B4.3 Distribution Theory MT20

Lecture 8:

- Mollification and approximation
- The adjoint identity scheme revisited
- Review of the divergence theorem and consequences

(pp. 39–43 in lecture notes)

Recall from previous lecture:

If $u \in \mathcal{D}'(\mathbb{R}^n)$ and $(\rho_\varepsilon)_{\varepsilon>0}$ is the standard mollifier, then

$$\rho_\varepsilon * u \in C^\infty(\mathbb{R}^n), \quad \rho_\varepsilon * u \xrightarrow[\varepsilon \searrow 0]{} u \text{ in } \mathcal{D}'(\mathbb{R}^n)$$

and we have

$$(\rho_\varepsilon * u)(x) = \langle u, \rho_\varepsilon(x-\cdot)\rangle$$

that is u applied to the test function

$$y \mapsto \rho_\varepsilon(x-y).$$

This is a very useful result. It turns out that by use of it many results about distributions can be proved by first establishing them for C^∞ functions and then extend them to distributions by approximation.

Examples will follow in lecture 9 and later lectures.

Approximation theorem

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open.
If $u \in \mathcal{D}'(\Omega)$, then there exists a sequence (u_j) of test functions $u_j \in \mathcal{D}(\Omega)$ so

$$u_j \to u \text{ in } \mathcal{D}'(\Omega).$$

See lecture notes for the proof when $\Omega = \mathbb{R}^n$. General case is similar. We skip the details here.

Instead we discuss the implica= tion for the adjoint identity scheme.

Recall:

$T : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ linear map

'operation on test functions that we would like to extend to distributions'

If there exists

$S : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ linear, $\mathcal{D}$-continuous map so the adjoint identity

$$\int_\Omega T(\phi)\psi \, dx = \int_\Omega \phi S(\psi) \, dx \quad , \forall \phi, \psi \in \mathcal{D}(\Omega)$$

holds, then $\overline{T} : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ defined by rule

$$\langle \overline{T}(u), \phi \rangle := \langle u, S(\phi) \rangle, \phi \in \mathcal{D}(\Omega)$$

does the job.

$\overline{T} : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is linear and $\mathcal{D}'$– continuous.

In fact by Approximation Theorem $\overline{T}$ is <u>the extension by $\mathcal{D}'$– continuity</u> of T:

Let $u \in \mathcal{D}'(\Omega)$.

Take $u_j \in \mathcal{D}(\Omega)$ so $u_j \to u$ in $\mathcal{D}'(\Omega)$.

By adjoint identity: for $\phi \in \mathcal{D}(\Omega)$

$$\langle T(u_j), \phi \rangle = \int_\Omega T(u_j)\phi \, dx = \int_\Omega u_j \cdot S(\phi) dx$$

$$= \langle u_j, S(\phi) \rangle \longrightarrow \langle u, S(\phi) \rangle$$

$$\langle u, S(\phi) \rangle = \langle \overline{T}(u), \phi \rangle.$$

The <u>Gauss-Green formula</u> or the <u>divergence theorem</u> can be seen as n-dimensional version of the fundamental theorem of calculus.

$\boxed{\text{FTC}}$: If $f : [a,b] \to \mathbb{C}$ is a C^1 function then $f(b) - f(a) = \int_a^b f'(x)\,dx$.
(In prelims you saw another version too but the above statement is the essence.)

When we combine FTC with Leibniz' rule we get the <u>integration-by-parts rule</u>:

If $f, g : [a,b] \to \mathbb{C}$ are C^1, then

$$\int_a^b f'(x)\,g(x)\,dx = \Big[f(x)g(x)\Big]_{x=a}^{x=b} - \int_a^b f(x)\,g'(x)\,dx.$$

It is the same with the divergence theorem as we'll see!

$\boxed{\text{DEF}}$ Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open.

An open subset ω of Ω is a <u>C^1 subset of Ω</u> if there exists $\psi \in C^1(\Omega)$ so

$$\omega = \{x \in \Omega : \psi(x) < 0\},$$

$$\Omega \cap \partial\omega = \{x \in \Omega : \psi(x) = 0\} \text{ and}$$

$$\nabla\psi(x) \neq 0 \quad \text{for all } x \in \Omega \cap \partial\omega.$$

<u>Note</u> ψ is called a <u>defining function for ω</u> and we have $\Omega \setminus \overline{\omega} = \{x \in \Omega : \psi(x) > 0\}$.

$\nabla\psi(x)$ outward normal on $x \in \Omega \cap \partial\omega$.

$$\nu(x) := \frac{\nabla\psi(x)}{|\nabla\psi(x)|}$$ <u>outward unit normal</u> on $\Omega \cap \partial\omega$

$\Omega \cap \partial\omega$ must be nice

need not be nice

$\boxed{\text{The divergence theorem}}$

also called the Gauss-Green formula

Let ω be a C^1 subset of Ω and $V = (V_1, \ldots, V_n) \in \mathcal{D}(\Omega)^n$.

Then

$$\int_{\omega} \operatorname{div} V \, dx = \int_{\partial\omega} V \cdot \nu \, dS_x$$

Remarks More general versions exist.

We do not prove it here and the surface integral on the right-hand side will be treated as though we had carefully defined it — we haven't, but see brief discussion in lecture notes. We shall mainly use the divergence theorem in simple situations where ω is a ball or the set between two concentric balls.

$\boxed{\text{EX}}$ $\Omega = \mathbb{R}$ and $\omega = (a,b)$.

Then ω is a C^1 subset of Ω:

for instance $\psi(x) = +\left(x - \frac{b+a}{2}\right)^2 \div \left(\frac{b-a}{2}\right)^2$

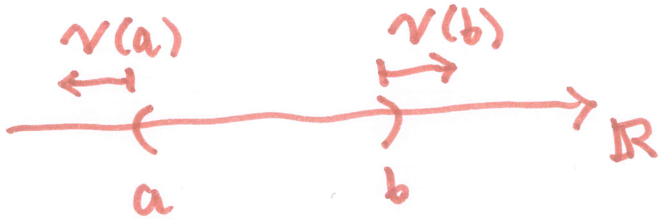
can be used as defining function:

clearly $\psi \in C^1(\mathbb{R})$,

$\omega = \{x \in \mathbb{R} : \psi(x) < 0\}$,

$\partial\omega = \{x \in \mathbb{R} : \psi(x) = 0\} = \{a, b\}$

and $\psi'(a) = -b + a$, $\psi'(b) = b - a$



$\nu(a) = -\frac{b-a}{|b-a|} = -1$,

$\nu(b) = \frac{b-a}{|b-a|} = 1$

If $V \in \mathcal{D}(\mathbb{R})$, then the divergence theorem yields, since for $n=1$, $\operatorname{div} V = V'$,

$$\int_\omega \operatorname{div} V' \, dx = \int_{\partial\omega} V \cdot \nu \, dS_x$$

$$\| \qquad\qquad\qquad \|$$

$$\int_a^b V'(x)\,dx \qquad V(b)\nu(b) + V(a)\nu(a) = V(b) - V(a)$$

$\boxed{\text{EX}}$ $\Omega = \mathbb{R}^2$ and $\omega = B_r(0)$.

Then ω is a C^1 subset of Ω:

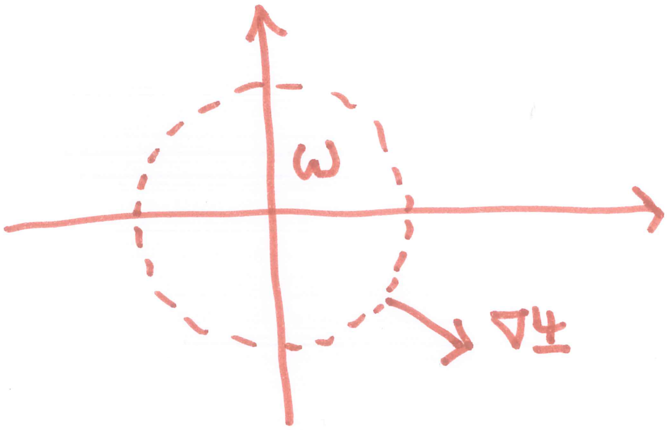
for instance $\Psi(x_1, x_2) = x_1^2 + x_2^2 - r^2$
can be used as defining function:

clearly $\Psi \in C^1(\mathbb{R}^2)$,

$$\omega = \{(x_1, x_2) \in \mathbb{R}^2 : \Psi(x_1, x_2) < 0\},$$

$$\partial\omega = \{(x_1, x_2) \in \mathbb{R}^2 : \Psi(x_1, x_2) = 0\} = \partial B_r(0)$$

and $\nabla\Psi(x_1, x_2) = (2x_1, 2x_2) \neq (0,0)$
for $x_1^2 + x_2^2 = r^2$.



$$\nu(x_1, x_2) = \left(\frac{x_1}{r}, \frac{x_2}{r}\right)$$

outward unit normal

If $V = (V_1, V_2) \in \mathcal{D}(\mathbb{R}^2)^2$, then the divergence theorem yields

$$\int_{B_r(0)} \left(\frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2}\right) d(x_1, x_2) = \int_{\partial B_r(0)} V \cdot \nu \, dS_x$$

Note that $(x_1, x_2) = (r\cos t, r\sin t), t \in [0, 2\pi]$, is a parametrization of $\partial B_r(0)$, so

$$\int_{\partial B_r(0)} V \cdot \nu \, dS_x = \int_0^{2\pi} V(r\cos t, r\sin t) \cdot \nu(r\cos t, r\sin t) \, r\,dt$$

where we used $dS_x = |(x_1'(t), x_2'(t))| \, dt$
$= r\,dt$.

Observe also that in this case

$$(x_1'(t), x_2'(t)) = (-r\sin t, r\cos t)$$

is a tangent vector to $\partial B_r(0)$ at $(x_1(t), x_2(t)) = (r\cos t, r\sin t)$ and that $\mathcal{I} = (-\sin t, \cos t)$ is the unit tangent vector for the parametrization $(x_1(t), x_2(t)) = (r\cos t, r\sin t)$. We have

$$\nu(r\cos t, r\sin t) = -\mathcal{I}^{\perp}.$$

Integration by parts using the divergence theorem

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be an open set and let ω be a C^1 subset of Ω.

Then for $\phi \in \mathcal{D}(\Omega)$, $V \in \mathcal{D}(\Omega)^n$,

$$\int_\omega \nabla\phi \cdot V \, dx = \int_{\partial\omega} \phi V \cdot \nu \, dS_x - \int_\omega \phi \operatorname{div} V \, dx.$$

Simply note that Leibniz yields

$$\operatorname{div}(\phi V) = \nabla\phi . V + \phi \operatorname{div} V$$

and then use the divergence theorem.

$\boxed{\text{EX}}$ Let ω be a C^1 subset of Ω. Clearly $\mathbb{1}_\omega(x) = \begin{cases} 1 & \text{if } x \in \omega \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \omega \end{cases}$

is a local L^1 function, hence a regular distribution on Ω.

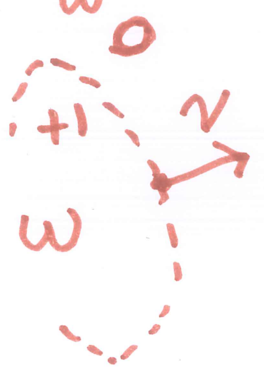
What is the distributional gradient of $\mathbb{1}_\omega$?

<u>Claim:</u>

$$\nabla \mathbb{1}_\omega = (\partial_1 \mathbb{1}_\omega, \dots, \partial_n \mathbb{1}_\omega)$$

$\mathbb{1}_\omega$

$$= -\nu \, dS_x$$



outward unit normal on $\partial\omega \cap \Omega$

(n-1)-dimensional surface area on $\Omega \cap \partial\omega$

<u>Note</u>: $-\nu$ is therefore the inward unit normal on $\partial\omega \cap \Omega$.

It is a generalization to n-dimensions of the formula: $H' = \delta_0$

Let $\phi \in \mathcal{D}(\Omega)$. Then for $j \in \{1, \dots, n\}$

$$\langle \partial_j \mathbb{1}_\omega , \phi \rangle = - \langle \mathbb{1}_\omega , \partial_j \phi \rangle$$
$$= - \int_\omega \partial_j \phi \, dx$$

Put $V = \phi e_j$, $(e_i)_{i=1}^n$ standard basis for $\mathbb{R}^n$

Calculate $\operatorname{div} V = \partial_j \phi$ and use the divergence theorem:

$$- \int_\omega \partial_j \phi \, dx = - \int_\omega \operatorname{div} V \, dx$$
$$= - \int_{\partial \omega} \phi \, \nu . e_j \, dS_x$$

and hence $\partial_j \mathbb{1}_\omega = - \nu . e_j \, dS_x$. $\square$

$\boxed{\text{EX}}$ Put for $x \in \mathbb{R}^n \setminus \{0\}$,

$$G_0(x) = G_0^n(x) := \begin{cases} -\dfrac{1}{(n-2)\omega_{n-1}} |x|^{2-n} & , n \in \mathbb{N} \setminus \{2\} \\ \dfrac{1}{\omega_1} \log|x| & , n = 2, \end{cases}$$

where ω_{n-1} is the $(n-1)$-dimensional area of the unit sphere S^{n-1} in $\mathbb{R}^n$ (so $\omega_0 = 2$, $\omega_1 = 2\pi$, $\omega_2 = 4\pi$).

G_0^2 is called <u>the logarithmic potential</u>

G_0^3 is called <u>the Newtonian potential</u>

In all dimensions $n \in \mathbb{N}$:

$$\partial_j G_0^n = \frac{1}{\omega_{n-1}} \frac{x_j}{|x|^n} \qquad \begin{array}{l} x \in \mathbb{R}^n \setminus \{0\} \\ j \in \{1, \dots, n\} \end{array}$$

G_0^n is C^∞ away from 0, $G_0^n \in L^1_{loc}(\mathbb{R}^n)$

and $\Delta G_0^n = \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$.

$\boxed{\text{Pf.}}$ It is clear that $G_0^n \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and that $G_0^1(x) = \frac{1}{2}|x| \in L^1_{loc}(\mathbb{R})$.

To see that also $G_0^n \in L^1_{loc}(\mathbb{R}^n)$ for $n>1$ we calculate in <u>polar coordinates</u>.

Assume $n>2$ ($n=2$ is an exercise)

Fix $R>0$. Then for $\varepsilon \in (0,R)$:

$$\int_{B_R(0)\setminus B_\varepsilon(0)} |G_0^n(x)|\,dx = \frac{1}{(n-2)\omega_{n-1}} \int_{B_R(0)\setminus B_\varepsilon(0)} \frac{dx}{|x|^{n-2}}$$

$$= \frac{1}{(n-2)\omega_{n-1}} \int_\varepsilon^R \int_{\partial B_r(0)} \frac{dS_x}{|\cancel{x}|^{n-2}}\,dr$$

$$= \frac{1}{(n-2)\omega_{n-1}} \int_\varepsilon^R \int_{\partial B_r(0)} \frac{dS_x}{r^{n-2}}\,dr$$

$$= \frac{1}{(n-2)\omega_{n-1}} \int_\varepsilon^R \frac{\text{area}(\partial B_r(0))}{r^{n-2}}\,dr$$

$$= \frac{1}{(n-2)\omega_{n-1}} \int_\varepsilon^R \frac{\omega_{n-1} r^{n-1}}{r^{n-2}} \, dr$$

$$= \frac{1}{n-2} \int_\varepsilon^R r \, dr \le \frac{1}{n-2} \cdot \frac{R^2}{2}$$

hence

$$\int_{B_R(0)} |G_0^n(x)| \, dx < \infty$$

for each $R > 0$, so $G_0^n \in L^1_{loc}(\mathbb{R}^n)$.

Next we calculate the distributional Laplacian of G_0^n. Assume $n=2$ (other cases is an exercise or see lecture notes):

$$G_0^2(x) = \frac{1}{\omega_1} \log|x| = \frac{1}{2\pi} \log|x|$$

and

$$\partial_j G_0^2(x) = \frac{1}{2\pi} \frac{x_j}{|x|^2} \qquad j = 1,2$$

Fix $\phi \in \mathcal{D}(\mathbb{R}^2)$. Now

$$\langle \Delta G_0^2, \phi \rangle = \langle G_0^2, \Delta\phi \rangle = \lim_{s \to 0+} \int_{\mathbb{R}^2 \setminus B_s(0)} G_0^2 \Delta\phi \, dx$$

and since G_0^2 is C^∞ away from 0

and $\Delta G_0^2 = 0$ there (usual derivatives)

so if $\operatorname{supp}\phi \subset B_t(0)$:

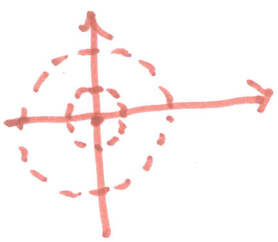
$$\int_{\mathbb{R}^2 \setminus B_s(0)} G_0^2 \Delta\phi \, dx = \int_{B_t(0) \setminus B_s(0)} (G_0^2 \Delta\phi - \Delta G_0^2 \phi) \, dx$$

$$= \int_{B_t(0) \setminus B_s(0)} \operatorname{div}(G_0^2 \nabla\phi - \phi \nabla G_0^2) \, dx$$

divergence theorem

$$= \int_{\partial B_t(0)} (G_0^2 \nabla\phi - \phi \nabla G_0^2) \cdot \frac{x}{t} \, dS_x$$

outward unit normal



$$+ \int_{\partial B_s(0)} (G_0^2 \nabla\phi - \phi \nabla G_0^2) \cdot \left(-\frac{x}{s}\right) dS_x$$

$\phi = 0$ on $\partial B_t(0)$

$$= \int_{\partial B_s(0)} \left(\phi \nabla G_0^2 - G_0^2 \nabla \phi \right) \cdot \frac{x}{s} \, dS_x$$

Insert $G_0^2 = \frac{1}{2\pi} \log|x|$, so

$$\nabla G_0^2 = \frac{1}{2\pi} \frac{x}{|x|^2}$$

to get

$$\langle \Delta G_0^2, \phi \rangle = \lim_{s \to 0+} \frac{1}{2\pi} \int_{\partial B_s(0)} \left(\phi(x) \frac{x}{|x|^2} - \log|x| \nabla\phi(x) \right) \cdot \frac{x}{s} \, dS_x$$

$$= \lim_{s \to 0+} \left[\frac{1}{2\pi s} \int_{\partial B_s(0)} \phi(x) \, dS_x - \frac{\log s}{2\pi} \int_{\partial B_s(0)} \nabla\phi(x) \cdot \frac{x}{s} \, dS_x \right]$$

$$= \phi(0)$$

since $\frac{1}{2\pi s} \int_{\partial B_s(0)} \phi \, dS_x = \phi(0)$ and

$$\left| \frac{-\log s}{2\pi} \int_{\partial B_s(0)} \nabla\phi(x) \cdot \frac{x}{s} \, dS_x \right| = \left| \frac{-\log s}{2\pi} \int_{\partial B_s(0)} \left(\nabla\phi(x) - \nabla\phi(0) \right) \cdot \frac{x}{s} \, dS_x \right|$$

$$\leq \frac{-\log s}{2\pi} \max|\nabla^2\phi| \cdot s \cdot 2\pi s \xrightarrow[s \to 0]{} 0 .$$

$\boxed{\text{EX}}$ The Cauchy-Riemann differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

act on distributions defined on open subsets of $\mathbb{C}$.

<u>Claim</u> $\frac{\partial}{\partial \bar{z}}\left(\frac{1}{\pi z}\right) = \delta_0$,

$$\frac{\partial}{\partial z}\left(\frac{1}{\pi \bar{z}}\right) = \delta_0 .$$

We do the latter:

recall $\Delta = \frac{4\partial^2}{\partial z \partial \bar{z}}$

and by previous example

$$\delta_0 = \Delta G_0^2 = \Delta\left(\frac{1}{2\pi}\log|z|\right)$$

$$= \frac{1}{4\pi} 4\frac{\partial^2}{\partial z \partial \bar{z}}\left(\log|z|^2\right)$$

$$= \frac{1}{\pi}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \bar{z}}\log(z\bar{z})\right)$$

$$= \frac{1}{\pi}\frac{\partial}{\partial z}\left(\frac{1}{z\bar{z}} z\right)$$

$$= \frac{\partial}{\partial z}\left(\frac{1}{\pi\bar{z}}\right). \quad \square$$