B4.3 Distribution Theory MT20

Lecture 13: Distributions of compact support and convolution

1. Characterization of distributions of compact support
2. Discussion of the boundedness property
3. Distributions supported at a single point
4. Convolution of distribution with test function revisited
5. Convolution of a compactly supported distribution with a C^∞ function

The material corresponds to pp. 59–65 in the lecture notes and should be covered in Week 7.

Distributions of compact support: Let $u \in \mathscr{D}'(\Omega)$ be a compactly supported distribution on Ω, a non-empty open subset of $\mathbb{R}^n$.

In the last lecture we saw that, given any $\psi \in \mathscr{D}(\Omega)$ satisfying $\psi = 1$ on $\mathrm{supp}(u)$ the functional

$$\mathrm{C}^\infty(\Omega) \ni \phi \mapsto \langle u, \psi\phi \rangle$$

is independent of the choice of ψ and defines an extension of u. Furthermore this is the unique such extension satisfying the condition that it vanishes on $\phi \in \mathrm{C}^\infty(\Omega)$ with $\mathrm{supp}(u) \cap \mathrm{supp}(\phi) = \emptyset$. We therefore also denote this extension by u.

Corresponding to the compact neighbourhood $K = \mathrm{supp}(\psi)$ of $\mathrm{supp}(u)$ we found constants $c = c_K \geq 0$, $m = m_K \in \mathbb{N}_0$ so

$$\left| \langle u, \phi \rangle \right| \leq c \sum_{|\alpha| \leq m} \sup_K \left| \partial^\alpha \phi \right| \tag{1}$$

holds for all $\phi \in \mathrm{C}^\infty(\Omega)$.

Definition of the class $\mathscr{E}'(\Omega)$: it is the set of all linear functionals $v\colon \mathrm{C}^\infty(\Omega) \to \mathbb{C}$ for which there exist a compact subset L of Ω and constants $c \geq 0$, $m \in \mathbb{N}_0$ such that

$$\bigl|v(\phi)\bigr| \leq c \sum_{|\alpha| \leq m} \sup_L |\partial^\alpha \phi| \tag{2}$$

holds for all $\phi \in \mathrm{C}^\infty(\Omega)$. (L. Schwartz who also wrote $\mathscr{E}(\Omega)$ for $\mathrm{C}^\infty(\Omega)$.)

We refer to a bound of the form (2) as an $\mathscr{E}'$ bound and note that any compactly supported distribution on Ω admits a unique extension to $\mathrm{C}^\infty(\Omega)$ that belongs to $\mathscr{E}'(\Omega)$.

It is clear that $\mathscr{E}'(\Omega)$ is a vector space under the usual definitions of *addition* and *multiplication with scalar*.

What are the functionals in the class $\mathscr{E}'(\Omega)$?

Assume $v \in \mathscr{E}'(\Omega)$, so $v\colon \mathrm{C}^\infty(\Omega) \to \mathbb{C}$ is a linear functional satisfying an $\mathscr{E}'$-bound (say (2) holds).

Then the restriction of v to $\mathscr{D}(\Omega)$, $v|_{\mathscr{D}(\Omega)}$, is a distribution on Ω:
it is clearly linear and by the $\mathscr{E}'$-bound (2) it has the boundedness property.

The restriction also has compact support contained in L:
if $\phi \in \mathscr{D}(\Omega)$ and $\mathrm{supp}(\phi) \cap L = \emptyset$, then we infer from the $\mathscr{E}'$-bound (2) that $v(\phi) = 0$. Therefore

$$\mathrm{supp}(v|_{\mathscr{D}(\Omega)}) \subseteq L, \text{ a compact subset of } \Omega.$$

Characterization of $\mathscr{E}'(\Omega)$ and compactly supported distributions: Let Ω be a non-empty open subset of $\mathbb{R}^n$. Then the set of all compactly supported distributions on Ω coincides with $\mathscr{E}'(\Omega)$ in the following sense:

(i) A compactly supported distribution $u \in \mathscr{D}'(\Omega)$ admits a unique extension to a functional $U \in \mathscr{E}'(\Omega)$. If $\psi \in \mathscr{D}(\Omega)$ satisfies $\psi = 1$ on $\mathrm{supp}(u)$, then $U(\phi) = \langle u, \psi\phi \rangle$ for all $\phi \in C^\infty(\Omega)$ and the definition is independent of ψ. In the $\mathscr{E}'$ bound for U we can take *any* compact neighbourhood K of $\mathrm{supp}(u)$ (and it is important to note that the corresponding constants c_K, m_K in general depend on K).

(ii) The restriction of a functional $U \in \mathscr{E}'(\Omega)$ to $\mathscr{D}(\Omega)$, $U|_{\mathscr{D}(\Omega)}$, is a compactly supported distribution on Ω. If L is the compact set appearing in the $\mathscr{E}'$ bound (2) for U, then the support of the distribution $U|_{\mathscr{D}(\Omega)}$ will be contained in L.

Notation We do not distinguish between u and U and denote both simply by u.

A subtle point about the $\mathscr{E}'$ bound

When $u \in \mathscr{D}'(\Omega)$ has compact support, then its unique extension to a functional in $\mathscr{E}'(\Omega)$ is found by taking $\psi \in \mathscr{D}(\Omega)$ so $\psi = 1$ on $\mathrm{supp}(u)$ and putting $u(\phi) = \langle u, \psi\phi \rangle$ for all $\phi \in C^\infty(\Omega)$. Using the boundedness property of u on the compact set $K = \mathrm{supp}(\psi)$ we get an $\mathscr{E}'$ bound for the extension:

$$|u(\phi)| \leq c \sum_{|\alpha| \leq m} \sup_K |\partial^\alpha \phi| \quad \forall\, \phi \in C^\infty(\Omega)$$

where the contants $c = c_K$ and $m = m_K$ will depend on K in general.

Here the set K is a compact neighbourhood of $\mathrm{supp}(u)$: *a compact set containing* $\mathrm{supp}(u)$ *in its interior.*

It is in general not possible to obtain an $\mathscr{E}'$ bound for the extension of u with the compact set K replaced by $\mathrm{supp}(u)$.

Example Define for $\phi \in \mathrm{C}^\infty(\mathbb{R})$,

$$\langle u, \phi \rangle = \sum_{j=1}^{\infty} \frac{\phi(\frac{1}{j}) - \phi(-\frac{1}{j})}{j}.$$

Because (use FTC)

$$\begin{aligned} \left| \frac{\phi(\frac{1}{j}) - \phi(-\frac{1}{j})}{j} \right| &= \left| \int_0^1 \left(\phi'\left(\tfrac{t}{j}\right) + \phi'\left(-\tfrac{t}{j}\right) \right) \frac{\mathrm{d}t}{j^2} \right| \\ &\leq \frac{2}{j^2} \max_{|x| \leq 1} \left| \phi'(x) \right| \end{aligned}$$

the series converges absolutely for each ϕ. It is then routine to check that $u \colon \mathrm{C}^\infty(\mathbb{R}) \to \mathbb{C}$ is a linear functional with the boundedness property

$$\left| \langle u, \phi \rangle \right| \leq \left(2 \sum_{j=1}^{\infty} \frac{1}{j^2} \right) \max_{[-1,1]} |\phi'| \quad \forall \, \phi \in \mathrm{C}^\infty(\mathbb{R})$$

Thus $u \in \mathscr{E}'(\mathbb{R})$ and so the restriction of u to $\mathscr{D}(\mathbb{R})$ is a distribution with compact support (and of order at most 1.)

Example continued... Note that if $x \notin A$, where

$$A = \left\{ \tfrac{1}{j} : j \in \mathbb{Z} \setminus \{0\} \right\} \cup \{0\}$$

then $r = \operatorname{dist}(x, A) > 0$ and so $A \cap (x - r, x + r) = \emptyset$. Thus for all $\phi \in \mathscr{D}(\mathbb{R})$ with $\operatorname{supp}(\phi) \subset (x - r, x + r)$ we have $\langle u, \phi \rangle = 0$. Therefore $\operatorname{supp}(u) \subseteq A$ and it is not difficult to see that in fact $\operatorname{supp}(u) = A$: if $x \in A$, then

$$\phi(y) = \rho_\varepsilon(y - x - \varepsilon)$$

is a test function with support $[x, x + 2\varepsilon]$ and $\langle u, \phi \rangle \neq 0$ for all sufficiently small $\varepsilon > 0$.

Could we have an $\mathscr{E}'$ bound for u with the compact set $K = \operatorname{supp}(u)$?

Example continued... Assume we did have such a bound: there exist constants $c \geq 0$, $m \in \mathbb{N}_0$ so (with $K = \mathrm{supp}(u)$)

$$\left|\langle u, \phi\rangle\right| \leq c \sum_{k=0}^{m} \sup_K \left|\phi^{(k)}\right|$$

holds for all $\phi \in \mathrm{C}^\infty(\mathbb{R})$. We claim this cannot be true! To see this fix $s \in \mathbb{N}$ so

$$\sum_{j=1}^{s} \frac{1}{j} > c.$$

Define

$$\phi = \rho_\varepsilon * \mathbf{1}_{[\frac{1}{s}-\varepsilon, 1+\varepsilon]}$$

with $\varepsilon \in (0, \frac{1}{2s(s+1)})$. Then $\phi \in \mathscr{D}(\mathbb{R})$ and

$$\begin{aligned} \phi &= 1 \text{ near } [\tfrac{1}{s}, 1] \\ \phi &= 0 \text{ near } \mathrm{supp}(u) \setminus [\tfrac{1}{s}, 1] \end{aligned}$$

Example continued... According to the assumed $\mathscr{E}'$ bound we have by construction of ϕ:

$$\left|\langle u, \phi\rangle\right| \leq c \sum_{k=0}^{m} \sup_{K} \left|\phi^{(k)}\right| = c,$$

but by the definition of u we also get

$$\langle u, \phi\rangle = \sum_{j=1}^{\infty} \frac{\phi(\frac{1}{j}) - \phi(-\frac{1}{j})}{j} = \sum_{j=1}^{s} \frac{1}{j}.$$

By our choice of s this is the required contradiction.

Conclusion: For a compactly supported distribution u we can for each compact neighbourhood K of its support obtain an $\mathscr{E}'$ bound:

$$|u(\phi)| \leq c \sum_{|\alpha| \leq m} \sup_{K} |\partial^{\alpha}\phi| \quad \forall\, \phi \in \mathrm{C}^{\infty}(\Omega)$$

The constants $c = c_K$, $m = m_K$ depend on the set K and will in general diverge when we shrink the compact neighbourhood K to $\mathrm{supp}(u)$.

But $\mathscr{E}'$ bounds with $K = \operatorname{supp}(u)$ hold in special cases:
(1) Let $x_0 \in \Omega$ and $\alpha \in \mathbb{N}_0^n$ be a multi-index. Then $u = \partial^\alpha \delta_{x_0}$ is a compactly supported distribution on Ω with $\operatorname{supp}(u) = \{x_0\}$. We clearly have the $\mathscr{E}'$ bound

$$\big|\langle u, \phi\rangle\big| \leq \big|\big(\partial^\alpha \phi\big)(x_0)\big| \quad \forall\, \phi \in \mathrm{C}^\infty(\Omega)$$

(2) Let $v \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$. It can be shown that

$$\operatorname{supp}(v) = \Big\{x \in \Omega : \int_{\Omega \cap B_r(x)} |v(y)| \,\mathrm{d}y > 0 \;\text{ for all }\; r > 0\Big\}.$$

If $K = \operatorname{supp}(v)$ is compact, then we have the $\mathscr{E}'$ bound

$$\big|\langle v, \phi\rangle\big| \leq \int_K |v(y)| \,\mathrm{d}y \max_K |\phi| \quad \forall\, \phi \in \mathrm{C}^\infty(\Omega)$$

Characterization of distributions supported at a single point

Theorem Let Ω be a non-empty open subset of $\mathbb{R}^n$ and $x_0 \in \Omega$. If $u \in \mathscr{D}'(\Omega)$ and $\operatorname{supp}(u) = \{x_0\}$, then there exist $m \in \mathbb{N}_0$ and $c_\alpha \in \mathbb{C}$ corresponding to multi-indices $\alpha \in \mathbb{N}_0^n$ of lengths at most m, so

$$u = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha \delta_{x_0}.$$

This an important result, but its proof is not examinable (though you can find it in the lecture notes).

Example Find all distributions $u \in \mathscr{D}'(\mathbb{R})$ satisfying $xu = 0$ in $\mathscr{D}'(\mathbb{R})$. Assume u is a solution. If $x_0 \neq 0$, say $x_0 > 0$, and $\phi \in \mathscr{D}(\mathbb{R})$ is supported in $(0, \infty)$, then $\psi(x) = \phi(x)/x$, $x \in \mathbb{R}$, is a test function. Therefore $0 = \langle xu, \psi \rangle = \langle u, x\psi \rangle = \langle u, \phi \rangle$, and so by definition of support, $x_0 \notin \mathrm{supp}(u)$. We argue similarly for $x_0 < 0$, so $\mathrm{supp}(u) \subseteq \{x_0\}$. If the support is empty then $u = 0$ and if the support is non-empty, then it must be $\{x_0\}$. But then the characterization of distributions of point support gives that

$$u \in \mathrm{span}\big\{\delta_0^{(k)} :\ k \in \mathbb{N}_0\big\}.$$

It is easy to check that δ_0 satisfies the equation. What about $\delta_0^{(k)}$ for $k \in \mathbb{N}$? We calculate for $\phi \in \mathscr{D}(\mathbb{R})$ using definitions:

$$\langle x\delta_0^{(k)}, \phi \rangle = (-1)^k \langle \delta_0, \tfrac{\mathrm{d}^k}{\mathrm{d}x^k}(x\phi(x)) \rangle = (-1)^k k \phi^{(k-1)}(0)$$

that clearly does not vanish for all ϕ. Therefore $\delta_0^{(k)}$ are not solutions for $k \in \mathbb{N}$. We conclude that u must have the form $c\delta_0$ for some $c \in \mathbb{C}$, and as it is easily seen that these distributions indeed all are solutions this must be the general solution.

Convolution of distribution with test function revisited

Recall that we used the adjoint identity scheme to define the convolution of $u \in \mathscr{D}'(\mathbb{R}^n)$ with $\theta \in \mathscr{D}(\mathbb{R}^n)$: $u * \theta \in \mathscr{D}'(\mathbb{R}^n)$ is given by rule

$$\langle u * \theta, \phi \rangle = \langle u, \tilde{\theta} * \phi \rangle \quad \text{for } \phi \in \mathscr{D}(\mathbb{R}^n)$$

where $\tilde{\theta}(x) := \theta(-x)$. Likewise we defined $\theta * u$, but since $\tilde{\theta} * \phi = \phi * \tilde{\theta}$ we have $u * \theta = \theta * u$. We furthermore proved that

(i) $u * \theta \in C^\infty(\mathbb{R}^n)$,

(ii) $(u * \theta)(x) = \langle u, \theta(x - \cdot) \rangle$, $x \in \mathbb{R}^n$,

(iii) $\partial^\alpha (u * \theta) = (\partial^\alpha u) * \theta = u * (\partial^\alpha \theta)$ for all $\alpha \in \mathbb{N}_0^n$.

Lemma about support of convolution For $u \in \mathscr{D}'(\mathbb{R}^n)$, $\theta \in \mathscr{D}(\mathbb{R}^n)$ we have

$$\operatorname{supp}(u * \theta) \subseteq \operatorname{supp}(u) + \operatorname{supp}(\theta).$$

(It is an exercise to show that the right-hand side is a closed set.)

Proof of Lemma about support of convolution: Since $u * \theta \in \mathrm{C}^\infty(\mathbb{R}^n)$ the support of $u * \theta$ as a distribution is the same as the support as a continuous function, so is the closure of the set $\{x \in \mathbb{R}^n : (u * \theta)(x) \neq 0\}$. Now note that

$$(u * \theta)(x) = \langle u, \theta(x - \cdot)\rangle = \langle u, \tilde{\theta}(\cdot - x)\rangle = \langle u, \tau_{-x}\tilde{\theta}\rangle,$$

so $(u * \theta)(x) = 0$ when $\mathrm{supp}(u) \cap \mathrm{supp}(\tau_{-x}\tilde{\theta}) = \emptyset$. Put

$$A = \big\{x \in \mathbb{R}^n : \, \mathrm{supp}(u) \cap \mathrm{supp}(\tau_{-x}\tilde{\theta}) = \emptyset\big\}.$$

We have shown that $u * \theta = 0$ on A.
We claim that A is an open set. Indeed, because $\mathrm{supp}(u)$ is closed and $\mathrm{supp}(\tau_{-x}\tilde{\theta})$ is compact we have that $\mathrm{supp}(u) \cap \mathrm{supp}(\tau_{-x}\tilde{\theta}) = \emptyset$ if and only if

$$\mathrm{dist}(\mathrm{supp}(u), \mathrm{supp}(\tau_{-x}\tilde{\theta})) > 0,$$

so A must be open (why?). Since $u * \theta = 0$ on A we must have $\mathrm{supp}(u * \theta) \subseteq \mathbb{R}^n \setminus A$.

Proof of Lemma about support of convolution continued...

We must relate $\mathrm{supp}(u) + \mathrm{supp}(\theta)$ to A.
If $x \notin \mathrm{supp}(u) + \mathrm{supp}(\theta)$, then $x - y \notin \mathrm{supp}(u)$ for all $y \in \mathrm{supp}(\theta)$, that is, $(x - \mathrm{supp}(\theta)) \cap \mathrm{supp}(u) = \emptyset$. But

$$x - \mathrm{supp}(\theta) = x + \mathrm{supp}(\tilde{\theta}) = \mathrm{supp}(\tau_{-x}\tilde{\theta}),$$

so it follows that $x \in A$. Consequently,

$$\mathbb{R}^n \setminus \big(\mathrm{supp}(u) + \mathrm{supp}(\theta)\big) \subseteq A,$$

and therefore

$$\mathbb{R}^n \setminus A \subseteq \mathrm{supp}(u) + \mathrm{supp}(\theta)$$

concluding the proof □

Convolution of compactly supported distribution with C^∞ function

We proceed similarly to the adjoint identity scheme and start by observing that for $\theta \in \mathrm{C}^\infty(\mathbb{R}^n)$ and ϕ, $\psi \in \mathscr{D}(\mathbb{R}^n)$ we have $\psi * \theta$, $\tilde{\theta} * \phi \in \mathrm{C}^\infty(\mathbb{R}^n)$. We may now apply Fubini's theorem to obtain the following adjoint identity variant:

$$\int_{\mathbb{R}^n} (\psi * \theta)\phi \, \mathrm{d}x = \int_{\mathbb{R}^n} \psi(\tilde{\theta} * \phi) \, \mathrm{d}x. \tag{3}$$

Definition Let $v \in \mathscr{E}'(\mathbb{R}^n)$ and $\theta \in \mathrm{C}^\infty(\mathbb{R}^n)$. We then define $v * \theta$ by the rule

$$\langle v * \theta, \phi \rangle = \langle v, \tilde{\theta} * \phi \rangle$$

for $\phi \in \mathscr{D}(\mathbb{R}^n)$.

Hereby $v * \theta \colon \mathscr{D}(\mathbb{R}^n) \to \mathbb{C}$ is a well-defined linear functional. To see that it is a distribution on $\mathbb{R}^n$ we must check that it is $\mathscr{D}$ continuous (or that it has the boundedness property). Suppose $\phi_j \to 0$ in $\mathscr{D}(\mathbb{R}^n)$ and consider

$$\langle v * \theta, \phi_j \rangle = \langle v, \tilde{\theta} * \phi_j \rangle.$$

Convolution of compactly supported distribution with C^∞ function

Because $v \in \mathscr{E}'(\mathbb{R}^n)$ it satisfies an $\mathscr{E}'$ bound: for a compact set L in $\mathbb{R}^n$ and constants $c \geq 0$, $m \in \mathbb{N}_0$ we have

$$\bigl|\langle v, \phi\rangle\bigr| \leq c \sum_{|\alpha|\leq m} \sup_L \bigl|\partial^\alpha \phi\bigr| \tag{4}$$

for all $\phi \in C^\infty(\mathbb{R}^n)$. Next, because $\phi_j \to 0$ in $\mathscr{D}(\mathbb{R}^n)$ we find a compact subset K of $\mathbb{R}^n$ so $\mathrm{supp}(\phi_j) \subseteq K$ for all j and

$$\sup_{\mathbb{R}^n} |\partial^\beta \phi_j| \to 0$$

for all multi-indices $\beta \in \mathbb{N}_0^n$. Now if $\alpha \in \mathbb{N}_0^n$ and $|\alpha| \leq m$, then

$$\begin{aligned} \sup_L \bigl|\partial^\alpha(\tilde{\theta} * \phi_j)\bigr| &\leq \sup_{x\in L} \int_K |\theta(y-x)(\partial^\alpha \phi_j)(y)|\,\mathrm{d}y \\ &\leq \int_{K-L} |\theta|\,\mathrm{d}y \sup_{\mathbb{R}^n} |\partial^\alpha \phi_j| \end{aligned}$$

and therefore in view of (4), $\langle v * \theta, \phi_j\rangle \to 0$, as required.

Convolution of compactly supported distribution with C^∞ function

So $v * \theta \in \mathscr{D}'(\mathbb{R}^n)$ when $v \in \mathscr{E}'(\mathbb{R}^n)$, $\theta \in \mathrm{C}^\infty(\mathbb{R}^n)$. We note that because of the adjoint identity variant (3) the distributional definition is consistent with the usual definition when $v \in \mathscr{D}(\mathbb{R}^n)$.

Defining $\theta * v$ by the rule $\langle \theta * v, \phi \rangle = \langle v, \phi * \tilde{\theta} \rangle$ for $\phi \in \mathscr{D}(\mathbb{R}^n)$ we get nothing new: $\theta * v = v * \theta$ because $\phi * \tilde{\theta} = \tilde{\theta} * \phi$ for all $\phi \in \mathscr{D}(\mathbb{R}^n)$.

Lemma Let $v \in \mathscr{E}'(\mathbb{R}^n)$, $\theta \in \mathrm{C}^\infty(\mathbb{R}^n)$. Then

(i) $v * \theta \in \mathrm{C}^\infty(\mathbb{R}^n)$ and $(v * \theta)(x) = \langle v, \theta(x - \cdot) \rangle$, $x \in \mathbb{R}^n$,

(ii) $\partial^\alpha (v * \theta) = (\partial^\alpha v) * \theta = v * (\partial^\alpha \theta)$ for all $\alpha \in \mathbb{N}_0^n$,

(iii) $\mathrm{supp}(v * \theta) \subseteq \mathrm{supp}(v) + \mathrm{supp}(\theta)$.

We omit the proofs that are, up to minor adaptations, the same as those we gave for $u * \theta$ when $u \in \mathscr{D}'(\mathbb{R}^n)$, $\theta \in \mathscr{D}(\mathbb{R}^n)$.

Looking ahead to next lecture...

If $v \in \mathscr{E}'(\mathbb{R}^n)$ and $\theta \in \mathscr{D}(\mathbb{R}^n)$, then the lemma gives $v * \theta \in \mathrm{C}^\infty(\mathbb{R}^n)$ and

$$\mathrm{supp}(v * \theta) \subseteq \mathrm{supp}(v) + \mathrm{supp}(\theta).$$

Since both supports on the right-hand side are compact, so is their sum, hence also $\mathrm{supp}(v * \theta)$ is compact and thus $v * \theta \in \mathscr{D}(\mathbb{R}^n)$ in this case. We also record that the operations *dilation with* $r > 0$, $d_r v$, L^1 *dilation with* $r > 0$, $v_r = r^{-n} d_{r^{-1}} v$, *translation by* $h \in \mathbb{R}^n$, $\tau_h v$, and *reflection in the origin*, $\tilde{v}$, all map $\mathscr{E}'(\mathbb{R}^n)$ to itself.

How should we define $u * v$ when $u \in \mathscr{D}'(\mathbb{R}^n)$ and $v \in \mathscr{E}'(\mathbb{R}^n)$?
It is the distribution defined by the rule

$$\langle u * v, \phi \rangle = \langle u, \tilde{v} * \phi \rangle, \quad \phi \in \mathscr{D}(\mathbb{R}^n).$$

Since $\tilde{v} * \phi \in \mathscr{D}(\mathbb{R}^n)$ the definition makes sense and we will show it is a distribution on $\mathbb{R}^n$ and investigate some of its properties in the next lecture.