(B4.3) Distribution Theory MT20

Lecture 9: Calculus for distributions

- The constancy theorem
- The fundamental theorem of calculus for distributions
- Exx of solution of DDEs within distributions

(pp. 43–47 in lecture notes)

The constancy theorem

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open & connected.
If $u \in \mathcal{D}'(\Omega)$ and $\nabla u = 0$ in $\mathcal{D}'(\Omega)^n$,
then $u = c$ for some constant $c \in \mathbb{C}$.

$\boxed{\text{Pf.}}$ We only give the proof for the case $\Omega = \mathbb{R}^n$. In this case we use mollification to deduce the result from the corresponding result for C^∞ functions.

Let $u \in \mathcal{D}'(\mathbb{R}^n)$ with $\nabla u = 0$ in $\mathcal{D}'(\mathbb{R}^n)$. For the standard mollifier $(\rho_\varepsilon)_{\varepsilon>0}$ on $\mathbb{R}^n$ we consider $\rho_\varepsilon * u$. From last lecture: $\rho_\varepsilon * u \in C^\infty(\mathbb{R}^n)$,

$$\nabla(\rho_\varepsilon * u) = \rho_\varepsilon * \nabla u = 0,$$

$$\rho_\varepsilon * u \to u \text{ in } \mathcal{D}'(\mathbb{R}^n) \text{ as } \varepsilon \searrow 0.$$

From the usual constancy theorem $\rho_\varepsilon * u = c_\varepsilon$ for some constant $c_\varepsilon \in \mathbb{C}$.

Here (with $\rho = \rho_1$, the standard mollifier kernel):

$$c_\varepsilon = c_\varepsilon \int_{\mathbb{R}^n} \rho \, dx = \int_{\mathbb{R}^n} (\rho_\varepsilon * u)\rho \, dx$$

$$= \langle \rho_\varepsilon * u, \rho \rangle \xrightarrow[\varepsilon \searrow 0]{} \langle u, \rho \rangle =: c$$

Now for $\phi \in \mathcal{D}(\mathbb{R}^n)$ we calculate

$\langle \rho_\varepsilon * u, \phi \rangle = c_\varepsilon \int_{\mathbb{R}^n} \phi \, dx$, so

taking $\varepsilon \searrow 0$ we find

$$\langle u, \phi \rangle = c \int_{\mathbb{R}^n} \phi \, dx$$

as required. $\square$

$\boxed{\text{EX}}$ Find the general solution to $y' + ay = 0$ in $\mathcal{D}'(\mathbb{R})$.

where $a \in C^\infty(\mathbb{R})$.

Solution: Let A be a primitive of a, say $A(x) = \int_0^x a(t)\,dt$, $x \in \mathbb{R}$.

Since $e^A \in C^\infty(\mathbb{R})$ we get by use of Leibniz if $y \in \mathcal{D}'(\mathbb{R})$ is a solution:

$$0 = e^A(y' + ay) = (e^A y)'.$$

Therefore the constancy theorem gives $e^A y = c$ for some $c \in \mathbb{C}$,

that is, $y = ce^{-A}$.

Conversely it's clear that this is a solution so GS is

$$y = ce^{-A}, \quad c \in \mathbb{C}.$$

The solution within $\mathcal{D}'(\mathbb{R})$ is in this case the same as within $C^\infty(\mathbb{R})$.

In order to solve the inhomogeneous ODE $y' + ay = f$ when $f \in \mathcal{D}'(\mathbb{R})$

we need

$\boxed{\text{The fundamental theorem of calculus for distributions} \quad (\mathrm{FTC}^d)}$

Let $f \in \mathcal{D}'(a,b)$. Then there exists $F \in \mathcal{D}'(a,b)$ so $F' = f$ in $\mathcal{D}'(a,b)$. Furthermore GS to $u' = f$ in $\mathcal{D}'(a,b)$ is $u = F + c$, $c \in \mathbb{C}$.

$\boxed{\text{Remark}}$ A formula for the primitive:

If $\chi \in \mathcal{D}(a,b)$ and $\int_a^b \chi \, dx = 1$,

then

$$E(\phi)(x) := \int_a^x \phi(t)\,dt - \int_a^b \phi(t)\,dt \int_a^x \chi(t)\,dt$$

for $x \in (a,b)$, $\phi \in \mathcal{D}(a,b)$ defines a linear $\mathcal{D}$-continuous map

$$E : \mathcal{D}(a,b) \longrightarrow \mathcal{D}(a,b).$$

For $\phi \in \mathcal{D}(a,b)$ we have $E(\phi') = \phi$ so E is a left-inverse to $\frac{d}{dx}$ on $\mathcal{D}(a,b)$

and we may take

$$\langle F, \phi \rangle = - \langle f, E(\phi) \rangle, \quad \phi \in \mathcal{D}(a,b).$$

$\boxed{\text{Pf.}}$ Clearly we have for $\phi \in \mathcal{D}(a,b)$ by usual FTC that

$$E(\phi)(x) = \int_a^x \phi(t)\,dt - \int_a^b \phi(t)\,dt \int_a^x \chi(t)\,dt$$

is $C^1(a,b)$ with

$$E(\phi)'(x) = \phi(x) - \int_a^b \phi(t)\,dt\, \chi(x),$$

so $E(\phi) \in C^\infty(a,b)$.

Take $[c,d] \subset (a,b)$ so large that

$$\mathrm{supp}(\phi), \ \mathrm{supp}(\chi) \subseteq [c,d].$$

Then we check that

$$E(\phi)(x) = 0 \quad \text{for} \quad x \in (a,c)$$

and for $x \in (d,b)$:

$$E(\phi)(x) = \int_a^b \phi(t)\,dt - \int_a^b \phi(t)\,dt \int_a^b \chi(t)\,dt = 0$$

Thus $\mathrm{supp}\, E(\phi) \subseteq [c,d]$, so $E(\phi) \in \mathcal{D}(a,b)$.

By inspection $E : \mathcal{D}(a,b) \to \mathcal{D}(a,b)$ is linear. We claim it's $\mathcal{D}$-continuous:

If $\phi_j \to 0$ in $\mathcal{D}(a,b)$, so

$\operatorname{supp} \phi_j \subseteq [c,d]$, some $[c,d] \subset (a,b)$,

and $\sup |\phi_j^{(k)}| \xrightarrow[j]{} 0$ for all k,

then enlarging $[c,d]$ if necessary so $\operatorname{supp} \chi \subseteq [c,d]$ too, we get

$\operatorname{supp} E(\phi_j) \subseteq [c,d]$.

Clearly also $\sup |E(\phi_j)^{(k)}| \xrightarrow[j]{} 0$ for all $k \in \mathbb{N}_0$, hence $E(\phi_j) \to 0$ in $\mathcal{D}(a,b)$.

By usual FTC: $E(\phi') = \phi$ for all $\phi \in \mathcal{D}(a,b)$.

Define $\langle F, \phi \rangle := -\langle f, E(\phi) \rangle$, $\phi \in \mathcal{D}(a,b)$.

It is not difficult to check that $F \in \mathcal{D}'(a,b)$ and for $\phi \in \mathcal{D}(a,b)$:

$$\langle F', \phi \rangle = -\langle F, \phi' \rangle = \langle f, E(\phi') \rangle$$
$$= \langle f, \phi \rangle$$

hence $F' = f$.

The last part about GS of $u' = f$ follows from the constancy theorem and above result. □

EX Let $a \in C^\infty(\mathbb{R})$ and $f \in \mathcal{D}'(\mathbb{R})$.

Find GS of ODE

$$y' + ay = f \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

A linear equation so we only need to find a particular solution to add to GS for corresponding homogeneous equation. The procedure for that is however almost the same as the way we found ~~GS~~ to the homogeneous equation:

Put $A(x) = \int_0^x a(t)\,dt$, $x \in \mathbb{R}$.

If $y \in \mathcal{D}'(\mathbb{R})$ is a solution, then

$$e^A f = e^A (y' + ay) = (e^A y)'.$$

By FTC^d we find $G \in \mathcal{D}'(\mathbb{R})$ so $G' = e^A f$. Indeed,

$$\langle G, \phi \rangle := -\langle e^A f, E(\phi) \rangle, \quad \phi \in \mathcal{D}(\mathbb{R}),$$

will do. Now from

$$G' = (e^A y)', \text{ that is, } (e^A y - G)' = 0$$

the constancy theorem yields a constant $c \in \mathbb{C}$ so $e^A y - G = c$,

or $y = c e^{-A} + G e^{-A}$.

Conversely if y has this form, then (recall $A' = a$, $G' = e^{A} f$)

$$y' + ay = (ce^{-A} + Ge^{-A})' + a(ce^{-A} + Ge^{-A})$$

$$= -ce^{-A}a + e^{A}fe^{-A} + Ge^{-A}(-a)$$

$$+ ace^{-A} + aGe^{-A} = f.$$

Thus GS is $y = (c+G)e^{-A}$, $c \in \mathbb{C}$.

$\boxed{\text{EX}}$ Find GS to $y'' + 4y' + 3y = 0$ in $\mathcal{D}'(a,b)$.

Characteristic equation has roots $-1, -3$ so we can factorize

$$\frac{d^2}{dx^2} + 4\frac{d}{dx} + 3I = \left(\frac{d}{dx} + I\right)\left(\frac{d}{dx} + 3I\right)$$

Assume $y \in \mathcal{D}'(a,b)$ is a solution.
Then

$$0 = y'' + 4y' + 3y = \left(\frac{d}{dx} + I\right)(y' + 3y)$$

Hence if $z = y' + 3y$, then

$z \in \mathcal{D}'(a,b)$ and $0 = z' + z$.

We solve as usual: multiply by e^x

and use Leibniz

$$0 = (z' + z)e^x = (e^x z)'$$

so by constancy theorem $e^x z = c$

for some $c \in \mathbb{C}$, that is, $z = ce^{-x}$.

Now $z = y' + 3y$, so consider

$$y' + 3y = ce^{-x}.$$

Multiply by e^{3x} and use Leibniz:

$$ce^{+2x} = (y' + 3y)e^{3x} = (e^{3x} y)'$$

hence

$$0 = \left(e^{3x} y - \frac{c}{2} e^{2x}\right)'$$

and so by constancy theorem again

$e^{3x} y - \frac{c}{2} e^{2x} = c_1$ for some $c_1 \in \mathbb{C}$.

Therefore y must be of form

$$Ae^{-3x} + Be^{-x} \qquad A, B \in \mathbb{C}.$$

Conversely it is clear that these functions are solutions, so GS is

$$y = Ae^{-3x} + Be^{-x}, \qquad A, B \in \mathbb{C}. \quad (*)$$

Again we note that this is exactly the same solution as we would have found when looking in $C^\infty(a,b)$.

Let $H : \mathbb{R} \to \mathbb{R}$ be Heaviside's function and consider

$$y'' + 4y' + 3y = H \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

What is GS?

A linear equation so we just need a particular solution to add to $(*)$

There are various approaches – you might have seen one in DE2 or Integral Transforms (from Part A) but it's better to wait with that until we have defined localization of distributions. Instead we retrace our calculation for finding GS to the homogeneous equation:

$$z = y' + 3y \quad \text{and} \quad z' + z = H.$$

Then $e^x H = e^x(z'+z) = (e^x z)'$ and we can use FTC^d to find a primitive to $e^x H$ ... but $e^x H$ is a regular distribution, so there is an easier way!

Put $h(x) = \int_0^x e^t H(t)\,dt = \begin{cases} e^x - 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$= (e^x - 1)H(x)$

Note that h is piecewise C^1, but not differentiable at 0. By Leibniz

$$h' = e^x H(x) + (e^x - 1)\delta_0 = e^x H(x)$$

as required.

Now calculate:

$$h' = e^x H = e^x(z' + z) = (e^x z)'$$

so $(e^x z - h)' = 0$ and therefore

$$z = e^{-x}(h + c) = (1 - e^{-x})H(x) + ce^{-x}.$$

Next $z = y' + 3y$, so consider

$$y' + 3y = (h + c)e^{-x}$$

As before:

$$(h+c)e^{2x} = e^{3x}(y' + 3y) = (e^{3x}y)'$$

a continuous function, so by usual

FTC $\quad e^{3x}y = \int_0^x (h(t) + c)e^{2t}\,dt + c_1$

$c_1 \in \mathbb{C}$

so

$$y = e^{-3x}\left(\left(\frac{e^{3x}-1}{3} + \frac{e^{2x}-1}{2}\right)H(x) + \frac{c}{2}(e^{2x}-1) + c_1 \right)$$

It is not difficult to check that these functions are solutions in $\mathcal{D}'(\mathbb{R})$ too, hence GS in $\mathcal{D}'(\mathbb{R})$ is

$$y = \left(\frac{1-e^{-3x}}{3} + \frac{e^{-x}-e^{-3x}}{2}\right)H(x)$$

$$+ Ae^{-3x} + Be^{-x} \qquad A, B \in \mathbb{C}$$

Note that none of these solutions are classical solutions — the GS within $C^2(\mathbb{R})$ is empty!

In fact we didn't have to solve the equation to see that — why?

For the same reason is GS within the class of twice differentiable functions empty.

Proposition Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open.

If $u \in \mathcal{D}'(\Omega)$ and $\nabla u \in C(\Omega)^n$, then $u \in C^1(\Omega)$.

Pf We only do the case $\Omega = \mathbb{R}$:

$u \in \mathcal{D}'(\mathbb{R})$ and $u' \in C(\Omega)$.

In this situation we use a molli= fier argument. For the standard mollifier $(\rho_\varepsilon)_{\varepsilon>0}$ in $\mathbb{R}$ note

$\rho_\varepsilon * u \in C^\infty(\mathbb{R})$ and $(\rho_\varepsilon * u)' = \rho_\varepsilon * u'$.

By usual FTC:

$$(*) \quad (\rho_\varepsilon * u)(x) = (\rho_\varepsilon * u)(0) + \int_0^x (\rho_\varepsilon * u')(t)\,dt$$

Multiply by $\rho(x)$ and integrate over $x \in \mathbb{R}$:

$$+\quad \langle \rho_\varepsilon * u, \rho \rangle = (\rho_\varepsilon * u)(0) + \int_{-\infty}^{\infty} \int_0^x (\rho_\varepsilon * u')(t)\,dt\,\rho(x)\,dx$$

Note that $\langle \rho_\varepsilon * u, \rho \rangle \xrightarrow[\varepsilon \searrow 0]{} \langle u, \rho \rangle$

and, since u' is continuous,

$(\rho_\varepsilon * u')(t) \xrightarrow[\varepsilon \searrow 0]{} u'(t)$ locally uniformly in $t \in \mathbb{R}$.

But then

$$\int_{-\infty}^{\infty} \int_0^x (\rho_\varepsilon * u')(t)\,dt\,\rho(x)\,dx \xrightarrow[\varepsilon \searrow 0]{} \int_{-\infty}^{\infty} \int_0^x u'(t)\,dt\,\rho(x)\,dx$$

and consequently from $+$

$$(\rho_\varepsilon * u)(0) \xrightarrow[\varepsilon \searrow 0]{} C := \langle u, \rho \rangle - \int_{-\infty}^{\infty} \int_0^x u'(t)\,dt\,\rho(x)\,dx.$$

Now using this in $*$ we see that $(\rho_\varepsilon * u)(x)$ converges locally uniformly in $x \in \mathbb{R}$ as $\varepsilon \searrow 0$. Since it also converges in $\mathcal{D}'(\mathbb{R})$ to u

it follows that u is a local uniform limit of continuous functions hence must be continuous. Now return to ⊛ and take $\varepsilon > 0$:

$$u(x) = u(0) + \int_0^x u'(t)\,dt$$

for $x \in \mathbb{R}$. Here u' is the distri=
butional derivative of u and since u' is assumed continuous it follows easily that u is C^1 with usual derivative u'. $\square$