B8.2: Continuous Martingales and Stochastic Calculus
Problem Sheet 4

The questions on this sheet are divided into three sections. Only those questions in Section B are compulsory and should be handed in for marking.

The questions are not in order of difficulty; if you are stuck on one question, move on to the next.

Section A

Let B be a standard one-dimensional Brownian motion.

1. Show that $e^{t/2}\cos(B_t)$ is a martingale.

   There are many ways to do this. Letting $M_t = e^{t/2}\cos(B_t)$ and using Itô we have (writing in differential form)
   $$de^{t/2}\cos(B_t) = -e^{t/2}\sin(B_t)dB_t.$$
   Hence M is a continuous local martingale. We could then compute the quadratic variation
   $$\mathbb{E}\langle M\rangle_t = \mathbb{E}\int_0^t e^s\cos^2(B_s)ds \leq e^t < \infty.$$
   Thus by Theorem 7.24 of the notes it is a martingale.

2. Let $\beta_k(t) = \mathbb{E}B_t^{2k}$ (which you can assume is finite for all $k \in \mathbb{N}$ and $t \geq 0$). Using Itô's formula show that
   $$\beta_k(t) = k(2k-1)\int_0^t \beta_{k-1}(s)ds.$$
   Hence show $\beta_k(t) = \frac{(2k)!}{2^k k!}t^k$.

   Find $\beta_k(t) = \mathbb{E}|B|^{2k}$ when B is a standard two-dimensional Brownian motion.

   By Itô with $B_0 = 0$ we have
   $$B_t^{2k} = \int_0^t 2kB_s^{2k-1}dB_s + \frac{1}{2}\int_0^t 2k(2k-1)B_s^{2k-2}ds.$$
   Taking expectations and using that the stochastic integral $\int_0^t B_s^m dB_s$ is a martingale (follows from the fact that $\beta_k(t)$ is finite) and hence has expectation 0, we have
   $$\beta_k(t) = \mathbb{E}\int_0^t k(2k-1)B_s^{2k-2}ds = k(2k-1)\int_0^t \beta_{k-1}(s)ds.$$
   Using that $\beta_0(t) = 1, \beta_1(t) = t$ we can recursively solve the equation to get the even moments of Brownian motion.

   For the two-dimensional case using Itô's formula again gives $\beta_k(t) = 2k^2\int_0^t \beta_{k-1}(s)ds$ and $\beta_k(t) = 2^k k! t^k$.

Section B (Compulsory)

1. Let $M, N \in \mathcal{H}^{2,c}$, $K \in L^2(M)$ and $F \in L^2(N)$. Show that for each $t \in [0,\infty]$ we have

$$\mathbb{E}\left[\left(\int_0^t K_s dM_s\right)\left(\int_0^t F_s dN_s\right)\right] = \mathbb{E}\left[\int_0^t K_s F_s d\langle M, N\rangle_s\right].$$

Since $(K \bullet M) \in \mathcal{H}^{2,c}$, $(F \bullet N) \in \mathcal{H}^{2,c}$, $(K \bullet M)(F \bullet N) - \langle K \bullet M, F \bullet N\rangle$ is a uniformly integrable martingale and thus using Theorem 8.8 of lectures ($\langle K \bullet M, N\rangle = K \cdot \langle M, N\rangle$) twice

$$\mathbb{E}[(K \bullet M)_t (F \bullet N)_t] = \mathbb{E}[\langle K \bullet M, F \bullet N\rangle_t] = \mathbb{E}\Big[\int_0^t K_s F_s d\langle M, N\rangle_s\Big], \quad t \in [0,\infty].$$

2. Suppose that M is a continuous local martingale and $K \in L^2_{\text{loc}}(M)$. Fix $t > 0$. Show that if $\mathbb{E}\left[\int_0^t K_s^2 d\langle M\rangle_s\right] < \infty$, then the stopped process $(K \bullet M)^t$ is a martingale and

$$\mathbb{E}\left[\int_0^t K_s dM_s\right] = 0, \quad \mathbb{E}\left[\left(\int_0^t K_s dM_s\right)^2\right] = \mathbb{E}\left[\int_0^t K_s^2 d\langle M\rangle_s\right] < \infty.$$

$(K \bullet M)$ is a continuous local martingale, so is $(K \bullet M)^t$. Further

$$\mathbb{E}[\langle (K \bullet M)^t\rangle_\infty] = \mathbb{E}[\langle K \bullet M\rangle_t] = \mathbb{E}\Big[\int_0^t K_s^2 d\langle M\rangle_s\Big] < \infty.$$

Hence by characterisation of martingales in $\mathcal{H}^2$ we have that $(K \bullet M)^t$ is a martingale and is in $\mathcal{H}^2$. Then

$$\mathbb{E}[(K \bullet M)^t_\infty] = \mathbb{E}[(K \bullet M)_t] = 0$$

and

$$\begin{aligned}\mathbb{E}\Big[\big((K \bullet M)^t_\infty\big)^2\Big] &= \mathbb{E}[\langle (K \bullet M)^t\rangle_\infty] \\ &= \mathbb{E}[\langle K \bullet M\rangle_t] = \mathbb{E}\Big[\int_0^t K_s^2 d\langle M\rangle_s\Big]\end{aligned}$$

3. Let f be a continuous function on $[0,\infty)$ and B a standard Brownian motion. Prove that the random variable

$$X_t := \int_0^t f(s) dB_s, \quad t \geq 0,$$

is Gaussian and compute the covariance of X_t and X_s.

(The same result holds true for locally bounded Borel functions f.)
Hint: You may use that the space of mean-zero Gaussian variables is a closed subspace of L^2.

Since f is continuous, the approximation via Riemann sums of the Lebesgue integral $\int f(u)du$ converges pointwise and in L^2.
The stopped process B^t is in $\mathcal{H}^2$ and $f \in \mathrm{L}^2(B^t)$, where $f_t(\omega) = f(t)$, and so Proposition 8.18 of lectures gives the limit in probability, but here we also know that everything lives in L^2 and so

$$X_t = (f \bullet B)_t = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} f\Big(\frac{tk}{2^n}\Big)\Big(B_{\frac{t(k+1)}{2^n}} - B_{\frac{tk}{2^n}}\Big),$$

where limit is in L^2.

Clearly each finite sum is Gaussian and hence the limit is also Gaussian (using the hint).

$$\mathbb{E}[X_t] = \mathbb{E}[(f \bullet B)_t] = 0,$$
$$\mathbb{E}[X_t^2] = \mathbb{E}[\langle f \bullet B\rangle_t] = \int_0^t f(u)^2 du.$$

Now $(X_{t_1}, \ldots, X_{t_n})$ is a limit in L^2 of similar sums, the approximating vector

$$\left(\sum f\left(\frac{tk}{2^n}\right)\left(B_{\frac{t_1(k+1)}{2^n}} - B_{\frac{tk}{2^n}}\right), \ldots, \sum f\left(\frac{t_n k}{2^n}\right)\left(B_{\frac{t(k+1)}{2^n}} - B_{\frac{tk}{2^n}}\right)\right)$$

is an image under a linear map of Gaussian vector (of Brownian increments). Thus it is Gaussian and so is the limit. Further, X is a martingale and hence $\mathbb{E}[(X_t - X_s)X_s] = 0$. It follows that

$$\mathbb{E}[X_s X_t] = \mathbb{E}[X_s^2 + X_s(X_t - X_s)] = \int_0^s f(u)^2 du = \int_0^{s\wedge t} f(u)^2 du, \quad \text{for } s \le t.$$

Big hammer method (not as intuitive, but nice to know):
It is enough to show the statement for $t \in [0, T]$, with arbitrary T. The statement is clearly true for $f = \mathbf{1}_{(s,t]}$, for any $0 \le s < t \le T$ and these functions generate the Borel σ-algebra. The statement then follows by the monotone class theorem (one should check carefully that the class of Borel measurable bounded functions for which the statement holds satisfies the assumptions of the monotone class theorem).

4. Suppose that $(B_t)_{t\ge 0}$ is standard Brownian motion and f and g are twice continuously differentiable real-valued functions. Using Itô's formula, decompose the semimartingale $X_t = \exp\left(f(B_t) - \int_0^t g(B_s)ds\right)$ into a local martingale and a bounded variation part and hence find an expression relating f and g which guarantees that $(X_t)_{t\ge 0}$ is a local martingale.

   We apply Ito's formula to X giving, in differential form,

$$dX_t = f'(B_t)X_t dB_t + \frac{1}{2}\left(f''(B_t) + f'(B_t)^2\right) X_t dt - g(B_t)X_t dt.$$

   Thus $X_t = X_0 + M_t + A_t$, where the local martingale part is

$$M_t = \int_0^t f'(B_s)X_s dB_s.$$

   The finite variation part is

$$A_t = \int_0^t \frac{1}{2}\left(f''(B_s) + f'(B_s)^2 - g(B_s)\right) X_s ds.$$

   Then to get a local martingale we need the finite variation term to be 0. Thus we require f, g to satisfy

$$\frac{1}{2}f''(x) + \frac{1}{2}f'(x)^2 = g(x), \ \ \forall x \in \mathbb{R}.$$

5. Let B be a standard Brownian motion. Recall that $M_t^\theta := \exp(\theta B_t - \frac{\theta^2}{2}t)$ is a local martingale. Expanding as a Taylor series in θ, around $\theta = 0$, we can write

$$M_t^\theta = \sum_{k=0}^{\infty} \theta^k H_k(t, B_t),$$

where $H_k(t, x)$ are polynomials.

(a) Find the first four of the $H_k(t, x)$ and show that $(H_k(t, B_t) : t \geq 0)$, $k = 0, 1, 2, 3, 4$ are local martingales. *(Hint: you may use the Itô formula to verify the local martingale property)*

$$\begin{aligned}
H_0(t,x) =& M_t^0 = 1, \\
H_1(t,x) =& \frac{\partial M_t^\theta}{\partial \theta}|_{\theta=0} = (x - \theta t) M_t^\theta|_{\theta=0} = x, \\
H_2(t,x) =& \frac{1}{2!}\frac{\partial^2 M_t^\theta}{\partial^2 \theta}|_{\theta=0} = \frac{1}{2!}\{(x-\theta t)^2 - t\} M_t^\theta|_{\theta=0} = \frac{1}{2}(x^2 - t), \\
H_3(t,x) =& \frac{1}{3!}\frac{\partial^3 M_t^\theta}{\partial^3 \theta}|_{\theta=0} = \frac{1}{6}\{(x-\theta t)^3 - 3t(x-\theta t)\} M_t^\theta|_{\theta=0} = \frac{1}{6}(x^3 - 3tx), \\
H_4(t,x) =& \frac{1}{4!}\frac{\partial^4 M_t^\theta}{\partial^4 \theta}|_{\theta=0} = \frac{1}{4!}\{(x-\theta t)^4 - 6t(x-\theta t)^2 - 3t^2\} M_t^\theta|_{\theta=0} \\
=& \frac{1}{24}(x^4 - 6tx^2 + 3t^2).
\end{aligned}$$

Hence it is easy to see that

$$H_0(t, B_t) = 1,\ H_1(t, B_t) = B_t,\ H_2(t, B_t) = \frac{1}{2}(B_t^2 - t)$$

are local martingales. In fact, they are martingales.
To see $H_3(t, B_t)$ and $H_4(t, B_t)$ are local martingales, we apply Ito's formula to H_3 and H_4 and find that, written in a differential notation for brevity,

$$\begin{aligned}
dH_3(t, B_t) =& \frac{1}{6}(3B_t^2 - 3t)dB_t - \frac{3B_t}{6}dt + \frac{1}{2}\frac{1}{6}3 \times 2B_t d\langle B\rangle_t = \frac{1}{6}(3B_t^2 - 3t)dB_t, \\
dH_4(t, B_t) =& \frac{1}{24}\left\{(4B_t^3 - 12tB_t)dB_t - 6B_t^2 dt + 6tdt + \frac{1}{2}(12B_t^2 - 12t)d\langle B\rangle_t\right\} \\
=& \frac{1}{24}(4B_t^3 - 12tB_t)dB_t.
\end{aligned}$$

The drift terms are cancelled out, hence $H_3(t, B_t)$ and $H_4(t, B_t)$ are indeed local martingales.

(b) We now show that in fact for any local martingale M, $H_k(\langle M\rangle_t, M_t)$ are local martingales and deduce a stochastic integral representation for them. Define h_k via

$$\sum_{k=0}^{\infty} u^k h_k(x) = \exp(ux - u^2/2), \quad u, x \in \mathbb{R}.$$

Let $f(x) = \exp(-x^2/2)$ and deduce that

$$h_k(x) = \frac{(-1)^k}{k! f(x)} f^{(k)}(x).$$

Note that for $a > 0$, we have

$$\exp(ux - au^2/2) = \exp\left(u\sqrt{a}\left(\frac{x}{\sqrt{a}}\right) - \frac{(u\sqrt{a})^2}{2}\right)$$

and deduce that $H_k(a, x) = a^{k/2} h_k(x/\sqrt{a})$. Give the value of $H_k(0, x)$.

We first note that

$$\begin{aligned} F(u,x) = \exp(ux - \frac{u^2}{2}) &= \exp(\frac{x^2}{2}) \exp(-\frac{1}{2}(x^2 - 2ux + u^2)) \\ &= \exp(\frac{x^2}{2}) \exp(-\frac{1}{2}(x-u)^2) = \frac{f(x-u)}{f(x)}. \end{aligned}$$

Therefore

$$\begin{aligned} h_k(x) :=& \frac{1}{k!} \frac{\partial^k F(u,x)}{\partial u^k}|_{u=0} \\ =& \frac{1}{k!}(-1)^k \frac{f^{(k)}(x-u)|_{u=0}}{f(x)} = \frac{(-1)^k}{k! f(x)} f^{(k)}(x). \end{aligned}$$

For $a > 0$, we have

$$\begin{aligned} G^u(a,x) = \exp(ux - au^2/2) =& \exp\left(u\sqrt{a}(\frac{x}{\sqrt{a}}) - \frac{(u\sqrt{a})^2}{2}\right) = F\left(u\sqrt{a}, \frac{x}{\sqrt{a}}\right) \\ \Rightarrow \quad H_k(a,x) =& \frac{1}{k!} \frac{\partial^k G^u(a,x)}{\partial u^k}|_{u=0} = (\sqrt{a})^k h_k(x/\sqrt{a}) \quad \forall k \geq 0. \end{aligned}$$

To compute $H_k(0, x)$, first note $G^\theta(0, x) = \exp(\theta x)$. Hence

$$H_k(0,x) = \frac{1}{k!} \frac{\partial^k G^\theta(0,x)}{\partial \theta^k}|_{\theta=0} = \frac{x^k}{k!}.$$

(c) Use Itô's formula and the above representation to show that if M is a continuous local martingale, then $(H_k(\langle M \rangle_t, M_t) : t \geq 0)$ is a continuous local martingale.

Applying Ito's formula to $H_k(\langle M \rangle_t, M_t)$, we derive that

$$dH_k(\langle M \rangle_t, M_t) = \frac{\partial H_k}{\partial x}(\langle M \rangle_t, M_t) dM_t + \left(\frac{\partial H_k}{\partial a} + \frac{1}{2}\frac{\partial^2 H_k}{\partial x^2}\right)(\langle M \rangle_t, M_t) d\langle M \rangle_t.$$

In order to conclude that $H_k(\langle M \rangle_t, M_t)$ is a local martingale, we need to show that

$$\frac{\partial H_k}{\partial a} + \frac{1}{2}\frac{\partial^2 H_k}{\partial x^2} = 0.$$

First we observe (or prove by induction)

$$f^{k+1}(x) + x f^k(x) + k f^{k-1}(x) = 0 \quad \forall k \geq 1.$$

This gives us

$$\begin{aligned} \frac{\partial H_k}{\partial x}(a,x) =& a^{\frac{k-1}{2}} h_k'(x/\sqrt{a}) \\ =& a^{\frac{k-1}{2}} \frac{(-1)^k}{k! f(x/\sqrt{a})} \left(f^{(k+1)}(x/\sqrt{a}) + \frac{x}{\sqrt{a}} f^{(k)}(x/\sqrt{a})\right) \\ =& a^{\frac{k-1}{2}} \frac{(-1)^k}{k! f(x/\sqrt{a})} (-k f^{(k-1)}(x/\sqrt{a})) = H_{k-1}(a,x). \end{aligned}$$

Hence

$$
\begin{aligned}
&\frac{\partial H_k}{\partial a} + \frac{1}{2}\frac{\partial^2 H_k}{\partial x^2}\\
=&\frac{k}{2}a^{\frac{k-1}{2}}h_k(x/\sqrt{a}) + x(-\frac{1}{2})a^{-\frac{3}{2}}a^{\frac{k}{2}}h_k'(x/\sqrt{a}) + \frac{1}{2}a^{\frac{k}{2}}a^{-1}h_k''(x/\sqrt{a})\\
=&\frac{a^{\frac{k}{2}-1}}{2}(h_k(x/\sqrt{a})k - \frac{x}{\sqrt{a}}h_{k-1}(x/\sqrt{a}) + \frac{1}{2}h_{k-2}(x/\sqrt{a}))\\
=&\frac{a^{\frac{k}{2}-1}}{2}\frac{(-1)^k}{(k-1)!f(x/\sqrt{a})}\{f^k(x/\sqrt{a}) + \frac{x}{\sqrt{a}}f^{k-1}(x/\sqrt{a}) + (k-1)f^{k-2}(x/\sqrt{a})\} = 0.
\end{aligned}
$$

So $dH_k(\langle M\rangle_t, M_t) = \frac{\partial H_k}{\partial x}(\langle M\rangle_t, M_t)dM_t$ and hence

$$
\begin{aligned}
H_k(\langle M\rangle_t, M_t) =&H_k(0,0) + \int_0^t \frac{\partial H_k}{\partial x}(\langle M\rangle_s, M_s)dM_s\\
=&0 + \int_0^t H_{k-1}(\langle M\rangle_s, M_s)dM_s.
\end{aligned}
$$

(d) Observe that $\frac{\partial H_k}{\partial x}(a,x) = H_{k-1}(a,x)$. Show by induction that

$$H_k(\langle M\rangle_t, M_t) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} 1\, dM_{s_n} \cdots dM_{s_2}\, dM_{s_1}.$$

As $H_0(a,x) = 1$, the result is a simple induction.

Section C (Extra practice questions, not for hand-in)

1. Use a heuristic argument based on a Taylor expansion to check that for Stratonovich stochastic calculus the chain rule takes the form of the classical (Newtonian) one.

   Taylor's expansion says that for a C^3 function (for simplicity),

$$
\begin{aligned}
f(t) &= f(s) + f'(s)(s-t) + \frac{1}{2}f''(s)(s-t)^2 + O((s-t)^3)\\
&= f(s) + f'(t)(s-t) - \frac{1}{2}f''(t)(s-t)^2 + O((s-t)^3)
\end{aligned}
$$

   The second expansion comes from expanding around t and rearranging. Averaging these two out, we obtain

$$f(t) = f(s) + \frac{f'(s)+f'(t)}{2}(t-s) + \frac{f''(s)-f''(t)}{2}(t-s)^2 + O((s-t)^3)$$

   To calculate $f(X)$, where X is a Brownian motion (or more generally a continuous semimartingale), we take a partition π with end points $t_0 = 0$ and $t_{N(\pi)} = t$,

$$
\begin{aligned}
f(X_t) &= f(X_0) + \sum_\pi \left(f(X_{t_{i+1}}) - f(X_{t_i})\right)\\
&= f(X_0) + \sum_\pi \left(\frac{f'(X_{t_{i+1}}) + f'(X_{t_i})}{2}(X_{t_{i+1}} - X_{t_i})\right)\\
&\quad + \sum_\pi \left(\frac{f''(X_{t_{i+1}}) - f''(X_{t_i})}{2}(X_{t_{i+1}} - X_{t_i})^2\right)\\
&\quad + \sum_\pi O((X_{t_{i+1}} - X_{t_i})^3)
\end{aligned}
$$

Heuristically taking the limit as $\|\pi\| \to 0$, we know that the 3-variation of X is zero and f' and f'' are continuous, so we get

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \circ dX_s + 0 + 0$$

which is the classical chain rule.

2. Let B be a three-dimensional Brownian motion with B_0 an independent random variable in $\mathbb{R}^3 \backslash \{0\}$. (You may assume that this proces does not hit 0 and is transient in that $|B_t| \to \infty$ as $t \to \infty$).

   (a) Show using Itô's formula and Levy's characterization of Brownian motion that the radial part, $X = |B|$, satisfies
   $$X_t = X_0 + \int_0^t \frac{1}{X_s} ds + W_t,$$
   where W is a Brownian motion.

   (b) Show using Itô's formula that $1/|B_t|$ is a local martingale.

   (c) Let $B_0 = y$, and set $M_t = |B_{t+1} - y|^{-1}$ for $t \geq 0$. Show that $\mathbb{E}M_t^2 = 1/(1+t)$ and hence that the process M is an L^2-bounded local martingale.

   (d) Show that M is a supermartingale.

   (e) Using the martingale convergence theorm, show that M is not a martingale.

(a) use Itô's formula with the SDE for X, the radial part of a 3-d Brownian motion,

$$dX = \frac{1}{X} dt + \sum_{i=1}^{3} \frac{B_i}{X} dB_i,$$

Then Levy's characterization shows that the local martingale part is a Brownian motion, so we have the required equation.

(b) To show that $1/|B_t| = 1/X$ is a local martingale we use Ito and the SDE for X and hence

$$d(\frac{1}{X}) = -\frac{1}{X^2} dW.$$

(c) From here on the solution to this question is in section VI.33 of Rogers and Williams. $E(M_t^2) = \frac{1}{t+1}$ can be calculated directly from the density

$$\begin{aligned}
EM_t^2 &= \int_{\mathbb{R}^3} |x-y|^{-2} \frac{1}{(\sqrt{2\pi(t+1)})^3} \exp(-\frac{|x-y|^2}{2(t+1)}) dx \\
&= \int_0^\infty \int_0^\pi \int_0^{2\pi} r^{-2} \frac{1}{(\sqrt{2\pi(t+1)})^3} \exp(-\frac{r^2}{2(t+1)}) r^2 \sin\theta d\phi d\theta dr \\
&= \frac{1}{t+1}.
\end{aligned}$$

It follows immediately that M is uniformly bounded in L^2, and this implies that it is uniformly integrable.

(d) Since $t \mapsto B_{t+1} - y$ is a Brownian motion, an application of part (a) tells us M_t is a local martingale, but don't forget to note that the filtration here is $\mathcal{G}_t = \mathcal{F}_{t+1}$. Since M is a nonnegative local martingale with $E|M_0| < \infty$, M is a supermartingale from basic properties of non-negative local martingales.

(e) If M was a martingale, then it would be a uniformly integrable continuous martingale, and the martingale convergence theorem would imply the existence of an L^1 random variable M_∞ such that $M_t \to M_\infty$ both a.s. and in L^1. Suppose this holds. Since $B_t \to \infty$ a.s. as $t \to \infty$, we must have $M_\infty = 0$. Hence, also by martingale convergence, $M_t = E\left[M_\infty | \mathcal{G}_t\right] = 0$, which is a contradiction and hence M cannot be a true martingale!

This shows that, even with good integrability properties a local martingale need not necessarily be a true martingale.

3. (Brownian local time at zero) Let B be a standard Brownian motion. Let $f(x) = |x|$ and f_n be a sequence of convex C^2 functions converging pointwise to $f(x)$ with $f_n'(x)$ increasing in n to $f_-'(x)$ (the left-hand derivative of f which is well defined everywhere).

   Such a sequence can be constructed quasi-explicitly. Indeed, take $h(x)$ a non-negative C^∞ function supported on $[-1, 0]$ and $\int_{-1}^0 h(x)dx = 1$. Put $f_n(x) := n \int_{-1}^0 f(x+y)h(ny)dy$ and verify it satisfies the required properties.

   Apply Itô's formula to $f_n(B_t)$ and denote by L_t^n the finite variation term in the resulting semimartingale decomposition of $f_n(B_t)$. Observe that L_t^n is a non-decreasing process.

   (a) Determine the region where $f_n''(x)$ is non-zero and hence comment when, along the paths of B, the process L^n is increasing and when it is constant and deduce what, if it existed, the limit would measure.

   (b) Define $sgn(x)$ to be 1 for $x > 0$ and -1 for $x \leq 0$. Use the stochastic dominated convergence theorem to show that for any $t > 0$, $\int_0^t f_n'(B_s)dB_s$ converge, in probability and uniformly in $s \leq t$, to $\int_0^t sgn(B_s)dB_s$.

   (c) Deduce that L_t^n converges in probability to some process L_t which is non-decreasing and in particular that $|B_t|$ is a semimartingale.
   Hint: to deduce monotonicity of L you may want to take a subsequence and pass to a.s. convergence.

   (d) Finally, using Itô on B and $|B|$ for a suitable choice of function, show that

   $$\forall t \geq 0 \quad \int_0^t |B_s| dL_s = 0 \quad a.s.$$

   from which you should deduce that L is supported on $\mathcal{Z}$ a.s.
   (i.e. for any $s < t$, $L_t(\omega) - L_s(\omega) = \int_s^t dL_u(\omega) = \int_{[s,t]\cap\mathcal{Z}(\omega)} dL_u(\omega)$ a.s.)
   The process L is called the local time in zero.

   (e) How would you go about defining local time at level a?

   (f) Does the above extend to an arbitrary continuous local martingale M?

We have

$$f_n(B_t) = f_n(0) + \int_0^t f_n'(B_s)dB_s + \frac{1}{2}\int_0^t f_n''(B_s)ds.$$

Note that by convexity of $|\cdot|$ we have

$$f_n(\lambda x+(1-\lambda)z)=n\int_{-1/n}^{0}|\lambda(x+y)+(1-\lambda)(z+y)|h(ny)dy\le\lambda f_n(x)+(1-\lambda)f_n(z) \quad (1)$$

so that f_n is convex, in particular $L_t^n:=\frac{1}{2}\int_0^t f_n''(B_s)ds$ is non-decreasing.

(a) Further, in (1) we had equality if $(x+y)$ and $(z+y)$ were of the same sign, i.e. if $x,z<0$ or $x,z>1/n$. It follows that f_n is affine on $(-\infty,0]\cup[1/n,\infty)$ and hence f_n'' is strictly positive only in the neighbourhood of zero and L_t^n is increasing when B visits the neighbourhood of zero. We might expect that L_t^n in the limit as $n\to\infty$ (if the normalisation is correct) would give a measure of time B spends in zero.

(b) We know that $f_n'(x)\to f_-'(x)=sgn(x)$. Further, by convexity of f_n and since $|f_n'(\cdot)|=1$ on $(-\infty,0]\cup[1/n,\infty)$ it follows that $|f_n'(\cdot)|\le 1$. In particular we may apply the stochastic dominated convergence theorem to deduce that for any $t>0$, as $n\to\infty$

$$\int_0^s f_n'(B_u)dB_u\longrightarrow\int_0^s sgn(B_u)dB_u,$$

in probability and uniformly in $s\le t$.

(c) It follows that we can take a subsequence on which the convergence above is a.s. for all $s\le t$. It then follows, since $f_n(x)\to|x|$ that L_s^n, $s\le t$, converges a.s. (on a subsequence) to a process $(L_s:s\le t)$. Repeating this argument for $t=1,2,\dots$ we obtain $(L_t:t\ge 0)$ a.s. well defined for all t (we may put $L_t\equiv 0$ on the remaining null set). $L$is then a.s. non-decreasing and satisfies

$$|B_t|=\int_0^t sgn(B_s)dB_s+L_t,\quad t\ge 0,\ a.s.$$

It follows that L_t is in fact continuous. It is of finite variation since it is non-decreasing and hence $(|B_t|:t\ge 0)$ is a continuous semimartingale (the above gives its semimartingale decomposition). Note that $X_t=\int_0^t sgn(B_s)dB_s$ is a continuous local martingale, $X_0=0$ and

$$\langle X\rangle_t=\int_0^t(sgn(B_s))^2ds=\int_0^t ds=t$$

so that by Lévy's characterisation theorem, X is a Brownian motion. In particular, $\langle|B|\rangle_t=\langle X\rangle_t=t$.

(d) We now apply Itô formula to B_t^2 and to $(|B_t|)^2$ to deduce, for all $t\ge 0$,

$$B_t^2=2\int_0^t B_s^2dB_s+t,$$
$$B_t^2=|B_t|^2=2\int_0^t|B_s|d|B_s|+\langle|B|\rangle_t=2\int_0^t|B_s|sgn(B_s)dB_s+2\int_0^t|B_s|dL_s+t$$

Subtracting we obtain the desired result:

$$\forall t\ge 0\quad\int_0^t|B_s|dL_s=0\quad a.s.$$

In particular, $B_s=0$ dL_s-a.e. a.s., i.e. dL_s is supported on the set of Brownian zeros $\mathcal{Z}$ a.s.

(e) The above readily generalises to $f(x) = |x - a|$ which will yield

$$|B_t - a| = |a| + \int_0^t sgn(B_s - a) dB_s + L_t^a,$$

for a non-decreasing continuous process L^a which we will call *the local time at the level a.*

(f) All the analysis also extends to a continuous local martingale M since

$$\langle \int sgn(M_s) dM_s \rangle_t = \int_0^t (sgn(M_s))^2 d\langle M \rangle_s = \langle M \rangle_t.$$