B8.2: Continuous Martingales and Stochastic Calculus
Problem Sheet 2

The questions on this sheet are divided into three sections. Only those questions in Section B are compulsory and should be handed in for marking.

The questions are not in order of difficulty; if you are stuck on one question, move on to the next.

Section A

1. Let B be a Brownian motion and set $S_t := \sup_{0\le u\le t} B_u$. Deduce from the reflection principle that the pair (S_t, B_t) has density given by

$$f_{S_t,B_t}(a,b) = \frac{2(2a-b)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2a-b)^2}{2t}\right) \mathbf{1}_{a>0,b<a}.$$

By the Corollary to the Reflection Principle, for $a \ge 0$ and $b \le a$, we have

$$\mathbb{P}(S_t \ge a,\ B_t \le b) = \mathbb{P}(B_t \ge 2a - b).$$

Let $f_{(S_t,B_t)}$ be the the density function of (S_t, B_t). For $a \ge 0$ and $b \le a$, we have

$$\begin{aligned} f_{(S_t,B_t)}(a,b) &= -\frac{\partial^2}{\partial a \partial b}\mathbb{P}(S_t \ge a,\ B_t \le b) \\ &= -\frac{\partial^2}{\partial a \partial b}\Big(1 - \Phi\Big(\frac{2a-b}{\sqrt{t}}\Big)\Big) \\ &= \frac{\partial}{\partial b}\Big(\frac{2}{\sqrt{t}}\Phi'\Big(\frac{2a-b}{\sqrt{t}}\Big)\Big) \\ &= \frac{2(2a-b)}{\sqrt{2\pi t^3}} \exp\Big(-\frac{(2a-b)^2}{2t}\Big). \end{aligned}$$

As $S_t \ge B_t$ and $S_t \ge 0$, it is straightforward to deduce that for $a < 0$ or $b > a$

$$f_{(S_t,B_t)}(a,b) = 0.$$

Thus,

$$f_{(S_t,B_t)}(a,b) = \frac{2(2a-b)}{\sqrt{2\pi t^3}} \exp\Big(-\frac{(2a-b)^2}{2t}\Big)\mathbf{1}_{a\ge 0,\, b\le a}.$$

2. Let τ be a stopping time relative to a given filtration $(\mathcal{F}_t)$. Show that $\mathcal{F}_\tau$ is a σ-algebra and τ is $\mathcal{F}_\tau$ measurable.

To see that $\mathcal{F}_\tau$ is a σ-algebra, note that $\emptyset \in \mathcal{F}_\tau$ since $\emptyset \cap \{\tau \le t\} = \emptyset \in \mathcal{F}_t$; then

$$\forall\ A \in \mathcal{F}_\tau, \quad A^{\complement} \cap \{\tau \le t\} = \{\tau \le t\} \backslash \big(A \cap \{\tau \le t\}\big) \in \mathcal{F}_t \Longrightarrow A^{\complement} \in \mathcal{F}_\tau;$$

and finally $\forall (A_n) \in \mathcal{F}_\tau$, $\big(\bigcup A_n\big) \cap \{\tau \le t\} = \bigcup (A_n \cap \{\tau \le t\}) \in \mathcal{F}_t$.

To see that τ is $\mathcal{F}_\tau$-measurable, note that $\{\tau \le t\}$ is $\mathcal{F}_\tau$-measurable for any $t \ge 0$ and these sets form a π-system which generates $\sigma(\tau)$ and hence $\sigma(\tau) \subset \mathcal{F}_\tau$ by Dynkin's π–λ systems Lemma.

Section B (Compulsory)

1. Let $(B_t)_{t\geq 0}$ be a Brownian motion. Show that for every $M > 0$,

$$\mathbb{P}[\sup_{s\geq 0} B_s > M] = 1,$$

and hence that

$$\limsup_{t\to\infty} B_t = -\liminf_{t\to\infty} B_t = \infty, \quad a.s.$$

Use time inversion of Brownian motion to show that the Brownian path is almost surely non-differentiable at $t = 0$.

This is a scaling argument much like the one used in the proof of Proposition 4.17. Recall that if B is a Brownian motion, then B^λ defined by $B_t^\lambda = B_{\lambda^2 t}/\lambda$ is also a Brownian motion. Then for any $M, \delta > 0$,

$$\begin{aligned}\mathbb{P}[\sup_{0\leq s\leq 1} B_s > M\delta] &= \mathbb{P}[\sup_{0\leq s\leq 1/\delta^2} B_{\delta^2 s}/\delta > M] \\ &= \mathbb{P}[\sup_{0\leq s\leq 1/\delta^2} \tilde{B}_s > M].\end{aligned}$$

Let $\delta \to 0$ and the left hand side tends to 1 (since $M\delta \to 0$) and the right hand side tends to $\mathbb{P}[\sup_{0\leq s<\infty} B_s > M]$ and the result follows.

By using $-B$ we also have

$$\mathbb{P}[\inf_{s\geq 0} B_s < -M] = 1.$$

As these hold for every M we have

$$\limsup_{t\to\infty} B_t = -\liminf_{t\to\infty} B_t = \infty, \quad a.s.$$

Now for the Brownian path to be differentiable at 0 we must have the existence of $\lim_{t\downarrow 0} \frac{B_t}{t}$. By time inversion

$$\lim_{t\downarrow 0} \frac{B_t}{t} = \lim_{1/s\downarrow 0} sB_{1/s} = \lim_{s\to\infty} \tilde{B}_s,$$

for a Brownian motion $\tilde{B}$, in distribution. Thus our previous result shows that

$$\limsup_{t\to 0} \frac{B_t}{t} = -\liminf_{t\to 0} \frac{B_t}{t} = \infty, \quad a.s.,$$

and the Brownian motion is not differentiable at 0 a.s.

2. Consider the following stochastic process

$$X_t := x(1-t) + yt + (B_t - tB_1); 0 \leq t \leq 1.$$

(a) Show that X is a continuous Gaussian process with $X_0 = x$ and $X_1 = y$.

It is clear that (X_t) has continuous paths because B_t does. Further, for any $0 \leq t_1 < t_2 < \ldots < t_n$, the vector $(X_{t_1}, \cdots, X_{t_n})$ is a linear map of $(B_{t_1}, \cdots, B_{t_n}, B_1)$. The latter is Gaussian and hence so is the former. It is also clear from definition that $X_0 = x$ and $X_1 = y + B_1 - B_1 = y$.

(b) Show that X cannot be adapted to $(\mathcal{F}_t)$. Is B also an $(\mathcal{F}_t^X)$-Brownian motion on $[0,1]$?

If X_t were adapted, then, since B_t is, we would have that B_1 is $\mathcal{F}_t$-measurable $\forall t \geq 0$. This would imply that $\forall 0 < t \leq 1$, $B_t = \mathbb{E}[B_1|\mathcal{F}_t] = B_1$ a.s. Combined with continuity of paths, this would give $B_t \equiv 0$, a clear contradiction to B being an $(\mathcal{F}_t)$-BM. We conclude that X is not adapted.

Likewise, B is not an $(\mathcal{F}_t^X)$-BM. In fact, it is not even adapted. If it were, then B_1 would be $\mathcal{F}_t^X$-measurable since B_t would be adapted. An analogous argument to the one above then shows it could not be an $(\mathcal{F}_t^X)$-BM.

(c) Calculate the mean and covariance function of $(X_t)_{0\leq t\leq 1}$.

We clearly have $\mathbb{E}[X_t] = x(1-t) + yt$.

Let $0 \leq s \leq t \leq 1$;

$$\begin{aligned}\texttt{cov}(X_t, X_s) =& \mathbb{E}[(B_t - tB_1)(B_s - sB_1)] = \mathbb{E}[B_tB_s - sB_tB_1 - tB_sB_1 + tsB_1^2] \\ =& s - st - ts + ts = s - st = s(1-t) \\ =& \min\{s,t\}(1 - \max\{s,t\})\end{aligned}$$

(d) Verify that X_t has the same law as the conditional process $(W_t|W_0 = x, W_1 = y)$, where W is a Brownian motion.

In the notation of Question C.2 on Sheet 1, take $\mathbf{Y} = [W_s \quad W_t]^\top$ and $\mathbf{X} = [W_1]$, where $s < t$. Then we have

$$\Gamma_X = [1], \qquad \Gamma_Y = \begin{bmatrix} s & s \\ s & t \end{bmatrix}, \qquad \Gamma_{XY} = \begin{bmatrix} s & t \end{bmatrix}, \qquad \mu_X = 0, \qquad \mu_Y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $\mathbf{Y}|\mathbf{X}$ has a normal distribution with mean

$$\mu_Y + \Gamma_{XY}^\top \Gamma_X^{-1}(\mathbf{X} - \mu_X) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} s & t \end{bmatrix}^\top [1] W_1 = \begin{bmatrix} sW_1 \\ tW_1 \end{bmatrix}$$

which in the notation of this question is $E[X_t] = yt$, and covariance

$$\Gamma_Y - \Gamma_{XY}^\top \Gamma_X^{-1} \Gamma_{XY} = \begin{bmatrix} s & s \\ s & t \end{bmatrix} - \begin{bmatrix} s & t \end{bmatrix}^\top \begin{bmatrix} s & t \end{bmatrix} = \begin{bmatrix} s - s^2 & s - st \\ s - st & t - t^2 \end{bmatrix}$$

which in the notation of this question is $\text{cov}(X_s, X_t) = s(1-t)$. This agrees with our earlier calculations, and a Gaussian process is defined by its mean and covariance, so the laws must agree.

X is called the Brownian bridge from x to y over $[0,1]$.

3. Show that if a stochastic process (X_t) adapted to a filtration $(\mathcal{F}_t)$ has

(a) right-continuous paths then for an open set Γ, $H_\Gamma := \inf\{t \geq 0 : X_t \in \Gamma\}$ is a stopping time relative to $(\mathcal{F}_{t+})$,

Let $t > 0$.

We need to show that $\{H_\Gamma \leq t\} \in \mathcal{F}_{t+}$. Observe that for this, it is enough to show that $\{H_\Gamma < t\} \in \mathcal{F}_t$ since then, for any $k \geq 1$,

$$\{H_\Gamma \leq t\} = \bigcap_{n=k}^{\infty} \{H_\Gamma < t + \frac{1}{n}\} \in \mathcal{F}_{t+\frac{1}{k}}$$

and hence $\{H_\Gamma \le t\} \in \mathcal{F}_{t+}$.
(Worth emphasizing that this is a general fact about checking that something is a stopping time with respect to $\mathcal{F}_{t+}$.)
Now, since (X_u) is right-continuous and Γ is open, if $X_u \in \Gamma$ then necessarily $X_s \in \Gamma$ for some $s > u$, $s \in \mathbb{Q}$. It follows that

$$\{H_\Gamma < t\} = \{\exists_{s<t} X_s \in \Gamma\} = \{\exists_{\substack{s<t \\ s\in\mathbb{Q}}} X_s \in \Gamma\},$$

which is in $\mathcal{F}_t$, as required, since X is adapted.

(b) continuous paths then for a closed set Γ, H_Γ is a stopping time relative to $(\mathcal{F}_t)$.
By definition, since Γ is closed, the event $\{H_\Gamma \le t\} = \{\inf_{s\in[0,t]} d(X_s, \Gamma) = 0\}$, and by continuity of paths that is $\{\inf_{s\in[0,t]\cap\mathbb{Q}} d(X_s,\Gamma) = 0\}$. Then

$$\{H_\Gamma \le t\} = \bigcap_{n>1} \bigcup_{\substack{s\le t \\ s\in\mathbb{Q}}} \{X_s \in \Gamma^n\},$$

where Γ^n is the $1/n$-expansion of Γ; that is $\Gamma^n := \{x \in \mathbb{R} : \inf_{y\in\Gamma} |x-y| < \frac{1}{n}\}$.
The event on the right is in $\mathcal{F}_t$ and so we are done.

4. Let τ and ρ be two stopping times relative to a given filtration $(\mathcal{F}_t)$. Show that

(a) $\tau \wedge \rho := \min\{\tau, \rho\}$, $\tau \vee \rho := \max\{\tau, \rho\}$ and $\tau + \rho$ are all also stopping times For any $t \ge 0$,

$$\{\tau \wedge \rho > t\} = \underbrace{\{\tau > t\}}_{\in\mathcal{F}_t} \cap \underbrace{\{\rho > t\}}_{\in\mathcal{F}_t} \in \mathcal{F}_t,$$
$$\{\tau \vee \rho \le t\} = \underbrace{\{\tau \le t\}}_{\in\mathcal{F}_t} \cap \underbrace{\{\rho \le t\}}_{\mathcal{F}_t} \in \mathcal{F}_t$$

and

$$\{\tau + \rho > t\} = \{\tau = 0, \rho > t\} \cup \{\rho = 0, \tau > t\} \cup \bigcup_{0<q<t, q\in\mathbb{Q}} \{\tau > q, \rho > t - q\} \in \mathcal{F}_t$$

So $\tau \wedge \rho$, $\tau \vee \rho$ and $\tau + \rho$ are all stopping times.

(b) $\mathcal{F}_{\tau\wedge\rho} = \mathcal{F}_\tau \cap \mathcal{F}_\rho$ and $\{\tau \le \rho\}$ is $\mathcal{F}_{\tau\wedge\rho}$ measurable
First note that if we have two stopping times $\tau \le \rho$ then for any $A \in \mathcal{F}_\tau$

$$A \cap \{\rho \le t\} = \underbrace{A \cap \{\tau \le t\}}_{\in\mathcal{F}_t} \cap \{\rho \le t\} \in \mathcal{F}_t,$$

so that $\mathcal{F}_\tau \subset \mathcal{F}_\rho$. In particular, $\tau \wedge t$ is $\mathcal{F}_t$ measurable.
It follows that if τ, ρ are arbitrary stopping times then

$$\mathcal{F}_{\tau\wedge\rho} \subset \mathcal{F}_\tau \quad \text{and} \quad \mathcal{F}_{\tau\wedge\rho} \subset \mathcal{F}_\rho.$$

On the other hand, if $A \in \mathcal{F}_\tau \cap \mathcal{F}_\rho$ then

$$A \cap \{\tau \wedge \rho \le t\} = \Big(A \cap \{\tau \le t\}\Big) \cup \Big(A \cap \{\rho \le t\}\Big) \in \mathcal{F}_t,$$

so that in the end $\mathcal{F}_{\tau\wedge\rho} = \mathcal{F}_\tau \cap \mathcal{F}_\rho$.

We have,

$$\{\tau \le \rho\} \cap \{\tau \le t\} = \{\tau \wedge t \le \rho \wedge t\} \cap \{\tau \le t\} \in \mathcal{F}_t$$

and similarly

$$\{\tau \le \rho\} \cap \{\rho \le t\} = \{\tau \le t\} \cap \{\rho \le t\} \cap \{\tau \wedge t \le \rho \wedge t\} \in \mathcal{F}_t.$$

We conclude that $\{\tau \le \rho\} \in \mathcal{F}_\tau \cap \mathcal{F}_\rho = \mathcal{F}_{\tau\wedge\rho}$.

5. Let $H_a = \inf\{t \ge 0 : B_t = a\}$ be the first hitting time of a.

   (a) Use the Optional Stopping Theorem to compute the distribution of $B_{H_a \wedge H_b}$ for $a < 0 < b$.

   Recall that

$$\limsup_{t\to\infty} B_t = +\infty \quad \text{and} \quad \liminf_{t\to\infty} B_t = -\infty \text{ a.s.}$$

   and in particular $H_a \wedge H_b < \infty$ a.s.. Then, by continuity of paths, $B_{H_a\wedge H_b} \in \{a, b\}$ and $B_t^{H_a\wedge H_b} := B_{H_a\wedge H_b\wedge t}$ is a bounded martingale (since $|B^{H_a\wedge H_b}| \le \max\{-a, b\}$) and hence is uniformly integrable. By the Optional Stopping Theorem, $\mathbb{E}[B_{H_a\wedge H_b\wedge t}] = 0$ for any $t \ge 0$. Letting $t \to \infty$ and applying the Dominated Convergence Theorem, we obtain $\mathbb{E}[B_{H_a\wedge H_b}] = 0$.

$$0 = \mathbb{E}[B_{H_a\wedge H_b}] = a\mathbb{P}[B_{H_a\wedge H_b} = a] + b(1 - \mathbb{P}[B_{H_a\wedge H_b} = a]).$$

   Rearranging,

$$\mathbb{P}[B_{H_a\wedge H_b} = a] = \frac{b}{b-a}, \quad \mathbb{P}(B_{H_a\wedge H_b} = b) = \frac{-a}{b-a}.$$

   (b) Use the Optional Stopping Theorem to show that the Laplace transform of $H_a \wedge H_{-a}$ is given by

$$\mathbb{E}\left[e^{-\lambda H_a \wedge H_{-a}}\right] = \frac{1}{\cosh(a\sqrt{2\lambda})}, \qquad a > 0, \lambda > 0.$$

   Recall first the argument from lectures that allows us to calculate the Laplace transform of H_a: If $M_t^{(\theta)} = \exp(\theta B_t - \theta^2 t/2)$, then by the OST we have $\mathbb{E}[M_{H_a\wedge t}^{(\theta)}] = 1$. Let $\theta > 0$ (so that $M_{H_a\wedge t}^{(\theta)}$ is bounded). By the Dominated Convergence Theorem we obtain

$$1 = \mathbb{E}[M_{H_a}^{(\theta)}] = \mathbb{E}[e^{\theta a - \frac{\theta^2}{2} H_a}],$$

   from which

$$\mathbb{E}[e^{-\frac{\theta^2}{2} H_a}] = e^{-\theta a}.$$

   Now set $\theta = \sqrt{2\lambda}$ to obtain

$$\mathbb{E}[e^{-\lambda H_a}] = e^{-a\sqrt{2\lambda}}.$$

   We use an analogous argument for $H_a \wedge H_{-a}$ only now we can take $\theta \in \mathbb{R}$ and still have a bounded martingale. By optional stopping and the DCT we have

$$1 = \mathbb{E}[M_{H_a}^{(\theta)}] = \mathbb{E}[e^{\theta B_{H_a\wedge H_{-a}} - \frac{\theta^2}{2} H_a \wedge H_{-a}}],$$

Conditioning on $H_a \wedge H_{-a}$ we have

$$1 = \mathbb{E}[\mathbb{E}\left(e^{\theta B_{H_a \wedge H_{-a}}}|\mathcal{F}_{H_a \wedge H_{-a}}\right) e^{-\frac{\theta^2}{2} H_a \wedge H_{-a}}].$$

Using part (a) we have that $B_{H_a \wedge H_{-a}}$ is a or $-a$ with probability $1/2$, giving

$$1 = \cosh(\theta a)\mathbb{E}[e^{-\frac{\theta^2}{2} H_a \wedge H_{-a}}].$$

Rearranging and setting $\theta = \sqrt{2\lambda}$ gives the result.

6. Let $H_a = \inf\{t \geq 0 : B_t = a\}$ be the first hitting time of a and $S_t = \sup_{0 \leq s \leq t} B_s$.

(a) For $a > 0$ show that $\{H_a \leq t\} = \{S_t \geq a\}$. Hence find the probability density function $f_{H_a}(t)$ of H_a for any $a \neq 0$.
The first part is obvious - we hit a before t if and only if the supremum up to t is greater than a. Thus

$$\mathbb{P}(H_a \leq t) = \mathbb{P}(S_t \geq a) = 2\mathbb{P}(B_t \geq a) = 2(1 - \Phi(a/\sqrt{t})),$$

and differentiating with respect to t gives the density. In the case $a < 0$ we have the same argument for $-B$ and hence

$$f_{H_a}(t) = \frac{|a|}{\sqrt{2\pi t^3}} \exp(-\frac{a^2}{2t}).$$

(b) Let $U_a = \sup\{t \geq 0 : B_t = at\}$ be the last time that Brownian motion hits the line at. Show that $U_a = 1/H_a$ in distribution. Is U_a a stopping time?
As U_a is a supremum over times we cannot determine this from information before that time, so it is not a stopping time. Now

$$\begin{aligned}
U_a &= \sup\{t \geq 0 : B_t = at\} \\
&= \sup\{t \geq 0 : \frac{B_t}{t} = a\} \\
&= \sup\{t \geq 0 : W_{1/t} = at\} \text{ (in distribution by time inversion)} \\
&= \sup\{1/s \geq 0 : W_s = a\} \\
&= 1/\inf\{s \geq 0 : W_s = a\} = 1/H_a.
\end{aligned}$$

(c) Calculate $\mathbb{E}U_a$ and $\mathbb{E}B_{U_a}$.
We can use previous results to compute the mean of U_a as

$$\mathbb{E}U_a = \mathbb{E}1/H_a = \int_0^\infty \frac{1}{t} f_{H_a}(t)dt.$$

Thus

$$\begin{aligned}
\mathbb{E}U_a &= \int_0^\infty \frac{a}{t\sqrt{2\pi t^3}} \exp(-\frac{a^2}{2t}) \\
&= \frac{1}{a^2}\int_0^\infty \frac{2}{\sqrt{2\pi}} v^2 e^{-v^2/2} dv = 1/a^2.
\end{aligned}$$

Finally, by continuity of the Brownian paths, we have $\mathbb{E}B_{U_a} = \mathbb{E}aU_a = 1/a$. So we can see the OST certainly does not hold at this random time.

Section C (Extra practice questions, not for hand-in)

A. Fix $t > 0$. Without reference to Lévy's modulus of continuity, show that a.s. Brownian motion is not differentiable with respect to t at time t.
Hint:

- *Argue that* $|B_{t+\epsilon} - B_t|/\epsilon$ *diverges to* $+\infty$ *with probability* 1 *as* $\epsilon \to 0$.
- *Recall that by Blumenthal's 0-1 law for any* $\epsilon > 0$

$$\sup_{0\le u\le \epsilon} B_{t+u} - B_t > 0 \quad \textit{and} \quad \inf_{0\le u\le \epsilon} B_{t+u} - B_t < 0 \textit{ a.s.}$$

- *Draw conclusions on a.s. behaviour of* lim sup *and* lim inf *of* $(B_{t+\epsilon} - B_t)/\epsilon$ *with* $\epsilon \to 0$.

In fact a stronger property holds: a.s. the sample paths are nowhere differentiable

We know

$$\begin{aligned}\mathbb{P}(|B_{t+\epsilon} - B_t| \le K\epsilon) =& \mathbb{P}\Big(\frac{|B_{t+\epsilon} - B_t|}{\sqrt{\epsilon}} \le K\sqrt{\epsilon}\Big) = \mathbb{P}(|B_1| \le K\sqrt{\epsilon}) \\ =& \int_{-K\sqrt{\epsilon}}^{K\sqrt{\epsilon}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \le \frac{2}{\sqrt{2\pi}} K\sqrt{\epsilon},\end{aligned}$$

which converges to 0 as $\epsilon \to 0$. In particular, taking $K = \epsilon^{-1/4}$,

$$\mathbb{P}\left(\frac{|B_{t+\epsilon} - B_t|}{\epsilon} > \epsilon^{-1/4}\right) \ge 1 - \frac{2}{\sqrt{2\pi}}\epsilon^{1/4} \to 1, \text{ as } \epsilon \to 0,$$

which shows that $\frac{|B_{t+\epsilon}-B_t|}{\epsilon}$ cannot converge to a finite r.v. with a positive probability.

In fact the above combined with Borel-Cantelli lemma and Blumenthal's 0-1 law allows us to draw more detailed conclusions. Let $A_n = \{|B_{t+\frac{1}{n^4}} - B_t| \le K/n^4\}$, then $\mathbb{P}(A_n) \le \frac{2K}{\sqrt{2\pi}}\frac{1}{n^2}$ by the above estimate.

It follows that $\sum_{i=1}^{\infty} \mathbb{P}(A_n) < \infty$ and we can apply Borel-Cantelli to show that A_n happens only finitely often a.s, which implies that for any $K > 0$

$$\limsup_{\epsilon\to 0} \frac{|B_{t+\epsilon} - B_t|}{\epsilon} \ge K \quad \text{ a.s.} \tag{1}$$

It follows that

$$\left\{\limsup_{\epsilon\to 0} \frac{|B_{t+\epsilon} - B_t|}{\epsilon} = +\infty\right\} = \bigcap_{K=1}^{\infty} \left\{\limsup_{\epsilon\to 0} \frac{|B_{t+\epsilon} - B_t|}{\epsilon} \ge K\right\}$$

has also probability one.

Recall that by Blumenthal's 0-1 law a.s. for any $\epsilon > 0$,

$$\sup_{0\le u\le \epsilon} (B_{u+\epsilon} - B_t) > 0 \text{ and } \inf_{0\le u\le \epsilon} (B_{u+\epsilon} - B_t) < 0.$$

Then, by (1)

$$\limsup_{\epsilon\to 0} \frac{B_{t+\epsilon} - B_t}{\epsilon} = +\infty \text{ and } \liminf_{\epsilon\to 0} \frac{B_{t+\epsilon} - B_t}{\epsilon} = -\infty.$$

B. Let τ be a stopping time relative to a given filtration $(\mathcal{F}_t)$. We write

$$\begin{aligned}\mathcal{F}_\tau &:= \{A \in \mathcal{F} \,:\, A \cap \{\tau \le t\} \in \mathcal{F}_t \,\forall t \ge 0\},\\ \mathcal{F}_{\tau+} &:= \{A \in \mathcal{F} \,:\, A \cap \{\tau < t\} \in \mathcal{F}_t\},\\ \mathcal{F}_{\tau-} &:= \sigma(\{A \cap \{\tau > t\} \,:\, t \ge 0,\, A \in \mathcal{F}_t\})\end{aligned}$$

Show that

(a) $\mathcal{F}_\tau \subset \mathcal{F}_{\tau+}$ and $\mathcal{F}_\tau = \mathcal{F}_{\tau+}$ if $(\mathcal{F}_t)$ is right-continuous;
To see that $\mathcal{F}_\tau \subset \mathcal{F}_{\tau+}$, take any $A \in \mathcal{F}_\tau$ and note that for any $t \ge 0$

$$A \cap \{\tau < t\} = \Big(\bigcup_{n \ge 1} \underbrace{(A \cap \{\tau \le t - \frac{1}{n}\})}_{\in \mathcal{F}_{t-\frac{1}{n}}} \Big) \in \mathcal{F}_t.$$

If $(\mathcal{F}_t)$ is right-continuous, i.e. $\mathcal{F}_{t+} = \mathcal{F}_t \ \forall t \ge 0$, then for any $A \in \mathcal{F}_{\tau_+}$ and $t \ge 0$

$$A \cap \{\tau \le t\} = \Big(\bigcap_{n \ge 1} \underbrace{(A \cap \{\tau < t + \frac{1}{n}\})}_{\in \mathcal{F}_{t+\frac{1}{n}}} \Big) \in \mathcal{F}_{t+} = \mathcal{F}_t.$$

So $\mathcal{F}_{\tau+} \subseteq \mathcal{F}_\tau$ and hence $\mathcal{F}_{\tau+} = \mathcal{F}_\tau$.

(b) τ is $\mathcal{F}_{\tau-}$-measurable;
Simply observe that $\{\tau \le t\} = \{\tau \ge t\}^c \in \mathcal{F}_{\tau-}$ for every t.

(c) if $\tau = t$ is deterministic then $\mathcal{F}_\tau = \mathcal{F}_t$ and $\mathcal{F}_{\tau+} = \mathcal{F}_{t+}$;
Let $\tau = t$. To show that $\mathcal{F}_\tau = \mathcal{F}_t$, first note that for any $A \in \mathcal{F}_t$ and $s \ge 0$

$$A \cap \{\tau \le s\} = \begin{cases} A \in \mathcal{F}_t \subseteq \mathcal{F}_s & \text{if } t \le s \\ \emptyset & \text{otherwise,} \end{cases}$$

which implies that $\mathcal{F}_t \subseteq \mathcal{F}_\tau$.
On the other hand, for any $A \in \mathcal{F}_\tau$, by definition $A \cap \{\tau \le t\} \in \mathcal{F}_t$, and $A \cap \{\tau \le t\} = A$. Thus $\mathcal{F}_\tau \subseteq \mathcal{F}_t$ and hence $\mathcal{F}_\tau = \mathcal{F}_t$.
To see that $\mathcal{F}_{\tau+} = \mathcal{F}_{t+}$, first note that for any $A \in \mathcal{F}_{t+}$ and $s \ge 0$

$$A \cap \{\tau < s\} = \begin{cases} A \in \mathcal{F}_{t+} \subseteq \mathcal{F}_s & \text{if } t < s \\ \emptyset & \text{otherwise,} \end{cases}$$

which implies $\mathcal{F}_{t+} \subseteq \mathcal{F}_{\tau+}$.
On the other hand, for any $A \in \mathcal{F}_{\tau+}$ and $n \ge 1$, $A = \big(A \cap \{\tau < t + \frac{1}{n}\}\big) \in \mathcal{F}_{t+\frac{1}{n}}$ by definition. Thus $A \in \mathcal{F}_{t+}$ and hence $\mathcal{F}_{\tau+} \subseteq \mathcal{F}_{t+}$. Therefore $\mathcal{F}_{\tau+} = \mathcal{F}_{t+}$.

(d) if τ_n is a non-decreasing sequence of stopping times (i.e. for any $\omega \in \Omega$ and $n < m$ $\tau_n(\omega) \le \tau_m(\omega)$) then $\tau := \lim_{n\to\infty} \tau_n$ is also a stopping time;

$$\{\tau \le t\} = \{\lim_{n\to\infty} \tau_n \le t\} = \Big(\bigcap_{n \ge 1} \underbrace{\{\tau_n \le t\}}_{\in \mathcal{F}_t} \Big) \in \mathcal{F}_t,$$

where we have used that $\{\tau_n \le t\} \subset \{\tau_{n-1} \le t\}$ so that the intersection is a decreasing limit of events.

(e) if τ_n is a non-increasing sequence of stopping times then $\tau := \lim_{n\to\infty} \tau_n$ is a stopping time relative to $(\mathcal{F}_{t+})$.

Let $\tau_n \to \tau$ (not necessarily monotone), then

$$\{\tau \le t\} = \bigcap_{k=1}^{\infty} \underbrace{\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left\{\tau_n \le t + \frac{1}{k}\right\}}_{\in \mathcal{F}_{t+1/k}} \in \mathcal{F}_{t+}.$$

C. Recall that in lectures we used Optional Stopping to show that the Laplace transform of H_a is given by

$$\mathbb{E}\left[e^{-\lambda H_a}\right] = e^{-|a|\sqrt{2\lambda}} \quad a \in \mathbb{R}, \lambda > 0.$$

Consider $a, b > 0$. Deduce that if ξ_a, ξ_b are independent and distributed as H_a and H_b respectively, then $\xi_a + \xi_b$ has the same distribution as H_{a+b}. Use the strong Markov property to find an alternative proof of this result.

Let $\xi_a \sim H_a$ and $\xi_b \sim H_b$ be independent random variables. Then

$$\mathbb{E}[e^{-\lambda(\xi_a+\xi_b)}] = \mathbb{E}[e^{-\lambda\xi_a}]\mathbb{E}[e^{-\lambda\xi_b}] = e^{-a\sqrt{2\lambda}}e^{-b\sqrt{2\lambda}} = e^{-(a+b)\sqrt{2\lambda}} = \mathbb{E}[e^{-\lambda H_{a+b}}], \quad \lambda \ge 0,$$

from which we deduce that, as required, $(\xi_a + \xi_b) \sim H_{a+b}$.

To see this from the Strong Markov property, it suffices to note that $H_{a+b}(B) = H_a(B) + H_b(\tilde{B})$ where, by the strong Markov property of B, $\tilde{B}_t := B_{H_a+t} - B_{H_a} = B_{H_a+t} - a$ is a standard Brownian motion independent of $(B_u : u \le H_a)$. In particular, $H_b(\tilde{B}) \sim H_b(B)$ and further $H_b(\tilde{B})$ and $H_a(B)$ are independent.

D. Let $H_a = \inf\{t \ge 0 : B_t = a\}$ be the first hitting time of a.

(a) Show that H_a has the same distribution as $\frac{a^2}{B_1^2}$ and deduce its density.

For any $a > 0$ and $u > 0$,

$$\begin{aligned}
\mathbb{P}(H_a \le u) =& \mathbb{P}(\sup_{0\le t\le u} B_t \ge a) \\
=& \mathbb{P}(|B_u| \ge a) \qquad \text{by reflection principle} \\
=& \mathbb{P}(\sqrt{u}|B_1| \ge a) \qquad \text{by scaling} \\
=& \mathbb{P}(\frac{a^2}{B_1^2} \le u).
\end{aligned}$$

Hence H_a has the same distribution as $\frac{a^2}{B_1^2}$. Let f_{H_a} be the density function of H_a. Then,

$$\begin{aligned}
f_{H_a}(u) =& \frac{d\mathbb{P}(\sqrt{u}|B_1| \ge a)}{du} \\
=& \frac{d}{du}\Big(2\int_{\frac{a}{\sqrt{u}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})dx\Big) \\
=& \frac{a}{\sqrt{2\pi u^3}} \exp(-\frac{a^2}{2u}).
\end{aligned}$$

(b) Using the strong Markov property show that for any continuous bounded function f we have

$$\mathbb{E}[f(H_b - H_a)|\mathcal{F}_{H_a}] = \mathbb{E}[f(H_{b-a})], \quad 0 \le a \le b.$$

Deduce that $(H_a)_{a\ge 0}$ has stationary and independent increments. Discuss the properties of its paths $a \to H_a(\omega)$.

The first bit is immediate. H_a is a 1/2-stable subordinator, so has increasing right continuous paths. They could calculate the moments of increments to see $\mathbb{E}H_a^\beta$ exists for all $\beta < 1/2$.

E. Suppose that $(Z_t : t \ge 0)$, defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$, is adapted and has independent increments i.e. for any $0 \le s < t$, $Z_t - Z_s$ is independent of $\mathcal{F}_s$. Show that

- if $\mathbb{E}[|Z_t|] < \infty$ for all $t \ge 0$, then $\tilde{Z}_t := Z_t - \mathbb{E}[Z_t]$ is an $(\mathcal{F}_t)$–martingale;

  Note that clearly all the processes in question are adapted.

  For any $t \ge 0$

$$\mathbb{E}[|\tilde{Z}_t|] = \mathbb{E}[|Z_t - \mathbb{E}[Z_t]|] \le \mathbb{E}[|Z_t|] + |\mathbb{E}[Z_t]| \le 2\mathbb{E}[|Z_t|] < \infty.$$

  As $Z_t - Z_s$ is independent of $\mathcal{F}_s$ for any $0 \le s < t$, $\mathbb{E}[Z_t - Z_s|\mathcal{F}_s] = \mathbb{E}[Z_t - Z_s]$. Hence

$$\begin{aligned}\mathbb{E}[\tilde{Z}_t|\mathcal{F}_s] =& \mathbb{E}[Z_s + (Z_t - Z_s)|\mathcal{F}_s] - \mathbb{E}[Z_t]\\ =& Z_s = \mathbb{E}[Z_t - Z_s] - \mathbb{E}[Z_t] = \tilde{Z}_s\end{aligned}$$

- if $\mathbb{E}[Z_t^2] < \infty$ for all $t \ge 0$, then $\tilde{Z}_t^2 - \mathbb{E}[\tilde{Z}_t^2]$ is an $(\mathcal{F}_t)$–martingale;

  For any $t \ge 0$,

$$\begin{aligned}\mathbb{E}[|\tilde{Z}_t^2 - \mathbb{E}[\tilde{Z}_t^2]|] \le& 2\mathbb{E}[\tilde{Z}_t^2]\\ =& 2\mathbb{E}[(Z_t - \mathbb{E}[Z_t])^2]\\ \le& 4\mathbb{E}[Z_t^2 + (\mathbb{E}[Z_t])^2]\\ \le& 8\mathbb{E}[Z_t^2] < \infty \quad \text{by Jensen's Inequality.}\end{aligned}$$

  From part (a), we know $(\tilde{Z}_t)_{t\ge 0}$ is a martingale. Therefore $\mathbb{E}[\tilde{Z}_s(\tilde{Z}_t - \tilde{Z}_s)|\mathcal{F}_s] = 0$ and thus

$$\mathbb{E}[\tilde{Z}_s(\tilde{Z}_t - \tilde{Z}_s)] = \mathbb{E}\big[\mathbb{E}[\tilde{Z}_s(\tilde{Z}_t - \tilde{Z}_s)|\mathcal{F}_s]\big] = 0.$$

  Further, by independence of increments, $\mathbb{E}[(\tilde{Z}_t - \tilde{Z}_s)^2|\mathcal{F}_s] = \mathbb{E}[(\tilde{Z}_t - \tilde{Z}_s)^2]$. We then have

$$\begin{aligned}\mathbb{E}[\tilde{Z}_t^2 - \mathbb{E}[\tilde{Z}_t^2]|\mathcal{F}_s] =& \mathbb{E}[(\tilde{Z}_s + \tilde{Z}_t - \tilde{Z}_s)^2|\mathcal{F}_s] - \mathbb{E}[(\tilde{Z}_s + \tilde{Z}_t - \tilde{Z}_s)^2]\\ =& \mathbb{E}[\tilde{Z}_s^2 + 2\tilde{Z}_s(\tilde{Z}_t - \tilde{Z}_s) + (\tilde{Z}_t - \tilde{Z}_s)^2|\mathcal{F}_s]\\ &- \mathbb{E}[\tilde{Z}_s^2] - 2\mathbb{E}[\tilde{Z}_s(\tilde{Z}_t - \tilde{Z}_s)] - \mathbb{E}[(\tilde{Z}_t - \tilde{Z}_s)^2]\\ =& \tilde{Z}_s^2 - \mathbb{E}[\tilde{Z}_s^2] + \mathbb{E}[(\tilde{Z}_t - \tilde{Z}_s)^2|\mathcal{F}_s] - \mathbb{E}[(\tilde{Z}_t - \tilde{Z}_s)^2]\\ =& \tilde{Z}_s^2 - \mathbb{E}[\tilde{Z}_s^2].\end{aligned}$$

- if for some $\theta \in \mathbb{R}$, $\mathbb{E}\left[e^{\theta Z_t}\right] < \infty$ for all $t \ge 0$, then $\frac{\exp(\theta Z_t)}{\mathbb{E}[\exp(\theta Z_t)]}$ is an $(\mathcal{F}_t)$–martingale.

  For any $t \ge 0$,

$$\mathbb{E}\Big[\frac{\exp(\theta Z_t)}{\mathbb{E}[\exp(\theta Z_t)]}\Big] = \mathbb{E}[\exp(\theta Z_t)]/\mathbb{E}[\exp(\theta Z_t)] = 1.$$

For any $0 \le s < t$, using that $Z_t - Z_s$ is independent of $\mathcal{F}_s$, we have

$$\begin{aligned}\mathbb{E}\Big[\frac{\exp(\theta Z_t)}{\mathbb{E}[\exp(\theta Z_t)]}\Big|\mathcal{F}_s\Big] &= \frac{\mathbb{E}[\exp(\theta Z_s + \theta(Z_t - Z_s))|\mathcal{F}_s]}{\mathbb{E}[\exp(\theta(Z_t - Z_s))\exp(\theta Z_s)]} \\ &= \frac{\exp(\theta Z_s)\mathbb{E}[\exp(\theta(Z_t - Z_s)))]}{\mathbb{E}[\exp(\theta(Z_t - Z_s))]\mathbb{E}[\exp(\theta Z_s)]} = \frac{\exp(\theta Z_s)}{\mathbb{E}[\exp(\theta Z_s)]}.\end{aligned}$$

F. Let $\mathcal{Z}(\omega) := \{t : B_t(\omega) = 0\}$ be the set of Brownian zeros. Show that $\mathcal{Z}$ is a closed set. Using Fubini's theorem show that the Lebesgue measure of $\mathcal{Z}$ is zero a.s.

Let $R_t = \inf\{u \ge t : B_u = 0\}$. Using the strong Markov property and known facts about Brownian paths show that for any $t \ge 0$

$$\mathbb{P}[\inf\{u > 0 : B_{R_t+u} = 0\} > 0] = 0$$

and deduce that

$$\mathbb{P}\left[\inf\{u > 0 : B_{R_t+u} = 0\} > 0 \text{ for some rational } t\right] = 0.$$

Conclude that a.s. if a point $t \in \mathcal{Z}(\omega)$ is isolated from the left, i.e. $(q, t) \cap \mathcal{Z}(\omega) = \emptyset$ for some rational $q < t$ then necessarily t is a decreasing limit of points in $\mathcal{Z}(\omega)$ and thus $\mathcal{Z}$ does not have isolated points.

Fact: It follows that a.s. $\mathcal{Z}$ is uncountable, as it is a 'perfect set', ie. it is a non-empty closed set with no isolated points (an example of a random Cantor set).

$\mathcal{Z}$ is a closed set as it is the preimage of the closed set [0] under a continuous path. Using Fubini, it is easy to see that

$$\mathbb{E}[\mu(\mathcal{Z})] = \mathbb{E}\Big[\int_0^\infty \mathbf{1}_{\mathcal{Z}} dx\Big] = \int_0^\infty \mathbb{E}[\mathbf{1}_{\mathcal{Z}}]dx = 0$$

as $\mathbb{P}(B_t = 0) = 0$ for all $t \neq 0$.

We know that R_t is a stopping time, so by the strong Markov property, B_{R_t+u} is a Brownian motion in its own filtration. From Lecture notes (Prop 3.18), we know that a Brownian motion hits zero infinitely many times near $t = 0$, so R_t is not an isolated point (from the right) in $\mathcal{Z}$. Write $\mathcal{Z}^R$ for the random set of these zeros of the Brownian motion.

Now any point in $\mathcal{Z}$ is either isolated from the left (in which case there is some rational t such that it agrees with R_t, in which case it is in $\mathcal{Z}^R$), or it is not isolated from the left. As points in $\mathcal{Z}^R$ are not isolated, we see that $\mathcal{Z}$ a.s. has no isolated points.