two-body problem. This is the simplest non-trivial closed system, consisting of two particles with Lagrangian

$$L \;=\; \frac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \frac{1}{2}m_2|\dot{\mathbf{r}}_2|^2 - V(|\mathbf{r}_1 - \mathbf{r}_2|)\ . \tag{2.92}$$

As explained in section 2.4, being a closed system this enjoys conservation of energy, momentum and angular momentum. Conservation of momentum $\mathbf{P} = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2$ is equivalent to the centre of mass

$$\mathbf{R} \;\equiv\; \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \tag{2.93}$$

moving at constant velocity. Thus we may write (choosing $\mathbf{R}(0) = \mathbf{0}$ without loss of generality)

$$\mathbf{R}(t) \;=\; \mathbf{V}_0 t\ , \qquad \text{where} \quad \mathbf{V}_0 \;=\; \frac{m_1\mathbf{v}_1^{(0)} + m_2\mathbf{v}_2^{(0)}}{m_1 + m_2} \tag{2.94}$$

and $\mathbf{v}_I^{(0)}$ are the initial velocities of the two particles.

The interesting dynamics is described by the separation vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. One computes

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{R} + \frac{m_2}{m_1+m_2}\mathbf{r} \ , \\ \mathbf{r}_2 &= \mathbf{R} - \frac{m_1}{m_1+m_2}\mathbf{r} \ . \end{aligned} \tag{2.95}$$

Notice we have simply changed coordinates from $(\mathbf{r}_1, \mathbf{r}_2) \to (\mathbf{R}, \mathbf{r})$, and the Lagrangian (2.92) becomes

$$L = \frac{1}{2}(m_1+m_2)|\dot{\mathbf{R}}|^2 + \frac{1}{2}\mu|\dot{\mathbf{r}}|^2 - V(|\mathbf{r}|) \ , \tag{2.96}$$

where we have defined the *reduced mass* $\mu = m_1 m_2/(m_1+m_2)$. Here $\mathbf{R}$ is ignorable, which leads to conservation of the conjugate momentum $p_{\mathbf{R}} = \mathbf{P}$. We thus effectively reduce the problem to a single particle of mass μ moving in an *external* central potential $V = V(|\mathbf{r}|)$:

$$L_{\text{reduced}} = \frac{1}{2}\mu|\dot{\mathbf{r}}|^2 - V(|\mathbf{r}|) \ . \tag{2.97}$$

Since V is rotationally invariant angular momentum is conserved, which follows from conservation of angular momentum for the original two-body problem. Notice we have "lost" translational invariance in the reduced problem (translation acts instead on the coordinate $\mathbf{R}$), and hence also Galilean invariance. The rest of the problem is now familiar from the Dynamics course. Conservation of angular momentum $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ implies that motion occurs in the fixed plane orthogonal to $\mathbf{L}$. Introducing polar coordinates (ϱ, ϕ) in this plane the Lagrangian (2.97) reads

$$L_{\text{reduced}} = \frac{1}{2}\mu(\dot{\varrho}^2 + \varrho^2\dot{\phi}^2) - V(\varrho) \ , \tag{2.98}$$

and since ϕ is ignorable (another consequence of angular momentum conservation) we have that the conjugate momentum $p_\phi = \mu\varrho^2\dot{\phi}$ is conserved. The equation of motion for ϱ is

$$\mu\ddot{\varrho} = -\frac{\partial V}{\partial \varrho} + \mu\varrho\dot{\phi}^2 = -\frac{\partial V}{\partial \varrho} + \frac{p_\phi^2}{\mu\varrho^3} = -\frac{\partial V_{\text{eff}}}{\partial \varrho} \ , \tag{2.99}$$

where $V_{\text{eff}}(\varrho) = V(\varrho) + \frac{p_\phi^2}{2\mu\varrho^2}$. Conservation of energy for the Lagrangian $L_{\text{radial}} = \frac{1}{2}\mu\dot{\varrho}^2 - V_{\text{eff}}(\varrho)$ that gives the equation of motion (2.99) is then

$$E = \frac{1}{2}\mu\dot{\varrho}^2 + V_{\text{eff}}(\varrho) = \text{constant} \ , \tag{2.100}$$

which leads to a first order ODE for $\varrho(t)$, and we have reduced the problem to quadratures. Notice that it is the large number of conserved quantities (energy, momentum, angular momentum) that have allowed us to reduce the original problem with six degrees of freedom to a single integral equation for $\varrho(t)$. For $V(\varrho) = -\kappa/\varrho$, corresponding to the inverse square law force (1.11), the solutions are conic sections, as shown in the Dynamics course.

2.6 * More on Hamilton's principle

In section 2.2 we introduced a new fundamental principle of classical mechanics: Hamilton's principle of least action. We saw that this is equivalent to Newton's second law for point particles, but given the elegance and universal nature of the principle it is interesting to ask why it should hold. The whole of this subsection is understood as starred.

Historically the principle of least action took some time to develop, and Lagrange's formulation generalized earlier related ideas. However, one intuitive way to think about it is that Nature is lazy. Assuming that energy $E = T + V$ is conserved, then the system is free to transfer energy between kinetic and potential so long as their sum remains constant. Minimizing the integral of $T - V$ then amounts to minimizing the kinetic energy over the path (*i.e.* not moving around too much). At the same time one is then maximizing the potential energy (*i.e.* maximizing the amount of energy that may be converted into other forms).[6] This is a nice picture, but as we already commented the action is in general *not* minimized by the solution to the equations of motion – the latter is just a stationary point.

One can in a sense *derive* the principle of least action from quantum mechanics, although in doing so this really just shifts the question elsewhere. Obviously it's not appropriate to go into a detailed discussion of this here, but the basic ideas (due to Feynman) are simple to state. In the classical principle of least action problem we fixed the endpoints $\mathbf{q}^{(1)} = \mathbf{q}(t_1)$, $\mathbf{q}^{(2)} = \mathbf{q}(t_2)$ at fixed times t_1, t_2, and sought a path $\mathbf{q}(t)$ which extremizes the action S subject to these boundary conditions. This is the *classical path* of the system. In quantum mechanics *all* possible paths between these endpoints play a role. More precisely, each path $\mathbf{q}(t)$ in the configuration space has associated to it the complex phase $\mathrm{e}^{\mathrm{i}S/\hbar}$, where $S = S[\mathbf{q}(t)]$ is the action of the path (2.6) and $\hbar \simeq 1.05 \times 10^{-34}\,\mathrm{Js}$ is (the reduced) *Planck's constant*. Notice that both S and $\hbar$ have units of angular momentum, so it makes sense to take the exponential of the dimensionless quantity $\mathrm{i}S/\hbar$. One then defines the *amplitude* A to get from $(\mathbf{q}^{(1)}, t_1)$ to $(\mathbf{q}^{(2)}, t_2)$ to be the *sum* of these phases over *all* paths. Given the system is observed at $\mathbf{q}^{(1)}$ at time t_1, the *probability* of finding it at $\mathbf{q}^{(2)}$ at time t_2 is given by $|A|^2$.

As you might imagine, defining these things precisely is not so straightforward. In particular we have to "sum" over all paths $\mathbf{q}(t)$ connecting $\mathbf{q}(t_1) = \mathbf{q}^{(1)}$ and $\mathbf{q}(t_2) = \mathbf{q}^{(2)}$. This is an infinite-dimensional space, and such sums are called *path integrals* in quantum mechanics – it's an integral/sum over a space of paths. The basic idea is then that at non-stationary points of S the phases from different paths are very different ($S/\hbar$ is very large for classical systems, as $\hbar$ is so small), and their contributions to the amplitude tend to cancel. On the other hand near to a stationary point of S the phases constructively interfere (since by definition near a stationary point S is constant to first order). Thus the leading contribution to the amplitude A comes from

[6] So the next time someone tells you to get out of bed/off the sofa, tell them that you're simply obeying Hamilton's principle.

the classical path, meaning one is most likely to observe the particle near to where it should be according to the classical equations of motion. Mathematically what we just described is called the *stationary phase approximation* to oscillatory integrals, which is part of the subject of asymptotic analysis.

Hopefully this at least gives a flavour of how the principle of least action fits into quantum mechanics, and why we observe stationary points of S as classical paths. For those that took Part A Quantum Mechanics: one can prove that the above formulation of quantum mechanics (and in particular the probability we defined) is equivalent to the Schrödinger equation and Born interpretation. The interested reader is referred to the book *Quantum Mechanics and Path Integrals*, by R. P. Feynman and A. R. Hibbs.

3 Small oscillations

Having set up appropriate generalized coordinates, derived the Lagrangian, and hence Lagrange equations of motion, the next task in analysing any physical system is to identify its *equilibrium configurations*. These are constant solutions $\mathbf{q}(t) = \mathbf{q}^{(0)}$ to the equations of motion. Of course there may be no such solutions, but sometimes there can be many, and having found an equilibrium point the next question is whether or not it is *stable*. That is, does a small perturbation of the system tend to return it to the equilibrium point, or drive it away? For stable equilibria we will show that small oscillations are described by a set of decoupled harmonic oscillators.

3.1 Equilibria and quadratic Lagrangians

Consider a general Lagrangian for a system with n degrees of freedom of the form

$$L \;=\; T - V \;=\; \frac{1}{2}\sum_{a,b=1}^{n} T_{ab}(\mathbf{q})\dot{q}_a\dot{q}_b - V(\mathbf{q}) \;. \tag{3.1}$$

Here without loss of generality we take $T_{ab}(\mathbf{q}) = T_{ba}(\mathbf{q})$ to be symmetric – compare to the discussion around (2.46). The Lagrange equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{b=1}^{n} T_{ab}\,\dot{q}_b\right) - \frac{1}{2}\sum_{b,c=1}^{n}\frac{\partial T_{bc}}{\partial q_a}\dot{q}_b\dot{q}_c \;=\; -\frac{\partial V}{\partial q_a} \;, \qquad a = 1,\ldots,n \;. \tag{3.2}$$

By definition an equilibrium point has $\mathbf{q}(t) = \mathbf{q}^{(0)} \in \mathcal{Q}$ independent of time t, so that $\dot{\mathbf{q}} = \ddot{\mathbf{q}} = \mathbf{0}$ throughout the motion. It follows that the left hand side of (3.2) is zero at an equilibrium point, and hence defining the *generalized force* as

$$\mathbf{F} \;\equiv\; -\frac{\partial V}{\partial \mathbf{q}} \;, \tag{3.3}$$

we see this is precisely a *critical point* of the potential V, or equivalently a point at which the generalized force is zero.

In order to analyse the dynamics near to such a point we make a Taylor expansion of the Lagrangian (3.1) around $\mathbf{q}^{(0)}$. Without loss of generality we choose coordinates so that $\mathbf{q}^{(0)} = \mathbf{0}$, so as to simplify the following formulae. We then expand

$$\begin{aligned} T &= \frac{1}{2}\sum_{a,b=1}^{n} T_{ab}(\mathbf{0})\dot{q}_a\dot{q}_b + O(q^3) \;, \\ V &= V(\mathbf{0}) + \frac{1}{2}\sum_{a,b=1}^{n}\frac{\partial^2 V}{\partial q_a \partial q_b}(\mathbf{0})q_a q_b + O(q^3) \;, \end{aligned} \tag{3.4}$$

in a hopefully obvious notation. In particular we assume that both $\mathbf{q}$ and $\dot{\mathbf{q}}$ are small, and have neglected all terms of order 3 and higher. Notice that the linear term in the expansion of V is zero precisely because we are at a critical point. We may effectively drop the constant potential energy

term $V(\mathbf{0})$ as it does not contribute to the Lagrange equations of motion. Thus the dynamics near to the equilibrium point is governed by the *quadratic Lagrangian*

$$L_{\text{quadratic}} \;=\; \frac{1}{2}\sum_{a,b=1}^{n} \mathcal{T}_{ab}\,\dot{q}_a\dot{q}_b - \frac{1}{2}\sum_{a,b=1}^{n} \mathcal{V}_{ab}\,q_a q_b \;, \tag{3.5}$$

where $\mathcal{T}_{ab} = T_{ab}(\mathbf{0})$, $\mathcal{V}_{ab} = \frac{\partial^2 V}{\partial q_a \partial q_b}(\mathbf{0})$ are the components of constant $n \times n$ symmetric matrices. We write these matrices as $\mathcal{T} = (\mathcal{T}_{ab})$, $\mathcal{V} = (\mathcal{V}_{ab})$. We shall study the solutions to (3.5) in the next subsection, but we should bear in mind that these solutions will only describe the approximate dynamics near to equilibrium if they remain small. Otherwise the higher order terms we have neglected will become relevant.

3.2 Normal frequencies and normal modes

The Lagrange equations of motion for the quadratic Lagrangian (3.5) are

$$\sum_{b=1}^{n} \mathcal{T}_{ab}\,\ddot{q}_b \;=\; -\sum_{b=1}^{n} \mathcal{V}_{ab}\,q_b \;, \qquad a \;=\; 1,\ldots,n \;, \tag{3.6}$$

which are a set of n linear homogeneous second order ODEs in $\mathbf{q}(t)$ with constant coefficients. We may write these in matrix form

$$\mathcal{T}\ddot{\mathbf{q}} \;=\; -\mathcal{V}\,\mathbf{q} \;. \tag{3.7}$$

Notice that in this notation the kinetic energy is $T = \frac{1}{2}\dot{\mathbf{q}}^T\mathcal{T}\dot{\mathbf{q}}$, while the potential energy is $V = \frac{1}{2}\mathbf{q}^T\mathcal{V}\,\mathbf{q}$. Provided $\mathcal{T}$ is invertible we may solve (3.7) in terms of $\ddot{\mathbf{q}}$ as

$$\ddot{\mathbf{q}} \;=\; -\mathcal{T}^{-1}\mathcal{V}\,\mathbf{q} \;. \tag{3.8}$$

In physical situations $\mathcal{T}$ will be positive definite, and hence invertible, because the kinetic energy T is positive definite. We thus henceforth assume that $\mathcal{T}$ is positive definite.

We may now solve (3.7) using generalized eigenvectors and eigenvalues, as follows. We seek solutions of the form

$$\mathbf{q}(t) \;=\; f(t)\boldsymbol{\alpha} \;, \tag{3.9}$$

where $\boldsymbol{\alpha}$ is a constant (non-zero) vector. With this ansatz the equation of motion (3.7) becomes

$$\ddot{f}(t)\mathcal{T}\boldsymbol{\alpha} \;=\; -f(t)\mathcal{V}\,\boldsymbol{\alpha} \;. \tag{3.10}$$

Since $\mathcal{T}$ is invertible, so $\mathcal{T}\boldsymbol{\alpha}$ is non-zero, this implies that

$$\ddot{f} \;=\; -\lambda f \;, \qquad \text{where} \qquad (\lambda\mathcal{T} - \mathcal{V})\boldsymbol{\alpha} \;=\; \mathbf{0} \;, \tag{3.11}$$

and λ is a constant. It follows that $\boldsymbol{\alpha}$ lies in the kernel of $\lambda\mathcal{T} - \mathcal{V}$, and hence the determinant of this matrix must be zero.

This motivates defining the *characteristic equation* of the system to be

$$\det(\lambda\mathcal{T} - \mathcal{V}) \quad = \quad 0 \ . \tag{3.12}$$

Since this is a polynomial in λ of degree n, we will have n solutions $\lambda_1, \ldots, \lambda_n$.[7] These are precisely the eigenvalues of $\mathcal{T}^{-1}\mathcal{V}$. The corresponding eigenvectors are $\boldsymbol{\alpha}_a$, $a = 1, \ldots, n$, satisfying

$$(\lambda_a\mathcal{T} - \mathcal{V})\,\boldsymbol{\alpha}_a \quad = \quad \mathbf{0} \ , \tag{3.13}$$

and the equation of motion (3.7) is solved by $\mathbf{q}(t) = f_a(t)\boldsymbol{\alpha}_a$, where

$$\ddot{f}_a \quad = \quad -\lambda_a f_a \ . \tag{3.14}$$

Next we note that the matrix $\mathcal{T}^{-1}\mathcal{V}$ is diagonalizable. This follows since it is similar to the matrix $\mathcal{T}^{-1/2}\mathcal{V}\mathcal{T}^{-1/2}$, which is symmetric as the square root $\mathcal{T}^{-1/2}$ is symmetric[8], and real symmetric matrices are diagonalizable. There is hence a basis of $\mathbb{R}^n$ consisting of eigenvectors $\{\boldsymbol{\alpha}_a\}$ of $\mathcal{T}^{-1}\mathcal{V}$. Hence $\mathbf{q}_a(t) = f_a(t)\boldsymbol{\alpha}_a$ are n linearly independent solutions to (3.7), and the general solution is a linear combination of these.[9]

An important physical fact is that the eigenvalues we just defined are *real*. This is a simple consequence of the fact that $\mathcal{T}$ and $\mathcal{V}$ are real and symmetric, and $\mathcal{T}$ is positive definite. Indeed, taking the dot product of (3.13) with the complex conjugate $\bar{\boldsymbol{\alpha}}_a$ of $\boldsymbol{\alpha}_a$ we have $\bar{\boldsymbol{\alpha}}_a^T\mathcal{V}\,\boldsymbol{\alpha}_a = \lambda_a\bar{\boldsymbol{\alpha}}_a^T\mathcal{T}\boldsymbol{\alpha}_a$. But for any real symmetric matrix $\mathcal{S}$ the quantity $\bar{\boldsymbol{\alpha}}_a^T\mathcal{S}\,\boldsymbol{\alpha}_a$ is real, as follows simply by showing it is equal to its complex conjugate. Thus $(\bar{\lambda}_a - \lambda_a)\bar{\boldsymbol{\alpha}}_a^T\mathcal{T}\boldsymbol{\alpha}_a = 0$, and since $\mathcal{T}$ is positive definite we deduce that $\bar{\lambda}_a = \lambda_a$.

If $\lambda > 0$ then the solution to $\ddot{f} = -\lambda f$ is

$$f(t) \quad = \quad A\cos(\omega t + \beta) \ , \tag{3.15}$$

where we have written $\lambda = \omega^2$, A, β are constants of integration, and without loss of generality we take $\omega > 0$, $A > 0$. Thus the corresponding eigenmode executes simple harmonic oscillations, with *amplitude* A and *angular frequency* ω. In fact we shall simply refer to ω as the *frequency* of the mode. The modulus of the solution (3.15) is bounded by A for all times t.

On the other hand if $\lambda = 0$ then one obtains a linearly growing solution $f(t) = A + Bt$, while $\lambda < 0$ leads to exponential growth $f(t) = A\exp(\sqrt{-\lambda}t) + B\exp(-\sqrt{-\lambda}t)$. In particular in the latter case the mode quickly becomes large and the quadratic approximations we made in section 3.1 break down. If the system possesses an eigenvalue $\lambda < 0$ we hence say that it has a

[7] Notice that $\det(\lambda\mathbb{1} - \mathcal{T}^{-1}\mathcal{V})$ is the *characteristic polynomial* of the matrix $\mathcal{T}^{-1}\mathcal{V}$ you defined in first year linear algebra, and $\lambda_1, \ldots, \lambda_n$ are its roots.

[8] ***** For completeness: since $\mathcal{T}$ is positive definite and symmetric, so is $\mathcal{T}^{-1}$. It is hence diagonalizable, $\mathcal{T}^{-1} = \mathcal{P}D\mathcal{P}^T$, with $\mathcal{P} \in O(n)$ orthogonal and $D = \mathrm{diag}(\mu_1, \ldots, \mu_n)$ diagonal with positive eigenvalues $\mu_a > 0$, $a = 1, \ldots, n$. Then $\mathcal{T}^{-1/2} = \mathcal{P}D^{1/2}\mathcal{P}^T$, where $D^{1/2} = \mathrm{diag}(\sqrt{\mu_1}, \ldots, \sqrt{\mu_n})$, satisfies $(\mathcal{T}^{-1/2})^2 = \mathcal{T}^{-1}$ and is clearly symmetric.

[9] If you recall that $\mathbf{q}(t) = (q_1(t), \ldots, q_n(t))$ then $\mathbf{q}_a(t)$ will have two subscripts when written out in components: $\mathbf{q}_a(t) = (q_{a,1}(t), \ldots, q_{a,n}(t)$. Don't get confused over the different meanings of these subscripts! One labels the generalized coordinates, while the other labels different solutions to the equations of motion.

linear instability. If $\lambda = 0$ then this may be a translational mode (often called a *flat direction* of the potential, meaning its derivative is identically zero in that direction), or else one needs to analyse the third order, or higher order, terms in the expansions (3.4) of section 3.1 to determine whether or not it is stable.

In particular if *all* the eigenvalues $\lambda_a > 0$, $a = 1, \ldots, n$, then we have a point of *stable equilibrium*. The frequencies $\omega_a = \sqrt{\lambda_a} > 0$ are called the *normal frequencies*, with the corresponding eigenvectors $\boldsymbol{\alpha}_a$ called *normal modes*. Since $\{\boldsymbol{\alpha}_a\}$ form a basis for $\mathbb{R}^n$, we may write any $\mathbf{q} = \sum_{a=1}^{n} f_a \, \boldsymbol{\alpha}_a$, and the $(f_1, \ldots, f_n)$ may then be used as a new set of generalized coordinates, called *normal coordinates*. Since each solution $f_a(t)$ to the equations of motion satisfies

$$\ddot{f}_a + \omega_a^2 f_a \;\; = \;\; 0 \;, \tag{3.16}$$

we see that the coordinate transformation $(q_1, \ldots, q_n) \to (f_1, \ldots, f_n)$ essentially diagonalizes the original quadratic Lagrangian (3.5) into a set of n decoupled harmonic oscillators of frequencies ω_a.

3.3 Examples

Having developed the general theory, we now apply it to some examples.

Mass-spring system

Consider three springs of natural length a and spring constant k lying in a line. One end of the first spring is fixed at a point A, while the other end is attached to a particle of mass m. This mass is in turn attached to one end of the second spring, with the other end attached to a second particle of mass m. Finally this second mass is attached to one end of the third spring, with the other end fixed at a point B. The distance between A and B is $3a$. We denote the horizontal displacement of the first mass from its equilibrium position by x, and similarly the horizontal displacement of the second mass by y – see Figure 9. The kinetic energy of the system is then

$$T \;\; = \;\; \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) \;, \tag{3.17}$$

while the potential energy is

$$V \;\; = \;\; \frac{1}{2} k \left[x^2 + (y - x)^2 + y^2 \right] \;. \tag{3.18}$$

Here we have used the fact that if a spring with spring constant k is displaced by an amount Δ from its natural length, then the corresponding potential energy is $\frac{1}{2} k \Delta^2$.

Notice that the equilibrium point $\partial_x V = \partial_y V = 0$ is at $x = y = 0$ (as one would expect), and moreover the Lagrangian $L = T - V$ is *already* quadratic in x, y. We may then write the kinetic

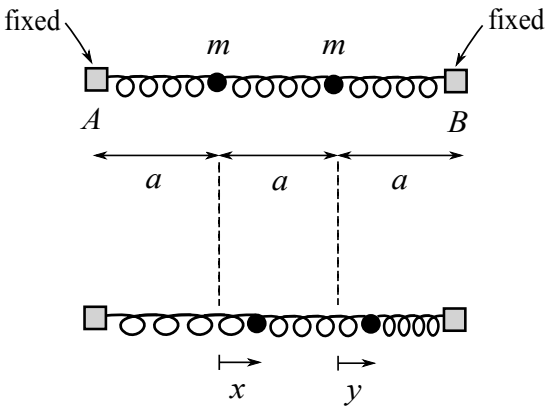

Figure 9: The spring-mass system. The upper diagram shows the equilibrium configuration. In the lower diagram we have shown the horizontal displacements x and y of the two masses from their equilibrium positions.

and potential energy terms in matrix form as

$$\begin{aligned} T &= \frac{1}{2}\,(\,\dot{x}\quad \dot{y}\,)\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}\ , \\ V &= \frac{1}{2}\,(\,x\quad y\,)\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\ . \end{aligned} \tag{3.19}$$

Thus

$$\lambda\mathcal{T} - \mathcal{V} \;=\; \begin{pmatrix} m\lambda - 2k & k \\ k & m\lambda - 2k \end{pmatrix}\ , \tag{3.20}$$

The solutions to the characteristic equation

$$0 \;=\; \det(\lambda\mathcal{T} - \mathcal{V}) \;=\; (m\lambda - k)(m\lambda - 3k)\ , \tag{3.21}$$

are hence $\lambda_1 = k/m$, $\lambda_2 = 3k/m$, giving normal frequencies

$$\omega_1 \;=\; \sqrt{\lambda_1} \;=\; \sqrt{\frac{k}{m}}\ , \qquad \omega_2 \;=\; \sqrt{\lambda_2} \;=\; \sqrt{\frac{3k}{m}}\ . \tag{3.22}$$

The corresponding eigenvectors are

$$\boldsymbol{\alpha}_1 \;=\; \begin{pmatrix} 1 \\ 1 \end{pmatrix}\ , \qquad \boldsymbol{\alpha}_2 \;=\; \begin{pmatrix} 1 \\ -1 \end{pmatrix}\ , \tag{3.23}$$

which give the normal modes. In particular the lower frequency mode ω_1 corresponds to the two masses oscillating in phase, while the higher frequency mode ω_2 corresponds to the two masses oscillating out of phase. Of course this is what one might have expected.

Double pendulum

A *double pendulum* consists of a simple pendulum of mass m_1 and length l_1 pivoted at the origin, together with another simple pendulum of mass m_2 and length l_2, pivoted at the mass m_1. The whole system moves freely in a vertical plane under gravity. If θ_1 and θ_2 denote the angles each pendulum makes with the vertical, then the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2\right] \\ &\quad + m_1 g l_1\cos\theta_1 + m_2 g(l_2\cos\theta_2 + l_1\cos\theta_1)\ . \end{aligned} \tag{3.24}$$

Deriving this is an exercise on Problem Sheet 1.

For simplicity we shall analyse the normal modes in the case where both pendula have the same parameters, *i.e.* $l_1 = l_2 = l$ and $m_1 = m_2 = m$. In particular the potential energy is then

$$V = -mgl(2\cos\theta_1 + \cos\theta_2)\ . \tag{3.25}$$

As one would expect $\theta_1 = \theta_2 = 0$ is a critical point of V, and is a point of stable equilibrium. To analyse this we Taylor expand V around $\theta_1 = \theta_2 = 0$:

$$V = -mgl\left[2(1-\tfrac{1}{2}\theta_1^2) + (1-\tfrac{1}{2}\theta_2^2) + \text{fourth order}\right]\ . \tag{3.26}$$

Dropping the constant term we thus obtain the quadratic potential

$$\begin{aligned} V_{\text{quadratic}} &= \frac{1}{2}mgl(2\theta_1^2+\theta_2^2) \\ &= \frac{1}{2}\begin{pmatrix}\theta_1 & \theta_2\end{pmatrix}\begin{pmatrix}2mgl & 0\\ 0 & mgl\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}\ . \end{aligned} \tag{3.27}$$

Similarly expanding the kinetic term T and dropping cubic and higher order terms we find

$$\begin{aligned} T_{\text{quadratic}} &= \frac{1}{2}ml^2\left(2\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)\ , \\ &= \frac{1}{2}\begin{pmatrix}\dot{\theta}_1 & \dot{\theta}_2\end{pmatrix}\begin{pmatrix}2ml^2 & ml^2\\ ml^2 & ml^2\end{pmatrix}\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix}\ . \end{aligned} \tag{3.28}$$

Thus

$$\lambda\mathcal{T} - \mathcal{V} = \begin{pmatrix}2ml^2\lambda - 2mgl & ml^2\lambda\\ ml^2\lambda & ml^2\lambda - mgl\end{pmatrix}\ , \tag{3.29}$$

The solutions to the characteristic equation

$$0 = \det(\lambda\mathcal{T}-\mathcal{V}) = m^2l^4\left[2\left(\lambda - \frac{g}{l}\right)^2 - \lambda^2\right]\ , \tag{3.30}$$

are then $\lambda_1 = (2-\sqrt{2})\frac{g}{l}$, $\lambda_2 = (2+\sqrt{2})\frac{g}{l}$. Both are positive, so the equilibrium is stable, as expected. The normal frequencies are hence

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{(2-\sqrt{2})\frac{g}{l}}\ , \qquad \omega_2 = \sqrt{\lambda_2} = \sqrt{(2+\sqrt{2})\frac{g}{l}}\ , \tag{3.31}$$

with corresponding normal modes

$$\boldsymbol{\alpha}_1 \;=\; \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} , \qquad \boldsymbol{\alpha}_2 \;=\; \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} . \tag{3.32}$$

Again the lower frequency normal mode has the pendula swinging in phase, while for the higher frequency mode they are out of phase.

4 Rigid body dynamics

In this section we discuss the extension of Lagrangian mechanics for point particles to cover the dynamics of extended bodies. A *rigid body* may be defined as any distribution of mass for which the distance between any two points is fixed. For example, we can consider a finite number of particles with position vectors $\mathbf{r}_I$ $(I = 1, \ldots, N)$ as a rigid body, provided that we impose the constraints $|\mathbf{r}_I - \mathbf{r}_J| = c_{IJ} =$ constant. One might imagine these as the positions of atoms in a solid, with the constraints arising from inter-molecular forces. However, we will more often model a rigid body as a continuous distribution of matter, which may be regarded as a limit of the point particle model in which the number of particles tends to infinity. However, before getting into the details of this, we first need to give a more precise description of rotating frames.

4.1 Rotating frames and angular velocity

Following on from our discussion in section 1.1, consider two reference frames $\mathcal{S}$, $\hat{\mathcal{S}}$, both with the same origin O. Focusing first on the frame $\mathcal{S}$, its coordinate axes correspond to three orthonormal vectors $\mathbf{e}_i$, $i = 1, 2, 3$. The $\{\mathbf{e}_i\}$ form a basis for $\mathbb{R}^3$, and being orthonormal means they have inner products $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ (the *Kronecker delta symbol* δ_{ij} is defined in the appendix). We may then write a position vector as $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = (x_1, x_2, x_3)$ with respect to this basis, where $x_i = \mathbf{r} \cdot \mathbf{e}_i$ are the *components of* $\mathbf{r}$ *in the frame* $\mathcal{S}$.

The frame $\hat{\mathcal{S}}$ will similarly have an orthonormal basis $\{\hat{\mathbf{e}}_i\}$, and we may write

$$\hat{\mathbf{e}}_i \;=\; \sum_{j=1}^{3} \mathcal{R}_{ji}\mathbf{e}_j \;. \tag{4.1}$$

Taking the dot product with $\mathbf{e}_k$, this is equivalent to defining $\mathcal{R}_{ki} = \mathbf{e}_k \cdot \hat{\mathbf{e}}_i$. The position vector $\mathbf{r}$ then has an expansion $\mathbf{r} = \hat{x}_1\hat{\mathbf{e}}_1 + \hat{x}_2\hat{\mathbf{e}}_2 + \hat{x}_3\hat{\mathbf{e}}_3 = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ in this basis. In section 1.1 we would have referred to this as $\hat{\mathbf{r}}$, as we wanted to emphasize that it is the components $\hat{x}_i = \mathbf{r} \cdot \hat{\mathbf{e}}_i$ that we measure in the frame $\hat{\mathcal{S}}$. However, mathematically $\mathbf{r}$ and $\hat{\mathbf{r}}$ are the same vector, just expressed in different bases. The *components* of the vector are related as

$$\hat{x}_i \;=\; \mathbf{r} \cdot \hat{\mathbf{e}}_i \;=\; \sum_{j=1}^{3} \mathcal{R}_{ji}(\mathbf{r} \cdot \mathbf{e}_j) \;=\; \sum_{j=1}^{3} \mathcal{R}_{ji}x_j \;. \tag{4.2}$$

The matrix $\mathcal{R}$ is *orthogonal*, as one sees by taking the dot product of (4.1) with $\hat{\mathbf{e}}_k$:

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_k \;=\; \delta_{ik} \;=\; \sum_{j=1}^{3} \mathcal{R}_{ji}\mathcal{R}_{jk} \;. \tag{4.3}$$

Thus $\mathbb{1} = \mathcal{R}^T\mathcal{R} = \mathcal{R}\mathcal{R}^T$, where $\mathbb{1}$ denotes the 3×3 identity matrix. The matrices in $GL(3, \mathbb{R})$ satisfying this condition form a subgroup called the *orthogonal group* $O(3)$. Notice that $\mathcal{R} \in O(3)$ implies that $\det \mathcal{R} = \pm 1$. Using the definition of the scalar triple product as a determinant it is straightforward to check that $\hat{\mathbf{e}}_1 \cdot (\hat{\mathbf{e}}_2 \wedge \hat{\mathbf{e}}_3) = (\det \mathcal{R})\, \mathbf{e}_1 \cdot (\mathbf{e}_2 \wedge \mathbf{e}_3)$. Since $\{\mathbf{e}_i\}$ is orthonormal

we must have $\mathbf{e}_1 \cdot (\mathbf{e}_2 \wedge \mathbf{e}_3) = \pm 1$, and we shall always choose a *right-handed frame* in which $\mathbf{e}_1 \cdot (\mathbf{e}_2 \wedge \mathbf{e}_3) = +1$. In order that a rotation maps a right-handed frame to another right-handed frame, we thus have $\det \mathcal{R} = 1$. The subgroup of $O(3)$ satisfying this condition is called the *special orthogonal group* $SO(3)$. (The elements in $O(3)$ that are not in $SO(3)$ involve a reflection.)

In general the rotation matrix $\mathcal{R} = \mathcal{R}(t)$ will depend on time t, so that the coordinate axes of $\mathcal{S}$ are rotating relative to those of $\hat{\mathcal{S}}$ (and *vice versa*). Differentiating the orthogonality relation $\mathcal{R}\mathcal{R}^T = \mathbb{1}$ with respect to t we obtain

$$\dot{\mathcal{R}}\mathcal{R}^T + \mathcal{R}\dot{\mathcal{R}}^T \;=\; 0 \;, \tag{4.4}$$

which since $\mathcal{R}\dot{\mathcal{R}}^T = (\dot{\mathcal{R}}\mathcal{R}^T)^T$ implies that the matrix

$$\Omega \;\equiv\; \mathcal{R}\dot{\mathcal{R}}^T \;=\; -\dot{\mathcal{R}}\mathcal{R}^T \;=\; \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \;, \tag{4.5}$$

is anti-symmetric. In the last equality we have introduced functions $\omega_i = \omega_i(t)$, $i = 1, 2, 3$, parametrizing the non-zero entries of this matrix. Notice that we may rewrite (4.5) as

$$\Omega_{ij} \;=\; -\sum_{k=1}^{3} \epsilon_{ijk}\omega_k \;, \tag{4.6}$$

using the *Levi-Civita alternating symbol* ϵ_{ijk} defined in the appendix. The *angular velocity* of the frame $\mathcal{S}$ relative to $\hat{\mathcal{S}}$ is then defined to be the vector $\boldsymbol{\omega} = \sum_{i=1}^{3} \omega_i \mathbf{e}_i$. In general this is a function of time $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$.

Now we come to the main point of this subsection. In section 1.1 we simply defined $\dot{\mathbf{r}}$ by differentiating its components in the frame. However, if one applies this definition in two different frames which are rotating relative to each other, then the resulting vector will depend on the choice of frame. In this situation the notation "$\dot{\mathbf{r}}$" is hence too ambiguous. In order to be clear, we will therefore define the *time derivative* of a vector $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ with respect to the orthonormal basis $\{\mathbf{e}_i\}$ of a frame $\mathcal{S}$ to be the vector

$$\mathrm{D}\mathbf{r} \;\equiv\; \dot{x}_1\mathbf{e}_1 + \dot{x}_2\mathbf{e}_2 + \dot{x}_3\mathbf{e}_3 \;. \tag{4.7}$$

For example, if $\mathbf{r}(t)$ is the position vector of a particle from the origin of $\mathcal{S}$, then $\mathrm{D}\mathbf{r}$ and $\mathrm{D}^2\mathbf{r}$ are its velocity and acceleration relative to $\mathcal{S}$. We may then derive

The Coriolis formula: The time derivatives $\mathrm{D}\mathbf{r}$ and $\hat{\mathrm{D}}\mathbf{r}$ of $\mathbf{r}$ relative to the two frames $\mathcal{S}$ and $\hat{\mathcal{S}}$ are related by

$$\hat{\mathrm{D}}\mathbf{r} \;=\; \mathrm{D}\mathbf{r} + \boldsymbol{\omega} \wedge \mathbf{r} \;, \tag{4.8}$$

where $\boldsymbol{\omega}$ is the angular velocity of $\mathcal{S}$ relative to $\hat{\mathcal{S}}$.

The proof is just a direct computation. Taking the time derivative of (4.2), in vector notation we have

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{pmatrix} = \mathcal{R}^T \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} + \dot{\mathcal{R}}^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} . \tag{4.9}$$

Multiplying on the left by $\mathcal{R}$ this reads

$$\mathcal{R} \begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} + \Omega \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} . \tag{4.10}$$

Given the formula (4.5), the components of the second term on the right hand side are precisely the components of $\boldsymbol{\omega} \wedge \mathbf{r}$ in the basis $\{\mathbf{e}_i\}$.[10] The components of $\hat{\mathrm{D}}\mathbf{r}$ in the basis $\{\hat{\mathbf{e}}_i\}$ for the frame $\hat{\mathcal{S}}$ are $\dot{\hat{x}}_1$, $\dot{\hat{x}}_2$, $\dot{\hat{x}}_3$, so the entries of the column vector on the left hand side of (4.10) are the components of $\hat{\mathrm{D}}\mathbf{r}$ in basis $\{\mathbf{e}_i\}$ for the frame $\mathcal{S}$. This proves (4.8).

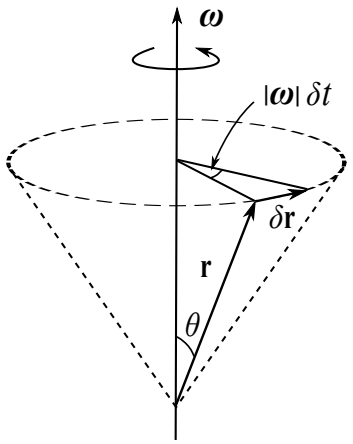

Figure 10: As seen in the frame $\hat{\mathcal{S}}$, the position vector $\mathbf{r}$ of a point P fixed in the frame $\mathcal{S}$ changes by $\delta\mathbf{r} = \boldsymbol{\omega} \wedge \mathbf{r}\, \delta t$ in a small time interval δt. This is a rotation of $\mathbf{r}$ through an angle $|\boldsymbol{\omega}|\delta t$ about an axis parallel to the vector $\boldsymbol{\omega}$.

In the application to rigid bodies, we will be interested in applying this formula to the case where $\hat{\mathcal{S}}$ is an inertial frame (which we regard as the frame in which we are viewing the motion), while the frame $\mathcal{S}$ rotates with the rigid body; that is, $\mathcal{S}$ is the *rest frame of the body*, which in general will not be an inertial frame. By definition any point P in the body is then at rest in $\mathcal{S}$, so its position vector $\mathbf{r}$ has $\mathrm{D}\mathbf{r} = \mathbf{0}$. But the time derivative of $\mathbf{r}$ in the inertial frame $\hat{\mathcal{S}}$ is given by

$$\hat{\mathrm{D}}\mathbf{r} = \boldsymbol{\omega} \wedge \mathbf{r} , \tag{4.11}$$

where $\boldsymbol{\omega}$ is the angular velocity of the rest frame of the rigid body. The formula (4.11) determines the way in which the vector $\mathbf{r}$ changes relative to the frame $\hat{\mathcal{S}}$. To get some geometric intuition for

[10] The computation is $(\Omega\,\mathbf{r})_i = \sum_{j=1}^3 \Omega_{ij} x_j = -\sum_{j,k=1}^3 \epsilon_{ijk}\omega_k x_j = \sum_{j,k=1}^3 \epsilon_{ijk}\omega_j x_k = (\boldsymbol{\omega} \wedge \mathbf{r})_i$, using (4.6).

this, consider the change $\delta\mathbf{r}$ in $\mathbf{r}$ in a small time interval δt, where we ignore quadratic and higher order terms. This is $\delta\mathbf{r} = \boldsymbol{\omega} \wedge \mathbf{r}\,\delta t$. This vector is orthogonal to both $\boldsymbol{\omega}$ and $\mathbf{r}$, and has modulus $|\boldsymbol{\omega}||\mathbf{r}|\sin\theta\,\delta t$, where θ is the angle between $\boldsymbol{\omega}$ and $\mathbf{r}$ – see Figure 10. As seen in $\hat{\mathcal{S}}$, the change in the position vector $\mathbf{r}$ of the point P fixed in the body in the time interval δt is hence obtained by rotating $\mathbf{r}$ through an angle $|\boldsymbol{\omega}|\delta t$ about an axis parallel to $\boldsymbol{\omega}$. The direction of $\boldsymbol{\omega}$ is thus the *instantaneous axis of rotation*, while its magnitude $|\boldsymbol{\omega}|$ is the rate of rotation.

Example: Consider the special case in which

$$\mathcal{R}(t) \;=\; \begin{pmatrix} \cos\varphi(t) & \sin\varphi(t) & 0 \\ -\sin\varphi(t) & \cos\varphi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \tag{4.12}$$

so that $\mathbf{e}_1 = \cos\varphi(t)\,\hat{\mathbf{e}}_1 + \sin\varphi(t)\,\hat{\mathbf{e}}_2$, $\mathbf{e}_2 = -\sin\varphi(t)\,\hat{\mathbf{e}}_1 + \cos\varphi(t)\,\hat{\mathbf{e}}_2$, $\mathbf{e}_3 = \hat{\mathbf{e}}_3$ is a rotation about the third axis. Then one computes $\boldsymbol{\omega} = \dot{\varphi}\,\mathbf{e}_3$.

As a final comment in this subsection, we note that $\boldsymbol{\omega}$ behaves as you would expect a vector to do. For example, if the angular velocity of $\mathcal{S}$ relative to $\hat{\mathcal{S}}$ is $\boldsymbol{\omega}$, then the angular velocity of $\hat{\mathcal{S}}$ relative to $\mathcal{S}$ is $-\boldsymbol{\omega}$. Moreover, if $\hat{\mathcal{S}}$ in turn has angular velocity $\hat{\boldsymbol{\omega}}$ relative to another frame $\mathcal{S}'$, then $\mathcal{S}$ has angular velocity $\boldsymbol{\omega} + \hat{\boldsymbol{\omega}}$ relative to $\mathcal{S}'$. One can prove these statements either by using the Coriolis formula (4.8) to characterize $\boldsymbol{\omega}$, or else by introducing rotation matrices for each change of basis and using the definition (4.5) (see Problem Sheet 2).

4.2 Motion in a non-inertial frame

In this subsection we take a brief detour from our main topic, and present Newton's equations in a general non-inertial frame.

Suppose that $\hat{\mathcal{S}}$ is an inertial frame with origin $\hat{O}$, and $\mathcal{S}$ is another frame whose origin O is at position vector $\mathbf{x} = \mathbf{x}(t)$ from $\hat{O}$. If $\mathbf{r}$ denotes the position vector of a particle measured from O, and $\hat{\mathbf{r}}$ is the position of the particle measured from $\hat{O}$, then

$$\hat{\mathbf{r}} \;=\; \mathbf{r} + \mathbf{x}\,. \tag{4.13}$$

As in the previous subsection we define the acceleration of the particle relative to $\mathcal{S}$ as $\mathbf{a} = \mathrm{D}^2\mathbf{r}$, and similarly relative to $\hat{\mathcal{S}}$ we have $\hat{\mathbf{a}} = \hat{\mathrm{D}}^2\hat{\mathbf{r}}$. Taking $\hat{\mathrm{D}}^2$ of (4.13) and using the Coriolis formula (4.8) we obtain

$$\begin{aligned} \hat{\mathbf{a}} \;&=\; \hat{\mathrm{D}}^2(\mathbf{r}+\mathbf{x}) \;=\; \hat{\mathrm{D}}(\mathrm{D}\mathbf{r} + \boldsymbol{\omega}\wedge\mathbf{r}) + \mathbf{A} \\ &=\; \mathbf{a} + (\mathrm{D}\boldsymbol{\omega})\wedge\mathbf{r} + 2\boldsymbol{\omega}\wedge\mathrm{D}\mathbf{r} + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}) + \mathbf{A}\,. \end{aligned} \tag{4.14}$$

Here we have defined $\mathbf{A} = \hat{\mathrm{D}}^2\mathbf{x}$, which is the acceleration of O relative to $\hat{\mathcal{S}}$, and it is simple enough to check from the definition that $\mathrm{D}(\mathbf{b}\wedge\mathbf{c}) = (\mathrm{D}\mathbf{b})\wedge\mathbf{c} + \mathbf{b}\wedge\mathrm{D}\mathbf{c}$ for any two vectors $\mathbf{b}$, $\mathbf{c}$.

Since $\hat{\mathcal{S}}$ is an inertial frame, Newton's second law holds and for a particle of mass m we may write

$$m\hat{\mathbf{a}} \;=\; \mathbf{F}\;, \tag{4.15}$$

where $\mathbf{F}$ is the force acting. Substituting from (4.14) we thus have

$$m\mathbf{a} \;=\; \mathbf{F} - m(\mathrm{D}\boldsymbol{\omega})\wedge\mathbf{r} - 2m\boldsymbol{\omega}\wedge\mathrm{D}\mathbf{r} - m\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}) - m\mathbf{A}\;. \tag{4.16}$$

The additional terms on the right hand side of (4.16) may be interpreted as "ficticious forces":

$$\begin{aligned} \mathbf{F}_1 &= -m(\mathrm{D}\boldsymbol{\omega})\wedge\mathbf{r}\;, & \mathbf{F}_2 &= -2m\boldsymbol{\omega}\wedge\mathrm{D}\mathbf{r}\;,\\ \mathbf{F}_3 &= -m\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r})\;, & \mathbf{F}_4 &= -m\mathbf{A}\;. \end{aligned} \tag{4.17}$$

That is, these may be regarded as corrections to the force in $\mathbf{F} = m\mathbf{a}$ due to the fact that the frame $\mathcal{S}$ is accelerating. The force $\mathbf{F}_1$ is known as the *Euler force*, and arises from the *angular acceleration* of $\mathcal{S}$. The Euler force is hence zero for a frame rotating at constant angular velocity, $\mathrm{D}\boldsymbol{\omega} = \mathbf{0}$. The force $\mathbf{F}_2$ is known as the *Coriolis force*, and is interesting in that it depends on the velocity $\mathrm{D}\mathbf{r}$ of the particle as measured in $\mathcal{S}$. The Coriolis force plays a role in weather, for example being responsible for the circulation of air around an area of low pressure, which is anti-clockwise in the northern hemisphere. The force $\mathbf{F}_3$ is the *centrifugal force*. It lies in a plane through $\mathbf{r}$ and $\boldsymbol{\omega}$, is perpendicular to the axis of rotation $\boldsymbol{\omega}$, and is directed away from the axis. This is the force you experience standing on a roundabout. Finally, $\mathbf{F}_4$ is simply due to the acceleration of the origin O. For example, this force effectively cancels the Earth's gravitational field in a freely-falling frame.

4.3 Rigid body motion and the inertia tensor

We turn now to our main topic for this section, namely rigid body motion. From the outset we fix a reference inertial frame $\hat{\mathcal{S}}$ with origin $\hat{O}$. A particular point in the body then has position vector $\mathbf{x} = \mathbf{x}(t)$ relative to $\hat{O}$. We denote this point by O, and take it to be the origin of the rest frame $\mathcal{S}$ of the body. Provided the matter distribution is not all along a line, this rest frame is defined uniquely by the body, up to a *constant* orthogonal rotation of its axes and a translation of the origin by a *constant* vector (relative to $\mathcal{S}$). In particular this latter freedom just shifts the choice of fixed point in the body. As we shall see shortly, the body itself provides a natural way to fix both of these freedoms.

As in section 4.1, the rotation of the frame $\mathcal{S}$ relative to the inertial frame $\hat{\mathcal{S}}$ is determined by the time-dependent orthogonal matrix $\mathcal{R} = \mathcal{R}(t)$. This matrix describes the rotation of the body about the point O. The *angular velocity of the body* is defined to be the angular velocity vector $\boldsymbol{\omega}$ of $\mathcal{S}$ defined in (4.5). An important fact is that this is independent of the point O we picked. To see this note that any other point O' fixed in the body will have position vector $\mathbf{r}'$ relative to O, and the axes $\{\mathbf{e}'_i\}$ for the corresponding frame $\mathcal{S}'$ will be *fixed* relative to the axes $\{\mathbf{e}_i\}$. The

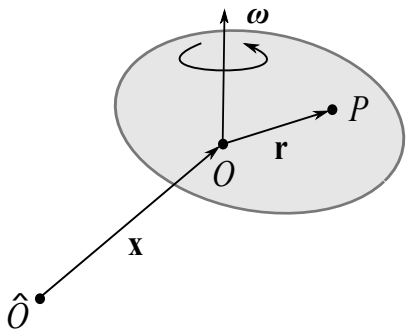

Figure 11: We fix a point O in the rigid body, which is taken to be the origin of the rest frame $\mathcal{S}$ of the body. The frame $\mathcal{S}$ has angular velocity $\boldsymbol{\omega}$, and its origin O has position vector $\mathbf{x}$ relative to the origin $\hat{O}$ of an inertial frame $\hat{\mathcal{S}}$. Any other point P fixed in the body has position vector $\mathbf{r}$ relative to O.

angular velocity $\boldsymbol{\omega}'$ of $\mathcal{S}'$ relative to $\hat{\mathcal{S}}$ is hence the same as the angular velocity $\boldsymbol{\omega}$ of $\mathcal{S}$ relative to $\hat{\mathcal{S}}$.

Since the rotation of the body is described entirely by the matrix $\mathcal{R}(t) \in SO(3)$, the configuration space of rotations is hence precisely the special orthogonal group $SO(3)$. A 3×3 matrix has 9 components and $\mathcal{R}\mathcal{R}^T = \mathbb{1}$ imposes 6 relations, so this rotational configuration space has dimension 3. We will introduce an explicit set of generalized angular coordinates for this space in section 4.5. Altogether the rigid body has 6 degrees of freedom: the 3 components of the position vector $\mathbf{x}$ of the point O, plus these three angles describing the orientation of the axes of $\mathcal{S}$.

If now $\mathbf{r}$ denotes the position vector of a point P fixed in the body, measured from O, then the velocity of this point as measured in the inertial frame $\hat{\mathcal{S}}$ is

$$\begin{aligned} \mathbf{v} &= \hat{\mathrm{D}}(\mathbf{r} + \mathbf{x}) \\ &= \boldsymbol{\omega} \wedge \mathbf{r} + \mathbf{v}_O \ . \end{aligned} \tag{4.18}$$

Here $\mathbf{v}_O = \hat{\mathrm{D}}\mathbf{x}$ is the velocity of the point O, and we have used the Coriolis formula (4.11).

Momentum and angular momentum

Let us suppose that the distribution of mass in the body is defined by a density $\rho(\mathbf{r})$, so that the mass δm in a small volume δV centred at $\mathbf{r}$ is $\delta m = \rho(\mathbf{r})\,\delta V$. Here $\mathbf{r}$ is measured from O. The total mass of the body is hence

$$M = \int_R \rho(\mathbf{r})\,\mathrm{d}V \ . \tag{4.19}$$

We will also sometimes want to treat two-dimensional bodies, such as a flat disc, or one-dimensional bodies such as a rigid rod. In this case one replaces ρ by a *surface density*, or *line density*, respectively, and integrates over the surface or curve, respectively.

An important quantity for the rigid body is its *centre of mass* G. With respect to the origin O this has position vector

$$\mathbf{r}_G \;=\; \frac{1}{M}\int_R \mathbf{r}\,\rho(\mathbf{r})\,\mathrm{d}V\ . \tag{4.20}$$

This is the obvious generalization of (2.93) for a collection of point masses, and is an average position weighted by mass. We will henceforth generally choose the origin O of the rest frame $\mathcal{S}$ of the body to be at the centre of mass, so $O = G$ and $\mathbf{r}_G = \mathbf{0}$. This is a natural choice, and as we will see leads to a number of simplifications in subsequent formulae.

We may now define the *total momentum* $\mathbf{P}$ of the body relative to the inertial frame $\hat{\mathcal{S}}$ by similarly dividing it into small masses $\delta m = \rho(\mathbf{r})\,\delta V$ centred at $\mathbf{r}$. Each element has velocity $\mathbf{v}$ given by (4.18), again relative to $\hat{\mathcal{S}}$, meaning that

$$\mathbf{P} \;=\; \int_R \rho\,\mathbf{v}\,\mathrm{d}V \;=\; \int_R \rho(\mathbf{r})(\boldsymbol{\omega}\wedge\mathbf{r}+\mathbf{v}_O)\,\mathrm{d}V\ . \tag{4.21}$$

If we now choose $O = G$ then the first term in (4.21) is zero (it is $\boldsymbol{\omega}\wedge M\mathbf{r}_G = \mathbf{0}$) and we obtain

$$\mathbf{P} \;=\; \int_R \rho(\mathbf{r})\,\mathbf{v}_G\,\mathrm{d}V \;=\; M\mathbf{v}_G\ . \tag{4.22}$$

Thus the total momentum is as if the whole mass M was concentrated at the centre of mass G.

We may examine the *total angular momentum* $\mathbf{L}$ *about* O in a similar way. We have

$$\mathbf{L} \;=\; \int_R \mathbf{r}\wedge\rho\,\mathbf{v}\,\mathrm{d}V \;=\; \int_R \rho(\mathbf{r})\,\mathbf{r}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}+\mathbf{v}_O)\,\mathrm{d}V\ . \tag{4.23}$$

If $O = G$ then the second term is zero, and using the vector triple product we may write

$$\mathbf{L} \;=\; \int_R \rho(\mathbf{r})\,\mathbf{r}\wedge(\boldsymbol{\omega}\wedge\mathbf{r})\,\mathrm{d}V \;=\; \int_R \rho(\mathbf{r})\,[(\mathbf{r}\cdot\mathbf{r})\boldsymbol{\omega}-(\mathbf{r}\cdot\boldsymbol{\omega})\mathbf{r}]\,\mathrm{d}V\ . \tag{4.24}$$

In components of the basis $\{\mathbf{e}_i\}$ for $\mathcal{S}$ this reads

$$L_i \;=\; \sum_{j=1}^{3}\mathcal{I}_{ij}\,\omega_j\ , \tag{4.25}$$

where we have introduced the *inertia tensor* $\mathcal{I} = (\mathcal{I}_{ij})$ with components

$$\mathcal{I}_{ij} \;=\; \int_R \rho(\mathbf{r})\ [(\mathbf{r}\cdot\mathbf{r})\delta_{ij} - r_i r_j]\,\mathrm{d}V\ . \tag{4.26}$$

Notice that this is defined with respect to the rest frame $\mathcal{S}$ of the body, so is independent of time t, and is symmetric $\mathcal{I} = \mathcal{I}^T$. In general the angular momentum $\mathbf{L} = \mathcal{I}\,\boldsymbol{\omega}$ is not in the same direction as the angular velocity $\boldsymbol{\omega}$, which is what leads to some of the peculiar properties of rigid body motion. In Cartesian coordinates $\mathbf{r} = (x, y, z)$ we have

$$\mathcal{I} \;=\; \int_R \rho(\mathbf{r}) \begin{pmatrix} y^2+z^2 & -xy & -zx \\ -xy & z^2+x^2 & -yz \\ -zx & -yz & x^2+y^2 \end{pmatrix} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ . \tag{4.27}$$

Note carefully the form of the terms in this matrix. The diagonal entries are called the *moments of inertia*, while the off-diagonal terms are the *products of inertia*. However, since $\mathcal{I}$ is a real symmetric matrix we may also diagonalize it via an orthogonal transformation. That is, there is a (constant) orthogonal matrix $\mathcal{P} \in O(3)$ such that $\mathcal{P}\mathcal{I}\mathcal{P}^T$ is diagonal. Thus in the new basis

$$\mathcal{I} \;=\; \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \;. \tag{4.28}$$

The eigenvalues I_i, $i = 1, 2, 3$, are called the *principal moments of inertia*. Notice that we have precisely used the freedom to rotate the rest frame $\mathcal{S}$ of the body by a constant orthogonal transformation, so that the axes $\mathbf{e}_i$ are the eigenvectors of the inertia tensor $\mathcal{I}$, called the *principal axes*, with corresponding eigenvalues I_i. These eigenvalues are non-negative: if $\boldsymbol{\alpha}$ is an eigenvector of $\mathcal{I}$ with eigenvalue I then

$$I|\boldsymbol{\alpha}|^2 \;=\; \boldsymbol{\alpha}^T\mathcal{I}\boldsymbol{\alpha} \;=\; \int_R \rho(\mathbf{r}) \left[|\mathbf{r}|^2|\boldsymbol{\alpha}|^2 - (\mathbf{r}\cdot\boldsymbol{\alpha})^2\right] \mathrm{d}V \;\geq\; 0, \tag{4.29}$$

where we have used the Cauchy-Schwarz inequality.

The principal axes may often be identified by symmetry considerations. For example, an *axisymmetric body* by definition has rotational symmetry about a particular axis. The centre of mass must then lie on this axis, which is one of the principal axes. One can see this directly from the formula (4.27), where without loss of generality we take the axis of symmetry to be the $\mathbf{e}_3$ direction, and write the integrals in cylindrical polar coordinates (2.3). Thus $x = \varrho\cos\phi$, $y = \varrho\sin\phi$ and $\rho = \rho(\varrho, z)$. In particular one deduces that $I_1 = I_2$, and all off-diagonal products of inertia are zero. Let's now look at some more explicit examples:

Example (uniform rectangular cuboid): We will mainly focus on *uniform* distributions of mass, in which the density $\rho =$ constant. If we take the cuboid to have side lengths $2a$, $2b$, $2c$ and mass M, then $\rho = M/(8abc)$. The centre of mass is the origin of the cuboid, and we take Cartesian axes aligned with the edges. It is then straightforward to see that the products of inertia in this basis are zero; for example

$$\mathcal{I}_{12} \;=\; -\frac{M}{8abc}\int_{x=-a}^{a}\int_{y=-b}^{b}\int_{z=-c}^{c} xy\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \;=\; 0\;. \tag{4.30}$$

Thus the edge vectors of the cuboid are the principal axes. We next compute

$$\int_{x=-a}^{a}\int_{y=-b}^{b}\int_{z=-c}^{c} \rho\,x^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \;=\; \frac{M}{8abc}\left[\frac{1}{3}x^3\right]_{-a}^{a} 2b\cdot 2c \;=\; \frac{Ma^2}{3}\;. \tag{4.31}$$

The integrals involving y^2 and z^2 are of course similar, and we deduce that $\mathcal{I}$ has principal moments of inertia $I_1 = \frac{1}{3}M(b^2+c^2)$, $I_2 = \frac{1}{3}M(c^2+a^2)$, $I_3 = \frac{1}{3}M(a^2+b^2)$.

Example (uniform disc): As a two-dimensional example, consider a uniform disc of radius r and mass M. Thus $\rho = M/(\pi r^2)$, and due to the rotational symmetry the centre of mass must be at

the origin of the disc. If we take the $\mathbf{e}_3$ axis perpendicular to the disc then due to the axisymmetry we know already that $\mathcal{I}$ is diagonal in this basis with $I_1 = I_2$. We compute in polar coordinates $x = \varrho\cos\phi$, $y = \varrho\sin\phi$

$$I_1 \;=\; \int \rho\, y^2\,\mathrm{d}x\,\mathrm{d}y \;=\; \frac{M}{\pi r^2}\int_{\varrho=0}^{r}\int_{\phi=0}^{2\pi} \varrho^2\sin^2\phi\,\varrho\,\mathrm{d}\varrho\,\mathrm{d}\phi \;=\; \frac{1}{4}Mr^2\ . \tag{4.32}$$

Notice here that the integrand is $\rho(x^2+y^2-x^2) = \rho\, y^2$, as the body is two-dimensional and lies in the plane $z = 0$. Since $I_1 = I_2$ (which is easy enough to check explicitly) we only have to compute

$$I_3 \;=\; \int \rho\,(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y \;=\; \frac{M}{\pi r^2}\cdot 2\pi\int_{\varrho=0}^{r}\varrho^2\,\varrho\,\mathrm{d}\varrho \;=\; \frac{1}{2}Mr^2\ . \tag{4.33}$$

The inertia matrix $\mathcal{I}$ defined by (4.26) in general depends on the choice of origin O. The formulae above were computed assuming this is at the centre of mass $O = G$. More generally the inertia tensor about another point Q fixed in the body is defined as

$$\mathcal{I}^{(Q)}_{ij} \;\equiv\; \int_R \rho(\mathbf{r})\left[(\mathbf{r}'\cdot\mathbf{r}')\delta_{ij} - r'_i r'_j\right]\mathrm{d}V\ . \tag{4.34}$$

Here $\mathbf{r}' = \mathbf{r} - \mathbf{c}$ is the position vector of a point as measured from Q, $\mathbf{r}$ is the position vector of the *same* point, but as measured from $O = G$, and $\mathbf{c} = (c_1, c_2, c_3)$ is the vector from G to Q. For example, with this definition we may write the *total angular momentum about* Q as

$$\begin{aligned}\mathbf{L}_Q \;&=\; \int_R \rho(\mathbf{r})\mathbf{r}'\wedge\left(\boldsymbol{\omega}\wedge\mathbf{r}' + \mathbf{v}_Q\right)\mathrm{d}V \\ &=\; \mathcal{I}^{(Q)}\boldsymbol{\omega} - \mathbf{c}\wedge M\mathbf{v}_Q\ .\end{aligned} \tag{4.35}$$

The steps used to show this are similar to those around equation (4.24), and we have used $\mathbf{r}' = \mathbf{r}-\mathbf{c}$ in the last step, together with $\mathbf{r}_G = \mathbf{0}$. With the above notation, we have the following result:

The parallel axes theorem: The inertia tensors about Q and G are related by

$$\mathcal{I}^{(Q)}_{ij} \;=\; \mathcal{I}^{(G)}_{ij} + M\left[(\mathbf{c}\cdot\mathbf{c})\delta_{ij} - c_i c_j\right]\ . \tag{4.36}$$

To prove this we then simply compute in Cartesian coordinates:

$$\mathcal{I}^{(Q)}_{ij} \;\equiv\; \int_R \rho(\mathbf{r})\left[(\mathbf{r}'\cdot\mathbf{r}')\delta_{ij} - r'_i r'_j\right]\mathrm{d}V \;=\; \int_R \rho(\mathbf{r})\left[(\mathbf{r}-\mathbf{c})\cdot(\mathbf{r}-\mathbf{c})\delta_{ij} - (r_i - c_i)(r_j - c_j)\right]\mathrm{d}V \tag{4.37}$$

If one now multiplies out all the brackets and recalls that by the definition of centre of mass we have $\int_R \rho\,\mathbf{r}\,\mathrm{d}V = \mathbf{0}$, then one obtains (4.36).

Notice that we may rephrase this by saying that the inertia matrix in any rest frame of the body (with origin at Q) is the sum of two terms: the inertia matrix in a frame with origin at the centre of mass G with axes parallel to the axes at Q, plus the inertia matrix of a single particle of mass M at $\mathbf{c}$, measured about the centre of mass G.

Kinetic energy

Finally, let us compute the kinetic energy of the body in the inertial frame $\hat{\mathcal{S}}$. Following the arguments we used above for momentum we have

$$T \;=\; \int_R \frac{1}{2}\rho\,|\mathbf{v}|^2 \mathrm{d}V \;, \tag{4.38}$$

where $\mathbf{v}$ is given by (4.18). Thus

$$T \;=\; \frac{1}{2}\int_R \rho\left(|\mathbf{v}_O|^2 + 2\mathbf{v}_O\cdot(\boldsymbol{\omega}\wedge\mathbf{r}) + |\boldsymbol{\omega}\wedge\mathbf{r}|^2\right)\mathrm{d}V \;. \tag{4.39}$$

Taking $O = G$ as the centre of mass, the middle term is zero while the first term may be integrated to give

$$T \;=\; \frac{1}{2}M|\mathbf{v}_G|^2 + \frac{1}{2}\int_R \rho\,(\boldsymbol{\omega}\wedge\mathbf{r})\cdot(\boldsymbol{\omega}\wedge\mathbf{r})\,\mathrm{d}V \;. \tag{4.40}$$

The first term is the kinetic energy due to the motion of the centre of mass relative to $\hat{O}$, and again is as though all the mass was concentrated at the centre of mass. The second term is the *rotational kinetic energy* of the body. Using the vector identity

$$(\boldsymbol{\omega}\wedge\mathbf{r})\cdot(\boldsymbol{\omega}\wedge\mathbf{r}) \;=\; \boldsymbol{\omega}\cdot(\mathbf{r}\wedge(\boldsymbol{\omega}\wedge\mathbf{r})) \;, \tag{4.41}$$

together with the formula (4.24) for the total angular momentum $\mathbf{L}$ about O we may write

$$T \;=\; \frac{1}{2}M|\mathbf{v}_G|^2 + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} \;. \tag{4.42}$$

Notice that the rotational kinetic energy may also be written in various ways as

$$T_{\mathrm{rot}} \;=\; \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} \;=\; \frac{1}{2}\sum_{i=1}^{3}\omega_i L_i \;=\; \frac{1}{2}\sum_{i,j=1}^{3}\mathcal{I}_{ij}\omega_i\omega_j \;=\; \frac{1}{2}\sum_{i=1}^{3} I_i\omega_i^2 \;. \tag{4.43}$$

4.4 Euler's equations

In this section we consider a *free* rigid body, meaning that no external forces act on it. From Newton's laws we thus have that the total momentum $\mathbf{P}$ given by (4.22) is constant, and hence $\dot{\mathbf{P}} = M\dot{\mathbf{v}}_G = \mathbf{0}$. One can deduce this from Newton's laws for point particles by considering the body as made up of many particles of small masses m_I at positions $\mathbf{r}_I$, with $M = \sum_{I=1}^{N} m_I$ being the total mass. The only forces acting are the constraint forces keeping the masses at fixed distances apart, $|\mathbf{r}_I - \mathbf{r}_J| =$ constant, and summing over the whole body these forces cancel, as in equation (1.16). We thus deduce that the centre of mass velocity $\mathbf{v}_G$ is constant. Via a Galilean boost we may take the inertial frame $\hat{\mathcal{S}}$ to be such that the centre of mass is at rest at the origin, $\hat{O} = G$, which we henceforth do.

The remaining dynamics is then entirely in the rotation of the basis $\{\mathbf{e}_i\}$ for the rest frame $\mathcal{S}$ of the body. Recall from section 4.1 that

$$\mathbf{e}_i(t) \;=\; \sum_{j=1}^{3} \mathcal{R}_{ij}(t)\,\hat{\mathbf{e}}_j \;, \tag{4.44}$$

where now $\{\hat{\mathbf{e}}_i\}$ is a time-independent basis for the inertial frame $\hat{\mathcal{S}}$. The Coriolis formula (4.11) applied to $\mathbf{r} = \mathbf{e}_i$ immediately gives

$$\dot{\mathbf{e}}_i \;=\; \boldsymbol{\omega} \wedge \mathbf{e}_i \;, \tag{4.45}$$

although we can also easily check this from (4.44): differentiating the latter with respect to time t we have

$$\dot{\mathbf{e}}_i \;=\; \sum_{j=1}^{3} \dot{\mathcal{R}}_{ij}\hat{\mathbf{e}}_j \;=\; \sum_{k=1}^{3} (\dot{\mathcal{R}}\mathcal{R}^T)_{ik}\mathbf{e}_k \;=\; -\sum_{k=1}^{3} \Omega_{ik}\mathbf{e}_k \;. \tag{4.46}$$

Using (4.6) we may then write this as

$$\dot{\mathbf{e}}_i \;=\; \sum_{j,k=1}^{3} \epsilon_{ikj}\omega_j\mathbf{e}_k \;=\; \sum_{j=1}^{3} \mathbf{e}_j \wedge \mathbf{e}_i\,\omega_j \;=\; \boldsymbol{\omega} \wedge \mathbf{e}_i \;, \tag{4.47}$$

where recall $\boldsymbol{\omega} = \sum_{j=1}^{3} \omega_j\mathbf{e}_j$. In the second equality in (4.47) we have used that the frame is right-handed, so $\mathbf{e}_i \wedge \mathbf{e}_j = \sum_{k=1}^{3} \epsilon_{ijk}\mathbf{e}_k$.

Choosing the basis $\{\mathbf{e}_i\}$ to be aligned with the principal axes of the body, we may write the total angular momentum as

$$\mathbf{L} \;=\; \sum_{i=1}^{3} I_i\omega_i\mathbf{e}_i \;, \tag{4.48}$$

where recall that I_i are the principal moments of inertia. This is likewise conserved, $\dot{\mathbf{L}} = \mathbf{0}$, as with no external forces acting there is no external torque (again, compare to the system of point particles in section 1.5). Using (4.48) we thus compute

$$\mathbf{0} \;=\; \dot{\mathbf{L}} \;=\; \sum_{i=1}^{3} I_i(\dot{\omega}_i\mathbf{e}_i + \omega_i\dot{\mathbf{e}}_i) \;, \tag{4.49}$$

which combining with (4.47) leads to the three equations

$$\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= 0 \;, \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= 0 \;, \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 &= 0 \;. \end{aligned} \tag{4.50}$$

This is a set of coupled first order ODEs for the three functions $\omega_i(t)$, $i = 1, 2, 3$, which determine how the rigid body rotates. The system also depends on the constants I_i, and collectively the

equations are known as *Euler's equations*. If we include an external torque $\boldsymbol{\tau}$ then the components τ_i simply appear on the right hand side of (4.50) via $\dot{\mathbf{L}} = \boldsymbol{\tau}$.

Let us examine (4.50) further. If we multiply the i^{th} equation by ω_i and then sum over i we find the resulting equation is equivalent to

$$2T \equiv \sum_{i=1}^{3} I_i \omega_i^2 \tag{4.51}$$

being constant. Of course we immediately recognize T as the rotational kinetic energy (4.43), which we expect to be conserved for this closed system. Similarly multiplying the i^{th} equation by $I_i\omega_i$ and summing we find

$$\mathbf{L}\cdot\mathbf{L} = L^2 = \sum_{i=1}^{3} I_i^2 \omega_i^2 \tag{4.52}$$

is constant. Here $\mathbf{L} = \sum_{i=1}^{3} I_i\omega_i \mathbf{e}_i$ is the total angular momentum (4.24). Again, on general grounds we expect this to be conserved.

We have thus found two conserved quantities. A little algebra allows us to rearrange the equations as

$$\begin{aligned} 2I_3T - L^2 &= I_1(I_3-I_1)\omega_1^2 + I_2(I_3-I_2)\omega_2^2 \ , \\ 2I_2T - L^2 &= I_1(I_2-I_1)\omega_1^2 + I_3(I_2-I_3)\omega_3^2 \ , \\ I_1^2\dot{\omega}_1^2 &= (I_2-I_3)^2\omega_2^2\omega_3^2 \ . \end{aligned} \tag{4.53}$$

We may thus algebraically eliminate ω_2, ω_3 using the first two equations to obtain a first order ODE for $\omega_1(t)$ of the form

$$\dot{\omega}_1^2 = F(\omega_1; I_i, T, L^2) \ , \tag{4.54}$$

where F is quadratic in ω_1^2. Thus the problem is *completely integrable*, since we may solve $t = \int \mathrm{d}\omega_1/\sqrt{F}$, and then substitute the solution for $\omega_1(t)$ into the remaining two equations in (4.53) to obtain $\omega_2(t)$, $\omega_3(t)$ algebraically. In general the integral that arises here is called an *elliptic integral of the first kind*, and cannot be written in terms of more elementary functions. However, one can gain some *qualitative* understanding of the general solutions to these equations using geometric methods. In particular notice that equations (4.51) and (4.52), when viewed as constraints on the angular momentum $L_i = I_i\omega_i$, say that $\mathbf{L} = (L_1, L_2, L_3)$ lies on an ellipsoid with semiaxes $\sqrt{2TI_1}$, $\sqrt{2TI_2}$, $\sqrt{2TI_3}$, and on a sphere of radius L, respectively. For futher discussion see Example 5.7 in the book by Woodhouse, or section 37 of the book by Landau and Lifschitz. We shall instead examine some special cases:

Example (axisymmetric body): Consider an axisymmetric body (also known as a *symmetric top*) with $I_1 = I_2$. Then the third Euler equation in (4.50) implies that $\dot{\omega}_3 = 0$, *i.e.* $\omega_3 =$ constant. We

may then combine the first two equations into the single *complex* equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\omega_1 + \mathrm{i}\omega_2) \;=\; \mathrm{i}\nu(\omega_1 + \mathrm{i}\omega_2)\;, \qquad \text{where} \qquad \nu \;\equiv\; \frac{(I_3 - I_1)}{I_1}\omega_3\;. \tag{4.55}$$

The solution is hence $\omega_1 + \mathrm{i}\omega_2 = A\exp(\mathrm{i}\nu t)$, where the constant amplitude A may be taken to be real by choosing a suitable origin for the time t. Thus

$$\boldsymbol{\omega}(t) \;=\; (A\cos\nu t, A\sin\nu t, \omega_3)\;. \tag{4.56}$$

Since the component ω_3 along the axis of symmetry $\mathbf{e}_3$ is constant, this shows that the vector $\boldsymbol{\omega}$ rotates uniformly with angular speed ν about the axis of symmetry, with constant modulus $|\boldsymbol{\omega}| = \sqrt{A^2 + \omega_3^2}$ and making a constant angle $\cos^{-1}(\omega_3/\sqrt{A^2 + \omega_3^2})$. Since the components of the angular momentum $\mathbf{L}$ in this basis are $L_i = I_i\omega_i$, the angular momentum vector $\mathbf{L}$ has a similar motion. That is, $\mathbf{L}$ rotates uniformly about the axis of symmetry $\mathbf{e}_3$, making a fixed angle $\theta = \cos^{-1}(I_3\omega_3/|\mathbf{L}|)$ with it.

Of course this description is as viewed from the rest frame $\mathcal{S}$ of the body. Viewed from the inertial frame $\hat{\mathcal{S}}$ we know that the angular momentum vector is constant. Viewed in this frame it is the axis of symmetry $\mathbf{e}_3$ that rotates uniformly about the constant direction of $\mathbf{L}$. We shall describe this in more detail in section 4.5.

Example: Suppose $I_1 < I_2$, $I_3 = I_1 + I_2$ and the body is set in rotation with $\omega_2 = 0$ and $\omega_3 = \sqrt{(I_2 - I_1)/(I_2 + I_1)}\,\omega_1$. Then the solution to the Euler equations is $\omega_1(t) = c_1\mathrm{sech}(c_2 t)$, where $c_1 = \sqrt{2T/I_2}$ and $c_2 = \sqrt{(I_2 - I_1)/(I_2 + I_1)}\,c_1$. Verifying the details is left to Problem Sheet 3.

Stability

Euler's equations (4.50) have three special solutions where $(\omega_1, \omega_2, \omega_3) = (\omega, 0, 0)$, $(0, \omega, 0)$, or $(0, 0, \omega)$, respectively, corresponding to rotation around each of the three principal axes with constant angular speed ω. When I_1, I_2, I_3 are all different, so that without loss of generality we may assume $I_1 < I_2 < I_3$, then the first and last of these solutions are stable, while the second is unstable.

To see this consider the first solution, and let us perturb it by writing $\boldsymbol{\omega} = (\omega + \varepsilon_1, \varepsilon_2, \varepsilon_3)$. Then ignoring quadratic order terms in ε_i, as in section 3, the Euler equations (4.50) read

$$\begin{aligned} I_1\dot{\varepsilon}_1 &= 0\;, \\ I_2\dot{\varepsilon}_2 &= (I_3 - I_1)\omega\varepsilon_3\;, \\ I_3\dot{\varepsilon}_3 &= (I_1 - I_2)\omega\varepsilon_2\;. \end{aligned} \tag{4.57}$$

In particular the last two equations imply that

$$I_2\ddot{\varepsilon}_2 \;=\; (I_3 - I_1)\omega\dot{\varepsilon}_3 \;=\; \frac{(I_3 - I_1)(I_1 - I_2)}{I_3}\omega^2\varepsilon_2\;. \tag{4.58}$$

Since $I_1 < I_2 < I_3$ this takes the form $\ddot{\varepsilon}_2 = -\lambda_2 \varepsilon_2$ where $\lambda_2 = \frac{(I_3 - I_1)(I_2 - I_1)}{I_2 I_3}\omega^2 > 0$. We thus find simple harmonic motion in the ε_2 and ε_3 directions, and the solution is stable.

The computation for the solution $\boldsymbol{\omega} = (\varepsilon_1, \omega + \varepsilon_2, \varepsilon_3)$ is very similar, only we now find that

$$I_1 \ddot{\varepsilon}_1 \;\; = \;\; \frac{(I_2 - I_3)(I_1 - I_2)}{I_3}\omega^2 \varepsilon_1 \;, \tag{4.59}$$

This takes the form $\ddot{\varepsilon}_1 = -\lambda_1 \varepsilon_1$, where now $\lambda_1 = -\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}\omega^2 < 0$ and the exponential solution means this is *unstable*.

We leave it as an exercise to check that the final case where $\boldsymbol{\omega} = (0, 0, \omega)$ is stable. One can demonstrate this very effectively by taking any rigid body with well-separated principal moments of inertia $I_1 \ll I_2 \ll I_3$, and spinning it about each principal axis. For example, we showed earlier that a rectangular cuboid of mass M and side lengths $2a$, $2b$, $2c$ has $I_1 = \frac{1}{3}M(b^2 + c^2)$, $I_2 = \frac{1}{3}M(c^2 + a^2)$, $I_3 = \frac{1}{3}M(a^2 + b^2)$. Then $a \gg b \gg c$ implies $I_1 \ll I_2 \ll I_3$, and rotation about the middle length axis is the unstable direction. Try it – preferably with something that will not break if you drop it!

4.5 Euler angles and $SO(3)$

As we've already mentioned, the motion of a rigid body may be described by three coordinates for its centre of mass G, together with three angles which determine the orientation of the axes $\{\mathbf{e}_i\}$ fixed in the body relative to the axes $\{\hat{\mathbf{e}}_i\}$ of the inertial frame $\hat{\mathcal{S}}$. A natural choice for these latter three generalized coordinates is *Euler angles*.

We might as well suppose that the origins O, $\hat{O}$ of $\mathcal{S}$, $\hat{\mathcal{S}}$ coincide, as we are only interested in the angles between coordinate axes. The axes of the two frames are then related as in (4.1), where $\mathcal{R} \in SO(3)$. We then have

Proposition: There exist angles $\theta \in [0, \pi]$, $\varphi, \psi \in [0, 2\pi)$ such that

$$\begin{aligned}
\mathcal{R} \;\; &= \;\; \begin{pmatrix} \cos\theta\cos\varphi\cos\psi - \sin\varphi\sin\psi & \cos\theta\sin\varphi\cos\psi + \cos\varphi\sin\psi & -\sin\theta\cos\psi \\ -\cos\theta\cos\varphi\sin\psi - \sin\varphi\cos\psi & -\cos\theta\sin\varphi\sin\psi + \cos\varphi\cos\psi & \sin\theta\sin\psi \\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \end{pmatrix} \\
&= \;\; \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} .
\end{aligned} \tag{4.60}$$

Moreover, θ, φ, ψ are determined uniquely by $\mathcal{R}$, provided that $|\mathcal{R}_{33}| \neq 1$.

N.B. There are different conventions for Euler angles. We're using the so-called "y-convention", in which the second rotation we describe below is about the "y-axis" (more precisely, the $\tilde{\mathbf{e}}_2$ axis). The other commonly used convention is called the "x-convention", where one instead rotates about the x-axis. This amounts to the simple replacement $\varphi \to \varphi - \frac{\pi}{2}$, $\psi \to \psi + \frac{\pi}{2}$.

Proof: There are essentially two approaches to deriving (4.60). One can either proceed algebraically, which is the method used on pages 9 and 10 of the book by Woodhouse, or else one can proceed geometrically. We shall follow the geometric route, as it gives a better intuition for the Euler angles.

Consider the two axes $\mathbf{e}_3$ and $\hat{\mathbf{e}}_3$. Each of these defines an orthogonal plane, namely the $(\mathbf{e}_1, \mathbf{e}_2)$ plane and the $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$ plane, respectively. If these planes coincide then $\mathbf{e}_3 = \pm\hat{\mathbf{e}}_3$, which is the special case $|\mathcal{R}_{33}| = 1$ mentioned above. In this special case the two frames simply differ by an $O(2)$ rotation of this plane. Otherwise the intersection of the two planes defines a unique line though the origin, called the *line of nodes*. We specify its direction by defining the unit vector

$$\mathbf{n} \;=\; \hat{\mathbf{e}}_3 \wedge \mathbf{e}_3 \;. \tag{4.61}$$

Since $\mathbf{n}$ is a unit vector lying in both the $(\mathbf{e}_1, \mathbf{e}_2)$ plane and the $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$ plane we may write

$$\mathbf{n} \;=\; \sin\psi\,\mathbf{e}_1 + \cos\psi\,\mathbf{e}_2 \;, \qquad \text{and} \qquad \mathbf{n} \;=\; -\sin\varphi\,\hat{\mathbf{e}}_1 + \cos\varphi\,\hat{\mathbf{e}}_2 \;. \tag{4.62}$$

This defines uniquely the two angles $\varphi,\ \psi \in [0, 2\pi)$. The angle $\theta \in [0, \pi]$ is simply the angle between $\mathbf{e}_3$ and $\hat{\mathbf{e}}_3$. This is probably the clearest description of the Euler angles – see Figure 12.

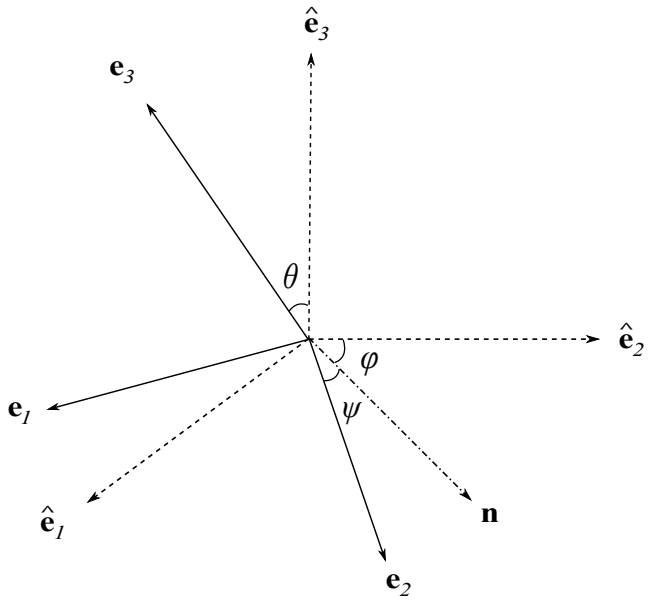

Figure 12: The Euler angles θ, φ, ψ between the two orthonormal bases $\{\hat{\mathbf{e}}_i\}$ and $\{\mathbf{e}_i\}$. The angle between $\hat{\mathbf{e}}_3$ and $\mathbf{e}_3$ is θ, while the vector $\mathbf{n} = \hat{\mathbf{e}}_3 \wedge \mathbf{e}_3$ along the line of nodes is orthogonal to both of these vectors. The angle between $\hat{\mathbf{e}}_2$ and $\mathbf{n}$ is denoted φ, while the angle between $\mathbf{e}_2$ and $\mathbf{n}$ is ψ.

Having given a geometric description of the angles, as suggested by the second line of (4.60) we may construct the rotation matrix $\mathcal{R}$ between the two frames as a sequence of three 2×2 rotations. We start with the $\{\hat{\mathbf{e}}_i\}$ basis. Clearly we'd like to rotate $\hat{\mathbf{e}}_3$ onto $\mathbf{e}_3$, but to do this we need to rotate about an axis perpendicular to both vectors, *i.e.* the line of nodes $\mathbf{n}$. Thus we first rotate

about the $\hat{\mathbf{e}}_3$ axis by an angle φ, which brings the $\hat{\mathbf{e}}_2$ vector into line with $\mathbf{n}$. Concretely the new orthonormal basis is

$$\tilde{\mathbf{e}}_1 \;=\; \cos\varphi\,\hat{\mathbf{e}}_1 + \sin\varphi\,\hat{\mathbf{e}}_2 \;, \qquad \tilde{\mathbf{e}}_2 \;=\; \mathbf{n} \;=\; -\sin\varphi\,\hat{\mathbf{e}}_1 + \cos\varphi\,\hat{\mathbf{e}}_2 \;, \qquad \tilde{\mathbf{e}}_3 \;=\; \hat{\mathbf{e}}_3 \;. \tag{4.63}$$

The line of nodes is the new basis vector $\tilde{\mathbf{e}}_2 = \mathbf{n}$, and rotating about this through an angle θ brings $\tilde{\mathbf{e}}_3 = \hat{\mathbf{e}}_3$ into line with $\mathbf{e}_3$. The new orthonormal basis is

$$\begin{aligned} \mathbf{e}'_1 \;&=\; \cos\theta\,(\cos\varphi\,\hat{\mathbf{e}}_1 + \sin\varphi\,\hat{\mathbf{e}}_2) - \sin\theta\,\hat{\mathbf{e}}_3 \;, \qquad \mathbf{e}'_2 \;=\; \mathbf{n} \;=\; -\sin\varphi\,\hat{\mathbf{e}}_1 + \cos\varphi\,\hat{\mathbf{e}}_2 \;, \\ \mathbf{e}'_3 \;&=\; \sin\theta\,(\cos\varphi\,\hat{\mathbf{e}}_1 + \sin\varphi\,\hat{\mathbf{e}}_2) + \cos\theta\,\hat{\mathbf{e}}_3 \;. \end{aligned} \tag{4.64}$$

Now that we have rotated $\hat{\mathbf{e}}_3$ onto $\mathbf{e}_3$, the last step is to ensure that in the final basis $\{\mathbf{e}_i\}$ we have $\mathbf{n} = \sin\psi\,\mathbf{e}_1 + \cos\psi\,\mathbf{e}_2$, which was the definition of ψ in (4.62). Since currently $\mathbf{n} = \mathbf{e}'_2$, we achieve this by rotating about the $\mathbf{e}'_3 = \mathbf{e}_3$ direction by an angle ψ. This hence explains the sequence of three rotations on the second line of (4.60), of course read right to left as we are mapping $\hat{\mathbf{e}}_i \to \mathbf{e}_i = \sum_{j=1}^{3} \mathcal{R}_{ij}\hat{\mathbf{e}}_j$. The case where $|\mathcal{R}_{33}| = 1$ (equivalently $\sin\theta = 0$) is a *coordinate singularity*. If one wants to analyse the behaviour near here one should really change coordinates (much like near the origin $\varrho = 0$ in plane polar coordinates, where the angle ϕ is not defined and one should change to Cartesian coordinates $x = \varrho\cos\phi$, $y = \varrho\sin\phi$).

Since the Euler angles are generalized coordinates for the rotation group $SO(3)$, we must be able to write the angular velocity $\boldsymbol{\omega}$ of $\mathcal{S}$ relative to $\hat{\mathcal{S}}$ in terms of these variables. Of course here $\theta = \theta(t)$, $\varphi = \varphi(t)$, $\psi = \psi(t)$. The brute force method would be to take the rather ugly 3×3 matrix on the first line of (4.60) and substitute it into the definition of $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ in (4.5). This works, but is a little tedious by hand.[11]

There is a much nicer way to get to the result though, again using the decomposition on the second line of (4.60). Let us denote the various frames in the proof above as $\hat{\mathcal{S}}$, $\tilde{\mathcal{S}}$, $\mathcal{S}'$ and $\mathcal{S}$, respectively. Then the rotation matrices from $\hat{\mathcal{S}} \to \tilde{\mathcal{S}}$, $\tilde{\mathcal{S}} \to \mathcal{S}'$ and $\mathcal{S}' \to \mathcal{S}$ are precisely the three 2×2 rotation matrices

$$\begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} , \quad \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{4.65}$$

respectively. Comparing to the example in (4.12), we thus see that the angular velocity of $\mathcal{S}$ relative to $\mathcal{S}'$ is $\dot{\psi}\,\mathbf{e}_3$, the angular velocity of $\mathcal{S}'$ relative to $\tilde{\mathcal{S}}$ is $\dot{\theta}\,\mathbf{n}$, and the angular velocity of $\tilde{\mathcal{S}}$ relative to $\hat{\mathcal{S}}$ is $\dot{\varphi}\,\hat{\mathbf{e}}_3$. Using the comment at the end of section 4.1, the angular velocity of $\mathcal{S}$ relative to $\hat{\mathcal{S}}$ is then simply given by the sum of these:

$$\begin{aligned} \boldsymbol{\omega} \;&=\; \dot{\psi}\,\mathbf{e}_3 + \dot{\theta}\,\mathbf{n} + \dot{\varphi}\,\hat{\mathbf{e}}_3 \\ &=\; \dot{\psi}\,\mathbf{e}_3 + \dot{\theta}\,(\sin\psi\,\mathbf{e}_1 + \cos\psi\,\mathbf{e}_2) + \dot{\varphi}\,(-\sin\theta\cos\psi\,\mathbf{e}_1 + \sin\theta\sin\psi\,\mathbf{e}_2 + \cos\theta\,\mathbf{e}_3) \;. \end{aligned} \tag{4.66}$$

[11] With a computer algebra package it takes less than a minute to type in this matrix and get the formula (4.67) for $\boldsymbol{\omega}$.

Thus writing $\boldsymbol{\omega} = \sum_{i=1}^{3} \omega_i \mathbf{e}_i$, the components ω_i in the frame $\mathcal{S}$ are

$$\begin{aligned} \omega_1 &= \dot\theta \, \sin\psi - \dot\varphi \, \sin\theta \cos\psi \ , \\ \omega_2 &= \dot\theta \, \cos\psi + \dot\varphi \, \sin\theta \sin\psi \ , \\ \omega_3 &= \dot\psi + \dot\varphi \, \cos\theta \ . \end{aligned} \tag{4.67}$$

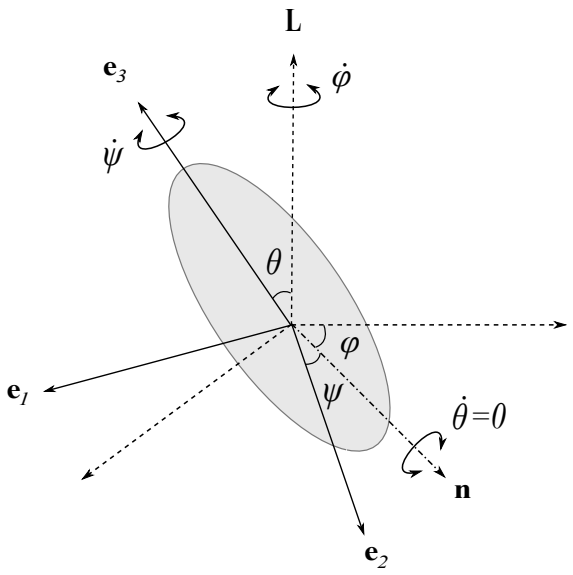

Figure 13: The precession of an axisymmetric body. The dotted axes are those of the fixed inertial frame $\hat{\mathcal{S}}$. We take the conserved angular momentum $\mathbf{L}$ to lie along the third axis $\hat{\mathbf{e}}_3$. The axis of symmetry $\mathbf{e}_3$ of the body makes a constant angle θ with $\mathbf{L}$, so that $\dot\theta = 0$, and precesses around $\mathbf{L}$ with rate $\dot\varphi$.

Example (axisymmetric body): As an application of these results, let's return to the free axisymmetric body discussed in the previous subsection. We showed there that in the *body frame* the angular momentum vector $\mathbf{L}$ rotated about the $\mathbf{e}_3$ axis at a fixed angle θ, with angular speed $\nu = (I_3 - I_1)\omega_3/I_1$. In the inertial frame $\hat{\mathcal{S}}$ the angular momentum $\mathbf{L}$ is constant, and for simplicity we take it to lie along the $\hat{\mathbf{e}}_3$ direction. From Figure 13 we thus see that $\dot\theta = 0$ (the angle between the axis of symmetry $\mathbf{e}_3$ and $\mathbf{L}$ is constant) while comparing (4.67) to the solution (4.56) we see that $\dot\psi = -\nu$ and $\omega_3 = \dot\psi + \dot\varphi \cos\theta$. Notice here that $\dot\psi = -\nu$ is the rate of rotation of the body about its symmetry axis $\mathbf{e}_3$, which since $\mathbf{L}$ is fixed in $\hat{\mathcal{S}}$ explains why $\mathbf{L}$ appears to rotate about $\mathbf{e}_3$ with angular speed $+\nu$ in the body frame. We may also deduce that

$$\dot\varphi \ = \ \frac{\omega_3 + \nu}{\cos\theta} \ = \ \frac{I_3 \omega_3}{I_1 \cos\theta} \ . \tag{4.68}$$

This is the constant rate of *precession* of the $\mathbf{e}_3$ axis around $\mathbf{L}$, as viewed in the inertial frame $\hat{\mathcal{S}}$.

4.6 The Lagrange top

Euler's equations (4.50) determine the time-dependence of the angular velocity $\boldsymbol{\omega}$, but in general these are not a useful description of the dynamical evolution, especially when we consider rigid

bodies moving under gravity (in (4.50) there are no external forces acting).

Consider a rigid body with rest frame $\mathcal{S}$, with the origin $O = G$ at the centre of mass and the axes aligned with the principal axes. We may use Cartesian coordinates (x, y, z) for the position of O with respect to the origin $\hat{O}$ of an inertial frame $\hat{\mathcal{S}}$. The orientation of the body may then be specified by giving the Euler angles θ, φ, ψ of the frame $\mathcal{S}$ relative to the fixed inertial frame $\hat{\mathcal{S}}$. Altogether $x, y, z, \theta, \varphi, \psi$ are six generalized coordinates for the configuration space. The kinetic energy of the body is given by (4.42), where the centre of mass velocity is $\mathbf{v}_G = (\dot{x}, \dot{y}, \dot{z})$. We may then use (4.67) to express the rotational kinetic energy (4.43) in terms of Euler angles. Altogether the kinetic energy of the body relative to $\hat{\mathcal{S}}$ is then

$$\begin{aligned} T &= \frac{1}{2}M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}I_1(\dot{\theta}\,\sin\psi - \dot{\varphi}\,\sin\theta\cos\psi)^2 \\ &\quad + \frac{1}{2}I_2(\dot{\theta}\,\cos\psi + \dot{\varphi}\,\sin\theta\sin\psi)^2 + \frac{1}{2}I_3(\dot{\psi} + \dot{\varphi}\,\cos\theta)^2 \ . \end{aligned} \tag{4.69}$$

The main application we will address in this section is the dynamics of a rigid body moving under gravity. An important fact is that if a body is placed in a uniform gravitational field, say of strength g in the downward z-direction, then the body acts as if all the mass was located at the centre of mass G. This follows since the total gravitational potential energy of the body is

$$V = \int_R \rho(\mathbf{r})\, gz \,\mathrm{d}V = Mg\, z_G \ , \tag{4.70}$$

where z_G is the z-coordinate of the centre of mass $\mathbf{r}_G$ in (4.20). In particular, the Lagrangian for a rigid body moving under gravity is $L = T - V$, where T is given by (4.69) and V is given by (4.70). The form of this Lagrangian immediately implies that the centre of mass motion $(x(t), y(t), z(t))$, behaves like a point particle of mass M at the centre of mass G, while the rotational motion about the centre of mass is entirely unaffected by the gravitational field.

A different situation has the body instead rotating about a general point P fixed both in $\hat{\mathcal{S}}$ *and* fixed in the body. In this case we may take the frame $\mathcal{S}$ to be centred at $O = P$, with axes aligned with the principal axes of $\mathcal{I}^{(P)}$, so that

$$T = \frac{1}{2}I_1(\dot{\theta}\,\sin\psi - \dot{\varphi}\,\sin\theta\cos\psi)^2 + \frac{1}{2}I_2(\dot{\theta}\,\cos\psi + \dot{\varphi}\,\sin\theta\sin\psi)^2 + \frac{1}{2}I_3(\dot{\psi} + \dot{\varphi}\,\cos\theta)^2 \ . \tag{4.71}$$

Here I_i are now the principal moments of inertia about the point P. This follows from the original expression (4.39) for the kinetic energy, where the middle term in the expression on the right hand side is now zero since $\mathbf{v}_O = \mathbf{0}$ (rather than because $O = G$). The form (4.71) of the kinetic energy is relevant for problems where the rigid body is suspended from a particular point P, but may move freely about this point.

In general this problem is not integrable, but certain special cases are. We will study the *Lagrange top*, which is a *constrained problem* in which an axisymmetric body is suspended from a point P lying on the axis of symmetry $\mathbf{e}_3$. If we denote by l the fixed distance between P and G along this axis, then from (4.70) the potential energy is

$$V = Mgl\cos\theta \ . \tag{4.72}$$

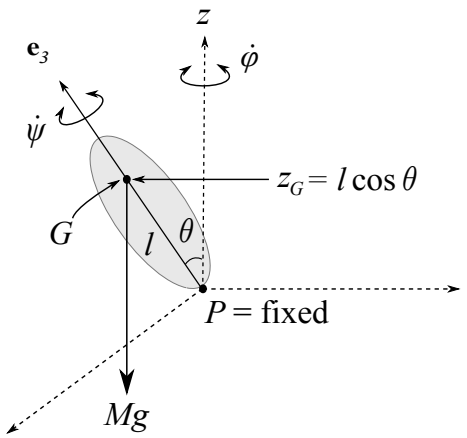

Figure 14: A Lagrange top.

In particular notice that θ and φ are spherical polar angles for the axis of symmetry $\mathbf{e}_3$. That is, $\mathbf{e}_3$ makes an angle θ with the upward vertical z-direction, while $\dot{\varphi}$ is the angular speed with which the axis $\mathbf{e}_3$ rotates about this vertical. The Lagrangian for the Lagrange top is hence given by

$$L \;=\; T - V \;=\; \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\varphi}^2\,\sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\varphi}\,\cos\theta)^2 - Mgl\cos\theta\ . \tag{4.73}$$

We stress that as in (4.71) I_i are the principal moments of inertia *about* P. We immediately see that both ψ and φ are ignorable coordinates, with conserved momenta

$$\begin{aligned} p_\psi &= \frac{\partial L}{\partial\dot{\psi}} \;=\; I_3(\dot{\psi} + \dot{\varphi}\,\cos\theta) \;=\; I_3\omega_3\ , \\ p_\varphi &= \frac{\partial L}{\partial\dot{\varphi}} \;=\; I_1\sin^2\theta\,\dot{\varphi} + p_\psi\cos\theta\ . \end{aligned} \tag{4.74}$$

Notice that p_ψ is simply the angular momentum of the top about its axis of symmetry, where the corresponding constant angular velocity ω_3 is called the *spin*. Since $\partial L/\partial t = 0$ the energy E is conserved, where

$$2E \;=\; I_1\dot{\theta}^2 + I_1\dot{\varphi}^2\,\sin^2\theta + I_3\omega_3^2 + 2Mgl\cos\theta\ . \tag{4.75}$$

It is then useful to introduce $u = \cos\theta$ and rearrange the equations as

$$\begin{aligned} \dot{\varphi} &= \frac{p_\varphi - I_3\omega_3 u}{I_1(1-u^2)}\ , \\ \dot{\psi} &= \omega_3 - \frac{(p_\varphi - I_3\omega_3 u)u}{I_1(1-u^2)}\ , \\ I_1\dot{u}^2 &= F(u)\ , \end{aligned} \tag{4.76}$$

where we have introduced

$$F(u) \;\equiv\; (2E - I_3\omega_3^2 - 2Mglu)(1-u^2) - \frac{(p_\varphi - I_3\omega_3 u)^2}{I_1}\ . \tag{4.77}$$

We have thus reduced the problem to quadratures, and as for the free Euler equations the u-integral is an elliptic integral. However, one can understand the *qualitative* behaviour of the dynamics without solving the equations directly.

Suppose we set the top in motion with starting value $\theta = \cos^{-1}(u_2)$ and $\dot{\theta} = 0$. We shall fix the angular speed $\omega_3 > 0$ and p_φ, and suppose that $u_c \equiv p_\varphi / I_3\omega_3$ satisfies $0 < u_c < 1$. This latter condition will lead to various interesting behaviour, as we shall see. Next consider the differential equation $I_1\dot{u}^2 = F(u)$, where $F(u)$ is given by (4.77). The constant E in this expression is fixed by the initial condition

$$0 \;=\; F(u_2) \;=\; (2E - I_3\omega_3^2 - 2Mglu_2)(1-u_2^2) - \frac{(p_\varphi - I_3\omega_3 u_2)^2}{I_1} \;. \tag{4.78}$$

The function $F(u)$ is a cubic with positive coefficient of u^3, and we also note that $F(\pm 1) = -(p_\varphi \mp I_3\omega_3)^2/I_1 < 0$. The physical range of interest is where $-1 \le u = \cos\theta \le 1$ and $F(u) \ge 0$. In Figure 15 we have sketched the graph of $F(u)$ as we vary u_2, with all other parameters kept fixed. Notice that $F(u)$ has two roots $u_1 \le u_2 \in (-1, 1)$ in the physical range.

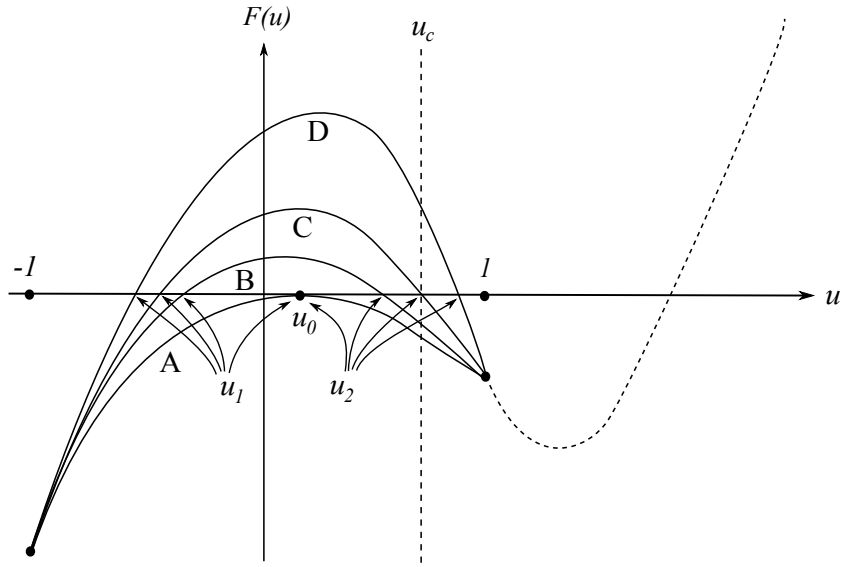

Figure 15: Sketch of the cubic function $F(u)$ for various values of initial condition u_2 (where $F(u_2) = 0$). The physical region is where $u = \cos\theta \in [-1, 1]$ and $F(u) \ge 0$. The motion of $u(t)$ is constrained to lie between two roots $u_1 \le u_2$ of $F(u)$. As we vary u_2 the endpoints $F(\pm 1) = -(p_\varphi \mp I_3\omega_3)^2/I_1 < 0$ of the curves are fixed. We have also noted the critical value $u_c \equiv p_\varphi / I_3\omega_3 \in (0, 1)$. There are then 4 distinct behaviours, corresponding to the 4 curves A, B, C and D.

Notice that to the left of the line $u = u_c$, meaning $-1 < u < u_c$, we see from (4.76) that $\dot{\varphi} > 0$ is positive. On the other hand to the right of $u = u_c$, meaning $u_c < u < 1$, we instead have $\dot{\varphi} < 0$ is negative. This leads to 4 distinct behaviours for the motion of the top. Notice that in this motion the axis of symmetry $\mathbf{e}_3$ traces out a curve on the unit sphere, with spherical polar coordinates (θ, φ), centred at P:

(A) Here there is a critical value $u_0 = u_1 = u_2$ which is a root of both $F(u)$ and $F'(u)$. The graph of $F(u)$ hence touches the u-axis at $u = u_0$, and the trajectory is simply $u = u_0 =$ constant. The axis thus steadily precesses about the vertical with θ and $\dot{\varphi} > 0$ both constant. This traces out a horizontal circle on the sphere with $\theta = \theta_0 = \cos^{-1}(u_0)$.

(B) Here u_2 satisfies $u_0 < u_2 < u_c$, and $u(t) \in [u_1, u_2]$ oscillates between the two roots $u_1 < u_2$ of $F(u)$ on either side of u_0. But in this motion we always have $\dot{\varphi} > 0$. The axis of symmetry hence rotates anticlockwise about the vertical (viewed from above the north pole $\theta = 0$), but at the same time its angle θ with the vertical oscillates between the two values corresponding to the roots of F. This oscillation in θ is called *nutation*.

(C) There is a limiting case where $u_2 = u_c$. Here $\dot{\varphi} = 0$ when θ reaches its minimum, corresponding to $u = u_c$. While this case might look fine-tuned, it's also exactly what happens when you simply let go of the rigid body from rest, *i.e.* both $\dot{\theta} = 0$ and $\dot{\varphi} = 0$ initially (of course it can still spin about its axis, $\dot{\psi} = \omega_3$ in this case).

(D) Perhaps most interesting is the final case when $u_c < u_2 < 1$. In this case $\dot{\varphi}$ is *negative* for appropriately small range of θ, and the curve traced out on the sphere loops back on itself.

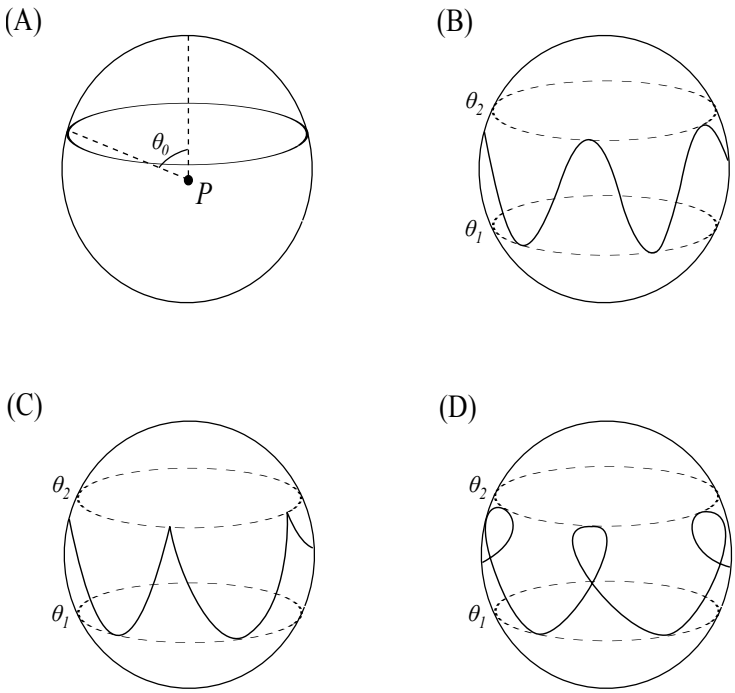

Figure 16: The distinct behaviours of the Lagrange top corresponding to the curves A, B, C and D in Figure 15. In each case we've shown the curve traced out by the axis of symmetry on the unit sphere centred at P.

5 Hamiltonian mechanics

We now leave behind the world of pendulums, springs, pulleys and tops, and develop the Hamiltonian formulation of classical mechanics. This is equivalent to the Lagrangian formulation but provides a different perspective, giving yet more insight into the structure of the theory. In particular the geometry underlying Hamiltonian mechanics is distinct from that in Lagrangian mechanics. Mathematically this leads on to the more abstract 20^{th} century subjects of (for example) symplectic geometry and integrable systems, both still very active areas of research. On the other hand, from a physics viewpoint Hamiltonian mechanics was very important in the development of quantum mechanics. For those who have studied the latter topic some of the structures we shall encounter may look familiar.

5.1 The Legendre transformation

In the Lagrangian formulation of classical mechanics we have the Lagrangian function $L(\mathbf{q}, \mathbf{v}, t)$, and the principle of least action leads to Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \frac{\partial L}{\partial \mathbf{q}} \;=\; \mathbf{0}\;, \qquad \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} \;=\; \mathbf{v}\;. \tag{5.1}$$

Here $\mathbf{q} = (q_1, \ldots, q_n)$ are generalized coordinates, and we have written the second order Lagrange equations in a first order form by writing $\mathbf{v} = \dot{\mathbf{q}}$ for the generalized velocity.

The form of Lagrange's equations is invariant under coordinate transformations $\mathbf{q} \to \tilde{\mathbf{q}} = \tilde{\mathbf{q}}(\mathbf{q}, t)$, as demonstrated in equation (2.13). Notice in particular that the generalized velocity $\mathbf{v} = (v_1, \ldots, v_n)$ correspondingly transforms as $v_a \to \tilde{v}_a = \tilde{v}_a(\mathbf{q}, \mathbf{v}, t) = \sum_{b=1}^{n} \frac{\partial \tilde{q}_a}{\partial q_b} v_b + \frac{\partial \tilde{q}_a}{\partial t}$. However, the form of the equations of motion is *not* preserved by more general transformations with $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}(\mathbf{q}, \mathbf{v}, t)$, $\tilde{\mathbf{v}} = \tilde{\mathbf{v}}(\mathbf{q}, \mathbf{v}, t)$. This is related to the fact that the coordinates $\mathbf{q}$ and velocities $\mathbf{v} = \dot{\mathbf{q}}$ are treated quite differently in the Lagrangian approach.

The Hamiltonian formulation rewrites the dynamical equations so that the variables appear on a more equal footing. The form of (5.1) suggests we might achieve this by introducing the *generalized momenta*

$$\mathbf{p} \;\equiv\; \frac{\partial L}{\partial \mathbf{v}}\;. \tag{5.2}$$

Here $\mathbf{p} = \mathbf{p}(\mathbf{q}, \mathbf{v}, t)$, and we saw this definition before in equation (2.48). The Lagrange equations (5.1) are then

$$\dot{\mathbf{p}} \;=\; \frac{\partial L}{\partial \mathbf{q}}\;, \qquad \dot{\mathbf{q}} \;=\; \mathbf{v}\;. \tag{5.3}$$

We aren't quite there yet though, because now we have $\mathbf{q}$, $\mathbf{v}$ *and* $\mathbf{p}$. What we'd like to do is eliminate $\mathbf{v}$ in favour of $\mathbf{p}$. This involves regarding L as a function of $\mathbf{q}$ and the partial derivative $\partial L/\partial \mathbf{v}$ (and time t). Mathematically the transformation that does this is called the *Legendre transform*.

The Legendre transformation

For simplicity we focus first on a function of a single variable $f(x)$, so that $f : \mathbb{R} \to \mathbb{R}$. We also introduce its derivative

$$s(x) \equiv \frac{\mathrm{d}f}{\mathrm{d}x} = \partial_x f \ . \tag{5.4}$$

Here we've used the partial derivative notation, even though we only have one variable, as this will generalize below. We'd like to regard f as a function of its derivative $s = \partial_x f$, rather than as a function of x. In particular this involves inverting the map $s : \mathbb{R} \to \mathbb{R}$. From first year Analysis we know a sufficient condition to be able to do this: if $\partial_x s > 0$ in some interval $[a, b] \subset \mathbb{R}$ then s is monotonic increasing on $[a, b]$ and will have a differentiable inverse that we call $x(s)$. (Of course we could replace this by $\partial_x s < 0$ so that s is monotonic decreasing, but it is the case $\partial_x s > 0$ that arises in classical mechanics, as we shall see shortly.)

We then define the *Legendre transform* of f to be the function

$$g(s) \equiv s \cdot x(s) - f(x(s)) \ . \tag{5.5}$$

There are various ways to understand why this is a natural definition. First notice that

$$\partial_s g = \frac{\mathrm{d}g}{\mathrm{d}s} = x(s) + s\frac{\mathrm{d}x}{\mathrm{d}s} - \frac{\mathrm{d}f}{\mathrm{d}x}(x(s)) \cdot \frac{\mathrm{d}x}{\mathrm{d}s} = x(s) \ . \tag{5.6}$$

Thus we have symmetry between $s(x) = \partial_x f$ and $x(s) = \partial_s g$. Another important fact is that the Legendre transform is its own inverse. That is, if we perform another Legendre transformation on $g(s)$, we simply recover $f(x)$. We may also more informally rewrite (5.5) as

$$f + g = s \cdot x \ , \tag{5.7}$$

where this is understood to mean either $f(x(s)) + g(s) = s \cdot x(s)$, or the equivalent $f(x) + g(s(x)) = s(x) \cdot x$. Equation (5.7) then has a simple geometric interpretation, shown in Figure 17.

Example: Taking $f(x) = x^\alpha/\alpha$, one finds that the Legendre transform is $g(s) = s^\beta/\beta$, where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ $(\alpha, \beta > 1)$.

The multivariable case is similar. Now $f = f(x_1, \ldots, x_n)$, and the dual variables are $\mathbf{s} = \partial_{\mathbf{x}} f$, where $\mathbf{s} = (s_1, \ldots, s_n)$. Then $\mathbf{s} = \mathbf{s}(x_1, \ldots, x_n)$ will be invertible in a neighbourhood of the point $(x_1, \ldots, x_n) \in \mathbb{R}^n$ provided the Jacobian determinant $\det(\partial_{x_a}\partial_{x_b} f)$ is non-zero (this is the *inverse function theorem* again). In particular this is true if the $n \times n$ symmetric matrix $(\partial_{x_a}\partial_{x_b} f)$, called the *Hessian* of f, is positive definite. There is then an inverse $\mathbf{x} = \mathbf{x}(s_1, \ldots, s_n)$ and the Legendre transform of $f = f(\mathbf{x})$ is

$$g(\mathbf{s}) = \sum_{a=1}^{n} s_a \cdot x_a(\mathbf{s}) - f(\mathbf{x}(\mathbf{s})) \ . \tag{5.8}$$

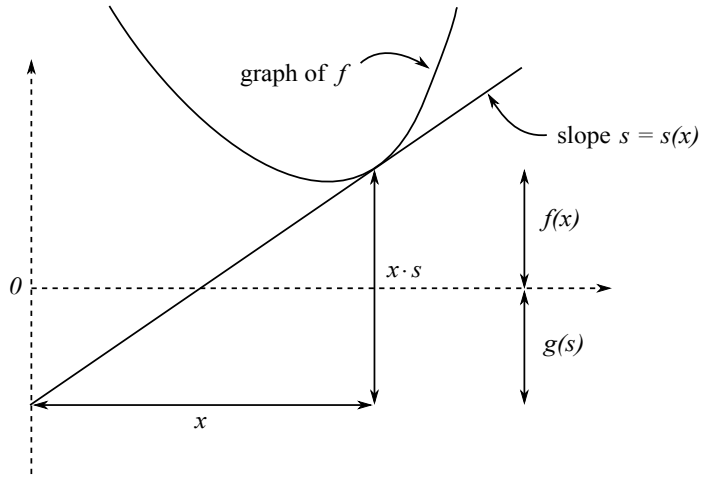

Figure 17: The Legendre transform $g(s)$ of the function $f(x)$.

Using the chain rule we compute

$$\partial_{\mathbf{s}} g \;=\; \mathbf{x}(\mathbf{s}) \;, \tag{5.9}$$

analogously to (5.6).

We may now go back to our classical mechanics problem. We define the *Hamiltonian function* $H = H(\mathbf{q}, \mathbf{p}, t)$ to be the Legendre transform of the Lagrangian with respect to the velocities $\mathbf{v} = \dot{\mathbf{q}}$. Thus

$$H(\mathbf{q}, \mathbf{p}, t) \;\equiv\; \sum_{a=1}^{n} p_a \dot{q}_a - L(\mathbf{q}, \dot{\mathbf{q}}, t)\Big|_{\dot{\mathbf{q}}=\dot{\mathbf{q}}(\mathbf{q},\mathbf{p},t)} \;. \tag{5.10}$$

Comparing to (5.8) we are taking $f = L$ (regarded as a function of $\mathbf{v} = \dot{\mathbf{q}}$ – the dependence on $\mathbf{q}$ and t goes along for the ride), $\mathbf{x} = \dot{\mathbf{q}}$ and hence $\mathbf{s} = \mathbf{p} = \partial_{\dot{\mathbf{q}}} L$. In particular in regarding the right hand side as a function of $\mathbf{p}$, we must invert $\dot{\mathbf{q}} = \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t)$. In classical mechanics notice that the $n \times n$ symmetric matrix $\partial_{\dot{q}_a} \partial_{\dot{q}_b} L$ is usually positive definite. For example for the Lagrangian (3.1) this is simply the kinetic energy tensor T_{ab} (whose positivity played a role in small perturbations of equilibria in section 3). The inverse function theorem then guarantees we can invert $\dot{\mathbf{q}} = \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t)$.

5.2 Hamilton's equations

It is now straightforward to derive *Hamilton's equations*:

$$\dot{\mathbf{p}} \;=\; -\frac{\partial H}{\partial \mathbf{q}} \;, \qquad \dot{\mathbf{q}} \;=\; \frac{\partial H}{\partial \mathbf{p}} \;. \tag{5.11}$$

The cleanest derivation uses the notation of differentials. We compute

$$\mathrm{d}L \;=\; \sum_{a=1}^{n} \left(\frac{\partial L}{\partial q_a} \mathrm{d}q_a + \frac{\partial L}{\partial \dot{q}_a} \mathrm{d}\dot{q}_a \right) + \frac{\partial L}{\partial t} \mathrm{d}t \;=\; \sum_{a=1}^{n} (\dot{p}_a \mathrm{d}q_a + p_a \mathrm{d}\dot{q}_a) + \frac{\partial L}{\partial t} \mathrm{d}t \;, \tag{5.12}$$

where we have used Lagrange's equations in the form (5.3) for the first term. Thus we find

$$\mathrm{d}H \;=\; \sum_{a=1}^{n}(\dot{q}_a \mathrm{d}p_a + p_a \mathrm{d}\dot{q}_a) - \mathrm{d}L \;=\; \sum_{a=1}^{n}(\dot{q}_a \mathrm{d}p_a - \dot{p}_a \mathrm{d}q_a) - \frac{\partial L}{\partial t}\mathrm{d}t \ . \tag{5.13}$$

But of course also

$$\mathrm{d}H \;=\; \sum_{a=1}^{n}\left(\frac{\partial H}{\partial q_a}\mathrm{d}q_a + \frac{\partial H}{\partial p_a}\mathrm{d}p_a\right) + \frac{\partial H}{\partial t}\mathrm{d}t \ . \tag{5.14}$$

Comparing these last two equations we thus deduce (5.11), together with the relation $\partial L/\partial t = -\partial H/\partial t$.

As an alternative derivation, notice that the second equation in (5.11) is precisely (5.9), where we identify the Legendre transformed function $g = H$, and the variables $\mathbf{x} = \dot{\mathbf{q}}$ and $\mathbf{s} = \mathbf{p} = \partial_{\dot{\mathbf{q}}} L$. Thus it is this particular property of the Legendre transform that has led to such a simple form for Hamilton's equations. The first equation in (5.11) is then simply the Lagrange equation written in the Hamiltonian variables.

Hamilton's equations (5.11) are a set of $2n$ coupled first order differential equations for the two sets of functions $\mathbf{q}(t)$, $\mathbf{p}(t)$. They are equivalent to Lagrange's equations, but are much more symmetric – in particular it is the property (5.9) of the Legendre transform that leads to this symmetry. The coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ are often referred to as *canonical coordinates*, and Hamilton's equations (5.11) are sometimes also called the *canonical equations*.

Example: Consider a particle of mass m moving in $\mathbb{R}^3$ under the influence of a potential $V(\mathbf{r})$. The Lagrangian is

$$L \;=\; L(\mathbf{r}, \dot{\mathbf{r}}) \;=\; \frac{1}{2}m|\dot{\mathbf{r}}|^2 - V(\mathbf{r}) \ . \tag{5.15}$$

The (generalized) momentum conjugate to $\mathbf{r}$ is

$$\mathbf{p} \;=\; \frac{\partial L}{\partial \dot{\mathbf{r}}} \;=\; m\dot{\mathbf{r}} \ , \tag{5.16}$$

which is the usual linear momentum of the particle. It is particularly trivial to invert this as $\dot{\mathbf{r}} = \mathbf{p}/m$, so that the Hamiltonian is

$$H \;=\; H(\mathbf{r}, \mathbf{p}) \;=\; \mathbf{p}\cdot\dot{\mathbf{r}} - L \;=\; \frac{|\mathbf{p}|^2}{2m} + V(\mathbf{r}) \ . \tag{5.17}$$

Of course this is $H = T + V$, where the kinetic energy is $T = |\mathbf{p}|^2/2m$. Hamilton's equations (5.11) read

$$\dot{\mathbf{p}} \;=\; -\frac{\partial V}{\partial \mathbf{r}} \ , \qquad \dot{\mathbf{r}} \;=\; \frac{\mathbf{p}}{m} \ , \tag{5.18}$$

which we recognize as Newton's second law, together with the standard relation between velocity and momentum.

Example: More generally we might consider Lagrangians of the form

$$L \;=\; L(\mathbf{q},\dot{\mathbf{q}},t) \;=\; \frac{1}{2}\sum_{a,b=1}^{n} T_{ab}(\mathbf{q},t)\dot{q}_a\dot{q}_b + \sum_{a=1}^{n} C_a(\mathbf{q},t)\dot{q}_a - V(\mathbf{q},t)\ , \tag{5.19}$$

where $T_{ab} = T_{ba}$ is symmetric. This is a general quadratic function of the generalized velocities $\dot{\mathbf{q}}$. We compute

$$p_a \;=\; \frac{\partial L}{\partial \dot{q}_a} \;=\; \sum_{b=1}^{n} T_{ab}\dot{q}_b + C_a\ , \tag{5.20}$$

and the Hamiltonian is hence

$$\begin{aligned} H \;&=\; \sum_{a=1}^{n} p_a\dot{q}_a - L \;=\; \frac{1}{2}\sum_{a,b=1}^{n} T_{ab}\dot{q}_a\dot{q}_b + V \\ &=\; \frac{1}{2}\sum_{a,b=1}^{n} (T^{-1})_{ab}(p_a - C_a)(p_b - C_b) + V\ , \end{aligned} \tag{5.21}$$

where $(T^{-1})_{ab}$ is the matrix inverse of T_{ab}. Notice that the last line of (5.21) is a function of $\mathbf{q}, \mathbf{p}, t$, as it should be.

As an example of this, recall from equation (2.84) that the Lagrangian of a particle of mass m and charge e moving in an electromagnetic field is

$$L \;=\; \frac{1}{2}m|\dot{\mathbf{r}}|^2 - e\,(\phi - \dot{\mathbf{r}}\cdot\mathbf{A})\ . \tag{5.22}$$

Here ϕ is the electric scalar potential and $\mathbf{A}$ is the magnetic vector potential. The momentum canonically conjugate to $\mathbf{r}$ is

$$\mathbf{p} \;=\; \frac{\partial L}{\partial \dot{\mathbf{r}}} \;=\; m\dot{\mathbf{r}} + e\,\mathbf{A}\ . \tag{5.23}$$

The Hamiltonian is hence

$$\begin{aligned} H(\mathbf{r},\mathbf{p}) \;&=\; \mathbf{p}\cdot\dot{\mathbf{r}} - L \\ &=\; \frac{1}{m}\mathbf{p}\cdot(\mathbf{p} - e\,\mathbf{A}) - \left[\frac{1}{2m}\,(\mathbf{p} - e\,\mathbf{A})^2 - e\,\phi + \frac{e}{m}\,(\mathbf{p} - e\,\mathbf{A})\cdot\mathbf{A}\right] \\ &=\; \frac{1}{2m}\,(\mathbf{p} - e\,\mathbf{A})^2 + e\,\phi\ , \end{aligned} \tag{5.24}$$

where we have inverted (5.23) to give $\dot{\mathbf{r}} = \frac{1}{m}(\mathbf{p} - e\,\mathbf{A})$. Comparing to (5.19), (5.21) we have $T = \mathrm{diag}(m,m,m)$, $\mathbf{C} = e\,\mathbf{A}$, $V = e\,\phi$. The Hamilton equation $\dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{1}{m}(\mathbf{p} - e\,\mathbf{A})$ simply re-expresses $\dot{\mathbf{r}}$ in terms of $\mathbf{p}$, while the other Hamilton equation reads (in components)

$$\dot{p}_a \;=\; -\frac{\partial H}{\partial r_a} \;=\; -e\,\frac{\partial\phi}{\partial r_a} + \frac{e}{m}\sum_{b=1}^{3}(p_b - e\,A_b)\frac{\partial A_b}{\partial r_a}\ . \tag{5.25}$$

Phase space

The canonical coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ parametrize the $2n$-dimensional *phase space* $\mathscr{P}$ of the system. Notice this has twice the dimension of the configuration space $\mathcal{Q}$, which is parametrized by $q_1, \ldots, q_n$. For example, a single particle moving in $\mathbb{R}^n$ has phase space $\mathscr{P} = \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$, with the points labelled by $(\mathbf{r}, \mathbf{p})$. In particular a particle moving in one dimension along the x-axis has phase space $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, with coordinates (x, p). A more interesting example is the simple pendulum we studied in section 2. Recall here that the configuration space is $\mathcal{Q} = S^1$, parametrized by the angle θ the pendulum makes with the vertical. Then $\theta \in [-\pi, \pi)$, with $\theta = -\pi$ identified with $\theta = \pi$. On the other hand the conjugate momentum $p = ml^2\dot{\theta}$ can take any real value, so the phase space is $\mathscr{P} = S^1 \times \mathbb{R}$.

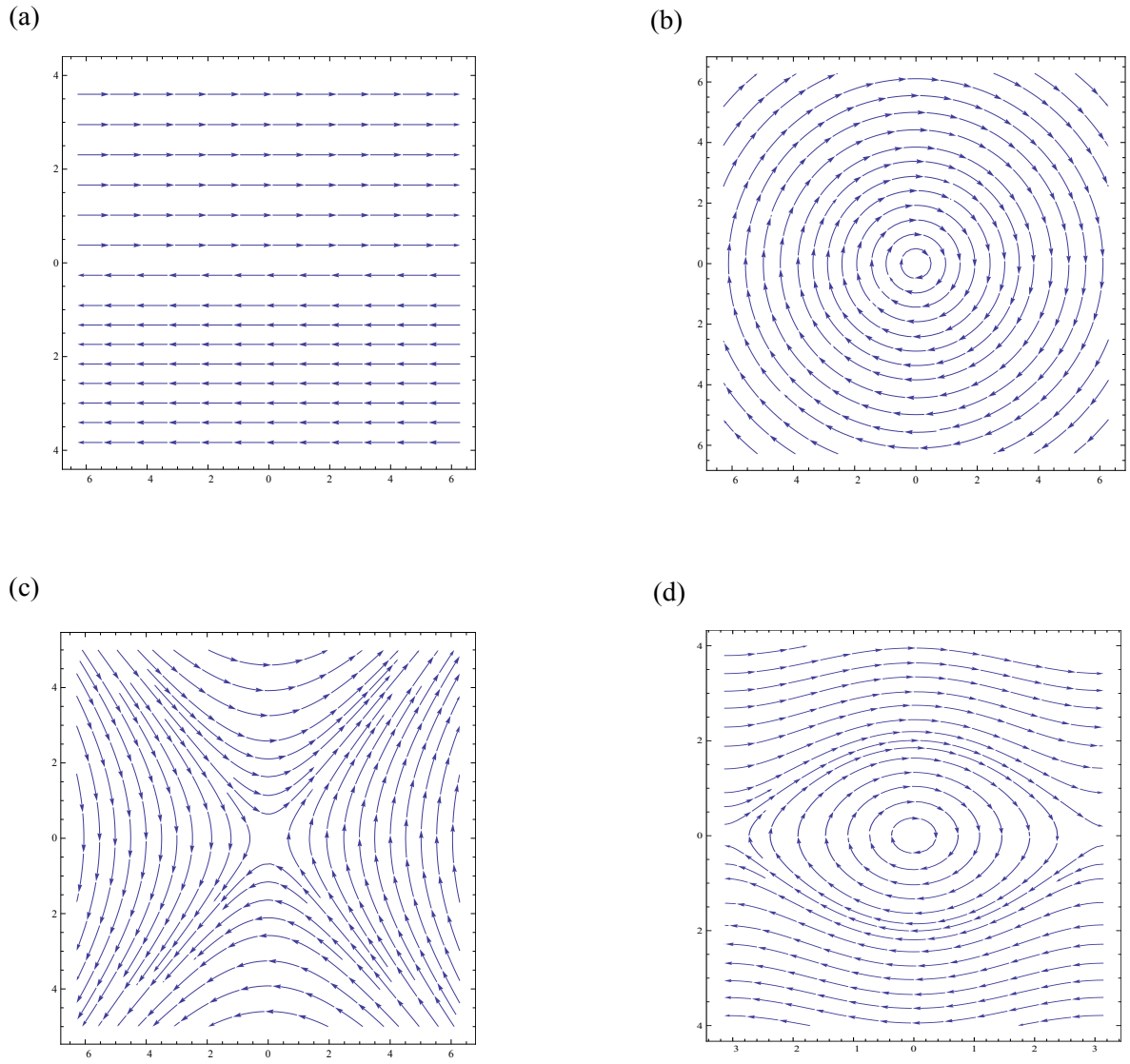

Figure 18: Hamiltonian flows on the phase spaces of a number of one-dimensional systems: (a) a free particle, (b) the harmonic oscillator, (c) the inverted harmonic oscillator, (d) the simple pendulum.

One can think of Hamilton's equations (5.11) as first order equations for a trajectory $\gamma(t) = (q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t))$ in $\mathscr{P}$. If we pick a point $(\mathbf{q}^{(0)}, \mathbf{p}^{(0)}) \in \mathscr{P}$, then general results from the theory of differential equations guarantee that under suitable conditions on the Hamiltonian function H we will have a unique solution for the path $\gamma(t) = (\mathbf{q}(t), \mathbf{p}(t))$ satisfying Hamilton's equations with initial condition $\gamma(0) = (\mathbf{q}^{(0)}, \mathbf{p}^{(0)})$. The set of all trajectories $\{\gamma(t)\}$ is called the *Hamiltonian flow* on phase space.

In order to make this more concrete, and to give some examples, let's focus on the case of dimension $n = 1$. The phase space is then two-dimensional, parametrized by (q, p). In Figure 18 we have shown parts of the phase spaces and Hamiltonian flow trajectories for 4 systems, where the spatial variable q is plotted horizontally and the momentum p vertically.

(a) For a free particle of mass m the Hamiltonian is simply $H = \frac{p^2}{2m}$. If the particle moves on the x-axis then we may use $q = x$ as a generalized coordinate, and the Hamiltonian flow equations are then $(\dot{x}(t), \dot{p}(t)) = (\partial_p H, -\partial_x H) = (\frac{1}{m}p(t), 0)$. In Figure 18(a) we have plotted some of the trajectories (setting $m = 1$). Notice that the entire x-axis at $p = 0$ is *fixed*.

(b) Using the same spatial coordinate $q = x$, the Hamiltonian for the harmonic oscillator is $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$, where ω is the frequency. The flow equations are hence $(\dot{x}(t), \dot{p}(t)) = (\partial_p H, -\partial_x H) = (\frac{1}{m}p(t), -m\omega^2 x)$. Some trajectories are plotted in Figure 18(b), where we have set $m = 1 = \omega$ so that $(\dot{x}, \dot{p}) = (p, -x)$. The origin $(x, p) = (0, 0)$ is a stable equilibrium point.

(c) The *inverted harmonic oscillator* has minus the potential energy of the harmonic oscillator, *i.e.* the Hamiltonian is $H = \frac{p^2}{2m} - \frac{1}{2}m\omega^2 x^2$. With $m = 1 = \omega$ the trajectories now obey $(\dot{x}, \dot{p}) = (p, x)$ – see Figure 18(c). The origin $(x, p) = (0, 0)$ is now an *unstable* equilibrium point.

(d) Finally recall that the simple pendulum has phase space $S^1 \times \mathbb{R}$, with the angle $\theta = q$ being the generalized coordinate on S^1. The Hamiltonian corresponding to the Lagrangian (2.24) is $H = \frac{p^2}{2ml^2} - mgl\cos\theta$, giving trajectories obeying $(\dot{\theta}, \dot{p}) = (\frac{1}{ml^2}p, -mgl\sin\theta)$. Some trajectories are shown in Figure 18(d), with $m = 1 = l = g$. The θ-axis is horizontal, with the left and right limits being $\theta = -\pi$, $\theta = \pi$, respectively, which should be identified. Notice that the trajectories around the origin (a stable equilibruim point) resemble those for the harmonic oscillator (as happens near any stable equilibrium point). As $|p|$ increases we begin to see trajectories that wind around the circle S^1, *i.e.* the pendulum swings all the way around. The point $(\theta, p) = (-\pi, 0) \sim (\pi, 0)$ is an unstable equilibrium point.

5.3 Poisson brackets

Suppose we have a function $f(\mathbf{q}, \mathbf{p}, t)$ on $\mathscr{P} \times \mathbb{R}$ (with $\mathbb{R}$ the time direction). Then its evolution in time, when evaluated on a solution $(\mathbf{q}(t), \mathbf{p}(t))$ to the Hamilton equations of motion, is computed

as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}f &= \sum_{a=1}^{n}\left(\frac{\partial f}{\partial q_a}\dot{q}_a + \frac{\partial f}{\partial p_a}\dot{p}_a\right) + \frac{\partial f}{\partial t} \\ &= \sum_{a=1}^{n}\left(\frac{\partial f}{\partial q_a}\frac{\partial H}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial H}{\partial q_a}\right) + \frac{\partial f}{\partial t}\ . \end{aligned} \tag{5.26}$$

This motivates defining the *Poisson bracket* of two functions f, g on phase space as[12]

$$\{f,g\} \equiv \sum_{a=1}^{n}\frac{\partial f}{\partial q_a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q_a}\ . \tag{5.27}$$

Here f and g may also depend on time t, *i.e.* $f = f(\mathbf{q},\mathbf{p},t)$, $g = g(\mathbf{q},\mathbf{p},t)$ are more precisely functions on $\mathscr{P}\times\mathbb{R}$. The Poisson bracket has the following properties, for any functions f,g,h on $\mathscr{P}\times\mathbb{R}$:

1. *anti-symmetry*: $\{f,g\} = -\{g,f\}$,
2. *linearity*: $\{\alpha f + \beta g, h\} = \alpha\{f,h\} + \beta\{g,h\}$, for all constant $\alpha,\beta\in\mathbb{R}$,
3. *Leibniz rule*: $\{fg,h\} = f\{g,h\} + \{f,h\}g$,
4. *Jacobi identity*: $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$.

The anti-symmetry and linearity follow almost immediately from the definition, while the Leibniz rule follows straightforwardly from the usual Leibniz rule of calculus. The Jacobi identity is a little more tedious to verify – you simply substitute from the definition and check that all the terms do indeed cancel (see Problem Sheet 4). Notice also that $\{f,c\} = 0$ for any constant c.

With this definition we may immediately derive some very nice formulae. First, the dynamical evolution of the funtion f in (5.26) is

$$\frac{\mathrm{d}}{\mathrm{d}t}f = \{f,H\} + \frac{\partial f}{\partial t}\ . \tag{5.28}$$

In particular when $\partial f/\partial t = 0$ we have simply $\dot{f} = \{f,H\}$, and we say that the Hamiltonian H *generates the time-evolution* of the function (via the Poisson bracket). Two special cases are Hamilton's equations themselves. Namely putting $f = p_a$ or $f = q_a$ we have

$$\begin{aligned}\dot{p}_a &= \{p_a,H\} = -\frac{\partial H}{\partial q_a}\ , \\ \dot{q}_a &= \{q_a,H\} = \frac{\partial H}{\partial p_a}\ . \end{aligned} \tag{5.29}$$

The Poisson bracket is clearly quite fundamental! Moreover, it is straightforward to compute the Poisson brackets between the canonical coordinates themselves:

$$\{q_a,q_b\} = 0 = \{p_a,p_b\}\ , \qquad \{q_a,p_b\} = \delta_{ab}\ . \tag{5.30}$$

[12] Some authors define this with the opposite overall sign.

These are often called the *fundamental Poisson brackets*, or *canonical Poisson brackets*. Those who have taken any quantum mechanics courses may recognize that these Poisson bracket formulae look *very* similar to commutation relations. This is no accident, but we'll postpone any further discussion of this to the starred subsection at the end of this section.

Conserved quantities

Let's begin by looking at the Hamiltonian itself. Using (5.28) we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}H \;=\; \dot{H} \;=\; \{H,H\} + \frac{\partial H}{\partial t} \;=\; \frac{\partial H}{\partial t}\;, \tag{5.31}$$

where $\{f,f\} = 0$ trivially for any function f due to the anti-symmetry of the Poisson bracket. Thus if H doesn't depend explicitly on time it is conserved. Of course this is simply the Hamiltonian version of the conservation of energy law we found in Lagrangian mechanics when $\partial L/\partial t = -\partial H/\partial t = 0$. In terms of the phase space picture we described in the last subsection, conservation of H means that H is constant along the flow trajectories $\gamma(t)$.

More generally a *constant of the motion*, or *conserved quantity*, is a function $f = f(\mathbf{q},\mathbf{p},t)$ that when evaluated on a solution to the Hamilton equations of motion is constant:

$$\frac{\mathrm{d}}{\mathrm{d}t}f \;=\; \dot{f} \;=\; \{f,H\} + \frac{\partial f}{\partial t} \;=\; 0\;. \tag{5.32}$$

In particular if $\partial f/\partial t = 0$, so that $f = f(\mathbf{q},\mathbf{p})$, then f will be conserved if

$$\dot{f} \;=\; \{f,H\} \;=\; 0\;. \tag{5.33}$$

We then say that f *Poisson commutes* with the Hamiltonian H. An example would be an ignorable coordinate q_a (with the index a fixed). Being ignorable means that $\partial H/\partial q_a = 0$, and from (5.29) we see that the conjugate momentum p_a is conserved, $\dot{p}_a = 0$.

One consequence of the Jacobi identity is that if we have two functions f, g that Poisson commute with H, then also their commutator $\{f,g\}$ Poisson commutes with H. Specifically, putting $h = H$ in the Jaboci identity above we have

$$\{\{f,g\},H\} \;=\; \{f,\{g,H\}\} + \{g,\{H,f\}\} \;=\; 0\;. \tag{5.34}$$

We may use this to prove

Poisson's theorem: If f, g are two constants of the motion, then $\{f,g\}$ is also conserved.

Proof: When $f = f(\mathbf{q},\mathbf{p})$, $g = g(\mathbf{q},\mathbf{p})$ don't explicitly depend on time t, this follows immediately from (5.33) and (5.34). More generally we can use (5.34) to rewrite (5.28) as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\{f,g\} \;&=\; \{\{f,g\},H\} + \frac{\partial}{\partial t}\{f,g\} \\
&=\; \left\{f,\{g,H\} + \frac{\partial g}{\partial t}\right\} + \left\{\{f,H\} + \frac{\partial f}{\partial t},g\right\} \\
&=\; \left\{f,\frac{\mathrm{d}g}{\mathrm{d}t}\right\} + \left\{\frac{\mathrm{d}f}{\mathrm{d}t},g\right\}\;,
\end{aligned} \tag{5.35}$$

from which the result follows. Notice that we have used anti-symmetry of the Poisson bracket on the second term, and $\partial_t\{f,g\} = \{\partial_t f, g\} + \{f, \partial_t g\}$ where $\partial_t = \partial/\partial t$.

Thus from two conserved quantities we can generate a third conserved quantity! Although notice that $\{f,g\}$ might simply turn out to be a constant linear combination of f and g (including perhaps 0), and thus give nothing new. Since conserved quantities are rather special, this is what tends to happen. If $\{f,g\} = 0$ then the functions f and g are said to be *in involution*.

Example (angular momentum): Consider a particle moving in $\mathbb{R}^3$ with angular momentum $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$. The Hamiltonian will thus be a function of $\mathbf{q} = \mathbf{r} = (x_1, x_2, x_3)$, and $\mathbf{p}$. The components of $\mathbf{L}$ are

$$L_i \;=\; \sum_{j,k=1}^{3} \epsilon_{ijk} x_j p_k \;. \tag{5.36}$$

Then one can compute the following Poisson bracket relations:

$$\{L_i, x_j\} \;=\; \sum_{k=1}^{3} \epsilon_{ijk} x_k \;, \qquad \{L_i, p_j\} \;=\; \sum_{k=1}^{3} \epsilon_{ijk} p_k \;, \qquad \{L_i, L_j\} \;=\; \sum_{k=1}^{3} \epsilon_{ijk} L_k \;. \tag{5.37}$$

Since the proofs of these are all quite similar, let's just look at the first identity:

$$\begin{aligned} \{L_i, x_j\} \;&=\; \sum_{k,l=1}^{3} \epsilon_{ikl} \{x_k p_l, x_j\} \;=\; \sum_{k,l=1}^{3} \epsilon_{ikl} \left(x_k \{p_l, x_j\} + \{x_k, x_j\} p_l \right) \\ &=\; \sum_{k,l=1}^{3} \epsilon_{ikl} \left(-x_k \delta_{lj} + 0 \right) \;=\; \sum_{k=1}^{3} \epsilon_{ijk} x_k \;. \end{aligned} \tag{5.38}$$

In the first equality we have substituted (5.36) and used linearity of the Poisson bracket to bring the alternating symbol outside of the bracket. In the second equality we have used the Leibniz rule. The third equality uses the canonical brackets (5.30) and anti-symmetry, while for the final equality $\epsilon_{ijk} = -\epsilon_{ikj}$ holds for all i, j, k.

We already know from section 2.4 that angular momentum is conserved for a particle moving under the influence of a central potential $V = V(|\mathbf{r}|)$. One can also derive this using the Poisson bracket formalism above (see Problem Sheet 4). Notice that the bracket $\{L_i, L_j\} = \sum_{k=1}^{3} \epsilon_{ijk} L_k$ implies that if we know that L_1 and L_2 are conserved, then it follows from Poisson's theorem that $L_3 = \{L_1, L_2\}$ is conserved.

Example (Laplace-Runge-Lenz vector): Another interesting example is the Laplace-Runge-Lenz vector, considered at the end of section 2.4. This was defined in (2.57) to be

$$\mathbf{A} \;\equiv\; \mathbf{p} \wedge \mathbf{L} - m\kappa \frac{\mathbf{r}}{|\mathbf{r}|} \;, \tag{5.39}$$

where κ is a constant. Using the brackets (5.37), similar computations to (5.38) lead to the following Poisson brackets:

$$\{L_i, A_j\} \;=\; \sum_{k=1}^{3} \epsilon_{ijk} A_k \;, \qquad \{A_i, A_j\} \;=\; -2m \left(\frac{|\mathbf{p}|^2}{2m} - \frac{\kappa}{|\mathbf{r}|} \right) \sum_{k=1}^{3} \epsilon_{ijk} L_k \;. \tag{5.40}$$

In particular for the central potential $V = -\kappa/|\mathbf{r}|$ the Hamiltonian is $H = |\mathbf{p}|^2/2m - \kappa/|\mathbf{r}|$, and the second equation above reads $\{A_i, A_j\} = -2mH \sum_{k=1}^{3} \epsilon_{ijk} L_k$. One can also then verify that

$$\{A_i, H\} \;=\; 0 \;, \qquad i \;=\; 1, 2, 3 \;, \tag{5.41}$$

from which we deduce that $\mathbf{A}$ is conserved (which you checked rather more directly on Problem Sheet 1). We then see that the constants of motion $\mathbf{L}$, $\mathbf{A}$ and H form a closed algebra under the Poisson bracket. In particular if we look at solutions with negative energy $E = H < 0$, corresponding to the closed elliptic orbits in the Kepler problem, then we may define the new quantities

$$\mathbf{L}^{\pm} \;\equiv\; \frac{1}{2} \left(\mathbf{L} \pm \frac{1}{\sqrt{-2mE}} \mathbf{A} \right) \;. \tag{5.42}$$

From the above expressions one checks these diagonalize the Poisson brackets as

$$\{L_i^{\pm}, L_j^{\pm}\} \;=\; \sum_{k=1}^{3} \epsilon_{ijk} L_k^{\pm} \;, \qquad \{L_i^{+}, L_j^{-}\} \;=\; 0 \;. \tag{5.43}$$

The elliptical orbits of the planets thus have two sets of Poisson commuting conserved angular momenta $\mathbf{L}^{\pm}$! Just as the usual angular momentum $\mathbf{L}$ is a conserved quantity arising from rotational symmetry under the group $SO(3)$, the full set of conserved vectors for the elliptic orbits in the Kepler problem is associated to a "hidden" action of the the rotation group $SO(4)$ (this actually follows immediately from (5.43), but to understand that we need to go into more group theory than we have time for). One can analyse the parabolic and hyperbolic orbits similarly (in particular, the hidden symmetry of the hyperbolic orbits is coincidentally the *Lorentz group* $SO(3,1)$!).

5.4 Canonical transformations

Recall that we are free to make any choice for the generalized coordinates $\mathbf{q}$ (as long as they define uniquely the configuration of the system at a fixed time). Moreover, Lagrange's equations take the same form under the coordinate transformation[13]

$$\mathbf{q} \;\to\; \mathbf{Q}(\mathbf{q}, t) \;. \tag{5.44}$$

The change of variables (5.44) is sometimes called a *point transformation*. Since Lagrange's equations are invariant, so too are Hamilton's equations (5.11). However, the form of Hamilton's

[13] We will use the notation $\mathbf{Q}(\mathbf{q}, t)$ rather than $\tilde{\mathbf{q}}(\mathbf{q}, t)$ in order to avoid a proliferation of tildes in what follows.

equations is invariant under a much larger class of transformations, called *canonical transformations*. This fact can be traced to the more symmetric way in which the coordinates and momenta appear in the Hamiltonian formalism, and is one of the advantages of it. In particular, judicious choices of transformation may sometimes be used to simplify the form of the Hamiltonian function, and hence the equations of motion.

There are a number of ways to both describe and construct canonical transformations. We shall begin with a formulation that is closest to the modern (geometric) viewpoint on the subject. We first introduce coordinates $\mathbf{y} = (y_1, \ldots, y_{2n}) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ on the whole phase space $\mathscr{P}$, and label the indices of y_α by $\alpha = 1, \ldots, 2n$. We then define the $2n \times 2n$ matrix[14]

$$\Omega \;\equiv\; \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} , \tag{5.45}$$

where $\mathbb{1}$ is the $n \times n$ identity matrix, and 0 is the $n \times n$ matrix with all entries 0. Notice that $\Omega^2 = -\mathbb{1}_{2n\times 2n}$ and $\Omega^T = -\Omega$. In this notation the Poisson bracket between two functions $f = f(\mathbf{y}, t), g = g(\mathbf{y}, t)$ on $\mathscr{P} \times \mathbb{R}$ becomes

$$\{f, g\} \;=\; \sum_{\alpha,\beta=1}^{2n} \frac{\partial f}{\partial y_\alpha} \Omega_{\alpha\beta} \frac{\partial g}{\partial y_\beta} \; . \tag{5.46}$$

The transformation

$$\mathbf{q} \;\to\; \mathbf{Q}(\mathbf{q}, \mathbf{p}, t) \; , \qquad \mathbf{p} \;\to\; \mathbf{P}(\mathbf{q}, \mathbf{p}, t) \; , \tag{5.47}$$

is called a *canonical transformation* if it leaves the Poisson bracket invariant. More precisely, in terms of the $\mathbf{y}$ coordinates the transformation (5.47) may be written

$$\mathbf{y} \;\to\; \mathbf{Y}(\mathbf{y}, t) \; , \tag{5.48}$$

and by invariance of the Poisson bracket we mean that

$$\{f, g\}_y \;\equiv\; \sum_{\gamma,\delta=1}^{2n} \frac{\partial f}{\partial y_\gamma} \Omega_{\gamma\delta} \frac{\partial g}{\partial y_\delta} \;=\; \sum_{\alpha,\beta=1}^{2n} \frac{\partial f}{\partial Y_\alpha} \Omega_{\alpha\beta} \frac{\partial g}{\partial Y_\beta} \;\equiv\; \{f, g\}_Y \; , \tag{5.49}$$

should hold for all functions f, g. Since by the chain rule

$$\frac{\partial f}{\partial y_\gamma} \;=\; \sum_{\alpha=1}^{2n} \frac{\partial f}{\partial Y_\alpha} \frac{\partial Y_\alpha}{\partial y_\gamma} \; , \tag{5.50}$$

this is equivalent to

$$\Omega_{\alpha\beta} \;=\; \sum_{\gamma,\delta=1}^{2n} \frac{\partial Y_\alpha}{\partial y_\gamma} \Omega_{\gamma\delta} \frac{\partial Y_\beta}{\partial y_\delta} \; . \tag{5.51}$$

[14] This is not to be confused with our use of the symbol Ω in (4.5). The latter use won't appear in the rest of these lectures.

To see this one first uses (5.50) and (5.51) to compute

$$\begin{aligned}\{f,g\}_y &\equiv \sum_{\gamma,\delta=1}^{2n} \frac{\partial f}{\partial y_\gamma}\Omega_{\gamma\delta}\frac{\partial g}{\partial y_\delta} \\ &= \sum_{\alpha,\beta,\gamma,\delta=1}^{2n} \frac{\partial f}{\partial Y_\alpha}\frac{\partial Y_\alpha}{\partial y_\gamma}\Omega_{\gamma\delta}\frac{\partial g}{\partial Y_\beta}\frac{\partial Y_\beta}{\partial y_\delta} \\ &= \sum_{\alpha,\beta=1}^{2n} \frac{\partial f}{\partial Y_\alpha}\Omega_{\alpha\beta}\frac{\partial g}{\partial Y_\beta} \equiv \{f,g\}_Y \ , \end{aligned} \tag{5.52}$$

so that (5.49) holds for all f and g. Conversely, if (5.49) holds for all f and g then simply choose $f = Y_\alpha$, $g = Y_\beta$: then (5.49) reads

$$\{Y_\alpha, Y_\beta\}_y = \sum_{\gamma,\delta=1}^{2n} \frac{\partial Y_\alpha}{\partial y_\gamma}\Omega_{\gamma\delta}\frac{\partial Y_\beta}{\partial y_\delta} = \{Y_\alpha, Y_\beta\}_Y = \Omega_{\alpha\beta} \ , \tag{5.53}$$

which is (5.51). In matrix notation (5.51) reads

$$\Omega = \mathcal{J}\,\Omega\,\mathcal{J}^T \ , \qquad \text{where} \qquad \mathcal{J}_{\alpha\beta} \equiv \frac{\partial Y_\alpha}{\partial y_\beta} \ . \tag{5.54}$$

Here $\mathcal{J}$ is the Jacobian matrix for the change of coordinates (5.48). $2n \times 2n$ matrices $\mathcal{J}$ satisfying $\Omega = \mathcal{J}\,\Omega\,\mathcal{J}^T$ form a subgroup of $GL(2n,\mathbb{R})$ called the *symplectic group* $Sp(2n,\mathbb{R})$. Thus canonical transformations are symplectic, in that the Jacobian matrix $\mathcal{J} \in Sp(2n,\mathbb{R})$ holds at each point in $\mathscr{P} \times \mathbb{R}$.

Notice that equation (5.53) says that a transformation is canonical if and only if it preserves the *canonical* Poisson brackets. Indeed, the canonical brackets (5.30) read

$$\{y_\alpha, y_\beta\} = \Omega_{\alpha\beta} \ , \tag{5.55}$$

and we may further rewrite (5.53) as

$$\{Y_\alpha, Y_\beta\}_y = \{y_\alpha, y_\beta\}_y \ . \tag{5.56}$$

Notice that in the original coordinates on phase space the Jacobian is

$$\mathcal{J} = \begin{pmatrix} \partial Q_a/\partial q_b & \partial Q_a/\partial p_b \\ \partial P_a/\partial q_b & \partial P_a/\partial p_b \end{pmatrix} \ , \tag{5.57}$$

where each entry is an $n \times n$ matrix ($a, b = 1, \ldots, n$), and

$$\mathcal{J}\,\Omega\,\mathcal{J}^T = \begin{pmatrix} \{Q_a, Q_b\} & \{Q_a, P_b\} \\ \{P_a, Q_b\} & \{P_a, P_b\} \end{pmatrix} \ , \tag{5.58}$$

where the Poisson brackets are evaluated in the original $\mathbf{q}, \mathbf{p}$ coordinates. We thus again see that if $\{Q_a, Q_b\} = 0 = \{P_a, P_b\}$ and $\{Q_a, P_b\} = \delta_{ab}$, then $\mathcal{J}$ is symplectic, and hence the transformation is canonical.

Example (point transformation): Of course we expect the point transformation (5.44) to be canonical. It is interesting to see in detail how this works. Denote the transformed Lagrangian by $\tilde{L}(\mathbf{Q}, \dot{\mathbf{Q}}, t) = L(\mathbf{q}(\mathbf{Q}, t), \dot{\mathbf{q}}(\mathbf{Q}, \dot{\mathbf{Q}}, t), t)$. Then the transformed momentum is by definition

$$\begin{aligned} P_a &= \frac{\partial \tilde{L}}{\partial \dot{Q}_a} = \sum_{b=1}^{n} \frac{\partial L}{\partial \dot{q}_b} \frac{\partial \dot{q}_b}{\partial \dot{Q}_a} = \sum_{b=1}^{n} \frac{\partial L}{\partial \dot{q}_b} \frac{\partial q_b}{\partial Q_a} \\ &= \sum_{b=1}^{n} p_b \frac{\partial q_b}{\partial Q_a} \,. \end{aligned} \tag{5.59}$$

Here notice that $\mathbf{q} = \mathbf{q}(\mathbf{Q}, t)$ means

$$\dot{q}_b = \sum_{a=1}^{n} \frac{\partial q_b}{\partial Q_a} \dot{Q}_a + \frac{\partial q_b}{\partial t} \qquad \Rightarrow \qquad \frac{\partial \dot{q}_b}{\partial \dot{Q}_a} = \frac{\partial q_b}{\partial Q_a} \,. \tag{5.60}$$

In particular (5.59) gives

$$\frac{\partial P_a}{\partial p_b} = \frac{\partial q_b}{\partial Q_a} \,. \tag{5.61}$$

If we now use the formula (5.57) for the Jacobian, and noting that $\partial Q_a / \partial p_b = 0$, we can directly compute

$$\mathcal{J}\, \Omega\, \mathcal{J}^T = \begin{pmatrix} 0 & \sum_{c=1}^{n} \frac{\partial Q_a}{\partial q_c} \frac{\partial P_b}{\partial p_c} \\ -\sum_{c=1}^{n} \frac{\partial P_a}{\partial p_c} \frac{\partial Q_b}{\partial q_c} & \{P_a, P_b\} \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \Omega \,, \tag{5.62}$$

where we use (5.61) in the second equality in (5.62). The remaining fiddly part in proving the second equality is to show that $\{P_a, P_b\} = 0$. To see this we again use (5.61) to compute

$$\sum_{c=1}^{n} \frac{\partial P_a}{\partial q_c} \frac{\partial P_b}{\partial p_c} = \sum_{c=1}^{n} \frac{\partial P_a}{\partial q_c} \frac{\partial q_c}{\partial Q_b} \,. \tag{5.63}$$

Here $\mathbf{q} = \mathbf{q}(\mathbf{Q}, t)$ and $\mathbf{P} = \mathbf{P}(\mathbf{q}, \mathbf{p}, t)$. Using (5.59) and the chain rule we may write

$$\frac{\partial}{\partial q_c} P_a = \sum_{d,e=1}^{n} p_d \frac{\partial^2 q_d}{\partial Q_a \partial Q_e} \frac{\partial Q_e}{\partial q_c} \,. \tag{5.64}$$

Substiting this into the right hand side of (5.63) we thus have

$$\begin{aligned} \sum_{c=1}^{n} \frac{\partial P_a}{\partial q_c} \frac{\partial P_b}{\partial p_c} &= \sum_{c,d,e=1}^{n} p_d \frac{\partial^2 q_d}{\partial Q_a \partial Q_e} \frac{\partial Q_e}{\partial q_c} \frac{\partial q_c}{\partial Q_b} \\ &= \sum_{d=1}^{n} p_d \frac{\partial^2 q_d}{\partial Q_a \partial Q_b} \,. \end{aligned} \tag{5.65}$$

The right hand side of (5.65) is symmetric in a and b, which thus shows that $\{P_a, P_b\} = 0$. This concludes the proof of (5.62), and hence that point transformations are canonical.

Example (swapping position and momentum): A more interesting canonical transformation is

$$\mathbf{Q} \;=\; \mathbf{p}\;, \qquad \mathbf{P} \;=\; -\mathbf{q}\;. \tag{5.66}$$

We thus swap the coordinates and momenta, with an appropriate sign. The Jacobian (5.57) for this transformation is simply Ω itself, and Ω is symplectic since $\Omega\,\Omega\,\Omega^T = -\Omega^T = \Omega$. Of course one can also see the invariance of the Poisson bracket (5.27) and Hamilton's equations (5.11) very simply. What we call "position" and "momentum" is therefore somewhat arbitrary. It is thus better simply to talk of $\mathbf{p}$ and $\mathbf{q}$ as being *canonically conjugate* variables, meaning they satisfy the canonical Poisson bracket relations (5.30).

Next we turn to Hamilton's equations (5.11). In the new notation these read

$$\dot{y}_\alpha \;=\; \sum_{\beta=1}^{2n} \Omega_{\alpha\beta} \frac{\partial H}{\partial y_\beta}\;. \tag{5.67}$$

From this equation and the transformation (5.48) we compute

$$\begin{aligned} \dot{Y}_\alpha \;&=\; \sum_{\gamma=1}^{2n} \frac{\partial Y_\alpha}{\partial y_\gamma}\dot{y}_\gamma + \frac{\partial Y_\alpha}{\partial t} \;=\; \sum_{\gamma,\delta=1}^{2n} \frac{\partial Y_\alpha}{\partial y_\gamma}\Omega_{\gamma\delta}\frac{\partial H}{\partial y_\delta} + \frac{\partial Y_\alpha}{\partial t} \\ &=\; \sum_{\beta=1}^{2n} \Omega_{\alpha\beta}\frac{\partial \tilde{H}}{\partial Y_\beta} + \frac{\partial Y_\alpha}{\partial t}\;, \end{aligned} \tag{5.68}$$

where in the last line we used the symplectic condition (5.51). We have also defined the Hamiltonian in the new coordinates as

$$\tilde{H}(\mathbf{Y},t) \;\equiv\; H(\mathbf{y}(\mathbf{Y},t),t)\;, \tag{5.69}$$

by inverting (5.48). Similarly, $\partial Y_\alpha/\partial t$ means more precisely $\partial_t Y_\alpha(\mathbf{y},t)\;|_{\mathbf{y}=\mathbf{y}(\mathbf{Y},t)}$, so that it is a function of $(\mathbf{Y},t)$.

We thus see from (5.68) that for *time-independent* canonical transformations (so that $\partial\mathbf{Y}/\partial t = \mathbf{0}$), Hamilton's equations take the same form in the transformed coordinates, with the Hamiltonian transforming simply as a scalar (5.69) – compare to the corresponding discussion of the Lagrangian around equation (2.15). In the more general time-dependent case we can still write the transformed equations in the form (5.67) by introducing a new Hamiltonian $K = K(\mathbf{Y},t)$. Specifically, one can show that there exists a function $\Lambda = \Lambda(\mathbf{Y},t)$ such that

$$\frac{\partial \Lambda}{\partial Y_\alpha} \;=\; -\sum_{\beta=1}^{2n} \Omega_{\alpha\beta}\frac{\partial Y_\beta}{\partial t}\;. \tag{5.70}$$

Given this, we may define the transformed Hamiltonian as

$$K \;\equiv\; \tilde{H} + \Lambda\;, \tag{5.71}$$

so that (5.68) takes the form of Hamilton's equations

$$\dot{Y}_\alpha \;=\; \sum_{\beta=1}^{2n} \Omega_{\alpha\beta} \frac{\partial K}{\partial Y_\beta} \;. \tag{5.72}$$

Since the existence of Λ satisfying (5.70) is a little technical, we have relegated it to the starred paragraph after the next example.

Example (Galilean boost): A simple example in dimension $n = 1$ that demonstrates the need for the additional Λ term in (5.71) is a Galilean boost $Q(q,t) = q + tv$, where v is constant. It follows that $\dot{Q} = \dot{q} + v$. Taking the Lagrangian for a free particle $L = \frac{1}{2}m\dot{q}^2$, the conjugate momentum is $p = m\dot{q}$ with Hamiltonian $H = p^2/2m$. The transformed Lagrangian is

$$\tilde{L} \;=\; \frac{1}{2}m\big(\dot{Q} - v\big)^2 \;=\; \frac{1}{2}m\dot{Q}^2 - \frac{\mathrm{d}}{\mathrm{d}t}\left(mvQ - \frac{1}{2}mv^2 t\right) \;. \tag{5.73}$$

The total derivative term doesn't contribute to the Lagrange equations of motion – see the comment after (2.17). So the free particle maps to a free particle, as it should do. We then compute the conjugate momentum $P = \partial\tilde{L}/\partial\dot{Q} = m(\dot{Q} - v) = p$. Of course this is in accord with our general formula (5.61) for point transformations. On the other hand the transformed Hamiltonian is

$$\begin{aligned} K \;&=\; P\dot{Q} - \tilde{L} \;=\; m(\dot{Q} - v)\dot{Q} - \frac{1}{2}m\dot{Q}^2 + mv\dot{Q} - \frac{1}{2}mv^2 \;, \\ &=\; \frac{1}{2}m\dot{Q}^2 - \frac{1}{2}mv^2 \;=\; \frac{P^2}{2m} + Pv \;=\; \tilde{H} + Pv \;. \end{aligned} \tag{5.74}$$

Here $\tilde{H}(P) = H(p)\,|_{p=P} = P^2/2m$. Thus we explicitly find $\Lambda = \Lambda(Q,P,t) = Pv$. In this notation (5.70) reads $\partial\Lambda/\partial Q = -\partial P/\partial t$ and $\partial\Lambda/\partial P = \partial Q/\partial t$, which are both seen to hold.

***** The existence of the function $\Lambda(\mathbf{Y},t)$ satisfying (5.70) is analogous to the more familiar fact that if $\nabla \wedge \mathbf{F} = \mathbf{0}$ then there exists (locally) a function $V = V(\mathbf{r})$ such that $\mathbf{F} = -\nabla V$. That is, a vector field with zero curl is (locally) the divergence of a function. In the current setting our space is $2n$-dimensional, and the "zero curl" condition means

$$\frac{\partial}{\partial Y_\gamma}\left(\sum_{\beta=1}^{2n}\Omega_{\alpha\beta}\frac{\partial Y_\beta}{\partial t}\right) - \frac{\partial}{\partial Y_\alpha}\left(\sum_{\beta=1}^{2n}\Omega_{\gamma\beta}\frac{\partial Y_\beta}{\partial t}\right) \;=\; 0 \;, \tag{5.75}$$

holds for all $\alpha, \gamma = 1, \ldots, 2n$. Notice that (5.75) is certainly a *necessary* condition for (5.70) to hold, simply using symmetry of the mixed partial derivatives with respect to Y_α. We shall assume it is also sufficient (this result goes under the general name of the *Poincaré Lemma*). We then check explicitly that (5.75) does indeed hold. Using the chain rule one sees that the left hand side is the anti-symmetric part of the matrix $\Omega\,\partial_t\mathcal{J}\,\mathcal{J}^{-1} = -\Omega\,\mathcal{J}\,\partial_t(\mathcal{J}^{-1})$, where $\mathcal{J}$ is the Jacobian matrix in (5.54). But we can also show that the latter is symmetric:

$$\begin{aligned} -\big(\Omega\,\mathcal{J}\,\partial_t\big(\mathcal{J}^{-1}\big)\big)^T \;&=\; \big(\partial_t(\mathcal{J}^{-1})\big)^T\,\mathcal{J}^T\Omega \\ &=\; -\Omega\,\partial_t\mathcal{J}\,\Omega\,\mathcal{J}^T\,\Omega \;=\; \Omega\,\partial_t\mathcal{J}\,\mathcal{J}^{-1} \;. \end{aligned} \tag{5.76}$$

Here in the first equality of (5.76) we have used $\Omega^T = -\Omega$, the second equality arises by differentiating the symplectic relation $(\mathcal{J}^{-1})^T = -\Omega\,\mathcal{J}\,\Omega$ with respect to time t, and the final equality again uses the symplectic relation (5.54). This proves (5.75).

5.5 Generating functions

We have so far seen the definition of canonical transformations, and some of their properties, but how does one actually *construct* transformations (5.47) that are canonical? In this section we describe two different approaches that both involve *generating functions*. The first is somewhat more mathematical/geometrical, while the second is the more traditional physics approach.

Hamiltonian vector fields

Given any function $f = f(\mathbf{y})$ on phase space, we define the following differential operator

$$\mathcal{D}_f \;\equiv\; \sum_{\alpha,\beta=1}^{2n} \frac{\partial f}{\partial y_\alpha} \Omega_{\alpha\beta} \frac{\partial}{\partial y_\beta} \;. \tag{5.77}$$

This maps functions on phase space to functions on phase space in the obvious way. In fact from (5.46) notice that if we apply $\mathcal{D}_f$ to a function $g = g(\mathbf{y})$ we simply obtain the Poisson bracket between f and g:

$$\mathcal{D}_f\, g \;=\; \{f, g\} \;. \tag{5.78}$$

We shall refer to $\mathcal{D}_f$ as the *Hamiltonian vector field associated to* f.[15] One can deduce some immediate properties:

- $\mathcal{D}_f$ is linear in f: $\mathcal{D}_{f_1+f_2} = \mathcal{D}_{f_1} + \mathcal{D}_{f_2}$,
- $\mathcal{D}_f\, g$ is linear in g: $\mathcal{D}_f\,(g_1 + g_2) = \mathcal{D}_f\, g_1 + \mathcal{D}_f\, g_2$,
- anti-symmetry of the Poisson bracket and (5.78) implies $\mathcal{D}_f\, g = -\mathcal{D}_g\, f$.

Another property of $\mathcal{D}_f$ is that it is invariant under canonical transformations; that is, for all f and g we have

$$(\mathcal{D}_f)_y\, g \;\equiv\; \{f, g\}_y \;=\; \{f, g\}_Y \;\equiv\; (\mathcal{D}_f)_Y\, g \;. \tag{5.79}$$

Said more simply, in (5.77) it doesn't matter whether we evaluate the partial derivatives with respect to $\mathbf{y}$ or $\mathbf{Y}$, provided they are related by a canonical transformation. Given two Hamiltonian vector fields $\mathcal{D}_f$, $\mathcal{D}_g$, we may consider their *commutator*

$$[\mathcal{D}_f, \mathcal{D}_g] \;\equiv\; \mathcal{D}_f\mathcal{D}_g - \mathcal{D}_g\mathcal{D}_f \;. \tag{5.80}$$

The composition of operators on the right hand side works in the obvious way, *i.e.* applied to another function $h = h(\mathbf{y})$ we have

$$(\mathcal{D}_f\mathcal{D}_g)(h) \;=\; \mathcal{D}_f(\mathcal{D}_g\, h) \;. \tag{5.81}$$

[15] This might seem like an odd name, because so far this seems to have nothing to do with either the Hamiltonian H or vector fields. The relation to the Hamiltonian and Hamilton's equations will become clear below. The term *vector field* is often used for (or conflated with) a *directional derivative*, since one can write $\mathcal{D}_f = \sum_{\alpha=1}^{2n} V_\alpha \partial_{y_\alpha}$, where V_α is a vector field on phase space.

Then a calculation shows that (acting on twice differentiable functions on phase space)

$$[\mathcal{D}_f, \mathcal{D}_g] \;=\; \mathcal{D}_{\{f,g\}} \;. \tag{5.82}$$

That is, the commutator of Hamiltonian vector fields equals the Hamiltonian vector field of the Poisson bracket. To prove (5.82) one simply applies both sides to an arbitrary function $h = h(\mathbf{y})$: on the left hand side there are *a priori* terms involving second derivatives of h, but one checks that these cancel, and the remaining terms on both sides can be rerranged and shown to be equal – we leave this as an exercise on Problem Sheet 4. The relation (5.82) implies that if f and g Poisson commute, then their Hamiltonian vector fields commute as differential operators.

Armed with this new technology, let us fix any function $f = f(\mathbf{y})$ and consider

$$\mathbf{Y}(\mathbf{y}, s) \;\equiv\; \mathrm{e}^{-s\mathcal{D}_f}\,\mathbf{y} \;\equiv\; \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} (\mathcal{D}_f)^n\,\mathbf{y} \;. \tag{5.83}$$

The operator $(\mathcal{D}_f)^n$ is understood to act as in (5.81), and we shall assume that everything is suitably differentiable/convergent for the right hand side of (5.83) to make sense. We claim that $\mathbf{Y}(\mathbf{y}, s)$ defines a one-parameter family of canonical transformations. To see this, note first that clearly $\mathbf{Y}(\mathbf{y}, 0) = \mathbf{y}$, giving the identity transformation on the coordinates $\mathbf{y}$. We next compute

$$\frac{\partial}{\partial s}\mathbf{Y}(\mathbf{y}, s) \;=\; -\sum_{n=1}^{\infty} \frac{(-s)^{n-1}}{(n-1)!} (\mathcal{D}_f)^n\,\mathbf{y} \;=\; -\mathcal{D}_f\,\mathbf{Y}(\mathbf{y}, s) \;=\; \{\mathbf{Y}(\mathbf{y}, s), f(\mathbf{y})\} \;. \tag{5.84}$$

The last equality here is simply (5.78). Writing $\mathbf{Y} = \mathbf{Y}(\mathbf{y}, s)$ we may hence compute

$$\begin{aligned} \frac{\partial}{\partial s}\{Y_\alpha, Y_\beta\} \;&=\; \left\{\frac{\partial}{\partial s} Y_\alpha, Y_\beta\right\} + \left\{Y_\alpha, \frac{\partial}{\partial s} Y_\beta\right\} \\ &=\; \{\{Y_\alpha, f\}, Y_\beta\} + \{Y_\alpha, \{Y_\beta, f\}\} \\ &=\; \{\{Y_\alpha, Y_\beta\}, f\} \;. \end{aligned} \tag{5.85}$$

Here we have used (5.84) in the second equality, and the Jacobi identity in the last equality. Notice that this equation takes the same form as (5.84). One can at this point turn things around, and regard (5.84) as a first order differential equation in s, whose solution is (5.83). We thus deduce from (5.85) that

$$\{Y_\alpha(\mathbf{y}, s), Y_\beta(\mathbf{y}, s)\} \;=\; \mathrm{e}^{-s\mathcal{D}_f}\{Y_\alpha(\mathbf{y}, 0), Y_\beta(\mathbf{y}, 0)\} \;=\; \mathrm{e}^{-s\mathcal{D}_f}\,\Omega_{\alpha\beta} \;=\; \Omega_{\alpha\beta} \;. \tag{5.86}$$

Here the second equality follows from $\mathbf{Y}(\mathbf{y}, 0) = \mathbf{y}$ and the canonical Poisson brackets, while the last equality follows since $\Omega_{\alpha\beta}$ is *constant* – hence $\mathcal{D}_f\,\Omega_{\alpha\beta} = 0$. Of course (5.86) is precisely what we wanted to prove: $\mathbf{Y}(\mathbf{y}, s)$ defines a canonical transformation for all $s \in \mathbb{R}$.

It follows that any function $f = f(\mathbf{y})$ on phase space generates an associated one-parameter family of canonical transformations. The simplest example would be to choose f constant, but clearly then $\mathcal{D}_f = 0$ and $\mathbf{Y}(\mathbf{y}, s) = \mathbf{y}$ for all s (a trivial one-parameter family). The next simplest

example is to consider f linear. Let's focus on $n = 1$ degree of freedom, and rewrite the Hamiltonian vector field in terms of the canonically conjugate q and p:

$$\mathcal{D}_f \;=\; \frac{\partial f}{\partial q}\frac{\partial}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial}{\partial q} \;. \tag{5.87}$$

Example: Let $f(q,p) = \lambda p - \mu q$, with λ, μ constant. Then we compute $\mathcal{D}_f = -\lambda\partial_q - \mu\partial_p$, and hence

$$\begin{aligned} Q(q,p,s) \;&\equiv\; \mathrm{e}^{-s\mathcal{D}_f}\, q \;=\; q + \lambda s \;, \\ P(q,p,s) \;&\equiv\; \mathrm{e}^{-s\mathcal{D}_f}\, p \;=\; p + \mu s \;. \end{aligned} \tag{5.88}$$

Notice here that the power series terminate after the first two terms. The resulting linear shifts in coordinates are clearly canonical transformations, albeit not very interesting ones.

Another interesting application of this formalism is to Hamilton's equations (5.29). These may be written

$$\dot{\mathbf{y}} \;=\; \{\mathbf{y}, H\} \;=\; -\mathcal{D}_H\,\mathbf{y} \;. \tag{5.89}$$

Here the function $f = H = H(\mathbf{y})$ is the Hamiltonian, which we assume here doesn't depend on time t. But (5.89) is just equation (5.84), with the variable s replaced by time t. We thus deduce that

$$\mathbf{y}(t) \;=\; \mathrm{e}^{-t\mathcal{D}_H}\,\mathbf{y} \tag{5.90}$$

solves Hamilton's equations, with initial condition $\mathbf{y}(0) = \mathbf{y}$. In this sense, the Hamiltonian vector field $\mathcal{D}_H$ generates the solution to Hamilton's equations. An immediate corollary of our results is that Hamiltonian flow, where we regard $\mathbf{y} = \mathbf{y}(0) \to \mathbf{y}(t)$ as a one-parameter family of coordinate transformations on phase space parametrized by time t, is in fact a family of *canonical transformations*!

Example: As a more interesting example, consider the one-dimensional harmonic oscillator with Hamiltonian $H(q,p) = \frac{1}{2}(p^2 + q^2)$. The Hamiltonian vector field is $\mathcal{D}_H = q\partial_p - p\partial_q$, and one computes

$$\begin{aligned} Q(q,p,t) \;&\equiv\; \mathrm{e}^{-t\mathcal{D}_H}\, q \;=\; q\cos t + p\sin t \;, \\ P(q,p,t) \;&\equiv\; \mathrm{e}^{-t\mathcal{D}_H}\, p \;=\; p\cos t - q\sin t \;. \end{aligned} \tag{5.91}$$

Of course these indeed solve Hamilton's equations $\partial_t Q = P$, $\partial_t P = -Q$, with initial condition $Q(q,p,0) = q$, $P(q,p,0) = p$.

The principle of least action

A different approach, that also involves a notion of generating functions, arises naturally if we think about the principle of least action in the Hamiltonian formalism. Recall that the action (2.6) is

$$S = \int_{t_1}^{t_2} L(\mathbf{q},\dot{\mathbf{q}},t)\mathrm{d}t = \int_{t_1}^{t_2}\left(\sum_{a=1}^{n} p_a\dot{q}_a - H\right)\mathrm{d}t = \int_{t_1}^{t_2}\left(\sum_{a=1}^{n} p_a\mathrm{d}q_a - H\mathrm{d}t\right) , \quad (5.92)$$

where we have written the Lagrangian in terms of the Hamiltonian. In fact Hamilton's equations follow from extremizing (5.92) with respect to both $\mathbf{q}$ *and* $\mathbf{p}$, *i.e.* we treat them as independent variables, with $S = S[\mathbf{q}(t),\mathbf{p}(t)]$. Indeed, writing $\delta\mathbf{q}(t) = \epsilon\,\mathbf{u}(t)$, $\delta\mathbf{p}(t) = \epsilon\,\mathbf{w}(t)$, as in our derivation of Lagrange's equations we compute the first order variation

$$\begin{aligned}\delta S &= \epsilon\sum_{a=1}^{n}\int_{t_1}^{t_2}\left(w_a\mathrm{d}q_a + p_a\mathrm{d}u_a - \frac{\partial H}{\partial q_a}u_a\mathrm{d}t - \frac{\partial H}{\partial p_a}w_a\mathrm{d}t\right)\\ &= \epsilon\sum_{a=1}^{n}\int_{t_1}^{t_2}\left[w_a\left(\mathrm{d}q_a - \frac{\partial H}{\partial p_a}\mathrm{d}t\right) - u_a\left(\mathrm{d}p_a + \frac{\partial H}{\partial q_a}\mathrm{d}t\right)\right] . \end{aligned} \quad (5.93)$$

Here we have integrated by parts the second term on the right hand side of the first line, and used the boundary condition $\mathbf{u}(t_1) = \mathbf{u}(t_2) = \mathbf{0}$. Hamilton's equations hence follow, much as we deduced Lagrange's equations.

Also as in our discussion of Lagrange's equations (see around equation (2.17)), adding a total time derivative to the integrand in (5.92) does not affect the equations of motion, as such a term depends only on the boundary data and so has zero variation. If we have two coordinate systems $(\mathbf{q},\mathbf{p})$ and $(\mathbf{Q},\mathbf{P})$ on phase space, with corresponding Hamiltonian's H and K, respectively, then our above discussion means that these will lead to the *same* equations of motion if

$$\sum_{a=1}^{n} P_a\mathrm{d}Q_a - K\mathrm{d}t = \sum_{a=1}^{n} p_a\mathrm{d}q_a - H\mathrm{d}t - \mathrm{d}F_1(\mathbf{q},\mathbf{Q},t) . \quad (5.94)$$

The function $F_1(\mathbf{q},\mathbf{Q},t)$ is called a *generating function of the first kind*. Writing out

$$\mathrm{d}F_1(\mathbf{q},\mathbf{Q},t) = \sum_{a=1}^{n}\frac{\partial F_1}{\partial q_a}\mathrm{d}q_a + \sum_{a=1}^{n}\frac{\partial F_1}{\partial Q_a}\mathrm{d}Q_a + \frac{\partial F_1}{\partial t}\mathrm{d}t , \quad (5.95)$$

equation (5.94) implies the relations

$$\mathbf{p} = \frac{\partial F_1}{\partial\mathbf{q}} , \qquad \mathbf{P} = -\frac{\partial F_1}{\partial\mathbf{Q}} , \qquad K = H + \frac{\partial F_1}{\partial t} . \quad (5.96)$$

This then effectively generates a change of coordinates. One must be very careful to read the dependences correctly in these equations. The first equation in (5.96) determines $\mathbf{p}$ as a function of $(\mathbf{q},\mathbf{Q},t)$. One then inverts this (assuming it is invertible) to find $\mathbf{Q}$ as a function of $(\mathbf{q},\mathbf{p},t)$. Similarly, the second equation in (5.96) determines $\mathbf{P}$ as a function of $(\mathbf{q},\mathbf{Q},t)$, which we may then in turn convert to a function of $(\mathbf{q},\mathbf{p},t)$. Notice that the final equation in (5.96) relates

the two Hamiltonians, and may be compared to equation (5.71) – although in the latter equation everything was a function of the new variables $(\mathbf{Q}, \mathbf{P})$, and we were careful to distinguish $H(\mathbf{q}, \mathbf{p}, t)$ from $\tilde{H}(\mathbf{Q}, \mathbf{P}, t)$. In this section we are being less precise, using the same name for a function and the function composed with a change of variable. This makes the notation cleaner and easier to read, but one must then be particularly careful when applying the chain rule. Notice that if the generating function is independent of time, so $\partial F_1/\partial t = 0$, then $K = H$ and the new Hamiltonian is obtained by simply substituting for $\mathbf{q}, \mathbf{p}$ in H their values in terms of the new variables $\mathbf{Q}, \mathbf{P}$ (so that more precisely $K = \tilde{H}$, as in (5.69)).

By construction, any two Hamiltonian systems related by the change of variables (5.96) will lead to the same Hamilton equations of motion, and be a canonical transformation. One can verify this directly by a brute force computation of Poisson brackets, being *very* careful with the chain rule, although this is rather long and not very enlightening. A more conceptually interesting way to think about the invariance of the Poisson brackets $\{f, g\}_{p,q} = \{f, g\}_{P,Q}$ is as follows. Since time t simply appears as a parameter in the Poisson bracket computation, we might as well consider only time-independent functions. We may then regard the arbitrary function g appearing in $\{f, g\}$ as the Hamiltonian of a ficticious dynamical system, so that via Hamilton's equations for this system $\{f, g\} = \frac{\mathrm{d}f}{\mathrm{d}t}$. Since Hamilton's equations in the two coordinate systems are the same, the time-evolution of the function f must be coordinate-independent, and we deduce that $\{f, g\}_{p,q} = \{f, g\}_{P,Q}$.

Example: The generating function $F_1 = \sum_{a=1}^{n} q_a Q_a$ generates the position and momentum swap $\mathbf{Q} = \mathbf{p}$, $\mathbf{P} = -\mathbf{q}$.

In (5.94) the generating function $F_1 = F_1(\mathbf{q}, \mathbf{Q}, t)$ is a function of the old and new generalized coordinates. However, it might be more convenient to express the generating function in terms of the old coordinate $\mathbf{q}$ and new *momenta* $\mathbf{P}$. The appropriate formulae are obtained via a Legendre transform. That is, we rewrite (5.94) as

$$\mathrm{d}F_1 \;=\; \sum_{a=1}^{n} p_a \mathrm{d}q_a - \sum_{a=1}^{n} P_a \mathrm{d}Q_a + (K - H)\mathrm{d}t \;, \tag{5.97}$$

which in turn may be written

$$\mathrm{d}(F_1 + \sum_{a=1}^{n} P_a Q_a) \;=\; \sum_{a=1}^{n} p_a \mathrm{d}q_a + \sum_{a=1}^{n} Q_a \mathrm{d}P_a + (K - H)\mathrm{d}t \;, \tag{5.98}$$

The argument of the differential on the left hand side may then be regarded as $F_2 = F_2(\mathbf{q}, \mathbf{P}, t) = F_1 + \sum_{a=1}^{n} P_a Q_a$. Up to an overall sign, this is a Legendre transform of $F_1 = F_1(\mathbf{q}, \mathbf{Q}, t)$ with respect to $\mathbf{Q}$, since $\partial F_1/\partial \mathbf{Q} = -\mathbf{P}$. From (5.98) we obtain the Legendre-transformed relations

$$\mathbf{p} \;=\; \frac{\partial F_2}{\partial \mathbf{q}} \;, \qquad \mathbf{Q} \;=\; \frac{\partial F_2}{\partial \mathbf{P}} \;, \qquad K \;=\; H + \frac{\partial F_2}{\partial t}. \tag{5.99}$$

We may similarly obtain formulae for generating functions which depend on $\mathbf{p}$ and $\mathbf{Q}$, or $\mathbf{p}$ and $\mathbf{P}$, respectively, which are generating functions of the third and fourth kind, respectively.

Example: As a less trivial example, consider the generating function

$$F_2(q,P) \;=\; \int^q \sqrt{2P-x^2}\,\mathrm{d}x \;. \tag{5.100}$$

It follows from (5.99) that

$$p \;=\; \frac{\partial F_2}{\partial q} \;=\; \sqrt{2P-q^2}\;, \qquad Q \;=\; \frac{\partial F_2}{\partial P} \;=\; \arctan\frac{q}{\sqrt{2P-q^2}}\;. \tag{5.101}$$

The coordinate transformation is obtained by inverting these appropriately. For example, we may write q,p in terms of Q,P as

$$q \;=\; \sqrt{2P}\sin Q\;, \qquad p \;=\; \sqrt{2P}\cos Q\;. \tag{5.102}$$

The new variables Q,P are essentially polar coordinates on the phase space $\mathscr{P}=\mathbb{R}^2$. Since F_2 is time-independent, the new Hamiltonian is simply $K=H$, written in terms of the new variables. For example, the harmonic oscillator Hamiltonian $H(q,p)=\frac{1}{2}(p^2+q^2)$ becomes simply $H(Q,P)=P$ in the new variables! In particular Q is an ignorable coordinate, leading to $\dot{P}=0$, while $\dot{Q}=\partial H/\partial P=1$. Of course this leads to the usual trigonometric solutions $q=\sqrt{2P}\sin(t-t_0)$ in the original variables, but the point is that the transformation essentially trivialises the Hamiltonian system.

5.6 Liouville's theorem

The formalism and results we've developed allow us to now prove easily the following result. Consider a region $V\subseteq\mathscr{P}$ in phase space. Its volume is

$$\mathrm{vol}(V) \;=\; \int_V \mathrm{d}q_1\cdots\mathrm{d}q_1\mathrm{d}p_1\cdots\mathrm{d}p_n\;. \tag{5.103}$$

Now consider a map from phase space to itself which in coordinates is described by $\mathbf{q}\to\mathbf{Q}=\mathbf{Q}(\mathbf{q},\mathbf{p},t)$, $\mathbf{p}\to\mathbf{P}=\mathbf{P}(\mathbf{q},\mathbf{p},t)$. Under this map the region V transforms to another region $\tilde{V}$ with volume

$$\mathrm{vol}(\tilde{V}) \;=\; \int_{\tilde{V}} \mathrm{d}Q_1\cdots\mathrm{d}Q_n\mathrm{d}P_1\cdots\mathrm{d}P_n\;. \tag{5.104}$$

If the map is a *canonical transformation* then the volume is *preserved*, *i.e.* $\mathrm{vol}(V)=\mathrm{vol}(\tilde{V})$. This is straightforward to see using the symplectic property of canonical transformations. Writing $(y_1,\ldots,y_{2n})=(q_1,\ldots,q_n,p_1,\ldots,p_n)$, $(Y_1,\ldots,Y_{2n})=(Q_1,\ldots,Q_n,P_1,\ldots,P_n)$, the transformed volume integral may be written as

$$\mathrm{vol}(\tilde{V}) \;=\; \int_{\tilde{V}} \mathrm{d}Y_1\cdots\mathrm{d}Y_{2n} \;=\; \int_V |\det\mathcal{J}|\,\mathrm{d}y_1\cdots\mathrm{d}y_{2n}\;, \tag{5.105}$$

where $\mathcal{J}_{\alpha\beta} = \partial Y_\alpha / \partial y_\beta$ is the Jacobian matrix. But the symplectic condition (5.54) immediately implies that $(\det \mathcal{J})^2 = 1$, and thus $|\det \mathcal{J}| = 1$. We thus conclude that $\mathrm{vol}(\tilde{V}) = \mathrm{vol}(V)$.

Recall that we may regard Hamiltonian flow as a one-parameter family of maps from phase space to itself, parametrized by time t. Thus $(\mathbf{q}, \mathbf{p}) = (\mathbf{q}(0), \mathbf{p}(0)) \to (\mathbf{Q}, \mathbf{P}) = (\mathbf{q}(t), \mathbf{p}(t))$. If one chooses a region $V = V(0) \subseteq \mathscr{P}$ at time $t = 0$ then under Hamiltonian flow this region will change shape to $V(t) \subseteq \mathscr{P}$. Since we showed in the last subsection that Hamiltonian flow is a canonical transformation for all t, we have thus proven

Liouville's Theorem: The volume of any region in phase space is invariant under Hamiltonian evolution.

In other words, $\frac{\mathrm{d}}{\mathrm{d}t} V = 0$.

The physical applications of this result take us in the direction of statistical physics. So far we've been describing the classical mechanics of a single particle, or more precisely the single "quasi-particle" described at the end of section 2.1. This traces out a single curve in phase space under Hamiltonian evolution. But we might instead want to think about a system with many such particles, each of which is described by the same Hamiltonian – for example, a gas. One is then not interested in keeping track of the individual behaviours of the particles, but rather wants to understand the average behaviour. In this situation we may introduce a *density function* $\rho(\mathbf{q}, \mathbf{p}, t)$ on $\mathscr{P} \times \mathbb{R}$, where the number of particles in a small region of phase space δV centred at $(\mathbf{q}, \mathbf{p}) \in \mathscr{P}$ at time t is $\delta N = \rho(\mathbf{q}, \mathbf{p}, t)\, \delta V$. The total number of particles is

$$N \;=\; \int_{\mathscr{P}} \rho(\mathbf{q}, \mathbf{p}, t)\, \mathrm{d}q_1 \cdots \mathrm{d}q_n \mathrm{d}p_1 \cdots \mathrm{d}p_n \;, \tag{5.106}$$

which is fixed, *i.e.* we assume that particles are not created or destroyed. In fact the latter is a *local* property. In particular if we follow the Hamiltonian evolution of a small number of particles δN centred at $(\mathbf{q}, \mathbf{p}) = (\mathbf{q}(0), \mathbf{p}(0))$ at time $t = 0$, then this will evolve to $\delta N = \rho(\mathbf{q}(t), \mathbf{p}(t), t)\, \delta V(t)$ at a later time t. Since particles are not created or destroyed, δN must be independent of time t. But we've also shown that $\delta V(t) = \delta V(0)$ is also independent of time. We thus conclude that

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} \;=\; 0 \;, \tag{5.107}$$

holds under Hamiltonian evolution. Using (5.28) we may rewrite this as

$$\{\rho, H\} + \frac{\partial \rho}{\partial t} \;=\; 0 \;, \tag{5.108}$$

which is often referred to as *Liouville's equation*. Equation (5.107) says that the density of states is constant along every trajectory in phase space.

5.7 The Hamilton-Jacobi equation

The Hamilton-Jacobi equation provides yet another approach to classical mechanics. Again, it has its own particular advantages – for example, this equation can be useful for finding conserved

quantities. On a more conceptual level, the Hamilton-Jacobi equation provides a particularly direct link with quantum mechanics and the Schrödinger equation.

We begin by reconsidering the action S, which we discussed briefly in the Hamiltonian formalism in section 5.5, equation (5.92). Recall that the action is extremized over all paths $\mathbf{q}(t)$ with fixed boundary conditions to find the equations of motion. More precisely, let's take the fixed boundary conditions for the paths to be $\mathbf{q}(t_0) = \mathbf{q}^{(0)}$ at time $t = t_0$, and $\mathbf{q}(t_f) = \mathbf{q}^f$ at time $t = t_f$. The action for any path $\mathbf{q}(t)$ satisfying these boundary conditions is

$$S[\mathbf{q}(t)] \;=\; \int_{t_0}^{t_f} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\, \mathrm{d}t \;, \tag{5.109}$$

and the classical path taken by the system extremizes this, resulting in a (we assume unique) solution $\mathbf{q}(t) = \mathbf{q}^{\text{classical}}(t)$. We now take this classical solution and substitute it back into the action (5.109). Of course doing this we get a number, but that number depends on the fixed final time t_f and fixed final point $\mathbf{q}^f$. We may then consider varying these final boundary conditions, which defines a function

$$\mathcal{S}(\mathbf{q}^f, t_f) \;=\; S[\mathbf{q}^{\text{classical}}(t)] \;. \tag{5.110}$$

Of course this also depends on the initial time t_0 and initial point $\mathbf{q}^{(0)}$, but we keep these absolutely fixed. Physicists usually refer to the action evaluated on the classical solution as the *on-shell action* (with the words "on-shell" really referring to a notion in special relavity, so this is another physics misnomer).

Now under *any* variation $\delta\mathbf{q}(t)$ of a path we have

$$\delta S \;=\; \sum_{a=1}^{n} \left\{ \int_{t_0}^{t_f} \left[\frac{\partial L}{\partial q_a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_a} \right) \right] \delta q_a(t)\, \mathrm{d}t + \left[\frac{\partial L}{\partial \dot{q}_a} \delta q_a(t) \right]_{t_0}^{t_f} \right\} . \tag{5.111}$$

If we vary the path so that the initial conditions are fixed, meaning $\delta\mathbf{q}(t_0) = \mathbf{0}$, but at the final point $\delta\mathbf{q}(t_f) = \delta\mathbf{q}^f$, then when (5.111) is evaluated on the classical path solving the Lagrange equations of motion with these boundary conditions we have simply

$$\delta S \;=\; \sum_{a=1}^{n} \left. \frac{\partial L}{\partial \dot{q}_a} \right|_{t=t_f} \delta q_a^f \;. \tag{5.112}$$

In terms of (5.110) this means that

$$\frac{\partial \mathcal{S}}{\partial \mathbf{q}^f} \;=\; \mathbf{p}^f \;, \tag{5.113}$$

where $\mathbf{p}^f = \partial L / \partial \dot{\mathbf{q}} \,|_{t=t_f}$ is the final momentum of the classical path. We may similarly consider varying the final time t_f. From the fundamental theorem of calculus

$$\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}t_f} \;=\; L \;=\; L(\mathbf{q}^{\text{classical}}(t_f), \dot{\mathbf{q}}^{\text{classical}}(t_f), t_f) \;=\; L(\mathbf{q}^f, \dot{\mathbf{q}}^f, t_f) \;, \tag{5.114}$$

where $\dot{\mathbf{q}}^f$ is the final velocity. On the other hand via the chain rule

$$\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}t_f} \;=\; \frac{\partial S}{\partial t_f} + \sum_{a=1}^{n} \frac{\partial \mathcal{S}}{\partial q_a^f}\dot{q}_a^f \;=\; \frac{\partial \mathcal{S}}{\partial t} + \sum_{a=1}^{n} p_a^f \dot{q}_a^f \;. \tag{5.115}$$

Combining these last two equations we find

$$\frac{\partial \mathcal{S}}{\partial t_f} \;=\; L(\mathbf{q}^f, \dot{\mathbf{q}}^f, t_f) - \sum_{a=1}^{n} p_a^f \dot{q}_a^f \;=\; -H(\mathbf{q}^f, \mathbf{p}^f, t_f) \;, \tag{5.116}$$

where $H(\mathbf{q}^f, \mathbf{p}^f, t_f)$ is the Hamiltonian evaluated on the final data.

Now that we have finished varying paths, we relabel the final data as $\mathbf{q}^f \to \mathbf{q}$, $t_f \to t$, so that (5.110) defines a function $\mathcal{S}(\mathbf{q}, t)$ on $\mathcal{Q} \times \mathbb{R}$. It is called *Hamilton's principal function*, and we've just shown that it satisfies

$$\frac{\partial \mathcal{S}}{\partial t} + H(\mathbf{q}, \mathbf{p}, t) \;=\; 0 \;, \qquad \text{where} \qquad \mathbf{p} \;=\; \frac{\partial \mathcal{S}}{\partial \mathbf{q}} \;. \tag{5.117}$$

In other words $\mathcal{S}(\mathbf{q}, t)$ satisfies the first order partial differential equation

$$\frac{\partial \mathcal{S}}{\partial t} + H\left(q_1, \ldots, q_n, \frac{\partial \mathcal{S}}{\partial q_1}, \ldots, \frac{\partial \mathcal{S}}{\partial q_n}, t\right) \;=\; 0 \;. \tag{5.118}$$

This is the *Hamilton-Jacobi equation*.

Since we have a single first order PDE in $n+1$ independent variables, general PDE theory says we should have $n+1$ constants of integration in the general solution (called a *complete integral*). Notice that one of these constants of integration is easy to identify: $\mathcal{S}$ enters (5.118) only through its partial derivatives, so if $\mathcal{S}(\mathbf{q}, t)$ is a solution then so is $\mathcal{S}(\mathbf{q}, t) + c$ for any constant c. We then write the general solution to (5.118) as

$$\mathcal{S} \;=\; \mathcal{S}_0(q_1, \ldots, q_n, t; \alpha_1, \ldots, \alpha_n) + A \;, \tag{5.119}$$

where $\alpha_1, \ldots, \alpha_n$ and A are the $n+1$ integration constants.

The equations (5.117) may look familiar from our discussion of generating functions in section 5.5. Indeed, consider a generating function of the second kind $F_2 = F_2(\mathbf{q}, \mathbf{P}, t)$, where $(\mathbf{q}, \mathbf{p})$ and $(\mathbf{Q}, \mathbf{P})$ are the old and new coordinates, respectively. As discussed around equation (5.99), this generates the canonical transformation

$$\mathbf{p} \;=\; \frac{\partial F_2}{\partial \mathbf{q}} \;, \qquad \mathbf{Q} \;=\; \frac{\partial F_2}{\partial \mathbf{P}} \;, \qquad K \;=\; H + \frac{\partial F_2}{\partial t}. \tag{5.120}$$

In particular, if we choose our generating function F_2 so that the new Hamiltonian $K = 0$ is identically zero, then in the new coordinates the dynamics is trivial: $\dot{\mathbf{P}} = \mathbf{0} = \dot{\mathbf{Q}}$. The transformed position and momenta are then conserved quantities, and we may write $\mathbf{Q} = \boldsymbol{\beta}$, and $\mathbf{P} = \boldsymbol{\alpha}$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ are altogether $2n$ constants. In particular notice that $F_2 = F_2(\mathbf{q}, \boldsymbol{\alpha}, t)$. Denoting this generating function by $F_2 = \mathcal{S}$, we see that (5.120) are simply the equations (5.117), while the generating function F_2 itself is simply Hamilton's principal function,

with the conserved momenta $\mathbf{P} = \boldsymbol{\alpha}$ in the new coordinates being interpreted as the integration constants in the general solution (5.119). Moreover, the second equation in (5.120) tells us that

$$\mathbf{Q} \;=\; \boldsymbol{\beta} \;=\; \frac{\partial \mathcal{S}}{\partial \boldsymbol{\alpha}} \tag{5.121}$$

are also conserved quantities.

What we've done might seem miraculous, so it's perhaps worth taking stock. Given a Hamiltonian function we construct the Hamilton-Jacobi equation (5.118), which is a first order PDE for $\mathcal{S}(\mathbf{q}, t)$ with general solution (5.119) depending on the $n+1$ integration constants $\alpha_1, \ldots, \alpha_n$ and A. We then impose the n algebraic equations (5.121), where $\boldsymbol{\beta}$ are constant. Having done this we may interpret the general solution (5.119) to the Hamilton-Jacobi equation as a generating function of the second kind. This generates a canonical transformation which trivialises the Hamiltonian $K = 0$ in the new coordinates, and where $\boldsymbol{\alpha} = \mathbf{P}$ are interpreted as the conserved momenta in the new coordinates. The constants $\boldsymbol{\beta} = \mathbf{Q}$ are correspondingly the conserved positions in the new coordinates. Since we've solved Hamilton's equations in the new coordinates, via the canonical transformation generated by $\mathcal{S}$ we have also solved Hamilton's equations in the original coordinates. Notice we have $2n$ integration constants in total, as expected.

We can also think of this slightly differently. If one manages to solve the Hamilton-Jacobi equation to find the complete integral (5.119), essentially one has solved half of the dynamical problem. Specifically, this determines the momenta

$$\mathbf{p} \;=\; \frac{\partial \mathcal{S}}{\partial \mathbf{q}} \tag{5.122}$$

as a function of $\mathbf{q}$, t and the integration constants $\boldsymbol{\alpha}$. One can then substitute this into the remaining n Hamilton equations

$$\dot{\mathbf{q}} \;=\; \left.\frac{\partial H}{\partial \mathbf{p}}\right|_{\mathbf{p}=\partial\mathcal{S}/\partial\mathbf{q}} \,. \tag{5.123}$$

These are now n *first order* differential equations for $\mathbf{q}(t)$, giving rise to another n integration constants (again, $2n$ in total). In this approach it's less clear that we have in fact solved Hamilton's equations. But this is easily checked from (5.122):

$$\dot{p}_a \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{S}}{\partial q_a}\right) \;=\; \frac{\partial^2 \mathcal{S}}{\partial t \partial q_a} + \sum_{b=1}^{n} \frac{\partial^2 \mathcal{S}}{\partial q_a \partial q_b}\dot{q}_b \,. \tag{5.124}$$

On the other hand taking the partial derivative of the Hamilton-Jacobi equation (5.117) with respect to q_a gives

$$\frac{\partial^2 \mathcal{S}}{\partial t \partial q_a} \;=\; -\frac{\partial H}{\partial q_a} - \sum_{b=1}^{n} \frac{\partial H}{\partial p_b}\frac{\partial^2 \mathcal{S}}{\partial q_a \partial q_b} \;=\; -\frac{\partial H}{\partial q_a} - \sum_{b=1}^{n} \frac{\partial^2 \mathcal{S}}{\partial q_a \partial q_b}\dot{q}_b \,, \tag{5.125}$$

where in the first equality we used (5.122), and in the second equality we used (5.123). Substituting this into (5.124) we thus find

$$\dot{\mathbf{p}} \;=\; -\frac{\partial H}{\partial \mathbf{q}} \,. \tag{5.126}$$

When the Hamiltonian is independent of time, so $\partial H/\partial t = 0$, one can solve the time-dependence of the Hamilton-Jacobi equation via

$$\mathcal{S}(\mathbf{q}, t) \quad = \quad W(\mathbf{q}) - Et \ , \tag{5.127}$$

where E is constant and the time-independent function W on configuration space $\mathcal{Q}$ is sometimes called *Hamilton's characteristic function*. This leads to a reduced Hamilton-Jacobi equation

$$H\left(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}\right) \quad = \quad E \ , \tag{5.128}$$

where E is then interpreted as the energy.

Separation of variables

Of course for this approach to be of any practical use, one needs to be able to solve the Hamilton-Jabobi equation (5.118), or in the time-independent setting the slightly easier (5.128). Either way, one has to solve a first order PDE. In general one could proceed via the method of characteristics, but this precisely converts the PDE back into ODEs, *i.e.* effectively Hamilton's equations! However, in certain cases we may solve the PDE via separation of variables. Of course whether or not there are separable solutions will depend both on the Hamiltonian and on the particular coordinates being used: a Hamiltonian may separate in one coordinate system, but not in another. There's a whole theory here, but we'll content ourselves with looking at some examples.

Suppose that the coordinate q_1 and derivative term $\partial_{q_1}\mathcal{S}$ appear in the Hamilton-Jacobi equation only through the combination $f(q_1, \partial_{q_1}\mathcal{S})$, for some function f. That is, the equation can be written in the form

$$\Phi\left[f(q_1, \partial_{q_1}\mathcal{S}), q_2, \ldots, q_n, \partial_{q_2}\mathcal{S}, \ldots, \partial_{q_n}\mathcal{S}, t\right] \quad = \quad 0 \ . \tag{5.129}$$

When then seek a separable solution

$$\mathcal{S}(q_1, \ldots, q_n, t) \quad = \quad \tilde{\mathcal{S}}(q_2, \ldots, q_n, t) + \mathcal{S}_1(q_1) \ . \tag{5.130}$$

With this ansatz the Hamilton-Jacobi equation (5.129) becomes

$$\Phi\left[f\left(q_1, \frac{\mathrm{d}\mathcal{S}_1}{\mathrm{d}q_1}\right), q_2, \ldots, q_n, \partial_{q_2}\tilde{\mathcal{S}}, \ldots, \partial_{q_n}\tilde{\mathcal{S}}, t\right] \quad = \quad 0 \ . \tag{5.131}$$

This must hold for all q_1, so taking the partial derivative of both sides with respect to q_1 gives

$$\partial_1\Phi \cdot \frac{\mathrm{d}}{\mathrm{d}q_1} f\left(q_1, \frac{\mathrm{d}\mathcal{S}_1}{\mathrm{d}q_1}\right) \quad = \quad 0 \ , \tag{5.132}$$

where $\partial_1\Phi$ denotes the partial derivative of Φ with respect to its first entry. Assuming that q_1 or $p_1 = \mathrm{d}\mathcal{S}_1/\mathrm{d}q_1$ actually appear in the Hamilton-Jacobi equation, then $\partial_1\Phi$ is non-zero, and hence we deduce

$$f\left(q_1, \frac{\mathrm{d}\mathcal{S}_1}{\mathrm{d}q_1}\right) \quad = \quad \alpha_1 \ , \tag{5.133}$$

where α_1 is constant. This is a first order ODE for $\mathcal{S}_1(q_1)$, which may be solved, and substituting $f = \alpha_1$ back into (5.118) the remaining Hamilton-Jacobi equation has one fewer independent variables.

In some cases one can successively solve for and eliminate all the variables in this way. In particular in the time-independent case (5.128) this means one can solve the Hamilton-Jacobi equation with the ansatz

$$\mathcal{S}(q_1, \ldots, q_n, t) \;=\; \sum_{a=1}^{n} \mathcal{S}_a(q_a) - Et \;, \tag{5.134}$$

where each $\mathcal{S}_a = \mathcal{S}_a(q_a)$ is a function of one variable only, leading to n first order ODEs in one variable. The Hamilton-Jacobi equation is then said to be *completely separable*. This results in n integration constants $\alpha_1, \ldots, \alpha_n$, as in (5.119). Notice that $E = E(\alpha_1, \ldots, \alpha_n)$ is not an independent integration constant, but rather is determined by substituting the solution $W = \sum_{a=1}^{n} \mathcal{S}_a(q_a)$ into (5.128).

A special case of the above discussion is an ignorable coordinate, *i.e.* q_1 doesn't appear in the Hamiltonian, $\partial H/\partial q_1 = 0$, and thus doesn't appear in the Hamilton-Jacobi equation either. This means that $f = f(\partial\mathcal{S}/\partial q_1)$ above, and (5.133) integrates to $\mathcal{S}_1(q_1) = \alpha_1 q_1$, giving

$$\mathcal{S}(q_1, \ldots, q_n, t) \;=\; \tilde{\mathcal{S}}(q_2, \ldots, q_n, t) + \alpha_1 q_1 \;. \tag{5.135}$$

Notice then that $p_1 = \partial\mathcal{S}/\partial q_1 = \alpha_1$ is simply the corresponding conserved momenta. Similarly, the $-Et$ term in (5.134) for a time-independent system corresponds to separation of the "ignorable coordinate" t.

Example: We consider the reduced two-body problem Lagrangian

$$L_{\text{reduced}} \;=\; \frac{1}{2}\mu(\dot{\varrho}^2 + \varrho^2\dot{\phi}^2) - V(\varrho) \;, \tag{5.136}$$

describing motion in the plane with polar coordinates (ϱ, ϕ). The Hamiltonian is

$$H \;=\; H(q_1, q_2, p_1, p_2) \;=\; \frac{1}{2\mu}\left(p_1^2 + \frac{p_2^2}{q_1^2}\right) + V(q_1) \;, \tag{5.137}$$

where $q_1 = \varrho$, $q_2 = \phi$. The Hamilton-Jacobi equation (5.118) hence reads

$$\frac{\partial\mathcal{S}}{\partial t} + \frac{1}{2\mu}\left(\frac{\partial\mathcal{S}}{\partial q_1}\right)^2 + \frac{1}{2\mu q_1^2}\left(\frac{\partial\mathcal{S}}{\partial q_2}\right)^2 + V(q_1) \;=\; 0 \;. \tag{5.138}$$

Since $\partial H/\partial t = 0$ and $\partial H/\partial q_2 = 0$ our general discussion above implies we can separate variables, seeking a solution

$$\mathcal{S}(q_1, q_2, t) \;=\; \mathcal{S}_1(q_1) + \mathcal{S}_2(q_2) - Et \;. \tag{5.139}$$

Substituting this into (5.138) and rearranging slightly gives

$$\left(\frac{\mathrm{d}\mathcal{S}_2}{\mathrm{d}q_2}\right)^2 \;=\; 2\mu q_1^2\left[E - V(q_1) - \frac{1}{2\mu}\left(\frac{\mathrm{d}\mathcal{S}_1}{\mathrm{d}q_1}\right)^2\right] \;. \tag{5.140}$$

Since the left hand side depends only on q_2, while the right hand side depends only on q_1, both sides must be constant and in particular we may integrate

$$\mathcal{S}_2(q_2) \;=\; \alpha_2 q_2 \;, \tag{5.141}$$

with α_2 constant (which is the conserved angular momentum), and (5.140) becomes the first order ODE

$$\frac{\mathrm{d}\mathcal{S}_1}{\mathrm{d}q_1} \;=\; \left(2\mu E - 2\mu V(q_1) - \frac{\alpha_2^2}{q_1^2}\right)^{1/2} \;\equiv\; F(q_1; E, \alpha_2) \;. \tag{5.142}$$

We have thus reduced the problem to quadratures, with the final solution to the Hamilton-Jacobi equation being

$$\mathcal{S}(q_1, q_2, t) \;=\; -Et + \alpha_2 q_2 + \int^{q_1} F(x; E, \alpha_2)\,\mathrm{d}x \;. \tag{5.143}$$

There is a trivial additive integration constant from integrating the final ODE. The two non-trivial integration constants are $\alpha_1 = E$ and α_2.

Recall that to solve Hamilton's equations we must now impose (5.121). This leads to the equations

$$\beta_1 \;=\; \frac{\partial \mathcal{S}}{\partial E} \;=\; -t + \int^{q_1} \frac{\mu}{F(x; E, \alpha_2)}\,\mathrm{d}x \;, \tag{5.144}$$

$$\beta_2 \;=\; \frac{\partial \mathcal{S}}{\partial \alpha_2} \;=\; q_2 - \int^{q_1} \frac{\alpha_2}{x^2 F(x; E, \alpha_2)}\,\mathrm{d}x \;. \tag{5.145}$$

It may not look like it, but we have in fact solved the equations of motion. To see this in a more familiar guise, recall that in the original variables the energy E is (see equation (2.100))

$$E \;=\; \frac{1}{2}\mu\dot{\varrho}^2 + \frac{p_\phi^2}{2\mu\varrho^2} + V(\varrho) \;, \tag{5.146}$$

where $\alpha_2 = p_\phi = \mu\varrho^2\dot{\phi}$. The equations of motion may then be written

$$\dot{\varrho} \;=\; \frac{1}{\mu}F(\varrho) \;, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}r} \;=\; \frac{\alpha_2}{\varrho^2 F(\varrho)} \;, \tag{5.147}$$

which integrate to (5.144) and (5.145).

5.8 * Quantum mechanics

We end these lectures with a few further remarks about the relationship between classical mechanics and quantum mechanics. We already briefly described how the principle of least action enters the Feynman formulation of quantum mechanics via the path integral (an integral over all paths) in section 2.6.

Those who have studied quantum mechanics will certainly have recognized the Poisson bracket structure we encountered in section 5.3. This similarity was first noticed by Dirac, who postulated

that a classical system may be "quantized" by taking the classical Poisson brackets and mapping them to commutator brackets of operators

$$\{f, g\} \quad \longrightarrow \quad -\frac{\mathrm{i}}{\hbar}[\hat{f}, \hat{g}] \ . \tag{5.148}$$

The time evolution then maps as

$$\frac{\mathrm{d}f}{\mathrm{d}t} \ = \ \{f, H\} \quad \longrightarrow \quad \mathrm{i}\hbar\frac{\mathrm{d}\hat{f}}{\mathrm{d}t} \ = \ [\hat{f}, \hat{H}] \ , \tag{5.149}$$

which is the equation for the time evolution of an operator in the Heisenberg picture. Of course for all this to make sense to each function f on classical phase space we must associate an operator $\hat{f}$ acting on an appropriate Hilbert space $\mathcal{H}$ of wave functions, so that (5.148) holds. There is a beautiful way to do this, under appropriate conditions, called *geometric quantization*. This mathematically constructs the Hilbert space directly from the phase space, and provides an explicit formula for mapping an appropriate class of classical observables f on phase space to operators $\hat{f}$ acting on $\mathcal{H}$. The geometric quantization procedure is engineered so that (5.148) and (5.149) then hold for the operators $\hat{f}$. This is all wonderful – when the method can be applied. But there is no general mathematical procedure called "quantization" that works in all circumstances. On the other hand, from a physics point of view it's not clear why one would expect there to be.

While the Poisson bracket representation of Hamiltonian mechanics relates to the Heisenberg picture of quantum mechanics, the Hamilton-Jacobi equation relates to the Schrödinger equation. In fact this was one of Schrödinger's motivations in finding his equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} \ = \ -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \ , \tag{5.150}$$

describing a quantum particle of mass m moving under the influence of a potential V. In the WKB approximation one substitutes the ansatz

$$\psi(\mathbf{q}, t) \ = \ A(\mathbf{q}, t, \hbar)\, \exp\left[\frac{\mathrm{i}}{\hbar}\mathcal{S}(\mathbf{q}, t)\right] \ , \tag{5.151}$$

where A is the real amplitude, and expands in an asymptotic series in $\hbar$, as we take $\hbar \to 0$. Substituting this into (5.150) the leading term gives

$$\frac{\partial\mathcal{S}}{\partial t} + H\left(\mathbf{q}, \mathbf{p} = \frac{\partial\mathcal{S}}{\partial\mathbf{q}}, t\right) \ = \ 0 \ , \tag{5.152}$$

where

$$H(\mathbf{q}, \mathbf{p}, t) \ = \ \frac{|\mathbf{p}|^2}{2m} + V(\mathbf{q}) \ , \tag{5.153}$$

is precisely the classical Hamiltonian. Thus the leading behaviour of the Schrödinger equation as $\hbar \to 0$ is governed by the Hamilton-Jacobi equation, with Hamilton's principal function $\mathcal{S}(\mathbf{q}, t)$ (which is also the classical action) determining the phase of the wave function.

A Appendix: Levi-Civita alternating symbol

The *Levi-Civita alternating symbol* ϵ_{ijk} is defined by

$$\epsilon_{ijk} \equiv \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of } 123 \ , \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123 \ , \\ 0 & \text{otherwise .} \end{cases}$$

Notice that $\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki}$ (cyclically permuting the indices), while $\epsilon_{ijk} = -\epsilon_{ikj}$. With this definition the cross product $\mathbf{A} \wedge \mathbf{B}$ of two vectors $\mathbf{A}$, $\mathbf{B}$ has i^{th} component

$$(\mathbf{A} \wedge \mathbf{B})_i = \sum_{j,k=1}^{3} \epsilon_{ijk} A_j B_k \ .$$

A useful identity is

$$\epsilon_{ijk}\epsilon_{abc} = \det \begin{pmatrix} \delta_{ia} & \delta_{ib} & \delta_{ic} \\ \delta_{ja} & \delta_{jb} & \delta_{jc} \\ \delta_{ka} & \delta_{kb} & \delta_{kc} \end{pmatrix} \ ,$$

where the *Kronecker delta symbol* is defined by

$$\delta_{ij} \equiv \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \ .$$

As a special case we have the identity

$$\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{ibc} = \delta_{jb}\delta_{kc} - \delta_{jc}\delta_{kb} \ .$$

As an application, let us compute the scalar quadruple product:

$$\begin{aligned} (\mathbf{A} \wedge \mathbf{B}) \cdot (\mathbf{C} \wedge \mathbf{D}) &= \sum_{i,j,k,b,c=1}^{3} \epsilon_{ijk} A_j B_k \, \epsilon_{ibc} C_b D_c \\ &= \sum_{i,j,k,b,c=1}^{3} (\delta_{jb}\delta_{kc} - \delta_{jc}\delta_{kb}) \, A_j B_k C_b D_c \\ &= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \ . \end{aligned}$$

Other vector product identities are similarly straightforward to derive (for example the scalar triple product identity follows immediately from cyclically permuting the indices on ϵ_{ijk}). Although we won't use the following identity, we present it for interest:

$$(\det M) \, \epsilon_{ijk} = \sum_{a,b,c=1}^{3} \epsilon_{abc} M_{ia} M_{jb} M_{kc} \ ,$$

where $M = (M_{ij})$ is any 3×3 matrix.