Numerical Solution of Partial Differential Equations: Sheet 4 (of 4)

Section A [background material]

1. Consider the second-order linear hyperbolic equation, the linear wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x,t),$$

where $c > 0$ is the wave speed and f is a given continuous function on $\mathbb{R} \times [0, \infty)$.

(a) Consider the function u defined by d'Alembert's formula:

$$u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(\xi)\,\mathrm{d}\xi.$$

We begin by checking the initial conditions:

$$u(x,0) = \frac{1}{2}[u_0(x) + u_0(x)] + \frac{1}{2c}\int_{x-0}^{x+0} u_1(\xi)\,\mathrm{d}xi = u_0(x) + 0 = u_0(x).$$

Next, we compute $\frac{\partial u}{\partial t}$ by using the chain rule and by differentiating the integral with respect to t appearing in $x - ct$ and $x + ct$:

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}[u_0'(x-ct)(-c) + u_0'(x+ct)(+c)] + \frac{1}{2c}[u_1(x+ct)(+c) - u_1(x-ct)(-c)].$$

Hence,

$$\frac{\partial u}{\partial t}(x,0) = \frac{1}{2}\,0 + \frac{1}{2c}[c\,u_1(x) + c\,u_1(x)] = 0 + u_1(x) = u_1(x).$$

To verify that u satisfies the PDE, consider

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{1}{2}[u_0''(x-ct)(-c)^2 + u_0''(x+ct)(+c)^2] + \frac{1}{2c}[u_1'(x+ct)(+c)^2 - u_1(x-ct)(-c)^2].$$

On the other hand,

$$\frac{\partial u}{\partial x} = \frac{1}{2}[u_0'(x-ct) + u_0'(x+ct)] + \frac{1}{2c}[u_1(x+ct) - u_1(x-ct)],$$

and therefore

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}[u_0''(x-ct) + u_0''(x+ct)] + \frac{1}{2c}[u_1'(x+ct) - u_1'(x-ct)].$$

Clearly,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

as is required.

That $u \in C^2(\mathbb{R} \times [0,\infty))$ follows from the assumptions that $u_0 \in C^2(\mathbb{R})$, $u_1 \in C^1(\mathbb{R})$; indeed, d'Alembert's formula implies that $u \in C(\mathbb{R} \times [0,\infty))$, and the expressions for $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial t^2}$, $\frac{\partial u}{\partial x}$, and $\frac{\partial^2 u}{\partial x^2}$ computed above imply that these partial derivatives also belong to $C(\mathbb{R} \times [0,\infty))$.

(b) Suppose that u_0 and u_1 are identically equal to zero outside an interval $[-A, A]$ of $\mathbb{R}$, where $A > 0$. Let $t > 0$; d'Alembert's formula implies that $u(\cdot, t)$ is identically equal to zero outside the interval $[-A - ct, A + ct]$. Therefore $A_t = A + ct$.

To show that the function u defined by d'Alembert's formula is the only solution in the class of functions contained in $C^2(\mathbb{R} \times [0, \infty))$ such that $u(x, t) = 0$ for all $x \in [-A - ct, A + ct]$, suppose that u_1 and u_2 are two such functions; then $u = u_1 - u_2$ is in the same class of functions and satisfies

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0,$$

with identically zero initial data. By multiplying the equation with $\frac{\partial u}{\partial t}$, integrating the resulting expression from $x = -\infty$ to $x = +\infty$, and performing partial integration with respect to x in the second integral, it follows that

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{+\infty} \left(\frac{\partial u}{\partial t}\right)^2 (x, t)\, \mathrm{d}x + \frac{c^2}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{+\infty} \left(\frac{\partial u}{\partial x}\right)^2 (x, t)\, \mathrm{d}x = 0.$$

Note that the integrals are finite since u and its partial derivatives vanish outside the bounded interval $[-A - ct, A + ct]$; in particular the terms that arise in the course of partial integration with respect to x vanish. Integration of the above equality with respect to t then yields (recall that the initial data are identically zero:

$$\frac{1}{2} \int_{-\infty}^{+\infty} \left(\frac{\partial u}{\partial t}\right)^2 (x, t)\, \mathrm{d}x + \frac{c^2}{2} \int_{-\infty}^{+\infty} \left(\frac{\partial u}{\partial x}\right)^2 (x, t)\, \mathrm{d}x = 0, \qquad t > 0.$$

Hence, u is a constant function; the requirement that $u(x, t) = 0$ for all x outside the interval $[-A - ct, A + ct]$ then implies that u is in fact identically zero, thus completing the proof of uniqueness of the solution.

(c) To verify that the function u defined by

$$u(x, t) = \frac{1}{2}\left[u_0(x - ct) + u_0(x + ct)\right] + \frac{1}{2c} \int_{x-ct}^{x+ct} u_1(\xi)\, \mathrm{d}\xi + \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s, \tau)\, \mathrm{d}s\, \mathrm{d}\tau.$$

is a solution to the initial-value problem in the case when $f \neq 0$, all that needs to be done is to show that the function F defined by

$$F(x, t) = \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s, \tau)\, \mathrm{d}s\, \mathrm{d}\tau$$

satisfies the nonhomogeneous equation, with $f \not\equiv 0$, but with zero initial data (because the sum of the first two terms in the expression for $u(x, t)$ above was already shown in part (a) of the question to satisfy the homogeneous equation, with $f \equiv 0$, and nonzero initial data).

Clearly, $F(x, 0) = 0$. Let

$$G(x, t, \tau) := \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s, \tau)\, \mathrm{d}s\, \mathrm{d}\tau.$$

Note that $G(x, t, t) = 0$. Then,

$$F(x, t) = \frac{1}{2c} \int_0^t G(x, t, \tau)\, \mathrm{d}\tau$$

and

$$\frac{\partial F}{\partial t}(x, t) = \frac{1}{2c} G(x, t, t) + \frac{1}{2c} \int_0^t \frac{\partial G}{\partial t}(x, t, \tau)\, \mathrm{d}\tau = \frac{1}{2} \int_0^t f(x + c(t - \tau), \tau) + f(x - (t - \tau), \tau)\, \mathrm{d}\tau,$$

whereby $\frac{\partial F}{\partial t}(x, 0) = 0$. Thus we have shown that F satisfies the zero initial data, as required. Differentiating again with respect to t, we have that

$$\begin{aligned}\frac{\partial^2 F}{\partial t^2}(x,t) &= \frac{1}{2}[f(x + c(t-t), t) + f(x - c(t-t), t)] \\ &\quad + \frac{c}{2}\int_0^t \left[\frac{\partial f}{\partial x}(x + c(t-\tau), \tau) - \frac{\partial f}{\partial x}(x - c(t-\tau), \tau)\right] \mathrm{d}\tau \\ &= f(x,t) + \frac{c}{2}\int_0^t \left[\frac{\partial f}{\partial x}(x + c(t-\tau), \tau) - \frac{\partial f}{\partial x}(x - c(t-\tau), \tau)\right] \mathrm{d}\tau.\end{aligned}$$

Analogously,

$$\frac{\partial F}{\partial x}(x,t) = \frac{1}{2c}\int_0^t [f(x + c(t-\tau), \tau) - f(x - c(t-\tau), \tau)]\, \mathrm{d}\tau$$

and therefore

$$\frac{\partial^2 F}{\partial x^2}(x,t) = \frac{1}{2c}\int_0^t \left[\frac{\partial f}{\partial x}(x + c(t-\tau), \tau) - \frac{\partial f}{\partial x}(x - c(t-\tau), \tau)\right] \mathrm{d}\tau.$$

Hence,

$$\frac{\partial^2 F}{\partial t^2}(x,t) - c^2\frac{\partial^2 F}{\partial x^2}(x,t) = f(x,t),$$

as required. The rest of the argument to complete part (c) proceeds in exactly the same way as in part (b).

Section C [optional]

7. For the first-order hyperbolic equation

$$u_t + a u_x = 0,$$

with $(x,t) \in \mathbb{R} \times [0, \infty)$, subject to the initial condition $u(x, 0) = u_0(x)$, the Lax–Wendroff finite difference scheme is defined by

$$U_j^{m+1} = U_j^m - \frac{1}{2}\mu(U_j^m - U_{j-1}^m) + \frac{1}{2}\mu^2(U_{j+1}^m - 2U_j^m + U_{j-1}^m),$$

for $m = 0, 1, \dots$ and $j \in \mathbb{Z}$, with $U_j^0 := u_0(x_j)$, where $\mu = a\Delta t/\Delta x$ is the CFL number, $x_j = j\Delta x$, $t_m = m\Delta t$.

(a) The Lax–Wendroff scheme can be rewritten as follows:

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} + a\frac{U_{j+1}^m - U_{j-1}^m}{2\Delta x} - \frac{a^2\Delta t}{2}\,\frac{U_{j+1}^m - 2U^m + j + U_{j-1}^m}{(\Delta x)^2} = 0.$$

Therefore, the consistency error of the scheme is

$$T_j^m := \frac{u_j^{m+1} - u_j^m}{\Delta t} + a\frac{u_{j+1}^m - u_{j-1}^m}{2\Delta x} - \frac{a^2\Delta t}{2}\,\frac{u_{j+1}^m - 2u^m + j + u_{j-1}^m}{(\Delta x)^2},$$

where $u_j^m := u(x_j, t_m)$, $u_{j\pm1}^m := u(x_{j\pm1}, t_m)$, $u_j^{m+1} := u(x_j, t_{m+1})$. By Taylor series expansion about the point (x_j, t_m), we have that

$$\begin{aligned}T_j^m &= \left(\frac{\partial u}{\partial t}(x_j, t_m) + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(x_j, t_m) + \mathcal{O}((\Delta t)^2)\right) + a\left(\frac{\partial u}{\partial x}(x_j, t_m) + \mathcal{O}((\Delta x)^2)\right) \\ &\quad - \frac{a^2\Delta t}{2}\left(\frac{\partial^2 u}{\partial x^2}(x_j, t_m) + \mathcal{O}((\Delta x)^2)\right) \\ &= \left(\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x}\right)(x_j, t_m) + \frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2}\right)(x_j, t_m) + \mathcal{O}((\Delta x)^2 + (\Delta t)^2).\end{aligned}$$

(b) By inserting the Fourier mode

$$U_j^m = \lambda^m e^{\imath\kappa\Delta x}$$

into the scheme (written in the form as in the statement of the question), where $\lambda = \lambda(\kappa) \in \mathbb{C}$ is the *amplification factor* and $\kappa \in [-\pi/\Delta x, \pi/\Delta x]$ is the wave number, we have that

$$\begin{aligned}\lambda(\kappa) &= 1 - \frac{\mu}{2}(e^{\imath\kappa h} - e^{-\imath\kappa h}) + \frac{\mu^2}{2}(e^{\imath\kappa h} - 2 + e^{-\imath\kappa h}) \\ &= 1 - \imath\mu\sin\kappa h + \mu^2(\cos\kappa h - 1) \\ &= 1 - \imath\mu\sin\kappa h - 2\mu^2\sin^2\frac{\kappa h}{2}.\end{aligned}$$

Hence,

$$\begin{aligned}|\lambda(\kappa)|^2 &= \left(1 - 2\mu^2\sin^2\frac{\kappa h}{2}\right)^2 + 4\mu^2\sin^2\frac{\kappa h}{2}\left(1 - \sin^2\frac{\kappa h}{2}\right) \\ &= 1 - 4\mu^2(1-\mu^2)\sin^2\frac{\kappa h}{2}.\end{aligned}$$

Clearly, $|\lambda(\kappa)|^2 \leq 1$ (for practical stability) if, and only if, $|\mu| \leq 1$, which is the required restriction on the CFL number to ensure that the Lax–Wendroff scheme is practically stable in the ℓ_2 norm.

8. We shall derive the first order upwind scheme for the initial value problem

$$\frac{\partial \mathbf{u}}{\partial t} + \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}\frac{\partial \mathbf{u}}{\partial x} = 0, \quad \mathbf{u}(x,0) = \mathbf{u}_0(x),$$

for $(x,t) \in \mathbb{R} \times [0,\infty)$. and identify the CFL stability condition for the scheme.

$$\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} = A = X\Lambda X^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

is the diagonalisation, so let

$$\begin{aligned}A^+ &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix},\end{aligned}$$

and

$$\begin{aligned}A^- &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix},\end{aligned}$$

and the first order upwind scheme as described in lectures is given by

$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n - \frac{\Delta}{\Delta x}\left[A^-(\mathbf{U}_{j+1}^n - \mathbf{U}_j^n) + A^+(\mathbf{U}_j^n - \mathbf{U}_{j-1}^n)\right], \quad \mathbf{U}_j^0 = \mathbf{u}_0(x_j).$$

Since the largest eigenvalue in absolute value is 2, the CFL limit is $\frac{2\Delta t}{\Delta x} \leq 1$.