B8.2: Continuous Martingales and Stochastic Calculus
Problem Sheet 1

The questions on this sheet are divided into three sections. Only those questions in Section B are compulsory and should be handed in for marking.

The questions are not in order of difficulty; if you are stuck on one question, move on to the next.

Section A

1. Suppose that under the probability measure $\mathbb{P}$, the discrete time process $(S_n)_{n\geq 0}$ is a simple symmetric random walk on $\mathbb{Z}$. Show that $(S_n)_{n\geq 0}$ is a $\mathbb{P}$-martingale (with respect to the natural filtration) and

$$\mathrm{cov}(S_n, S_m) = n \wedge m.$$

Since the random walk can be a distance at most n from its starting point at time n, the expectation $\mathbb{E}\left[|S_n|\right] < \infty$ is evidently finite. Moreover,

$$\begin{aligned}
\mathbb{E}\left[S_{n+1} | \mathcal{F}_n\right] &= \mathbb{E}\left[S_n + \xi_{n+1} | \mathcal{F}_n\right] \\
&= S_n + \mathbb{E}\left[\xi_{n+1} | \mathcal{F}_n\right] \\
&= S_n + \mathbb{E}\left[\xi_{n+1}\right],
\end{aligned}$$

where we have used independence of the $\{\xi_n\}_{n\geq 0}$ in the last line and the martingale property then follows since $\mathbb{E}[\xi_{n+1}] = 0$.

It remains to calculate the covariance.

$$\begin{aligned}
cov(S_n, S_m) &= \mathbb{E}[S_n S_m] - \mathbb{E}[S_n]\mathbb{E}[S_m] \\
&= \mathbb{E}\left[\mathbb{E}[S_n S_m | \mathcal{F}_{m\wedge n}]\right] \qquad \text{(tower property)} \\
&= \mathbb{E}\left[S_{m\wedge n}\mathbb{E}\left[S_{m\vee n} | \mathcal{F}_{m\wedge n}\right]\right] \\
&= \mathbb{E}[S_{m\wedge n}^2] \qquad \text{(martingale property)} \\
&= var(S_{m\wedge n}) = m \wedge n.
\end{aligned}$$

2. Suppose that ξ is normally distributed with mean zero and variance one and that $x > 0$. Show that

$$\mathbb{P}[\xi \geq x] \leq \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}.$$

This is just integration by parts:

$$\int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = \left[\frac{-e^{-y^2/2}}{\sqrt{2\pi} y}\right]_x^\infty - \int_x^\infty \frac{1}{\sqrt{2\pi} y^2} e^{-y^2/2} dy.$$

Section B (Compulsory)

1. Let $(X_i : i \in I)$ be a collection of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{H} = \sigma(X_i : i \in I)$ and $\mathcal{G} \subset \mathcal{F}$ a σ-algebra. Use the monotone class theorem (see Appendix in the lecture notes for a reminder) to argue that:

(a) In order to verify that $\mathcal{G}$ and $\mathcal{H}$ are independent it is enough to verify that $(X_{i_1}, \dots, X_{i_k})$ is independent of $\mathcal{G}$ for any finite set of indices $\{i_1, \dots, i_k\} \subset I$.

In fact the Appendix in the lecture notes tells them how to do this. We'll use monotone class arguments quite a bit, so I want them to be reminded of the idea at least.

First observe that $\mathcal{M} := \{A \in \mathcal{F} : A \text{ is independent of } \mathcal{G}\}$ is a monotone class (/Dynkin system/ λ-system). Make sure that they know what to check for this:

i. Whole space is in $\mathcal{M}$

ii. If A, $B \in \mathcal{M}$ and $A \subset B$, then $B \backslash A \in \mathcal{M}$;

iii. If A_n increasing sequence of sets in $\mathcal{M}$, then $\cup_n A_n \in \mathcal{M}$.

(i) is obvious. For (ii), take $G \in \mathcal{G}$ and A, B with $A \subset B$. Write $\mathbb{P}[B] = \mathbb{P}[A] + \mathbb{P}[B \backslash A]$, and $\mathbb{P}[B \cap G] = \mathbb{P}[A \cap G] + \mathbb{P}[(B \backslash A) \cap G]$. So, if A and B are independent of G, substituting in this second equation and then using the first, we find

$$\begin{aligned} \mathbb{P}[(B \backslash A) \cap G] &= \mathbb{P}[B \cap G] - \mathbb{P}[A \cap G] \\ &= \mathbb{P}[B]\mathbb{P}[G] - \mathbb{P}[A]\mathbb{P}[G] = \mathbb{P}[B \backslash A]\mathbb{P}[G]. \end{aligned}$$

For (iii), take an increasing sequence of sets A_n, independent of $\mathcal{G}$, and write A for their limit. Suppose that $G \in \mathcal{G}$, then by the Monotone Convergence Theorem,

$$\mathbb{P}[A \cap G] = \lim_{n \to \infty} \mathbb{P}[A_n \cap G] = \lim_{n \to \infty} \mathbb{P}[A_n]\mathbb{P}[G] = \mathbb{P}[A]\mathbb{P}[G].$$

Now note that $\mathcal{C}$, defined to be the class of events which that depend on only finitely many X_i is a π-system. So if $\mathcal{C} \subseteq \mathcal{M}$, the Monotone Class Lemma tells us that $\sigma(\mathcal{C}) \subseteq \mathcal{M}$. And since $\sigma(\mathcal{C})$ is precisely $\mathcal{H}$ the result follows.

(b) In order to verify that for a bounded real random variable Y, and a fixed $i_0 \in I$, we have $\mathbb{E}[Y|\sigma(X_{i_0})] = \mathbb{E}[Y|\mathcal{H}]$, it is enough to show that $\mathbb{E}[Y|\sigma(X_{i_0})] = \mathbb{E}[Y|\sigma(X_{i_0}, X_{i_1}, \dots, X_{i_k})]$ for any finite set of indices $\{i_1, \dots, i_k\} \subset I$.

The argument is almost the same. Clearly $\mathbb{E}[Y|\sigma(X_{i_0})]$ is $\mathcal{H}$-measurable. The collection $\mathcal{M}$ of sets $A \in \mathcal{F}$ such that $\mathbb{E}[\mathbf{1}_A Y] = \mathbb{E}[\mathbf{1}_A \mathbb{E}[Y|\sigma(X_{i_0})]]$ is a monotone class. To check that it is closed under monotone limits, use the Dominated Convergence Theorem.

As before, if we can show that the π-system $\mathcal{C}$ of events which depend on finitely many X_i is contained in $\mathcal{M}$, then by the Monotone Class Lemma, so is $\mathcal{H} = \sigma(\mathcal{C})$ and so it suffices to check that $\mathcal{C} \subset \mathcal{M}$ as claimed.

2. Given two stochastic processes $X = (X_t : t \geq 0)$ and $Y = (Y_t : t \geq 0)$ on a common probability space we say that Y is a modification of X if for all $t \geq 0$, $X_t = Y_t$ a.s. Show that if X and Y have a.s. right-continuous paths then, if they are modifications of each other, they are also *indistinguishable*; i.e. $\mathbb{P}[X_t = Y_t \text{ for all } t \geq 0] = 1$.

Let

$$A_1 = \{\omega \in \Omega : X_t(\omega) = \lim_{s \downarrow t,\, s \in \mathbb{Q}_+} X_s(\omega) \text{ for any } t \geq 0\},$$

$$A_2 = \{\omega \in \Omega : Y_t(\omega) = \lim_{s \downarrow t,\, s \in \mathbb{Q}_+} Y_s(\omega) \text{ for any } t \geq 0\},$$

$$A_3 = \{\omega \in \Omega : X_t(\omega) = Y_t(\omega) \text{ for any } t \in \mathbb{Q}_+\}.$$

Since X and Y are a.s right-continuous, $\mathbb{P}[A_1] = \mathbb{P}[A_2] = 1$.

As $\mathbb{P}[X_t \neq Y_t] = 0$ for any $t \geq 0$, and a countable union of null sets is null, it follows that

$$\mathbb{P}[A_3] = 1 - \mathbb{P}[X_t \neq Y_t \text{ for some } t \in \mathbb{Q}] = 1.$$

Finally, it suffices to observe that for $\omega \in \bigcap_{i=1}^3 A_i$ we have $X_t(\omega) = Y_t(\omega)$ for all $t \geq 0$, so that

$$\mathbb{P}[X_t = Y_t \text{ for any } t \geq 0] \geq \mathbb{P}\left[\bigcap_{i=1}^3 A_i\right] = 1.$$

3. Suppose that $(B_t)_{t\geq 0}$ is a Brownian motion. Fix $0 \leq s < t < \infty$. Show that conditionally on $\{B_s = x, B_t = z\}$ the intermediate value $B_{\frac{t+s}{2}}$ has Gaussian distribution with mean $\frac{x+z}{2}$ and variance $\frac{t-s}{4}$.

   I am perfectly happy if they just use Bayes' rule and the Markov property without too much attention to rigour for this one. The calculation is one we need.

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   Here is Jan's solution based on properties of Gaussian processes (which we've removed from the main lectures now):

   Recall that $\Gamma(t,s) := \mathtt{cov}(B_t, B_s) = \min\{t,s\}$. Consider a Gaussian r.v

$$Y := B_{\frac{s+t}{2}} - \frac{1}{2}(B_t + B_s) = (B_{\frac{s+t}{2}} - B_s) - \frac{1}{2}(B_t - B_s).$$

   Notice that

$$\begin{aligned}\mathtt{cov}(Y, B_s) =& \mathtt{cov}(B_{\frac{s+t}{2}} - B_s, B_s) - \mathtt{cov}((B_t - B_s)/2, B_s)\\ =&\Gamma\Big(\frac{s+t}{2}, s\Big) - \Gamma(s,s) - \frac{1}{2}\Big(\Gamma(t,s) - \Gamma(s,s)\Big)\\ =&0\end{aligned}$$

   Similarly,

$$\begin{aligned}\mathtt{cov}(Y, B_t) =& \mathtt{cov}(B_{\frac{s+t}{2}} - B_s, B_t) - \mathtt{cov}((B_t - B_s)/2, B_t)\\ =&\Gamma\Big(\frac{s+t}{2}, t\Big) - \Gamma(s,t) - \frac{1}{2}\Big(\Gamma(t,t) - \Gamma(s,t)\Big)\\ =&\frac{s+t}{2} - s - \frac{t-s}{2} = 0.\end{aligned}$$

   As Y, B_s and B_t are jointly normally distributed, it follows that Y is independent of B_s and B_t. Therefore, we find that

$$\mathbb{E}_{\mathbb{P}}[B_{\frac{s+t}{2}} | B_s, B_t] = \frac{B_t + B_s}{2} + \mathbb{E}_{\mathbb{P}}[Y | B_s, B_t] = \frac{B_t + B_s}{2},$$

   where we used $\mathbb{E}[Y] = 0$. Note that

$$B_{\frac{s+t}{2}} = \frac{1}{2}(B_s + B_t) + Y$$

so that conditionally on $B_s = x$ and $B_t = z$ the intermediate value $B_{\frac{s+t}{2}}$ has Gaussian distribution with mean $(x+z)/2$ and variance equal to the variance of Y.

$$\begin{aligned}\texttt{var}(Y) =& \texttt{var}(B_{\frac{s+t}{2}}) + \texttt{var}((B_s + B_t)/2) - \texttt{cov}(B_{\frac{s+t}{2}}, (B_s + B_t)) \\ =& \frac{s+t}{2} + \frac{3s+t}{4} - s - \frac{s+t}{2} = \frac{t-s}{4}.\end{aligned}$$

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Here is a solution using Question C.2:

Let $Y = B_{(t+s)/2}$, $X = [B_s \quad B_t]^\top$. These are mean-zero jointly Gaussian random variables, with covariance matrices

$$\Gamma_X = \begin{bmatrix} s & s \\ s & t \end{bmatrix}, \qquad \Gamma_Y = (t+s)/2, \qquad \Gamma_{XY} = \begin{bmatrix} s \\ (s+t)/2 \end{bmatrix}.$$

So

$$\Gamma_{XY}^\top \Gamma_X^{-1} = \frac{1}{s(t-s)} \begin{bmatrix} s & (s+t)/2 \end{bmatrix} \begin{bmatrix} t & -s \\ -s & t \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$$

From QB, we have that $Y|X$ is Gaussian, with mean $\Gamma_{XY}^\top \Gamma_X^{-1} X = (B_s + B_t)/2$ and variance $\Gamma_Y - \Gamma_{XY}^\top \Gamma_X^{-1} \Gamma_{XY} = (t-s)/4$, as desired.

4. Suppose that $(B_t : t \geq 0)$ is a Brownian motion.

   (a) Show that for any $c \neq 0$, $(cB_{t/c^2} : t \geq 0)$ is also a Brownian motion.

   (b) Show, using the Strong Law of Large Numbers, that $\frac{1}{n} B_n \to 0$ a.s. when $n \to \infty$, $n \in \mathbb{N}$. Using the Borel-Cantelli lemma, the fact that $\mathbb{P}(\sup_{0 \leq s \leq t} B_s > \lambda) = 2\mathbb{P}(B_t > \lambda)$ for all $\lambda > 0$ (which will be proved in Section 5 using the reflection principle) and the estimate in Question A.2, show that

$$\lim_{n \to \infty} \sup_{t \in [n, n+1)} \frac{B_t - B_n}{n} = 0, \quad a.s.$$

   Hence show that $\lim_{t \to \infty} \frac{1}{t} B_t = 0$ a.s. and that $(tB_{1/t} : t \geq 0)$ is a Brownian motion.

(a) $Y := (cB_{t/c^2})_{t \geq 0}$ has continuous paths and $Y_0 = 0$ a.s. Further, for any $0 \leq s \leq t$, $(Y_t - Y_s) = c(B_{t/c^2} - B_{s/c^2})$ is centred Gaussian with variance $c^2(t/c^2 - s/c^2) = (t-s)$. Finally, $B_{t/c^2} - B_{s/c^2}$ is independent of $\sigma(B_{u/c^2} : u \leq s)$. We conclude that Y is a Brownian motion.

(b) We have

$$\frac{B_n}{n} = \frac{\sum_{i=1}^n (B_i - B_{i-1})}{n} \to 0, \quad a.s.$$

by the strong law of large numbers since $(B_i - B_{i-1})$, $i \in \mathbb{N}$, is a sequence of i.i.d. centred random variables.

Now let $M_n = \sup_{n \leq t < n+1} B_t - B_n$ which, as Brownian motion has independent increments, is a sequence of iid random variables with the distribution of $M = \sup_{0 \leq t < 1} B_t$. Now, by the reflection principle result and question A.2

$$\mathbb{P}(\sup_{n \leq m} M_n > xm) \leq m\mathbb{P}(M > xm) \leq \frac{2m}{\sqrt{2\pi} xm} e^{-x^2 m^2/2}.$$

This is summable over m and thus the first Borel-Cantelli Lemma gives that for any $x > 0$

$$\mathbb{P}(\sup_{n\leq m} M_n > xm \ \ i.o.) = 0.$$

As this holds for all x we have

$$\limsup_{m\to\infty} \frac{\sup_{n\leq m} M_n}{m} = 0,$$

which gives the result. Now

$$\begin{aligned}\frac{B_t}{t} &= \frac{B_t - B_{\lfloor t\rfloor} + B_{\lfloor t\rfloor}}{t} \\ &\leq \frac{\lfloor t\rfloor}{t}\left(\sup_{t\in[\lfloor t\rfloor,\lceil t\rceil]} \frac{B_t - B_{\lfloor t\rfloor}}{\lfloor t\rfloor} + \frac{B_{\lfloor t\rfloor}}{\lfloor t\rfloor}\right)\end{aligned}$$

Thus, as $\lfloor t\rfloor/t \to 1$, and as the above two results show that the two terms in the bracket tend to 0, we have the result.

To show that $(tB_{1/t} : t \geq 0)$ is a Brownian motion we use the definition as a Gaussian process. Let $X_t = tB_{1/t}$. The Gaussianity of X follows from the contruction in terms of Brownian motion. The continuity is trivial except at 0. From the result we have just proved we see that $0 = \lim_{t\to\infty} B_t/t = \lim_{t\to 0} tB_{1/t}$ which shows that X is continous at 0. It is clearly centred so we just need to check the covariance:

$$\mathrm{Cov}(X_s, X_t) = ts\mathrm{Cov}(B_{1/s}, B_{1/t} = ts\min(1/s, 1/t) = \min(s,t).$$

Thus X is a Brownian motion.

5. Use the Kolmogorov continuity criterion to show that Brownian motion admits a modification that is (a.s.) locally Hölder continuous of order γ for every $0 < \gamma < 1/2$.

   Recall the Kolmogorov Continuity Criterion: if X_t satisfies

   $$\mathbb{E}[|X_t - X_s|^\alpha] \leq C|t-s|^{1+\beta}, \quad 0 \leq s,t \leq T,$$

   for strictly positive constants α, β, C, then there exists a modification of X whose paths are γ-locally Hölder continuous for all $\gamma \in (0, \beta/\alpha)$.

   Using scaling, $\mathbb{E}[|B_t - B_s|^\alpha] = |t-s|^{\alpha/2}\mathbb{E}[|Z|]$ where Z is standard normal and so Kolomogorov's criterion tells us that for any α, we have local Hölder continuity with parameter

   $$0 < \gamma < \frac{\alpha/2 - 1}{\alpha} = \frac{\alpha - 2}{2\alpha}.$$

   Letting $\alpha \to \infty$ gives the result.

Section C (Extra practice questions, not for hand-in)

1. Using the monotone class theorem or otherwise, show that any set $A \subset \mathbb{R}^{[0,\infty)}$ in the Borel cylinder σ-algebra is of the form

   $$A = \bigcup_{i\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} \{X : X_{t_{i,j}} \in B_{i,j}\}$$

for some countable set of times $\{t_{i,j}\}_{i,j\in\mathbb{N}}$ and sets $\{B_{i,j}\in\mathcal{B}(\mathbb{R})\}_{i,j\in\mathbb{N}}$. Consequently, show that the set $\{X\in\mathbb{R}^{[0,\infty)}: X \text{ is continuous}\}$ is not in $\mathcal{B}(\mathbb{R}^{[0,\infty)})$

Using the usual union-intersection laws, it's easy to see that the collection of A of this form is closed under complementation, countable unions and contains the entire set. Hence it is a monotone class. It also contains the finite-dimensional sets that generate the cylinder σ-algebra; therefore, by the monotone class theorem, it must contain the cylinder σ-algebra. Conversely, every set of the stated form can be constructed from countable operations on the finite-dimensional sets, so is clearly an element of the cylinder σ-algebra.

As continuity cannot be verified using a countable collection of times, it follows that the set of continuous paths cannot be in the cylinder σ-algebra.

2. Let X and Y be jointly Gaussian random variables in $\mathbb{R}^d$ and $\mathbb{R}^k$ respectively, where $k,d\in\mathbb{N}$. Suppose $\mathbb{E}[X]=\mu_X$, $\mathbb{E}[Y]=\mu_Y$, $\text{Var}(X)=\Gamma_X$, $\text{Var}(Y)=\Gamma_Y$ and

$$\text{cov}(X,Y)=\mathbb{E}[(X-\mu_X)(Y-\mu_Y)^\top]=\Gamma_{XY}\in\mathbb{R}^{d\times k}.$$

Suppose Γ_X,Γ_Y and $\Gamma_Y-\Gamma_{XY}^\top\Gamma_X^{-1}\Gamma_{XY}$ are all strictly positive definite. By considering the joint density or otherwise, show that

$$Y|X\sim N\Big(\mu_Y+\Gamma_{XY}^\top\Gamma_X^{-1}(X-\mu_X),\quad \Gamma_Y-\Gamma_{XY}^\top\Gamma_X^{-1}\Gamma_{XY}\Big).$$

Hint: The following matrix identity (from block-matrix inversion) may simplify calculations:

$$\begin{bmatrix} x\\ y\end{bmatrix}^\top\begin{bmatrix}\Gamma_X & \Gamma_{XY}\\ \Gamma_{XY}^\top & \Gamma_Y\end{bmatrix}^{-1}\begin{bmatrix} x\\ y\end{bmatrix}=x\Gamma_X^{-1}x+(y-\Gamma_{XY}^\top\Gamma_X^{-1}x)^\top(\Gamma_Y-\Gamma_{XY}^\top\Gamma_X^{-1}\Gamma_{XY})^{-1}(y-\Gamma_{XY}^\top\Gamma_X^{-1}x).$$

For notational simplicity, write $x=X-\mu_X$ and $y=Y-\mu_Y$. Then writing our the joint density of (x,y) in $\mathbb{R}^{d+k}$, we have (ignoring constants which must integrate out)

$$\begin{aligned}f(x,y)&\propto\exp\Big\{-\frac{1}{2}\begin{bmatrix} x\\ y\end{bmatrix}^\top\begin{bmatrix}\Gamma_X & \Gamma_{XY}\\ \Gamma_{XY}^\top & \Gamma_Y\end{bmatrix}^{-1}\begin{bmatrix} x\\ y\end{bmatrix}\Big\}\\ &=\exp\Big\{-\frac{1}{2}x\Gamma_X^{-1}x\Big\}\exp\Big\{-\frac{1}{2}(y-\Gamma_{XY}^\top\Gamma_X^{-1}x)^\top(\Gamma_Y-\Gamma_{XY}^\top\Gamma_X^{-1}\Gamma_{XY})^{-1}(y-\Gamma_{XY}^\top\Gamma_X^{-1}x)\Big\}\\ &=f_x(x)f_{y|x}(y)\end{aligned}$$

where we use the fact that $f_x(x)=\exp\big\{-\frac{1}{2}x\Gamma_X^{-1}x\big\}$ is the density of x to justify the final line. It follows that $y|x\sim N(\Gamma_{XY}^\top\Gamma_X^{-1}x,\Gamma_Y-\Gamma_{XY}^\top\Gamma_X^{-1}\Gamma_{XY})$, so

$$Y|X\sim N\Big(\mu_Y+\Gamma_{XY}^\top\Gamma_X^{-1}(X-\mu_X),\quad \Gamma_Y-\Gamma_{XY}^\top\Gamma_X^{-1}\Gamma_{XY}\Big).$$

It's worth noting that these are the same equations that arise in linear regression, where the regression coefficient is estimated by $\Gamma_{XY}^\top\Gamma_X^{-1}=\text{cov}(Y,X)/\text{Var}(X)\approx\sum(x-\bar{x})(y-\bar{y})/\sum(x-\bar{x})^2$, and the variance estimate is corrected accordingly.

3. A Gaussian process X is a stochastic process for which the joint distribution of $(X_{t_1},X_{t_2},...,X_{t_n})$ is Gaussian, for any finite collection of times $t_1,...,t_n$, $n\in\mathbb{N}$. Let X be a Gaussian process, and

suppose X has expectation $\mu(t) = \mathbb{E}[X_t]$ and covariance $\operatorname{cov}(X_s, X_t) = \Gamma(s,t)$, and has continuous paths. Let $Y_t = \int_0^t X_s ds$. Show that Y is a Gaussian process, with parameters

$$\mathbb{E}[Y_t] = \int_0^t \mu(s) ds, \qquad \operatorname{cov}(Y_s, Y_t) = \int_0^s \int_0^t \Gamma(s', t') dt' \, ds'.$$

Hence conclude that, for any Gaussian process with continuously differentiable mean and covariance, there is a construction of the process with differentiable paths.

(The stronger statement that every construction of such a process has a differentiable modification is true but is more difficult to prove.)

As we have continuous paths, we know that Riemann sums will converge (almost surely). It is easy to check that Riemann sums are Gaussian, and convergence a.s. implies convergence in distribution, so the integral is a Gaussian process.

Linearity of integrals gives us the expectation statement (Fubini). Similarly, after writing out the covariance as an expectation (and more Fubini). The result follows.

If we are given a GP with differentiable mean and covariance, then we can construct (using Kolmogorov's Extension Theorem) a continuous version of its derivative X. Integrating this, we obtain Y, which is a.s. differentiable and has the correct law.

4. Show that Brownian sample paths are of infinite variation over any time interval $[0,t]$ with $t > 0$ almost surely.

Suppose for a contradiction that B_s had finite variation over $[0,t]$. Write π for a generic partition of $[0,t]$, $N(\pi)$ for the number of intervals that make up π and $\delta(\pi)$ for the mesh of π (the length of the longest interval in π). We write $0 = t_0 < t_1 < \ldots < t_{N(\pi)} = t$ for the endpoints of the intervals that make up π. Then the variation of B over $[0,t]$ is

$$\lim_{\delta \to 0} \left\{ \sup_{\pi : \|\pi\| = \delta} \sum_{j=1}^{N(\pi)} |B_{t_j} - B_{t_{j-1}}| \right\}.$$

If this quantity is finite, then the quadratic variation is bounded by

$$\lim_{\|\pi\| \to 0} \sum_{j=1}^{N(\pi)} |B_{t_j} - B_{t_{j-1}}|^2 \le \lim_{\delta \to 0} \sup_{0 \le s_1 \le s_2 \le t; |s_2 - s_1| < \delta} |B_{s_2} - B_{s_1}| \sup_{\pi : \|\pi\| = \delta} \left\{ \sum_{j=1}^{N(\pi)} |B_{t_j} - B_{t_{j-1}}| \right\} = 0,$$

since Brownian motion is uniformly continuous on $[0,t]$ (which we know from Q2, as it is Hölder continuous). But from lectures, we know that B has quadratic variation t over $[0,t]$, and this is the limit in probability of the LHS above, so we have a contradiction.