B4.4 Fourier Analysis HT22

Lecture 3: Schwartz test functions

1. The norms $S_{\alpha,\beta}$ and $\overline{S}_{k,l}$ on the Schwartz space $\mathscr{S}$
2. $\mathscr{S}$ convergence with examples
3. The inclusion $\mathscr{S} \subset \mathrm{L}^p$ with bounds
4. The Fourier transform on $\mathscr{S}$ and the Fourier bounds

The material corresponds to pp. 12–16 in the lecture notes and should be covered in Week 2.

The norms $S_{\alpha,\beta}$ and $\overline{S}_{k,l}$ on $\mathscr{S}(\mathbb{R}^n)$

Recall from last lecture that for $\phi \in C^\infty(\mathbb{R}^n)$ we defined for multi-indices $\alpha,\ \beta \in \mathbb{N}_0^n$ the quantity

$$S_{\alpha,\beta}(\phi) := \sup_{x\in\mathbb{R}^n}\left|x^\alpha\left(\partial^\beta\phi\right)(x)\right|$$

and for $k,\ l \in \mathbb{N}_0$ the quantity

$$\overline{S}_{k,l}(\phi) := \max\left\{S_{\alpha,\beta}(\phi) : |\alpha| \le k,\ |\beta| \le l\right\}.$$

Here we have

$$0 \le S_{\alpha,\beta}(\phi) \le \infty \text{ and } 0 \le \overline{S}_{k,l}(\phi) \le \infty$$

and the value ∞ is not excluded.

The norms $S_{\alpha,\beta}$ and $\overline{S}_{k,l}$ on $\mathscr{S}(\mathbb{R}^n)$

We have directly from the definitions that

$$\begin{aligned} \mathscr{S}(\mathbb{R}^n) &= \left\{\phi \in C^\infty(\mathbb{R}^n) : \, S_{\alpha,\beta}(\phi) < \infty \, \forall \, \alpha, \, \beta \in \mathbb{N}_0^n \right\} \\ &= \left\{\phi \in C^\infty(\mathbb{R}^n) : \, \overline{S}_{k,l}(\phi) < \infty \, \forall \, k, \, l \in \mathbb{N}_0 \right\} \end{aligned}$$

Furthermore, it is easy to see that $S_{\alpha,\beta}$ and $\overline{S}_{k,l}$ are norms on $\mathscr{S}(\mathbb{R}^n)$.

We observed that $\mathscr{S}(\mathbb{R}^n)$ is a commutative algebra without multiplicative unit. It clearly contains $\mathscr{D}(\mathbb{R}^n)$ as a proper subalgebra.

Definition We say that a sequence (ϕ_j) in $\mathscr{S}(\mathbb{R}^n)$ converges to $\phi \in \mathscr{S}(\mathbb{R}^n)$ if

$$S_{\alpha,\beta}(\phi_j - \phi) \to 0 \text{ as } j \to \infty$$

for all multi-indices $\alpha, \, \beta \in \mathbb{N}_0^n$.
Clearly this can be stated in terms of the norms $\overline{S}_{k,l}$ too.

$\mathscr{S}$ convergence versus $\mathscr{D}$ convergence and $\mathscr{D}(\mathbb{R}^n)$

How does $\mathscr{S}$ convergence compare to $\mathscr{D}$ convergence? – If (ϕ_j) is a sequence in $\mathscr{D}(\mathbb{R}^n)$ that converges to $\phi \in \mathscr{D}(\mathbb{R}^n)$ in $\mathscr{D}(\mathbb{R}^n)$, then also $\phi_j \to \phi$ in $\mathscr{S}(\mathbb{R}^n)$. This follows by comparing the definitions.

Proposition Let $\phi \in \mathscr{S}(\mathbb{R}^n)$. Then there exists a sequence (ϕ_j) in $\mathscr{D}(\mathbb{R}^n)$ so $\phi_j \to \phi$ in $\mathscr{S}(\mathbb{R}^n)$.

Proof. Put $\chi_j := \rho * \mathbf{1}_{B_{j+1}(0)}$ for $j \in \mathbb{N}$. Then $\chi_j \in \mathscr{D}(\mathbb{R}^n)$ and $\chi_j = 1$ on $B_j(0)$. If we let $\phi_j = \phi\chi_j$, then $\phi_j \in \mathscr{D}(\mathbb{R}^n)$ and for all α, $\beta \in \mathbb{N}_0^n$ we can use the Leibniz rule to check that $S_{\alpha,\beta}(\phi - \phi_j) \to 0$ as $j \to \infty$. The details are left as an exercise on problem sheet 2. □

Examples of $\mathscr{S}$ convergence

Example 1: Let $p(x) \in \mathbb{C}[x]$ be a polynomial. If $\phi_j \to \phi$ in $\mathscr{S}(\mathbb{R}^n)$, then also $p(x)\phi_j \to p(x)\phi$ in $\mathscr{S}(\mathbb{R}^n)$ and $p(\partial)\phi_j \to p(\partial)\phi$ in $\mathscr{S}(\mathbb{R}^n)$.
To prove it one can just write out the definitions. This is straight forward for the latter, but requires the Leibniz rule and some book-keeping for the former. It is worthwhile to be more systematic. Assume that $p(x) = \sum_{|\gamma| \leq d} c_\gamma x^\gamma$. Then we have for all $\phi \in \mathscr{S}(\mathbb{R}^n)$ and $k, l \in \mathbb{N}_0$:

$$\overline{S}_{k,l}(p(x)\phi) \leq (l+1)^d \left(\sum_{|\gamma| \leq d} |c_\gamma| \right) \overline{S}_{k+d,l}(\phi) \tag{1}$$

and

$$\overline{S}_{k,l}(p(\partial)\phi) \leq \left(\sum_{|\gamma| \leq d} |c_\gamma| \right) \overline{S}_{k,l+d}(\phi) \tag{2}$$

See the lecture notes for the proofs.

Examples of $\mathscr{S}$ convergence

Example 2: Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ and $1 \le j \le n$. Then

$$\frac{\Delta_{he_j}\phi}{h} \to \partial_j \phi \text{ in } \mathscr{S}(\mathbb{R}^n) \text{ as } h \to 0.$$

To see this fix multi-indices α, $\beta \in \mathbb{N}_0^n$ and let $0 < |h| < 1$. By two applications of the fundamental theorem of calculus

$$\partial_x^\beta\left(\frac{\Delta_{he_j}\phi(x)}{h} - (\partial_j\phi)(x)\right) = \int_0^1\int_0^1 \big(\partial^{\beta+2e_j}\phi\big)(x + sthe_j)\,\mathrm{d}s\, th\mathrm{d}t,$$

hence, using $|x_j^{\alpha_j}| \le \big(|x_j + sth| + st|h|\big)^{\alpha_j} \le 2^{\alpha_j - 1}\big(|x_j + st|^{\alpha_j} + 1\big)$, we get

$$S_{\alpha,\beta}\left(\frac{\Delta_{he_j}\phi}{h} - \partial_j\phi\right) \le 2^{\alpha_j-2}\big(S_{\alpha,\beta+2e_j}(\phi) + S_{\alpha-\alpha_j e_j,\beta+2e_j}(\phi)\big)|h|$$

The desired conclusion follows from this inequality.

Examples of $\mathscr{S}$ convergence

Example 3: Let $(\rho_\varepsilon)_{\varepsilon>0}$ be the standard mollifier. Then for $\phi \in \mathscr{S}(\mathbb{R}^n)$ we have $\rho_\varepsilon * \phi \in \mathscr{S}(\mathbb{R}^n)$ and $\rho_\varepsilon * \phi \to \phi$ in $\mathscr{S}(\mathbb{R}^n)$ as $\varepsilon \searrow 0$.

Fix multi-indices α, $\beta \in \mathbb{N}_0^n$. For $\varepsilon \in (0,1)$ the fundamental theorem of calculus yields:

$$\begin{aligned}\partial_x^\beta\Big((\rho_\varepsilon * \phi)(x) - \phi(x)\Big) &= (\rho_\varepsilon * \partial^\beta \phi)(x) - (\partial^\beta \phi)(x) \\ &= \int_{\mathbb{R}^n} \rho(y) \int_0^1 \nabla(\partial^\beta \phi)(x + t\varepsilon y) \cdot y \,\mathrm{d}t\,\mathrm{d}y \varepsilon.\end{aligned}$$

Next we use the inequality

$$\begin{aligned}|x^\alpha| &\leq \prod_{j=1}^n 2^{\alpha_j - 1}\big(|x_j + t\varepsilon y_j|^{\alpha_j} + \varepsilon^{\alpha_j}\big) = 2^{|\alpha|-n}\prod_{j=1}^n \big(|x_j + t\varepsilon y_j|^{\alpha_j} + \varepsilon^{\alpha_j}\big) \\ &\leq 2^{|\alpha|-n}\prod_{j=1}^n \big(|x_j + t\varepsilon y_j|^{\alpha_j} + 1\big) \leq 2^{|\alpha|-n}\sum_{\gamma\leq\alpha}\big|(x + t\varepsilon y)^\gamma\big|.\end{aligned}$$

Examples of $\mathscr{S}$ convergence

Hereby we find, using also the Cauchy-Schwarz inequality,

$$\left|\nabla\left(\partial^{\beta}\phi\right)\right| \leq \sum_{j=1}^{n}\left|\partial^{\beta+e_j}\phi\right|,$$

and $\int_{\mathbb{R}^n}\rho(y)\,\mathrm{d}y = 1$ that

$$\begin{aligned} S_{\alpha,\beta}\left(\rho_\varepsilon * \phi - \phi\right) &\leq \sum_{j=1}^{n} 2^{|\alpha|-n}\sum_{\gamma\leq\alpha} S_{\gamma,\beta+e_j}(\phi)\varepsilon \\ &\leq c\overline{S}_{|\alpha|,|\beta|+1}(\phi)\varepsilon, \end{aligned}$$

where $c = n2^{|\alpha|-n}\sum_{\gamma\leq\alpha}1$ is a constant (whose precise value is unimportant here). This inequality allows us to conclude.

Remark about approach

Note that the estimates in terms of the norms $S_{\alpha,\beta}$ and $\overline{S}_{k,l}$ can be quite technical. We therefore advocate a systematic approach where we prove a few key inequalities for these norms that allow many other such inequalities to be deduced. The inequalities (1), (2) and those derived in Examples 2 and 3 are examples of key inequalities. Let us summarize the inequalities from examples 2 and 3 as follows: Let α, $\beta \in \mathbb{N}_0^n$ be two multi-indices. Then there exists a constant $c = c(n, |\alpha|, |\beta|)$ with the following properties. For each $\phi \in \mathscr{S}(\mathbb{R}^n)$, direction $1 \leq j \leq n$, increment $0 < |h| < 1$ and mollifier radius $\varepsilon \in (0,1)$ we have

$$S_{\alpha,\beta}\left(\frac{\Delta_{he_j}\phi}{h} - \partial_j\phi\right) \leq c\overline{S}_{|\alpha|,|\beta|+2}(\phi)|h| \tag{3}$$

and

$$S_{\alpha,\beta}\Big(\rho_\varepsilon * \phi - \phi\Big) \leq c\overline{S}_{|\alpha|,|\beta|+1}(\phi)\varepsilon. \tag{4}$$

Very often the actual value of the constant 'c' is not important.

The inclusion $\mathscr{S} \subset \mathrm{L}^p$ and a corresponding bound

Proposition Let $p \in [1, \infty]$. Then $\mathscr{S}(\mathbb{R}^n) \subset \mathrm{L}^p(\mathbb{R}^n)$ and the inclusion map is continuous. More precisely, there exists a constant $c = c(n, p)$ so

$$\|\phi\|_p \leq c\overline{S}_{n+1,0}(\phi) \tag{5}$$

holds for all $\phi \in \mathscr{S}(\mathbb{R}^n)$. When $p = \infty$ we have simply $\|\phi\|_\infty = S_{0,0}(\phi)$.

Proof. The claim is evidently true for $p = \infty$ so assume $p < \infty$. The main point to observe here is that the function $x \mapsto \left(1 + |x|^2\right)^{-\frac{n+1}{2}}$ is integrable over $\mathbb{R}^n$. To see that we integrate using polar coordinates:

$$\begin{aligned} I_{n+1} := \int_{\mathbb{R}^n} \frac{\mathrm{d}x}{\left(1 + |x|^2\right)^{\frac{n+1}{2}}} &= \int_0^\infty \int_{\partial B_r(0)} \frac{\mathrm{d}S_x}{\left(1 + |x|^2\right)^{\frac{n+1}{2}}} \, \mathrm{d}r \\ &= \omega_{n-1} \int_0^\infty \frac{r^{n-1}}{\left(1 + r^2\right)^{\frac{n+1}{2}}} \, \mathrm{d}r < \infty. \end{aligned}$$

The inclusion $\mathscr{S} \subset \mathrm{L}^p$ and a corresponding bound

Now we can estimate as

$$\begin{aligned}
\|\phi\|_p^p &= \int_{\mathbb{R}^n} \left(1+|x|^2\right)^{-\frac{n+1}{2}} \left(\left(1+|x|^2\right)^{\frac{n+1}{2}} |\phi(x)|^p \right) \mathrm{d}x \\
&\leq \int_{\mathbb{R}^n} \frac{\mathrm{d}x}{\left(1+|x|^2\right)^{\frac{n+1}{2}}} \sup_{x\in\mathbb{R}^n} \left(\left(1+|x|^2\right)^{\frac{n+1}{2p}} |\phi(x)| \right)^p \\
&\leq I_{n+1} \left(\sup_{x\in\mathbb{R}^n} \left(1+|x|^2\right)^{\frac{n+1}{2}} |\phi(x)| \right)^p
\end{aligned}$$

Here we have

$$\begin{aligned}
\left(1+|x|^2\right)^{\frac{n+1}{2}} &\leq \left(1+|x_1|+\ldots+|x_n|\right)^{n+1} \\
&\leq (n+1)^n \left(1+|x_1|^{n+1}+\ldots+|x_n|^{n+1}\right)
\end{aligned}$$

and so we obtain the desired bound with $c = c(n,p) = (n+1)^{\frac{n+1}{p}} I_{n+1}^{\frac{1}{p}}$. □

An improved bound for $\mathscr{S} \subset \mathrm{L}^p$

Inspection of the previous estimation shows that it is far from optimal. For instance we can use that

$$I_q := \int_{\mathbb{R}^n} \frac{\mathrm{d}x}{\left(1+|x|^2\right)^{\frac{q}{2}}} < \infty \text{ when } q > n,$$

whereby we get for $p \in [1, \infty)$ and $\phi \in \mathscr{S}(\mathbb{R}^n)$:

$$\|\phi\|_p \leq I_q^{\frac{1}{p}} \left(\sup_{x \in \mathbb{R}^n} \left(1+|x|^2\right)^{\frac{q}{2p}} |\phi(x)| \right) \tag{6}$$

This is obviously better than the bound we derived before. However, it is usually not necessary to use this sharper bound and we shall therefore rely on (5) in the following.

The Fourier transform on $\mathscr{S}(\mathbb{R}^n)$ and the Fourier bounds

Theorem The Fourier transform $\mathcal{F}\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is a linear and $\mathscr{S}$ continuous map. The $\mathscr{S}$ continuity is expressed more precisely through the **Fourier bounds**: for each k, $l \in \mathbb{N}_0$ there exists a constant $c = c(n, k, l)$ so

$$\overline{S}_{k,l}(\widehat{\phi}) \leq c\overline{S}_{l+n+1,k}(\phi)$$

holds for all $\phi \in \mathscr{S}(\mathbb{R}^n)$.

Note how the numbers k, l swap places in the Fourier bound.

Proof. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$. We want to show that $\widehat{\phi} \in \mathscr{S}(\mathbb{R}^n)$.
Now $x_j\phi$, $\partial_j\phi \in \mathscr{S}(\mathbb{R}^n)$ and by the differentiation rules

$$\partial_j\widehat{\phi} = \mathcal{F}_{x\to\xi}\bigl(-\mathrm{i}x_j\phi(x)\bigr) \text{ and } \xi_j\widehat{\phi}(\xi) = -\mathrm{i}\widehat{\partial_j\phi}(\xi).$$

Because $\mathscr{S}(\mathbb{R}^n) \subset \mathrm{L}^1(\mathbb{R}^n)$ the Riemann-Lebesgue lemma implies that $\partial_j\widehat{\phi}$, $\xi_j\widehat{\phi}$ belong to $\mathrm{C}_0(\mathbb{R}^n)$.

The Fourier transform on $\mathscr{S}(\mathbb{R}^n)$ and the Fourier bounds

By induction on the length of multi-indices we find for α, $\beta \in \mathbb{N}_0^n$ that

$$\partial^\beta \widehat{\phi}(\xi) = \mathcal{F}_{x \to \xi}\big((-\mathrm{i}x)^\beta \phi(x)\big) \text{ and } \xi^\alpha \widehat{\phi}(\xi) = (-\mathrm{i})^{|\alpha|} \widehat{\partial^\alpha \phi}(\xi)$$

both belong to $C_0(\mathbb{R}^n)$, and hence that

$$\xi^\alpha \partial^\beta \widehat{\phi}(\xi) = (-\mathrm{i})^{|\alpha|} \mathcal{F}_{x \to \xi}\Big(\partial^\alpha \big((-\mathrm{i}x)^\beta \phi(x)\big)\Big) \tag{7}$$

belongs to $C_0(\mathbb{R}^n)$. Therefore $S_{\alpha,\beta}(\widehat{\phi}) < \infty$ and since in particular β was arbitrary it follows that $\widehat{\phi} \in C^\infty(\mathbb{R}^n)$, hence $\widehat{\phi} \in \mathscr{S}(\mathbb{R}^n)$.

We turn to the proof of the Fourier bounds, and start with the bound

$$\|\widehat{\psi}\|_\infty \leq \|\psi\|_1 \leq c\overline{S}_{n+1,0}(\psi)$$

valid for all $\psi \in \mathscr{S}(\mathbb{R}^n)$, where $c = c(n,1)$.

The Fourier transform on $\mathscr{S}(\mathbb{R}^n)$ and the Fourier bounds

Combine the previous inequality with (7) to get

$$\sup_{\xi\in\mathbb{R}^n}\left|\xi^\alpha\partial^\beta\phi(\xi)\right| \leq c\overline{S}_{n+1,0}\left(\partial^\alpha\big((-\mathrm{i}x)^\beta\phi\big)\right).$$

Here the right-hand side can be estimated by use of (1), (2), whereby

$$S_{\alpha,\beta}(\widehat{\phi}) \leq c\big(|\alpha|+1\big)^{|\beta|}\overline{S}_{n+1+|\beta|,|\alpha|}(\phi)$$

which implies the desired bound. □

Remark In particular it follows that the Fourier transform of the standard mollifier kernel on $\mathbb{R}^n$, $\widehat{\rho}$, is a Schwartz test function on $\mathbb{R}^n$. On problem sheet 1 you will show that it cannot have compact support.

Smoothness versus decay at infinity

The Fourier bounds

$$\overline{S}_{k,l}(\widehat{\phi}) \leq c\overline{S}_{l+n+1,k}(\phi)$$

express a useful principle: the smoother a function f is, the faster its Fourier transform $\widehat{f}$ decays to 0 at infinity. Of course we established the Fourier bounds only for Schwartz test functions, that by definition are, together with all their derivatives, rapidly decreasing, but the proof is easily adapted to also give the following.

(a) Let $m \in \mathbb{N}_0$. If $f \in \mathrm{W}^{m,1}(\mathbb{R}^n)$ (so $\partial^\alpha f \in \mathrm{L}^1(\mathbb{R}^n)$ for each $|\alpha| \leq m$), then

$$\sup_{\xi \in \mathbb{R}^n} \left(1+|\xi|^2\right)^{\frac{m}{2}} |\widehat{f}(\xi)| \leq c\|f\|_{\mathrm{W}^{m,1}}$$

where $c = c(m,n)$ is a constant. In fact the Riemann-Lebesgue lemma yields that the function $\xi \mapsto \left(1+|\xi|^2\right)^{\frac{m}{2}} \widehat{f}(\xi)$ belongs to $\mathrm{C}_0(\mathbb{R}^n)$.

Smoothness versus decay at infinity

(b) Let $m \in \mathbb{N}$ with $m \geq n+1$. If $\left(1+|x|^2\right)^{\frac{m}{2}} f(x) \in \mathrm{L}^\infty(\mathbb{R}^n)$, then $\widehat{f} \in \mathrm{C}^{m-n-1}(\mathbb{R}^n)$ and $\partial^\alpha \widehat{f} \in \mathrm{C}_0(\mathbb{R}^n)$ for multi-indices α with $|\alpha| \leq m-n-1$.

There is a gap of $n+1$ derivatives between (a) and (b). It is incurred when we go from L^∞ to L^1 using the bound from the inclusion $\mathscr{S} \subset \mathrm{L}^p$. As we remarked that bound is not optimal. However, it might be worthwhile to note that if we replace the L^∞ hypotheis with an L^1 hypothesis in (b), then we get instead:

(b1) If $\left(1+|x|^2\right)^{\frac{m}{2}} f(x) \in \mathrm{L}^1(\mathbb{R}^n)$, then $\widehat{f} \in \mathrm{C}^m(\mathbb{R}^n)$ and $\partial^\alpha \widehat{f} \in \mathrm{C}_0(\mathbb{R}^n)$ for all multi-indices α of length at most m.

Smoothness versus decay at infinity

Looking at how the Fourier transform $\widehat{f}$ of f decays at infinity tells us something about how smooth f is. Conversely, if we know that f decays to 0 at infinity with a certain rate, then its Fourier transform $\widehat{f}$ must enjoy a certain smoothness. Here *smoothness* is often measured on the scale C^k or on the Sobolev scale $\mathsf{W}^{k,p}$, where the smoothness increases with k. A first quantitative expression of this principle is contained in (a), (b) and (b1).

Example Recall that we calculated $\widehat{\mathbf{1}_{(-1,1)}} = 2\mathrm{sinc}$, which is in $\mathsf{C}_0(\mathbb{R})$ but is not integrable on $\mathbb{R}$. Note that (a) does not apply because $\mathbf{1}_{(-1,1)}$ is discontinuous. However, from (b) we infer that the Fourier transform is C^m for *every m*, that is, it is C^∞. The compactness of the support of $\mathbf{1}_{(-1,1)}$ is responsible for this! But note that also all Schwartz test functions Fourier transform to a C^∞ function (in fact, a Schwartz test function again), so the property *compact support* is not necessary for having a C^∞ Fourier transform. Later when we discuss Paley-Wiener theory we shall be able to characterize the case of compact support in terms of the Fourier transform.