Numerical Solution of Partial Differential Equations: Sheet 1 (of 4)

Section A [background material]

1. We have $\phi(x) = (1 - |x|)_+$ for $x \in [-2, 2]$; hence,

$$\phi(x) \;=\; \begin{cases} 0 & -2 \le x \le -1, \\ 1 + x & -1 \le x \le 0, \\ 1 - x & 0 \le x \le 1, \\ 0 & 1 \le x \le 2, \end{cases}$$

as shown in Figure 1.

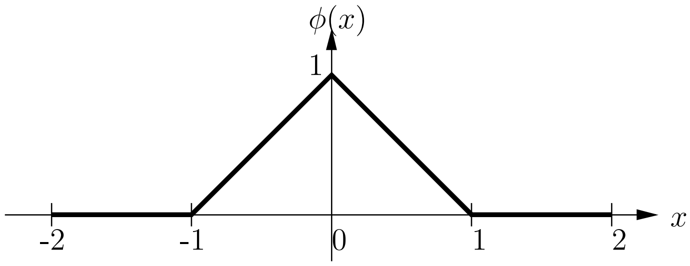

Figure 1: Graph of $\phi(x) = (1 - |x|)_+$.

The gradient of ϕ has a discontinuity at $x = 0$, hence $\phi \notin C([-2, 2]) \cap C^1((-2, 2))$. (A function f is in C^k if $D^\ell f$ is continuous for $\ell = 0, \ldots, k$.)

$D\phi$ is the first weak derivative of ϕ on the interval $[-2, 2]$ if

$$\int_{-2}^{2} \phi v' \,\mathrm{d}x \;=\; -\int_{-2}^{2} v D\phi \,\mathrm{d}x \qquad \forall v \in C_0^\infty((-2, 2)).$$

We have, for any such v,

$$\begin{aligned} \int_{-2}^{2} \phi v' \,\mathrm{d}x &= \int_{-2}^{2} (1 - |x|)_+ v' \,\mathrm{d}x \\ &= \int_{-1}^{0} (1 + x) v' \,\mathrm{d}x + \int_{0}^{1} (1 - x) v' \,\mathrm{d}x \\ &= -\int_{-1}^{0} v \,\mathrm{d}x + \int_{0}^{1} v \,\mathrm{d}x \\ &= -\int_{-2}^{2} v D\phi \,\mathrm{d}x, \end{aligned}$$

where

$$D\phi \;=\; \begin{cases} 0 & -2 \le x \le -1, \\ 1 & -1 \le x \le 0, \\ -1 & 0 \le x \le 1, \\ 0 & 1 \le x \le 2. \end{cases}$$

Next, $\phi \in L_p((-2, 2))$ if $\int_{-2}^{2} |\phi(x)|^p \,\mathrm{d}x < \infty$ for $1 \le p < \infty$. Now,

$$\begin{aligned} \int_{-2}^{2} |\phi(x)|^p \,\mathrm{d}x &= \int_{-1}^{0} (1 + x)^p \,\mathrm{d}x + \int_{0}^{1} (1 - x)^p \,\mathrm{d}x \;=\; \frac{2}{p+1} \;<\; \infty, \\ \int_{-2}^{2} |\phi'(x)|^p \,\mathrm{d}x &= \int_{-1}^{0} |1|^p \,\mathrm{d}x + \int_{0}^{1} |-1|^p \,\mathrm{d}x \;=\; 2 \;<\; \infty. \end{aligned}$$

Hence we have $\phi, \phi' \in L_p((-2, 2))$ for all $p \in [1, \infty)$ and so $\phi \in W_p^1((-2, 2))$ for all $p \in [1, \infty)$.

2. If $u(x) = x^\alpha$ for $x \in [0,1]$ where $0 < \alpha < 1$ we have

$$D^k u = \alpha(\alpha-1)\cdots(\alpha-k+1)\, x^{\alpha-k}, \qquad k \in \mathbb{N}.$$

Hence $D^k u \in C((0,1))$ for all $k \in \mathbb{N}$, i.e., $u \in C^\infty((0,1))$.

Consider

$$\int_0^1 |u'(x)|^p \, \mathrm{d}x = \int_0^1 (\alpha x^{\alpha-1})^p \, \mathrm{d}x = \frac{\alpha^p}{p(\alpha-1)+1}[x^{p(\alpha-1)+1}]_0^1,$$

which is only finite if $p(\alpha-1)+1 > 0$. Thus $u' \notin L_p((0,1))$ and $u \notin W_p^1((0,1))$ for $p \geq (1-\alpha)^{-1}$.

Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < \frac{1}{4}\}$ and let $w(x,y) = \log|\log\sqrt{x^2+y^2}|$. Then $w(x,y)$ has a singularity at the point $(0,0)$ and so $w \notin C(\Omega)$. To show $w \in W_2^1(\Omega)$ consider

$$\int_\Omega |w(x,y)|^2 \, \mathrm{d}x\,\mathrm{d}y = \int_0^{1/2}\int_0^{2\pi} |\log|\log r||^2 r \, \mathrm{d}\theta\,\mathrm{d}r = 2\pi \int_0^{1/2} r|\log|\log r||^2 \, \mathrm{d}r$$

which is finite if the integrand is continuous. In particular we must consider the value as $r \to 0$:

$$\begin{aligned}\lim_{r\to 0} r|\log|\log r||^2 &= \lim_{t\to\infty} \mathrm{e}^{-t}|\log|t||^2 \quad (r = \mathrm{e}^{-t}) \\ &= \lim_{t\to\infty}\frac{|\log t|^2}{\mathrm{e}^t} = \lim_{t\to\infty}\frac{2\log t}{t\mathrm{e}^t} = \lim_{t\to\infty}\frac{2}{t(\mathrm{e}^t + t\mathrm{e}^t)} = 0\end{aligned}$$

(using L'Hôpital's rule). Hence $w \in L_2(\Omega)$. Now we consider

$$\begin{aligned}\int_\Omega \left|\frac{\partial w}{\partial x}\right|^2 + \left|\frac{\partial w}{\partial y}\right|^2 \mathrm{d}x\,\mathrm{d}y &= \int_0^{1/2}\int_0^{2\pi}\left|\frac{\partial w}{\partial r}\right|^2 r\,\mathrm{d}\theta\,\mathrm{d}r \\ &= 2\pi\int_0^{1/2}\frac{1}{r(\log r)^2}\,\mathrm{d}r = 2\pi\left[-\frac{1}{\log r}\right]_0^{1/2} = \frac{2\pi}{\log 2},\end{aligned}$$

and since this is finite we see that $w \in W_2^1(\Omega) = H^1(\Omega)$, but $w \notin C(\Omega)$.

3. (a) Let $v \in H^1_{E_0}((a,b))$. Then since $v(x) = \int_a^x v'(\xi)\,\mathrm{d}\xi$ we have

$$\begin{aligned}\|v\|^2_{L_2((a,b))} &= \int_a^b |v(x)|^2\,\mathrm{d}x = \int_a^b \left|\int_a^x v'(\xi)\,\mathrm{d}\xi\right|^2 \mathrm{d}x \\ &\leq \int_a^b \left(\int_a^x 1\,\mathrm{d}\xi\right)\left(\int_a^x |v'(\xi)|^2\,\mathrm{d}\xi\right)\mathrm{d}x \quad \text{(Cauchy–Schwarz ineq.)} \\ &= \int_a^b (x-a)\int_a^x |v'(\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}x \\ &\leq \int_a^b (x-a)\,\mathrm{d}x \int_a^b |v'(\xi)|^2\,\mathrm{d}\xi \\ &= \frac{1}{2}(b-a)^2|v|^2_{H^1((a,b))}.\end{aligned}$$

(b) We have

$$\begin{aligned}[v(x)]^2 &= 2\int_a^x v(\xi)v'(\xi)\,\mathrm{d}\xi \\ &\leq 2\left(\int_a^x |v(\xi)|^2\,\mathrm{d}\xi\right)^{1/2}\left(\int_a^x |v'(\xi)|^2\,\mathrm{d}\xi\right)^{1/2} \quad \text{(Cauchy–Schwarz ineq.)} \\ &\leq 2\left(\int_a^b |v(\xi)|^2\,\mathrm{d}\xi\right)^{1/2}\left(\int_a^b |v'(\xi)|^2\,\mathrm{d}\xi\right)^{1/2} \\ &= 2\|v\|_{L_2((a,b))}|v|_{H^1((a,b))}\end{aligned}$$

for all $x \in (a,b)$. Hence taking the maximum of the left-hand-side we have

$$\|v\|^2_{L_\infty((a,b))} \;\leq\; 2\|v\|_{L_2((a,b))}|v|_{H^1((a,b))}.$$

4. The Lax–Milgram theorem states: Suppose V is a real Hilbert space equipped with a norm $\|\cdot\|_V$. Let $a(\cdot,\cdot)$ be a bilinear form on $V \times V$ such that

(i) $\exists c_0 > 0$ such that $\forall v \in V \; a(v,v) \geq c_0\|v\|_V^2$;

(ii) $\exists c_1 > 0$ such that $\forall v,w \in V \; |a(v,w)| \leq c_1\|v\|_V\|w\|_V$;

and let $\ell(\cdot)$ be a linear functional on V such that

(iii) $\exists c_2 > 0$ such that $\forall v \in V \; |\ell(v)| \leq c_2\|v\|_V$.

Then, there exists a unique $u \in V$ such that $a(u,v) = \ell(v) \; \forall v \in V$.

(a) Let $v \in H^1_0((0,1)) = \{v : v \in H^1((0,1)) \text{ and } v(0) = v(1) = 0\}$; then

$$\int_0^1 (-u''v + uv)\,\mathrm{d}x \;=\; \int_0^1 fv\,\mathrm{d}x.$$

Integrating the first term by parts gives

$$\int_0^1 (u'v' + uv)\,\mathrm{d}x - [u'v]_0^1 \;=\; \int_0^1 fv\,\mathrm{d}x.$$

Noting that $v(0) = v(1) = 0$ we have the weak formulation of the problem: find $u \in H^1_0((0,1))$ such that

$$\int_0^1 (u'v' + uv)\,\mathrm{d}x \;=\; \int_0^1 fv\,\mathrm{d}x \qquad \forall v \in H^1_0(0,1).$$

We define V to be the space $H^1_0((0,1))$,

$$a(u,v) \;=\; \int_0^1 (u'v' + uv)\,\mathrm{d}x$$

which is a bilinear form on $V \times V$ and

$$\ell(v) \;=\; \int_0^1 fv\,\mathrm{d}x$$

which is a linear functional on V. Now check conditions (i)–(iii) of the Lax–Milgram theorem.

$$a(v,v) \;=\; \int_0^1 (|v'|^2 + |v|^2)\,\mathrm{d}x \;=\; \|v\|^2_{H^1((0,1))}$$

and so (i) holds with $c_0 = 1$. Using the Cauchy-Schwarz inequality we have

$$\begin{aligned}
|a(v,w)| &\leq \|w'\|_{L_2((0,1))}\|v'\|_{L_2((0,1))} + \|w\|_{L_2((0,1))}\|v\|_{L_2((0,1))} \\
&\leq (\|w\|_{L_2((0,1))} + \|w'\|_{L_2((0,1))})(\|v\|_{L_2((0,1))} + \|v'\|_{L_2((0,1))}) \\
&\quad \text{(since } ab + cd \leq (a+c)(b+d) \text{ for all } a,b,c,d \geq 0) \\
&\leq 2(\|w\|^2_{L_2((0,1))} + \|w'\|^2_{L_2((0,1))})^{1/2}(\|v\|^2_{L_2((0,1))} + \|v'\|^2_{L_2((0,1))})^{1/2} \\
&\quad \text{(since } (a+b)^2 \leq 2(a^2+b^2)) \\
&= 2\|w\|_{H^1((0,1))}\|v\|_{H^1((0,1))}
\end{aligned}$$

and so (ii) holds with $c_1 = 2$.
(Could also get $c_1 = 1$ by just using the Cauchy–Schwarz inequality with the H^1 inner product.)
Finally, using the Cauchy–Schwarz inequality we have

$$\begin{aligned} |\ell(v)| &\leq \left(\int_0^1 |f|^2 \,\mathrm{d}x\right)^{1/2} \left(\int_0^1 |v|^2 \,\mathrm{d}x\right)^{1/2} \\ &\leq \left(\int_0^1 |f|^2 \,\mathrm{d}x\right)^{1/2} \left(\int_0^1 |v|^2 + |v'|^2 \,\mathrm{d}x\right)^{1/2} \\ &= \|f\|_{L_2((0,1))} \|v\|_{H^1((0,1))}, \end{aligned}$$

and so (iii) holds with $c_2 = \|f\|_{L_2((0,1))}$. Hence the Lax–Milgram theorem tells us that this problem possesses a unique weak solution.
(b) Let $v \in H^1_{E_0}((0,1)) = \{v : v \in H^1((0,1)) \text{ and } v(0) = 0\}$; then, as in part (a),

$$\int_0^1 (u'v' + uv)\,\mathrm{d}x - [u'v]_0^1 = \int_0^1 fv\,\mathrm{d}x.$$

Noting that $v(0) = u'(1) = 0$ we have the weak formulation of the problem: find $u \in H^1_{E_0}((0,1))$ such that

$$\int_0^1 (u'v' + uv)\,\mathrm{d}x = \int_0^1 fv\,\mathrm{d}x \qquad \forall v \in H^1_{E_0}((0,1)).$$

The proof of existence of a unique weak solution is then as in part (a) but we use $V = H^1_{E_0}((0,1))$ (so the norm is unchanged).
(c) As in (b) for $v \in H^1_{E_0}((0,1))$ we have

$$\int_0^1 (u'v' + uv)\,\mathrm{d}x - [u'v]_0^1 = \int_0^1 fv\,\mathrm{d}x.$$

Now we use the fact that $v(0) = 0$ and that $u'(1) = -u(1)$ to get the weak formulation of the problem: find $u \in H^1_{E_0}((0,1))$ such that

$$\int_0^1 (u'v' + uv)\,\mathrm{d}x + u(1)v(1) = \int_0^1 fv\,\mathrm{d}x \qquad \forall v \in H^1_{E_0}((0,1)).$$

Clearly,

$$a(u,v) = \int_0^1 (u'v' + uv)\,\mathrm{d}x + u(1)v(1)$$

defines a bilinear form on $V \times V$ where $V = H^1_{E_0}((0,1))$. We have

$$a(v,v) = \int_0^1 (|v'|^2 + |v|^2)\,\mathrm{d}x + |v(1)|^2 \geq \|v\|^2_{H^1((0,1))}$$

so (i) holds with $c_0 = 1$. Using the Cauchy–Schwarz inequality we have

$$\begin{aligned} |a(w,v)| &\leq |w|_{H^1((0,1))}|v|_{H^1((0,1))} + \|w\|_{L_2((0,1))}\|v\|_{L_2((0,1))} + \max_{x\in(0,1)} |w(x)| \max_{x\in(0,1)} |v(x)| \\ &\leq |w|_{H^1((0,1))}|v|_{H^1((0,1))} + \|w\|_{L_2((0,1))}\|v\|_{L_2((0,1))} \\ &\quad +2(\|w\|_{L_2((0,1))}\|v\|_{L_2((0,1))}|w|_{H^1((0,1))}|v|_{H^1((0,1))})^{1/2} \\ &\quad \text{(using Agmon's inequality from question 3b)} \\ &= (|w|^{1/2}_{H^1((0,1))}|v|^{1/2}_{H^1((0,1))} + \|w\|^{1/2}_{L_2((0,1))}\|v\|^{1/2}_{L_2((0,1))})^2 \\ &\leq 2(|w|_{H^1((0,1))}|v|_{H^1((0,1))} + \|w\|_{L_2((0,1))}\|v\|_{L_2((0,1))})^2 \\ &\quad \text{(since } (a+b)^2 \leq 2(a^2+b^2)\text{).} \end{aligned}$$

The analysis then follows as in part (a) to see that (ii) holds with $c_1 = 4$. Also (iii) holds as in part (a) so this problem also has a unique weak solution.

Section C [optional]

9. (a) Using Taylor series expansions,

$$u(x_i \pm 2h) = u(x_i) \pm 2hu'(x_i) + \frac{4h^2}{2}u''(x_i) \pm \ldots + \frac{16h^4}{24}u''''(x_i) \pm \ldots + \frac{64}{6!}u^{(vi)}(\xi_i),$$

so

$$u(x_{i+2}) + u(x_{i-2}) = 2u(x_i) + 4h^2u''(x_i) + \frac{4h^4}{3}u''''(x_i) + \mathcal{O}(h^6). \quad (1)$$

Similarly,

$$u(x_{i+1}) + u(x_{i-1}) = 2u(x_i) + h^2u''(x_i) + \frac{h^4}{12}u''''(x_i) + \mathcal{O}(h^6). \quad (2)$$

So

$$-\frac{1}{12}(1) + \frac{4}{3}(2) = \frac{5}{2}u(x_i) + h^2u''(x_i) + 0\,h^4u''''(x_i) + \mathcal{O}(h^6).$$

Thus, the given finite difference approximation has consistency error $T_i = \mathcal{O}(h^4)$.

(b) We have a total of $N-1$ unknowns, $U_1, \ldots, U_{N-1}$ (note that U_0 and U_N are known from the boundary conditions: $U_0 = 0$ and $U_N = 0$). Unfortunately, because now the finite difference equations are valid for $i = 2, \ldots, N-2$ only, providing a total of $N-3$ equations, we are 2 equations short.

We circumvent the problem by noting that

$$u''(x_0) = u''(0) = f(0) \qquad \text{and} \qquad u''(x_N) = u''(1) = f(1),$$

which we can discretise, using the 'ghost points' $x_{-1} = -h$ and $x_{N+1} = 1 + h$, with associated values U_{-1} and U_{N+1}, resulting in two additional difference equations

$$\frac{U_{-1} - 2U_0 + U_1}{h^2} = f(0) \qquad \text{and} \qquad \frac{U_{N+1} - 2U_N + U_{N-1}}{h^2} = f(1).$$

As $U_0 = 0$ and $U_N = 0$ these can be simplified to

$$\frac{U_{-1} + U_1}{h^2} = f(0) \qquad \text{and} \qquad \frac{U_{N+1} + U_{N-1}}{h^2} = f(1).$$

Thus we have created two further equations; however, we now also have two new unknowns, U_{-1} and U_{N+1}; so while the total number of equations has increased from $N-3$ to $N-1$, the total number of unknowns has also increased: instead of $N-1$ unknowns we now have $N+1$ unknowns.

We rectify this by extending the range of the index i for the difference equation stated in the question from $i = 2, \ldots, N-2$ to $i = 1, \ldots, N-1$, — whereby the equations corresponding to $i = 1$ and $i = N-1$ will also involve the 'ghost values' U_{-1} and U_{N+1}. Thus, finally, we end up with $(N-1)+2 = N+1$ equations for the $N+1$ unknowns: $U_{-1}, U_1, U_2, \ldots, U_{N-2}, U_{N-1}, U_{N+1}$; $U_0 = 0$ and $U_N = 0$ do not feature in the list of unknowns, of course.