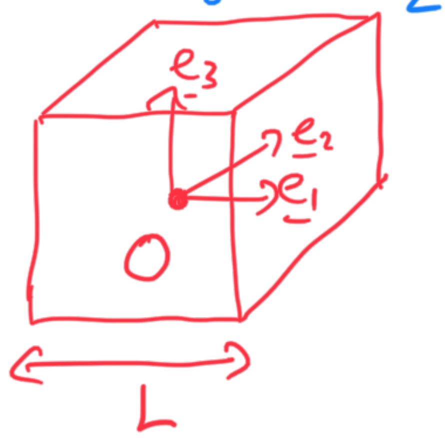
Stress tensor properties:

(i) It is symmetric, $\sigma_{ij} = \sigma_{ji}$

Proof:

Consider a small material volume $V(t)$ instantaneously forming a cube with centre O and with faces at $x_j = \pm L/2$



Conservation of angular momentum about O at position $\underline{x}$ inside the cube gives

$$\frac{d}{dt}\iiint_{V(t)} \underline{x} \wedge \rho \underline{u} \, dV = \iint_{\partial V(t)} \underline{x} \wedge \underline{t}(\underline{n}) \, dS + \iiint_{V(t)} \underline{x} \wedge \rho \underline{F} \, dV$$

(note: this is $\underline{x} \wedge$ momentum equation).

Applying Reynolds Transport theorem (componentwise, then summing), we have:

$$\frac{d}{dt}\iiint_{V(t)} \underline{x} \wedge \rho \underline{u} \, dV = \iiint_{V(t)} \underline{x} \wedge \rho \frac{D\underline{u}}{Dt} \, dV$$

NB: $\underline{x}$ and t are independent variables

$$\therefore \iiint_{V(t)} \underline{x} \wedge \rho \left(\frac{D\underline{u}}{Dt} - \underline{F}\right) dV = \iint_{\partial V(t)} \underline{x} \wedge \underline{t}(\underline{n}) \, dS$$

Since $|\underline{x}| = O(L)$, LHS $= O(L^4)$.

The RHS $= \iint_{\partial V(\epsilon)} \frac{L}{2} \underline{e}_k \wedge \underline{t}(\underline{e}_k) - \frac{L}{2} \underline{e}_k \wedge \underline{t}(-\underline{e}_k) dS$

NB: 6 faces,
3 in +ve directions,
3 in -ve directions

But $\underline{t}(\underline{e}_k) = -\underline{t}(-\underline{e}_k)$

Thus RHS $= \iint_{\partial V(\epsilon)} L \underline{e}_k \wedge \underline{t}(\underline{e}_k) ds = L^3 \underline{e}_k \wedge \underline{t}(\underline{e}_k) + O(L^4)$

$\therefore$ in the limit $L \to 0$, $\underline{e}_k \wedge \underline{t}(\underline{e}_k) = \underline{0}$

Now, $\underline{t}(\underline{e}_k) = \underline{e}_i \sigma_{ik}$, so $\underline{e}_k \wedge \underline{e}_i \sigma_{ik} = \underline{0}$

Expanding this out, we have

$$\underline{e}_1 \wedge (\underline{e}_1 \sigma_{11} + \underline{e}_2 \sigma_{12} + \underline{e}_3 \sigma_{13}) + \underline{e}_2 (\underline{e}_1 \sigma_{12} + \underline{e}_2 \sigma_{22} + \underline{e}_3 \sigma_{32})$$
$$+ \underline{e}_3 (\underline{e}_1 \sigma_{13} + \underline{e}_2 \sigma_{23} + \underline{e}_3 \sigma_{33}) = \underline{0},$$

which we rearrange as:

$$(\underline{e}_1 \wedge \underline{e}_2)(\sigma_{21} - \sigma_{12}) + (\underline{e}_2 \wedge \underline{e}_3)(\sigma_{32} - \sigma_{23}) + (\underline{e}_1 \wedge \underline{e}_3)(\sigma_{13} - \sigma_{31}) = \underline{0}$$

ie $\underline{e}_3 (\sigma_{21} - \sigma_{12}) + \underline{e}_1 (\sigma_{32} - \sigma_{23}) + \underline{e}_2 (\sigma_{13} - \sigma_{31}) = \underline{0}$.

Dot with $\underline{e}_1, \underline{e}_2, \underline{e}_3$ separately $\Rightarrow$

$$\sigma_{12} = \sigma_{21}, \quad \sigma_{23} = \sigma_{32}, \quad \sigma_{13} = \sigma_{31}$$

and thus the stress tensor is symmetric
(and the number of components is thus 6 not 9)

(ii) It is a tensor.

Definition: A tensor of rank n is a mathematical object $\underline{\underline{T}}$ with components involving n suffices which obeys the transformation law

$$T_{ijk\ldots\ldots} = L_{ip} L_{jq} L_{kr} \ldots\ldots T_{pqr\ldots\ldots}$$

where L is the rotation matrix between the frames.

Notes:
- (i) a scalar is a tensor of rank 0 $(v = v)$
- (ii) a vector is a tensor of rank 1 $(v_i' = L_{ij} v_j)$
- (iii) any rank 2 tensor can be represented as a matrix

We need to show that σ_{ij} transforms like a tensor.

Proof:

We recall that $\underline{t}$ and $\underline{n}$ are both vectors.
A rotation of the coordinate axes from OX_1, OX_2, OX_3 (with basis $\underline{e}_1, \underline{e}_2, \underline{e}_3$) to OX_1', OX_2', OX_3' (with basis $\underline{e}_1', \underline{e}_2', \underline{e}_3'$) transforms the vector $\underline{r} = x_i \underline{e}_i$ to $\underline{r} = x_j' \underline{e}_j'$.

Thus, $x_i \underline{e}_i = x_j' \underline{e}_j'$.

Dotting with $\underline{e}_j'$, we have

$$x_j' = x_i \underline{e}_i \cdot \underline{e}_j'.$$

Writing $L_{ij} = \underline{e}_i \cdot \underline{e}_j'$, $x_j' = L_{ij} x_i$, and the matrix L, of which L_{ij} are the components, is orthogonal $(LL^T = I)$.

Now let's apply the rotation to the stress vector $\underline{t}$ and normal $\underline{n}$ linked through Cauchy's Stress theorem.

Since $\underline{t}(\underline{n}) = \underline{e}_i \sigma_{ij} n_j$, we have $t_i = \sigma_{ij} n_j$

so $t_k' = L_{ik} t_i = L_{ik} \sigma_{ij} n_j$. However, in the rotated frame we can also write $t_k' = \sigma_{kj}' n_j'$ and since $n_j' = L_{ij} n_i$, we have

$$t_k' = \sigma_{kj}' L_{ij} n_i.$$

Thus, $L_{ik} \sigma_{ij} n_j = \sigma_{kj}' L_{ij} n_j = \sigma_{kj}' L_{ji} n_j$

$$\therefore L_{ik} \sigma_{ij} = \sigma_{kj}' L_{ji}$$

$$= \sigma_{jk}' L_{ij}$$ since σ is symmetric and L is orthogonal

$$\therefore L\sigma = \sigma' L$$

$$\Rightarrow \sigma' = L \sigma L^T,$$ which is equivalent to $\sigma_{ij}' = L_{ip} L_{jq} \sigma_{pq}$ ($\frac{1}{2}$ page algebra for 2x2 system).

Thus σ is indeed a tensor and represents something which is independent of the frame chosen.