B8.2: Continuous Martingales and Stochastic Calculus
Problem Sheet 3

The questions on this sheet are divided into three sections. Only those questions in Section B are compulsory and should be handed in for marking.

The questions are not in order of difficulty; if you are stuck on one question, move on to the next.

Section A

1. Let a be a function of finite variation, $a(0) = 0$ and $f : [0, T] \to \mathbb{R}$ a continuous function. Show that

$$\int_0^T f(s)da(s) = \lim_{n\to\infty} \sum_{i=0}^{m_n-1} f(t_i^n)(a(t_{i+1}^n) - a(t_i^n)),$$

where $\pi_n = \{0 = t_0 < t_1 < \ldots < t_{m_n} = T\}$ is a sequence of partitions of $[0, T]$ with $mesh(\pi_n) \to 0$ as $n \to \infty$.
(*Hint: use dominated convergence theorem for the associated measures* μ_+ *and* μ_-, *where* $\mu([0, t]) = a(t)$ *and* $\mu = \mu_+ - \mu_-$).

Let $f_n(t) := \sum f(t_i^n)\mathbf{1}_{(t_i^n, t_{i+1}^n]}(t)$. Then

$$\sum_{i=0}^{m_n-1} f(t_i^n)(a(t_{i+1}^n) - a(t_i^n)) = \int_0^T f_n(s)da(s) = \int_0^T f_n(s)(\mu_+(ds) - \mu_-(ds)).$$

We have pointwise convergence $f_n(t) \to f(t)$ by continuity. We can apply the DCT (f is bounded since continuous on a compact interval)

$$\lim_{n\to\infty} \sum_{i=0}^{m_n-1} f(t_i^n)(a(t_{i+1}^n) - a(t_i^n)) = \int_0^T f(s)(\mu_+(ds) - \mu_-(ds)) = \int_0^T f(s)da(s).$$

2. Let M be a continuous square integrable martingale with $M_0 = 0$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_t, \mathbb{P})$. Show that for any partition $\pi_n = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ we have

$$\mathbb{E}M_t^2 = \mathbb{E}\sum_{i=1}^n (M_{t_i} - M_{t_{i-1}})^2.$$

We just use the martingale property

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^n (M_{t_i} - M_{t_{i-1}})^2 &= \mathbb{E}\left(\sum M_{t_i}^2 - 2M_{t_i}M_{t_{i-1}} + M_{t_{i-1}}^2\right) \\
&= \mathbb{E}\left(\sum M_{t_i}^2 - 2\mathbb{E}\left(M_{t_i}M_{t_{i-1}}|\mathcal{F}_{t_{i-1}}\right) + M_{t_{i-1}}^2\right) \\
&= \mathbb{E}\sum M_{t_i}^2 - M_{t_{i-1}}^2 = \mathbb{E}M_t^2.
\end{aligned}$$

Section B (Compulsory)

1. Let $(x_n)_{n\geq 1}$ be a sequence of real numbers. For real numbers $a < b$, let $U([a,b],(x_n)_{n\geq 1})$ be the number of upcrossings of $[a,b]$ by the sequence. Show that $(x_n)_{n\geq 1}$ converges to a limit in $[-\infty,\infty]$ as $n\to\infty$ if and only if $U([a,b],(x_n)_{n\geq 1}) < \infty$ for all $a < b$ with $a,b\in\mathbb{Q}$.

   Recall that a sequence of real numbers converges if and only if it is a Cauchy sequence. Evidently if $U([a,b],(x_n)_{n\geq 1})$ is unbounded for some $a<b$, then for any $N\in\mathbb{N}$, we can find $n,m>N$ such that $|x_n - x_m| > b-a > 0$ and so the sequence is not Cauchy.

   In the opposite direction, suppose that $U([a,b],(x_n)_{\geq 1}) < \infty$ for all $a<b$. We consider two cases: either there exists $M>0$ such that $x_n\in[-M,M]$ infinitely often, or not.

   In the second case, eventually, for any M, either $x_n > M$ or $x_n < -M$ (since otherwise $U([-M,M],(x_n)_{n\geq 1}) = \infty$), and so either $x_n\to\infty$ or $x_n\to-\infty$.

   Suppose then that there exists M such that $x_n\in[-M,M]$ infinitely often. Then the subsequence of those $x_n\in[-M,M]$ is a bounded infinite sequence and so has a convergent subsequence that we denote by $(x_{n_k})_{k\geq 1}$, converging to a limit l, say, as $k\to\infty$. Then given $\epsilon>0$, there exists K such that $x_{n_k}\in[l-\epsilon,l+\epsilon]$ for all $k>K$. But then $x_n\in[l-2\epsilon,l+2\epsilon]$ for all but finitely many n, since otherwise $U([l+\epsilon,l+2\epsilon],(x_n)_{n\geq 1})$ or $U([l-2\epsilon,l-\epsilon],(x_n)_{n\geq 1})$ is infinite. In other words, there exists N such that $n>N$ implies $|x_n - l| < 2\epsilon$. Since ϵ was arbitrary the sequence converges as required.

2. Let M be a positive continuous martingale converging a.s. to zero as $t\to\infty$. Let $M^* := \sup_{t\geq 0} M_t$. Note that we do not assume that $\mathcal{F}_0$ is trivial or that M_0 is deterministic.

   (a) For $x>0$, prove that
   $$\mathbb{P}\left[M^*\geq x|\mathcal{F}_0\right] = 1\wedge\frac{M_0}{x}.$$
   Conclude that M^* has the same distribution as M_0/U, where U is independent of M_0 and uniformly distributed on $[0,1]$.
   (*Hint: stop M when it becomes larger than x*).
   Let $H_x = H_x(M) := \inf\{t\geq 0 : M_t\geq x\}$. Then
   $$M_0 = \mathbb{E}[M_{H_x\wedge t}|\mathcal{F}_0] = M_0\mathbf{1}_{x\leq M_0} + x\mathbb{P}(H_x\leq t|\mathcal{F}_0) - x\mathbf{1}_{x\leq M_0} + \mathbb{E}[M_t\mathbf{1}_{H_x\geq t}|\mathcal{F}_0]$$
   $$x\mathbb{P}(H_x\leq t|\mathcal{F}_0) \xrightarrow{\text{Monotone Convergence}} x\mathbb{P}(M^*\geq x|\mathcal{F}_0), \qquad \mathbb{E}[M_t\mathbf{1}_{H_x\geq t}|\mathcal{F}_0] \to 0$$
   where the last convergence follows from the DCT because $M_t\to 0$ and $0\leq M_t\mathbf{1}_{H_x\geq t} < x$. Rearranging this yields
   $$x\mathbf{1}_{x\leq M_0} + M_0\mathbf{1}_{x>M_0} = x\mathbb{P}(M^*\geq x|\mathcal{F}_0).$$
   i.e.
   $$\mathbb{P}(M^*\geq x|\mathcal{F}_0) = \frac{M_0}{x}\wedge 1.$$
   Now average over the distribution of M_0 to recover
   $$\begin{aligned}\mathbb{P}[M^*\geq x] = \mathbb{E}\left[\mathbb{E}[\mathbf{1}_{M^*\geq x}|\mathcal{F}_0]\right] &= \mathbb{E}\Big[\frac{M_0}{x}\wedge 1\Big] = \mathbb{E}\Big[\mathbb{E}\Big[\mathbf{1}_{U\leq\frac{M_0}{x}}|\mathcal{F}_0\Big]\Big]\\ &= \mathbb{P}\Big[\frac{M_0}{U}\geq x\Big],\end{aligned}$$
   where U is independent of M_0 and uniformly distributed on $[0,1]$.

(b) Let $a > 0$ and $B_t^a := a + B_t$ be a Brownian motion started at a. Let $\tau = H_0(B^a) = H_{-a}(B) = \inf\{t \geq 0 : B_t^a = 0\}$. Find the distribution of the random variable $Y := \sup_{t \leq \tau} B_t^a$.

Fix $a > 0$ and set $B_t^a := a + B_t$. Then $M_t := B_{t \wedge \tau}^a$ is a martingale, $M_t \geq 0$ and $M_t \to 0$ a.s., and $Y = M^*$. It follows from the first part of the question that

$$Y = M^* \sim \frac{M_0}{U} = \frac{a}{U}.$$

3. Suppose that $(B_t)_{t \geq 0}$ is Brownian motion under $\mathbb{P}$. For a partition π of $[0, T]$, write $\|\pi\|$ for the mesh of the partition and $0 = t_0 < t_1 < t_2 < \ldots < t_{N(\pi)} = T$ for the endpoints of the intervals of the partition. Calculate

(a)

$$\lim_{\|\pi\| \to 0} \sum_{j=0}^{N(\pi)-1} B_{t_{j+1}} \left(B_{t_{j+1}} - B_{t_j} \right),$$

It's actually easiest to do part (b) first. If they read far enough ahead in the lecture notes, they will know that

$$\begin{aligned} t &= \lim_{\|\pi\| \to 0} \sum_{j=0}^{N(\pi)-1} \left(B_{t_{j+1}} - B_{t_j} \right)^2 \\ &= \lim_{\|\pi\| \to 0} \sum_{j=0}^{N(\pi)-1} \left(B_{t_{j+1}} + B_{t_j} \right) \left(B_{t_{j+1}} - B_{t_j} \right) - 2 \lim_{\|\pi\| \to 0} \sum_{j=0}^{N(\pi)-1} B_{t_j} \left(B_{t_{j+1}} - B_{t_j} \right) \\ &= B_T^2 - B_0^2 - 2 \lim_{\|\pi\| \to 0} \sum_{j=0}^{N(\pi)-1} B_{t_j} \left(B_{t_{j+1}} - B_{t_j} \right). \end{aligned}$$

Subtracting this quantity from

$$\lim_{\|\pi\| \to 0} \sum_{j=0}^{N(\pi)-1} \frac{1}{2} \left(B_{t_{j+1}} + B_{t_j} \right) \left(B_{t_{j+1}} - B_{t_j} \right) = \frac{1}{2} \left(B_T^2 - B_0^2 \right),$$

we obtain

$$\lim_{\|\pi\| \to 0} \sum_{j=0}^{N(\pi)-1} B_{t_{j+1}} \left(B_{t_{j+1}} - B_{t_j} \right) = \frac{1}{2} \left(B_T^2 - B_0^2 + T \right).$$

(b)

$$\int_0^T B_s \circ dB_s \triangleq \lim_{\|\pi\| \to 0} \sum_{j=0}^{N(\pi)-1} \frac{1}{2} \left(B_{t_{j+1}} + B_{t_j} \right) \left(B_{t_{j+1}} - B_{t_j} \right).$$

This is the *Stratonovich integral* of $\{B_s\}_{s \geq 0}$ with respect to itself over $[0, T]$.

Since the summand on the right is the difference of two squares, this is a telescoping sum with value

$$\frac{1}{2} \left(B_{t_{N(\pi)}}^2 - B_0^2 \right) = \frac{1}{2} \left(B_T^2 - B_0^2 \right).$$

4. Let M and N be continuous local martingales and τ a stopping time. We write M^τ and N^τ for the stopped processes, $M_t^\tau = M_{\tau\wedge t}$, $N_t^\tau = N_{\tau\wedge t}$. Show that $M^\tau(N - N^\tau)$ is a continuous local martingale.
Hint: use the properties of quadratic co-variation

$$M^\tau(N - N^\tau) = M^\tau N - M^\tau N^\tau = M^\tau N - \langle M^\tau, N\rangle + \langle M^\tau, N^\tau\rangle - M^\tau N^\tau$$

is a sum of two local martingales, where we used Proposition 8.26(iii) from lectures which says

$$\langle M, N\rangle^\tau = \langle M^\tau, N\rangle = \langle M^\tau, N^\tau\rangle.$$

5. Show that if M, N are two martingales in $\mathcal{H}^{2,c}$ (that is they are continuous and bounded in L^2), then $MN - \langle M, N\rangle$ is a uniformly integrable martingale.

First method:
Recall that for $X \in \mathcal{H}^{2,c}$, $X^2 - \langle X\rangle$ is a uniformly integrable martingale (Theorem 7.24i). We have $M + N \in \mathcal{H}^{2,c}$ and hence

$$2(MN - \langle M, N\rangle) = (M + N)^2 - \langle M + N\rangle - (M^2 - \langle M\rangle) - (N^2 - \langle N\rangle)$$

is a sum of three uniformly integrable martingales and hence a uniformly integrable martingale.

Second (brute force) method:
Recall that $|\langle M, N\rangle_\infty| \leq \sqrt{\langle M\rangle_\infty}\sqrt{\langle N\rangle_\infty}$ (Kunita–Watanabe). Then by Hölder's inequality

$$\mathbb{E}[|\langle M, N\rangle_\infty|] \leq \sqrt{\mathbb{E}[\langle M\rangle_\infty]}\sqrt{\mathbb{E}[\langle N\rangle_\infty]} < \infty,$$

where $\mathbb{E}[\langle M\rangle_\infty] < \infty$ and $\mathbb{E}[\langle N\rangle_\infty] < \infty$ since $M, N \in \mathcal{H}^{2,c}$.

Also note that $\sup_{s\geq 0}|M_sN_s| \leq \sup_{s\geq 0}|M_s|\sup_{s\geq 0}|N_s|$. Then by Hölder's inequality

$$\mathbb{E}\Big[\sup_{s\geq 0}|M_sN_s|\Big] \leq \sqrt{\mathbb{E}\Big[\sup_{s\geq 0}|M_s|^2\Big]}\sqrt{\mathbb{E}\Big[\sup_{s\geq 0}|N_s|^2\Big]} < \infty.$$

Then, by the Dominated Convergence Theorem, $M_tN_t - \langle M, N\rangle_t$ converges in L^1 and the result follows.

6. Let X be a positive random variable independent of a standard Brownian motion B on a probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t), \mathbb{P})$. Let $M_t = B_{tX}$ for $t \geq 0$. We assume that the filtration $(\mathcal{F}_t)$ is the smallest filtration (satisfying the usual conditions) to which M is adapted.

   (a) Show that $M = (M_t)_{t\geq 0}$ is a local martingale with respect to $(\mathcal{F}_t)$.
   (b) Show that M is a martingale if and only if $E(X^{1/2}) < \infty$.
   (c) Find the quadratic variation process $\langle M\rangle_t$.
   (d) Let $A = (A_t)_{t\geq 0}$ be a continuous increasing process with $A_0 = 0$ which is independent of the Brownian motion B. Assuming A is adapted to $(\mathcal{F}_t) = (\mathcal{G}_{A_t})$ show that $(B_{A_t})_{t\geq 0}$ is a local martingale and find conditions which ensure it is a martingale and determine its quadratic variation.

(a) To see that M is a local martingale take the set of stopping times $T_n = \inf\{t : |M_t| > n\}$. We just need to show that the sequence of stopped processes $M^{T_n} = (M_{t\wedge T_n})_{t\in\mathbb{R}_+}$ are martingales. Clearly by construction as B_t is adapted to $\mathcal{G}_t$, M is adapted to $\mathcal{F}_t$ and by definition of T_n we have $|M^{T_n}| \leq n$, so that M^{T_n} is integrable. The martingale property then follows from that of Brownian motion

$$\mathbb{E}(M_t|\mathcal{F}_s) = \mathbb{E}(B_{tX}|\mathcal{F}_s) = \mathbb{E}(B_{tX}|\mathcal{G}_{sX}) = B_{sX} = M_s, \quad 0 \leq s \leq t.$$

(b) To see when M is a martingale it is clear that all we need to establish is the integrability condition as the process is adapted and satisfies the martingale property. As X is independent of B we have, writing $\mathbb{E}_X$ for the expectation over the random variable X and $\mathbb{E}$ for the expectation for the Brownian motion

$$\begin{aligned}\mathbb{E}|M_t| &= \mathbb{E}\left(\mathbb{E}_X|B_{tX}|\right)\\ &= \mathbb{E}_X \int_{-\infty}^{\infty} \frac{|x|}{\sqrt{2\pi tX}} \exp(-\frac{x^2}{tX})dx\\ &= \mathbb{E}_X\sqrt{2tX/\pi}.\end{aligned}$$

Hence it is integrable iff $\mathbb{E}_X(X^{1/2}) < \infty$.

(c) The quadratic variation is easily seen to be $\langle M\rangle_t = tX$ from the result for Brownian motion.

(d) We can do the same thing with an adapted increasing process A. The local martingale argument is the same as we use the usual stopping times to get bounded paths - the martingale property holds using that of Brownian motion.

The condition for a martingale is that $\mathbb{E}A_t^{1/2} < \infty$ by the same argument as that given in (b).

Finally the quadratic variation process for B_{A_t} is the process A_t.

Section C (Extra practice questions, not for hand-in)

1. Suppose that the real-valued function a is of bounded variation and that f is a-integrable. Show that the function $(f \cdot a)$ defined by

$$(f \cdot a)(t) = \int_0^t f(s)da(s)$$

is right continuous and of finite variation.

It is enough to check that it is the difference of two non-decreasing right continuous functions (by a Lemma of lectures, a function is of bounded variation if and only if it can be written as the difference of two non-decreasing funtions).

If μ is a measure and $f \geq 0$ is integrable with respect to μ, then right continuity of $\int_0^t f(s)\mu(ds)$ is clear. Let μ be the generalised measure associated with a and $\mu = \mu_+ - \mu_-$ its Jordan decomposition. Then

$$(f \cdot a)(t) = \big((f^+ \cdot \mu_+)(t) + (f^- \cdot \mu_-)(t)\big) - \big((f^- \cdot \mu_+)(t) + (f^+ \cdot \mu_-)(t)\big)$$

is the difference of two non-decreasing functions as required.

A direct proof of finite variation is easy too: Let t_i, t_{i+1} be the endpoints of an interval in a partition π of $[0,T]$, then

$$|(f\cdot a)_{t_{i+1}} - (f\cdot a)_{t_i}| = \left|\int_{t_i}^{t_{i+1}} f(s)da(s)\right| \leq \int_{t_i}^{t_{i+1}} |f(s)|\,|da(s)|$$

and summing over intervals in the partition we find

$$V(f\cdot a)_t \leq \int_0 t|f(s)||da(s)|,$$

and so by integrability of f with respect to a, we see that $(f\cdot a)$ has finite variation.

2. Let M be a continuous L^2-bounded martingale. Show that

   (a) For every deterministic $t\in[0,\infty)$, the time $T_t = \inf\{s>t : M_s \neq M_t\}$ is a stopping time (and similarly with M replaced by $\langle M\rangle$)
   By now this should be an easy exercise. The simplest thing is to take the process $X_s = \mathbf{1}_{s>t}(M_s - M_t)$, then T_t is the first hitting time of the open set $\mathbb{R}\setminus\{0\}$.

   (b) The intervals of constancy for M and $\langle M\rangle$ coincide a.s., that is to say, if (S,S') are random times with $S\leq S'$, then for almost every ω,

   $$M_t(\omega) = M_{S(\omega)}(\omega) \quad \forall t\in[S(\omega),S'(\omega)]$$

   if and only if

   $$\langle M\rangle_t(\omega) = \langle M\rangle_{S(\omega)}(\omega) \quad \forall t\in[S(\omega),S'(\omega)].$$

   Hint: Begin with times $(S,S')=(t,T_t)$ for $t\in\mathbb{Q}$.
   Following the hint, as $M^2-\langle M\rangle$ is a uniformly integrable martingale (Q2/Thm 7.24i) and T_t is a stopping time, we can use the optional stopping theorem to write

   $$\begin{aligned}\mathbb{E}[(M_{T_t}-M_t)^2] &= \mathbb{E}\left[\left(\int_t^{T_t} dM_s\right)^2\right] \overset{\text{OST}}{=} \mathbb{E}\left[\int_t^{T_t} d\langle M\rangle_s\right]\\ &= \mathbb{E}[\langle M\rangle_{T_t} - \langle M\rangle_t]\end{aligned}$$

   and the result follows since if two non-negative r.v. have equal expectations then one is zero iff the other one is. As the rationals are countable, this result holds for all $t\in\mathbb{Q}$ simultaneously a.s, that is, except on a null set $\mathcal{N}$.
   Now suppose we have random times S,S'. Fix any $\omega\notin\mathcal{N}$. Suppose $s\mapsto M_s(\omega)$ is constant on $[S(\omega),S'(\omega)]$. Either this interval is a single point, or it contains a rational t, and hence $S'(\omega)\leq T_t(\omega)$. However as $\omega\notin\mathcal{N}$, we know that $s\mapsto\langle M\rangle_s(\omega)$ is constant on $[t,T_t(\omega)]$. As t is an arbitrary rational, this implies that $s\mapsto\langle M\rangle_s(\omega)$ is constant on $(S(\omega),S'(\omega)]$. As $s\mapsto\langle M\rangle_s(\omega)$ is continuous, it cannot vary at a single point, so it is constant on $[S(\omega),S'(\omega)]$. Similarly if we reverse the roles of M and $\langle M\rangle$.

3. Recall that an adapted right-continuous stochastic process M is called a *local martingale* if there exists a sequence of stopping times $\tau_n\uparrow\infty$ a.s. such that for any n the stopped process M^{τ_n} is a martingale, where $M^{\tau_n}_t = M_{\tau_n\wedge t}$. The sequence (τ_n) is called a *reducing sequence* or a *localising sequence*. Show that if M is a local martingale then

(a) if M is non-negative (i.e. $\forall t \geq 0$ $M_t \geq 0$ a.s.) then it is a supermartingale;
We have $\mathbb{E}[M_{\tau_n \wedge t} | \mathcal{F}_t] = M_{\tau_n \wedge s} \to M_s$ a.s. as $n \to \infty$. By Fatou then

$$M_s = \lim_{n\to\infty} \mathbb{E}[M_{\tau_n \wedge t} | \mathcal{F}_s] \geq \mathbb{E}[\lim_{s\to\infty} M_{\tau_n \wedge t} | \mathcal{F}_s] = \mathbb{E}[M_t | \mathcal{F}_s]$$

and hence (M_t) is a supermartingale.

(b) M is a (true) martingale if and only if for any $a > 0$ the family

$$\{M_\tau : \tau \text{ a stopping time with } \tau \leq a\}$$

is uniformly integrable.
Recall that (M_t) is a martingale iff

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0] \qquad \forall \tau \text{ bounded stopping time}$$

We also know that if M is a martingale then the statement holds.
For the reverse, suppose that M is a local martingale s.t. $\{M_\tau : \tau$ is a stopping time $\leq a\}$ is UI $\forall\, a \geq 0$. Take τ is a bounded stopping time, say $\tau \leq a$, and (τ_n) a reducing sequence for M.

$$\mathbb{E}[M_{\tau_n \wedge \tau}] = \mathbb{E}[M_0] \qquad \text{since } M^{\tau_n} \text{ is a martingale.}$$

and $\{M_{\tau_n \wedge \tau} : n \geq 0\} \subseteq \{M_\rho : \rho \leq a\}$ is UI. Hence

$$M_{\tau_n \wedge \tau} \to M_\tau \text{ a.s. and in } L^1.$$

Then $\mathbb{E}[M_\tau] = \lim_{n\to\infty} \mathbb{E}[M_{\tau_n \wedge \tau}] = \mathbb{E}[M_0]$ as required.

4. *(**A primer in stochastic integration**:* We define here the stochastic integral of a simple process w.r.t. to a nice martingale. The problem may appear long but is elementary and its aim is primarily for you to reflect on desirable properties a *stochastic integral* should have.*)*
Let $M = (M_t : t \geq 0)$ be a uniformly integrable martingale bounded in L^2: $\sup_{t\geq 0} \mathbb{E}[M_t^2] < \infty$. Let $\mathcal{E}$ be the space of simple bounded process of the form

$$\varphi_t = \sum_{i=0}^{m} \varphi^{(i)} \mathbf{1}_{(t_i, t_{i+1}]}(t), \quad t \geq 0$$

for some $n \in \mathbb{N}$, $0 \leq t_0 < t_1 < \ldots < t_{m+1}$ and where $\varphi^{(i)}$ are bounded $\mathcal{F}_{t_i}$ measurable random variables. Define the stochastic integral $\varphi \cdot M$ of such φ with respect to M via

$$(\varphi \cdot M)_t := \sum_{i=0}^{m} \varphi^{(i)} (M_{t \wedge t_{i+1}} - M_{t \wedge t_i}), \quad t \geq 0.$$

(a) *Show that $\varphi \to \varphi \cdot M$ is linear on $\mathcal{E}$;*
Note that $\varphi + \psi \in \mathcal{E}$ for $\varphi, \psi \in \mathcal{E}$. This follows since $\xi \mathbf{1}_{(t,s]} + \eta \mathbf{1}_{(u,v]}$ is either of the required form if (case (1)) $s \leq u$ or $v \leq t$ or else (case (2)) if e.g. $t < u < s < v$ then

$$\xi \mathbf{1}_{(t,s]} + \eta \mathbf{1}_{(u,v]} = \xi \mathbf{1}_{(t,u]} + (\xi + \eta) \mathbf{1}_{(u,s]} + \eta \mathbf{1}_{(s,v]}$$

is of the required form. Then $(\varphi+\psi\cdot M)_t=(\varphi\cdot M)_t+(\psi\cdot M)_t$. Indeed it is enough to show this for $\varphi=\xi\mathbf{1}_{(t,s]}$, $\psi=\eta\mathbf{1}_{(u,v]}$, where ξ (resp. η) is $\mathcal{F}_t$- (resp. $\mathcal{F}_u$-) measurable. Then e.g.

$$\begin{aligned}(\varphi+\psi\cdot M)_r=&\begin{cases}\xi(M_{s\wedge r}-M_{t\wedge r})+\eta(M_{v\wedge r}-M_{u\wedge r}) & \text{in case (1)}\\ \xi(M_{u\wedge r}-M_{t\wedge r})+(\xi+\eta)(M_{s\wedge r}-M_{u\wedge r})+\eta(M_{v\wedge r}-M_{s\wedge r}) & \text{in case (2).}\end{cases}\\ =&\xi(M_{s\wedge r}-M_{t\wedge r})+\eta(M_{v\wedge r}-M_{u\wedge r})\\ =&(\varphi\cdot M)_r+(\psi\cdot M)_r,\end{aligned}$$

as required. We clearly have $(\alpha\varphi\cdot M)_t=\alpha(\varphi\cdot M)_t$ $\forall\, \alpha\in\mathbb{R}$, $\varphi\in\mathcal{E}$.

(b) *Show that $\varphi\cdot M$ is a martingale for all $\varphi\in\mathcal{E}$;*

$$\mathbb{E}[(\varphi\cdot M)_t|\mathcal{F}_s]=\sum_{i=0}^{m}\mathbb{E}[\varphi^{(i)}(M_{t_{i+1}\wedge t}-M_{t_i\wedge t})|\mathcal{F}_s].$$

We have

$$\begin{aligned}&\mathbb{E}[\varphi^{(i)}(M_{t_{i+1}}-M_{t_i})|\mathcal{F}_s]=\varphi^{(i)}(M_{t_{i+1}}-M_{t_i})=\varphi^{(i)}(M_{t_{i+1}\wedge s}-M_{t_i\wedge s}) && \text{for } t_{i+1}\le s\\ &\mathbb{E}[\varphi^{(i)}(M_{t_{i+1}\wedge t}-M_{t_i})|\mathcal{F}_s]=\varphi^{(i)}(M_s-M_{t_i})=\varphi^{(i)}(M_{t_{i+1}\wedge s}-M_{t_i}) && \text{for } t_i\le s<t_{i+1}\\ &\mathbb{E}[\mathbb{E}[\varphi^{(i)}(M_{t_{i+1}\wedge t}-M_{t_i\wedge t})|\mathcal{F}_{t_i\wedge t}]|\mathcal{F}_s]=0=\varphi^{(i)}(M_{t_{i+1}\wedge s}-M_{t_i\wedge s}) && \text{for } t_i\ge s.\end{aligned}$$

Thus

$$\mathbb{E}[(\varphi\cdot M)_t|\mathcal{F}_s]=\sum_{i=0}^{m}\mathbb{E}[\varphi^{(i)}(M_{t_{i+1}\wedge t}-M_{t_i\wedge t})|\mathcal{F}_s]=(\varphi\cdot M)_s.$$

(c) *Let τ be a stopping time and recall that for a process X, X^τ is the stopped process $X^\tau_t=X_{\tau\wedge t}$. Show that we have equality between the following three processes:*

$$\varphi\cdot M^\tau=(\varphi\cdot M)^\tau=(\mathbf{1}_{[0,\tau]}\varphi\cdot M),$$

and where to define the last integral we would extend the definition of $\mathcal{E}$ to the case of (t_i) being a sequence of bounded stopping times.

$$\begin{aligned}(\varphi\cdot M^\tau)_t=&\sum\varphi^{(i)}(M^\tau_{t_{i+1}\wedge t}-M^\tau_{t_i\wedge t})\\ =&\sum\varphi^{(i)}(M_{t_{i+1}\wedge t\wedge\tau}-M_{t_i\wedge t\wedge\tau})\\ =&(\varphi\cdot M)_{t\wedge\tau}=(\varphi\cdot M)^\tau_t\\ =&\Big(\sum\varphi^{(i)}\mathbf{1}_{(t_i\wedge\tau,\, t_{i+1}\wedge\tau]}\cdot M\Big)_t=\Big(\sum\varphi^{(i)}\mathbf{1}_{[0,\tau]}\mathbf{1}_{(t_i,\, t_{i+1}]}\cdot M\Big)_t\\ =&(\varphi\mathbf{1}_{[0,\tau]}\cdot M)_t\end{aligned}$$

(d) *Compute $\mathbb{E}[(\varphi\cdot M)_t^2]$ for a $\varphi\in\mathcal{E}$. Conclude that $\sup_{t\ge0}\mathbb{E}[(\varphi\cdot M)_t^2]<\infty$.*

Now assume that there is an adapted non-decreasing process $\langle M\rangle$, $\langle M\rangle_0 = 0$ a.s. and such that $(M_t^2 - \langle M\rangle_t : t \geq 0)$ is a martingale. Show that

$$\mathbb{E}[(\varphi \cdot M)_t^2] = \mathbb{E}\left[\int_0^t \varphi_s^2 d\langle M\rangle_s\right].$$

$$\mathbb{E}[(\varphi \cdot M)_t^2] = \mathbb{E}\left[\sum\sum \varphi^{(i)}\varphi^{(j)}(M_{t_{i+1}\wedge t} - M_{t_i\wedge t})(M_{t_{j+1}\wedge t} - M_{t_j\wedge t})\right]$$

But

$$\mathbb{E}[\varphi^{(i)}\varphi^{(j)}(M_{t_{i+1}\wedge t} - M_{t_i\wedge t})(M_{t_{j+1}\wedge t} - M_{t_j\wedge t})] = \begin{cases} 0 & \text{if } i \neq j \\ \mathbb{E}[(\varphi^{(i)})^2(M_{t_{i+1}\wedge t} - M_{t_i\wedge t})^2] & \text{if } i = j \end{cases}$$

and

$$\begin{aligned} &\mathbb{E}[(\varphi^{(i)})^2(M_{t_{i+1}\wedge t} - M_{t_i\wedge t})^2] \\ =&\mathbb{E}[(\varphi^{(i)})^2\mathbb{E}[(M_{t_{i+1}\wedge t} - M_{t_i\wedge t})^2|\mathcal{F}_{t_i}]] \\ =&\mathbb{E}[(\varphi^{(i)})^2(M^2_{t_{i+1}\wedge t} - M^2_{t_i\wedge t})] \\ =&\mathbb{E}[(\varphi^{(i)})^2\big(M^2_{t_{i+1}\wedge t} - \langle M\rangle_{t_{i+1}\wedge t} - (M^2_{t_i\wedge t} - \langle M\rangle_{t_i\wedge t})\big)] + \mathbb{E}[(\varphi^{(i)})^2(\langle M\rangle_{t_{i+1}\wedge t} - \langle M\rangle_{t_i\wedge t})] \\ =&\mathbb{E}[(\varphi^{(i)})^2(\langle M\rangle_{t_{i+1}\wedge t} - \langle M\rangle_{t_i\wedge t})]. \end{aligned}$$

We conclude that

$$\mathbb{E}[(\varphi \cdot M)_t^2] = \sum \mathbb{E}[(\varphi^{(i)})^2(M_{t_{i+1}\wedge t} - M_{t_i\wedge t})^2] = \mathbb{E}\Big[\int_0^t \varphi_u^2 d\langle M\rangle_u\Big].$$

(e) *Let $\varphi, \psi \in \mathcal{E}$. Show that $\psi\varphi \in \mathcal{E}$ and that $\psi \cdot (\varphi \cdot M) = (\psi\varphi \cdot M)$.*
For $\psi = \sum_j \psi^{(j)}\mathbf{1}_{(s_j,s_{j+1}]}$ and $\varphi = \sum_i \varphi^{(i)}\mathbf{1}_{(t_i,t_{i+1}]}$

$$\begin{aligned} \Big(\psi \cdot (\varphi \cdot M)\Big)_t =& \sum_j \psi^{(j)}\Big((\varphi \cdot M)_{s_{j+1}\wedge t} - (\varphi \cdot M)_{s_j\wedge t}\Big) \\ =&\sum_j\sum_i \psi^{(j)}\varphi^{(i)}\Big(M_{t_{i+1}\wedge s_{j+1}\wedge t} - M_{t_i\wedge s_{j+1}\wedge t} - (M_{t_{i+1}\wedge s_j\wedge t} - M_{t_i\wedge s_j\wedge t})\Big) \\ =&(\psi\varphi \cdot M)_t \end{aligned}$$

since

$$\varphi^{(i)}\mathbf{1}_{(t_i,t_{i+1}]} \cdot \psi^{(j)}\mathbf{1}_{(s_j,s_{j+1}]} = \varphi^{(i)}\psi^{(j)}\mathbf{1}_{(t_i\vee s_j,\, t_{i+1}\wedge s_{j+1}]}$$

can be non-zero only if $t_i \vee s_j < t_{j+1} \wedge s_{j+1}$ but then $t_i < s_{j+1}$ and $s_j < t_{i+1}$. On the other hand, when $t_i < s_{j+1}$ and $s_j < t_{i+1}$

$$\begin{aligned} &M_{t_{i+1}\wedge s_{j+1}\wedge t} - M_{t_i\wedge s_{j+1}\wedge t} - (M_{t_{i+1}\wedge s_j\wedge t} - M_{t_i\wedge s_j\wedge t}) := (*) \\ =&M_{t_{i+1}\wedge s_{j+1}\wedge t} - M_{t_i\wedge t} - M_{s_j\wedge t} + M_{t_i\wedge s_j\wedge t} \\ =&M_{t_i\wedge t} - M_{t_{i+1}\wedge s_{j+1}\vee t} - M_{t_i\wedge s_j\vee t} + M_{s_j\vee t} \\ =&M_{t_{i+1}\wedge s_{j+1}\wedge t} - \underbrace{(M_{t_i\wedge t} + M_{s_j\wedge t} - M_{t_i\wedge s_j\wedge t})}_{M_{(t_i\vee s_j)\wedge t}}, \end{aligned}$$

and otherwise $(*) = 0$. We conclude that $\Big(\psi \cdot (\varphi \cdot M)\Big)_t = (\psi\varphi \cdot M)_t$ as required.