B4.3 Distribution Theory MT20

Lecture 3:
- Definition of distribution
- First examples, local Lebesgue spaces
- The boundedness property
- Order of distribution

(pp. 22–26 in lecture notes)

Recall from previous lecture:

* Test functions on Ω are

$$C_c^\infty(\Omega) = \mathcal{D}(\Omega)$$

* Convergence in sense of test functions on Ω, $\phi_j \to \phi$ in $\mathcal{D}(\Omega)$ iff for some compact $K \subset \Omega$,
$\operatorname{supp}\phi_j, \operatorname{supp}\phi \subseteq K$ all j, and
$\partial^\alpha \phi_j \to \partial^\alpha \phi$ uniformly for all α.

$\mathcal{D}(\Omega)$ clearly a vector space (& a commutative ring) under

$(\phi + t\psi)(x) := \phi(x) + t\psi(x)$, $x \in \Omega$

for $\phi, \psi \in \mathcal{D}(\Omega)$, $t \in \mathbb{R}$,

(and $(\phi\psi)(x) := \phi(x)\psi(x)$ ...).

Can show there is a topology $\mathcal{T}$ on $\mathcal{D}(\Omega)$ so vector space operations are $\mathcal{T}$-continuous and $\phi_j \to \phi$ in $\mathcal{D}(\Omega)$ iff $\phi_j \to \phi$ in $(\mathcal{D}(\Omega), \mathcal{T})$.

$(\mathcal{D}(\Omega), \mathcal{T})$ is an example of a topological vector space. We won't need this here.

DEF <u>Distributions corresponding to $\mathcal{D}(\Omega)$</u>

A functional $u : \mathcal{D}(\Omega) \to \mathbb{C}$ (or into $\mathbb{R}$) is a distribution on Ω if

① <u>u is linear</u>: $u(\phi + t\psi) = u(\phi) + t u(\psi)$ holds for all $\phi, \psi \in \mathcal{D}(\Omega)$ and $t \in \mathbb{C}$ (or $\mathbb{R}$).

and

② <u>u is $\mathcal{D}$-continuous</u>:

if $\phi_j \to \phi$ in $\mathcal{D}(\Omega)$, then

$u(\phi_j) \to u(\phi)$.

The set of all distributions on Ω denoted by $\mathcal{D}'(\Omega)$.

(Laurent Schwartz)

Remarks

• When $u : \mathcal{D}(\Omega) \to \mathbb{C}$ is linear, then u is $\mathcal{D}$-continuous iff u is $\mathcal{D}$-continuous at 0. Check it by writing out defs!

•• $\mathcal{D}'(\Omega)$ is a vector space:

$$(u+tv)(\phi) := u(\phi) + t v(\phi), \phi \in \mathcal{D}(\Omega)$$

$$u, v \in \mathcal{D}'(\Omega) \quad , \quad t \in \mathbb{C}$$

••• If $u : \mathcal{D}(\Omega) \to \mathbb{C}$ is linear and defined on <u>all</u> of $\mathcal{D}(\Omega)$, then only exx of $\mathcal{D}$-discontinuous functionals known are coming via the Axiom of Choice.

Bracket notation

If $u \in \mathcal{D}'(\Omega)$, $\phi \in \mathcal{D}(\Omega)$ write

$$u(\phi) =: \langle u, \phi \rangle$$

$\boxed{\text{EX}}$ Let $f \in L^p(\Omega)$, $1 \leq p \leq \infty$.

Put
$$\langle T_f, \phi \rangle := \int_\Omega \varphi(x) f(x)\, dx$$

for $\phi \in \mathcal{D}(\Omega)$.

Well-defined since if $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_\Omega |\varphi f|\, dx \leq \|\varphi\|_q \, \|f\|_p$$

by Hölder's inequality.

T_f linear by linearity of the integral, and $\mathcal{D}$-continuous because if $\phi_j \to \phi$ in $\mathcal{D}(\Omega)$, then in particular $\|\phi_j - \phi\|_{\mathcal{I}} \to 0$.

Thus $T_f \in \mathcal{D}'(\Omega)$

$\boxed{\text{EX}}$ Dirac's delta function

at $x_0 \in \Omega$:

$$\langle \delta_{x_0}, \phi \rangle := \phi(x_0), \quad \phi \in \mathcal{D}(\Omega)$$

Then $\delta_{x_0} \in \mathcal{D}'(\Omega)$.

Various generalizations possible:

- $x_1, \ldots, x_k \in \Omega$, $\alpha_1, \ldots, \alpha_k \in \mathbb{N}_0^n$

$$\langle T, \phi \rangle := \sum_{j=1}^{k} (\partial^{\alpha_j} \phi)(x_j), \quad \phi \in \mathcal{D}(\Omega).$$

•• and if also $c_1, \dots, c_k \in \mathbb{C}$,

then

$$\langle S, \phi \rangle := \sum_{j=1}^{k} c_j (\partial^{\alpha_j} \phi)(x_j), \ \phi \in \mathcal{D}(\Omega).$$

••• $f \in L^p(\Omega)$ $\quad 1 \leq p \leq \infty$

$\alpha \in \mathbb{N}_0^n$

$$\langle R, \phi \rangle := \int_\Omega f(x)(\partial^\alpha \phi)(x)\,dx$$

$$\phi \in \mathcal{D}(\Omega)$$

and many more ....

An important extension of L^p :

local L^p functions.

$\boxed{\text{DEF}}$ **Local Lebesgue spaces**

Fix $p \in [1, \infty]$.

Then a measurable function

$$f : \Omega \to \mathbb{C}$$

is locally in L^p if for all compact $K \subset \Omega$,

$$\begin{cases} \int_K |f(x)|^p \, dx < \infty & \text{if } p < \infty, \\ \underset{x \in K}{\operatorname{ess.sup}} |f(x)| < \infty & \text{if } p = \infty. \end{cases}$$

Put $\mathcal{L}^p_{loc}(\Omega) := \{ f \text{ locally in } L^p \}$

and

$$L^p_{loc}(\Omega) := \mathcal{L}^p_{loc}(\Omega) \Big/ \text{'}{=}\text{ a.e.'}$$

<u>Note</u>

- $L^p_{loc}(\Omega)$ is a vector space (for the same reasons as L^p is)

- $L^p_{loc}(\Omega)$ is strictly descending in $p \in [1,\infty]$:

$$L^1_{loc}(\Omega) \supsetneq L^p_{loc}(\Omega) \supsetneq L^{p+\varepsilon}_{loc}(\Omega) \supsetneq L^\infty_{loc}(\Omega)$$

for $1 < p < p+\varepsilon < \infty$

This is false without subscript 'loc' when $\mathcal{L}^n(\Omega) = \infty$:

$$\mathbb{1}_{(0,\infty)} \in L^\infty(0,\infty) \setminus L^p(0,\infty)$$

for $p \in [1,\infty)$.

$\boxed{\text{EX}}$ In def of $L^p_{loc}(\Omega)$ the set Ω defines what 'local' means:

$$\frac{1}{x} \notin L^1(0,\infty), \text{ but } \frac{1}{x} \in L^1_{loc}(0,\infty).$$

In fact: $\frac{1}{x} \in L^\infty_{loc}(0,\infty)$

But $\frac{1}{x} \notin L^1_{loc}(-1,1)$

$\boxed{\text{EX}}$ Let $p \in [1,\infty]$ and $f \in L^p_{loc}(\Omega)$.

Put
$$\langle T_f, \phi \rangle := \int_\Omega f(x)\phi(x)\,dx, \quad \phi \in \mathcal{D}(\Omega).$$

Claim: $T_f \in \mathcal{D}'(\Omega)$

Well-defined: Let $\phi \in \mathcal{D}(\Omega)$.

Then $K := \operatorname{supp}(\phi) \subset \Omega$ compact

so

$$\int_\Omega |f(x)\phi(x)|\,dx = \int_K |f(x)\phi(x)|\,dx$$

$\frac{1}{p}+\frac{1}{q}=1$

$$\leq \left(\int_K |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_K |\phi(x)|^q\,dx\right)^{\frac{1}{q}} < \infty$$

Then T_f linear because integral is, and if $\phi_j \to 0$ in $\mathcal{D}(\Omega)$, then for some compact $K \subset \Omega$, $\operatorname{supp}(\phi_j) \subset K$ all j, and

$$|T_f(\phi_j)| \leq \left(\int_K |f|^p\,dx\right)^{\frac{1}{p}} \|\phi_j\|_K \to 0.$$

$u \in \mathcal{D}'(\Omega)$ if

1. $u : \mathcal{D}(\Omega) \to \mathbb{C}$ linear

&

2. u $\mathcal{D}$-continuous

... and when 1 holds with u defined on <u>all</u> of $\mathcal{D}(\Omega)$ chances are that 2 holds too ... still it is worthwhile to reformulate 2 !

$\boxed{\text{TH}}$ The boundedness property

Assume $u : \mathcal{D}(\Omega) \to \mathbb{C}$ is linear.

Then u is $\mathcal{D}$-continuous iff

for every compact $K \subset \Omega$
there exist constants

$$c = c_K > 0, \quad m = m_K \in \mathbb{N}_0$$

so

$$\textcircled{*} \quad |\langle u, \phi \rangle| \leqq c \sum_{|\alpha| \leqq m} \sup |\partial^\alpha \phi|$$

holds for all $\phi \in \mathcal{D}(\Omega)$ with
$\text{supp}(\phi) \subseteq K$.

<u>Notation</u>: $\mathcal{D}(K) := \{\phi \in \mathcal{D}(\Omega) : \text{supp}\,\phi \subseteq K\}$

$\boxed{\text{Pf.}}$ '$\Leftarrow$' Assume u has the boundedness property. Let $\phi_j \to 0$ in $\mathcal{D}(\Omega)$. Then for some compact $K \subset \Omega$,

$$\operatorname{supp} \phi_j \subseteq K \quad \text{for all } j,$$

and

$$\sup |\partial^\alpha \phi_j| \to 0 \quad \text{for all } \alpha \in \mathbb{N}_0^n.$$

Use boundedness property of u with K: we find constants $c = c_K > 0$, $m = m_K \in \mathbb{N}_0$ so

$$\text{(*)} \quad |\langle u, \phi \rangle| \leq c \sum_{|\alpha| \leq m} \sup |\partial^\alpha \phi|$$

for $\phi \in \mathcal{D}(K)$.

Apply with $\phi = \phi_j$ to get

$$\langle u, \phi_j \rangle \to 0.$$

'$\Rightarrow$' Assume u $\mathcal{D}$-continuous.

To prove boundedness property we argue by contradiction: if u doesn't have it we find compact $K \subset \Omega$ and for each $c = m = j \in \mathbb{N}$ a $\phi_j \in \mathcal{D}(K)$ so

$$|\langle u, \phi_j \rangle| > j \sum_{|\alpha| \leq j} \sup |\partial^\alpha \phi_j| .$$

Put $\lambda_j = |\langle u, \phi_j \rangle|$. Clearly $\lambda_j > 0$ and if $\psi_j := \dfrac{\phi_j}{\lambda_j}$, then

$\psi_j \in \mathcal{D}(K)$, $|\langle u, \psi_j \rangle| = 1$ and

$$1 > j \sum_{|\alpha| \leq j} \sup |\partial^\alpha \psi_j| .$$

Claim. $\psi_j \to 0$ in $\mathcal{D}(\Omega)$

and hence contradiction

$1 = |\langle u, \psi_j \rangle| \longrightarrow 0$!!

For 'claim' note

$\operatorname{supp} \psi_j \subseteq K$ for all j.

Fix $\beta \in \mathbb{N}_0^n$. Take $j \geq |\beta|$

and note

$$\sup |\partial^\beta \psi_j| \leq \sum_{|\alpha| \leq j} \sup |\partial^\alpha \psi_j| < \frac{1}{j} \to 0$$

Hence claim, and thus $\lightning$ $\square$

$\boxed{\text{DEF}}$ Order of distribution

Let $u \in \mathcal{D}'(\Omega)$.

* u has order at most $m \in \mathbb{N}_0$

if for each compact $K \subset \Omega$ there exists a constant $c = c_K \geqq 0$ so

$$|\langle u, \phi \rangle| \leqq c \sum_{|\alpha| \leqq m} \sup |\partial^\alpha \phi|$$

for all $\phi \in \mathcal{D}(K)$.

** u has infinite order if u does not have order at most m for any $m \in \mathbb{N}_0$.

*** u has order 0 if it has order at most 0. u has order $m \in \mathbb{N}$ if it has order at most m and not order at most $m-1$.

Remarks The boundedness property says, loosely speaking, that a distribution on Ω must have locally finite order.

A distribution of order m on Ω depends, loosely speaking, only of derivatives up to m-th order.

We shall make both statements precise later in course.

Notation

$$\mathcal{D}'_m(\Omega) := \left\{ u \in \mathcal{D}'(\Omega) : \begin{array}{l} u \text{ has order} \\ \text{at most } m \end{array} \right\}$$

$\boxed{\text{EX}}$ Let $f \in L^1_{loc}(\Omega)$.

Then $T_f \in \mathcal{D}'_0(\Omega)$:

Let $K \subset \Omega$ be compact.

Put $c = c_K := \int_K |f|\,dx$.

Then for $\phi \in \mathcal{D}(K)$,

$$|\langle T_f, \phi \rangle| \leq \int_K |f\phi|\,dx \leq c \sup|\phi|.$$

Generalization: Let μ be a locally finite Borel measure on Ω.

Put $T_\mu(\phi) := \int_\Omega \phi \, d\mu$, $\phi \in \mathcal{D}(\Omega)$.

Then $T_\mu \in \mathcal{D}'_0(\Omega)$: exercise.

Particular case: δ_{x_0} Dirac's delta function at $x_0 \in \Omega$

$\boxed{\text{EX}}$ Let $x_0 \in \Omega$ and $\alpha \in \mathbb{N}_0^n$.

Put $\langle S, \phi \rangle := (\partial^\alpha \phi)(x_0)$, $\phi \in \mathcal{D}(\Omega)$.

Claim. $S \in \mathcal{D}'_{|\alpha|}(\Omega)$, in fact, it has order $|\alpha|$.

For $K \subset \Omega$ compact, $\phi \in \mathcal{D}(K)$,

$|\langle S, \phi \rangle| = |(\partial^\alpha \phi)(x_0)|$.

This is 0 when $x_0 \notin K$.

When $x_0 \in K$, then

$$|\langle S, \phi \rangle| \leqq \sup |(\partial^\alpha \phi)|$$

and thus order is at most $|\alpha|$.

If $\alpha = 0$, then the order is 0.

Assume $\alpha \neq 0$. Could S have order at most $|\alpha| - 1$?

Suppose it did. Take $r>0$

so $K := \overline{B_r(x_0)} \subset \Omega$.

Because S has order at most $|\alpha|-1$ we find $c = c_K \geq 0$ so

$$\circledast \quad |\langle S, \phi \rangle| \leq c \sum_{|\beta| \leq |\alpha|-1} \sup |\partial^\beta \phi|$$

for $\phi \in \mathcal{D}(K)$. Here $\langle S, \phi \rangle = (\partial^\alpha \phi)(x_0)$.

Take for $\varepsilon \in (0,r)$,

$$\phi(x) = \frac{(x-x_0)^\alpha}{\alpha!} \frac{\rho_\varepsilon(x-x_0)}{\rho_\varepsilon(0)}, \quad x \in \Omega.$$

Then $\phi \in \mathcal{D}(K)$ and since

$$\partial^\beta \left(\frac{(x-x_0)^\alpha}{\alpha!} \right) \Big|_{x=x_0} = \begin{cases} 1 & \text{if } \beta = \alpha \\ 0 & \text{if } \beta \neq \alpha \end{cases}$$

the generalized Leibniz rule gives for $|\beta| \leqq |\alpha|-1$

$$|\partial^\beta \phi(x)| = \Big|\sum_{\gamma \leqq \beta} \binom{\beta}{\gamma} \partial^\gamma \frac{(x-x_0)^\alpha}{\alpha!} \partial^{\beta-\gamma} \frac{\rho_\varepsilon(x-x_0)}{\rho_\varepsilon(0)}\Big|$$

$$\leqq \sum_{\gamma \leqq \beta} \binom{\beta}{\gamma} |x-x_0|^{|\alpha|-|\gamma|} \varepsilon^{|\gamma|-|\beta|} \frac{|\partial^{\beta-\gamma} \rho(\frac{x-x_0}{\varepsilon})|}{\rho(0)}$$

$$\leqq \sum_{\gamma \leqq \beta} \binom{\beta}{\gamma} \frac{\sup |\partial^{\beta-\gamma} \rho|}{\rho(0)} \varepsilon^{|\alpha|-|\beta|}$$

$$\leqq \underbrace{\sum_{\gamma \leqq \beta} \binom{\beta}{\gamma} \frac{\sup |\partial^{\beta-\gamma} \rho|}{\rho(0)}}_{C_\beta} \cdot \varepsilon$$

a constant whose value isn't important.

An upper bound for RHS of ⊛:

$$C \sum_{|\beta| \leqq |\alpha|-1} C_\beta \cdot \varepsilon$$

LHS of (*) is

$$|(\partial^\alpha \phi)(x_0)| = 1$$

can't be bounded above by

$$c \sum_{|\beta| \leq |\alpha|-1} c_\beta \cdot \varepsilon$$

for all $\varepsilon \in (0,r)$! $\lightning$

S can't have order at most $|\alpha|-1$, and so must have order $|\alpha|$.