Numerical Solution of Partial Differential Equations

Why?

Endre Süli

Mathematical Institute
University of Oxford
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Lecture 0

Numerical solution of differential equations is a rich and active field of modern applied mathematics.

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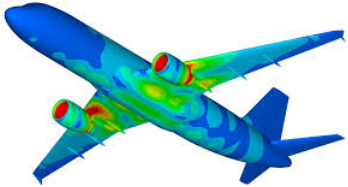
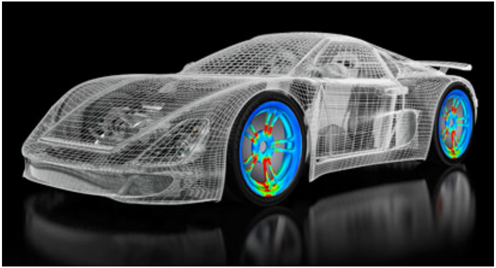
The growth of the subject is stimulated by ever-increasing demands from the natural sciences, engineering and economics to provide accurate and reliable approximations to mathematical models involving ODEs & PDEs whose exact solutions are either too complicated to determine in closed form or are not known to exist.





Evolution of crack-fields

Numerical approximation of the Francfort–Marigo model of brittle fracture.

Evolution of the computational grids

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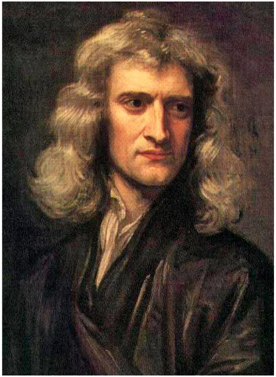
The foundations of the theory of differential equations were laid by Leibniz, the Bernoulli brothers, and others from the 1680s, not long after Newton introduced his ‘fluxional equations’ in the 1670s.



Sir Isaac Newton
1643–1727

Gottfried Wilhelm von Leibniz
1646–1716

Around 1671, Newton wrote his, then unpublished, *The Method of Fluxions and Infinite Series* (published in 1736), in which he classified first-order differential equations, known to him as fluxional equations, into three classes, as follows (using modern notation):

$$\underbrace{\frac{\mathrm{d}y}{\mathrm{d}x} = f(y), \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y),}_{\text{Ordinary Differential Equations (ODEs)}}$$

$$\underbrace{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.}_{\text{Partial Differential Equation (PDE)}}$$

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In the same year, Leibniz introduced the term *differential equation* (aequatio differentialis, Latin).

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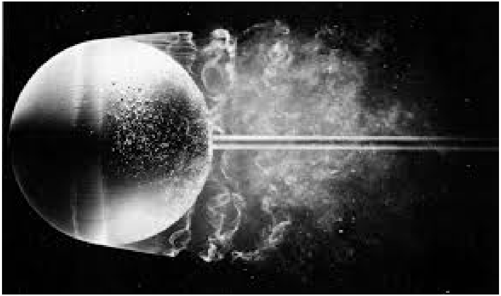
Sphere in a turbulent flow:



Source:

Milton Van Dyke, An Album of Fluid Motion, Parabolic Press, 12th ed., 1982.

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The compressible Navier–Stokes equations:

Suppose that $\Omega \subset \mathbb{R}^3$. Given

$$\rho_0 = \rho_0(\mathbf{x}), \qquad \mathbf{u}_0 = \mathbf{u}(\mathbf{x}), \qquad \mathbf{f} = \mathbf{f}(\mathbf{x}, t),$$

find

$$\rho = \rho(\mathbf{x}, t), \qquad \mathbf{u} = \mathbf{u}(\mathbf{x}, t),$$

such that:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{u}\,\rho) &= 0 && \text{in } \Omega \times (0, \infty),\\
\rho(\mathbf{x}, 0) &= \rho_0(\mathbf{x}) && \text{for } \mathbf{x} \in \Omega,\\
\frac{\partial(\rho\,\mathbf{u})}{\partial t} + \nabla \cdot (\rho\,\mathbf{u} \otimes \mathbf{u}) - \nabla \cdot \mathbf{S}(\mathbf{u}, \rho) + \nabla\, p(\rho) &= \rho\,\mathbf{f} && \text{in } \Omega \times (0, \infty),\\
\mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega \times (0, \infty),\\
(\rho\,\mathbf{u})(\mathbf{x}, 0) &= (\rho_0\,\mathbf{u}_0)(\mathbf{x}) && \text{for } \mathbf{x} \in \Omega.
\end{aligned}$$

P.-L. Lions (OUP, 1996, 1998).

Mathematics of numerical algorithms?

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The sinking of the Sleipner A offshore platform in Gandsfjorden near Stavanger, Norway, on August 23, 1991, resulted in a loss of nearly one billion dollars.

It was found to be the result of inaccurate numerical simulation.



A landmark contribution to the foundations of the mathematical theory of numerical methods for PDEs is *Über die partiellen Differenzengleichungen der mathematischen Physik* by Richard Courant, Karl Friedrichs, and Hans Lewy, (Mathematische Annalen, 1928).

The basic idea

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... is extremely simple. Suppose that y is differentiable at $x \in \mathbb{R}$; then,

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}.$$

Thus,

$$\frac{y(x+h) - y(x)}{h} = y'(x) + o(1) \qquad \text{as } h \to 0.$$

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Analogously, if y' is differentiable, then

$$y''(x) \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} \qquad \text{as } h \to 0.$$

Euler's method

Leonhard Euler (1707–1783)

Euler's method for $y'(x) = f(x, y(x))$ subject to the i.c. $y(x_0) = y_0$:

$$\frac{y(x_k + h) - y(x_k)}{h} \approx f(x_k, y(x_k)), \qquad y(x_0) = y_0, \qquad x_k = x_0 + kh,$$

for $k = 0, 1, \ldots$.

Cosmological simulation of the evolution of the Universe

Click here

Volker Springel (Max Planck Institute of Astrophysics, Garching, Germany)
Millennium-XXL project: $6720^3 \approx 303 \times 10^9$ particles over the equivalent of more than 13×10^9 years. Largest N-body simulation ever: required the equivalent of 300 years of CPU time and used more than 12000 computer cores and 30 TB of RAM on the Juropa Machine at the Jülich Supercomputer Centre in Germany.