B4.4 Fourier Analysis HT22

Lecture 7: Multiplication with moderate C^∞ functions

1. Definition of moderate C^∞ functions
2. Multiplication with moderate C^∞ functions
3. The convolution of a tempered distribution and a Schwartz test function is a moderate C^∞ function
4. Approximation and mollification in the tempered context
5. The convolution rule: the basic case
6. Examples

The material corresponds to pp. 27–30 in the lecture notes and should be covered in Week 4.

Functions of polynomial growth

Definition A function $f\colon \mathbb{R}^n \to \mathbb{C}$ is said to be of polynomial growth if there exist constants $c \geq 0$ and $m \in \mathbb{N}_0$ so

$$\bigl|f(x)\bigr| \leq c\bigl(1+|x|^2\bigr)^{\frac{m}{2}}$$

holds for all $x \in \mathbb{R}^n$.

Note: f is of polynomial growth if and only if there exists a polynomial $p(x) \in \mathbb{C}[x]$ so $\bigl|f(x)\bigr| \leq \bigl|p(x)\bigr|$ holds for all $x \in \mathbb{R}^n$. As it should be!

Example Let $f\colon \mathbb{R}^n \to \mathbb{C}$ be of polynomial growth. When f is measurable it is (representative of) a tempered L^∞ function, and if $g\colon \mathbb{R}^n \to \mathbb{C}$ is a continuous rapidly decreasing function, then $f(x)g(x)$ is integrable on $\mathbb{R}^n$. In particular, we may view f as the tempered distribution $\phi \mapsto \int_{\mathbb{R}^n} f\phi\,\mathrm{d}x$. In order to get a function we can multiply on a tempered distribution we must require that the function is C^∞ and that all its partial derivatives have polynomial growth.

Moderate C^∞ functions

Definition A function $a\colon \mathbb{R}^n \to \mathbb{C}$ is said to be a moderate C^∞ function if it is C^∞ and it and all its partial derivatives have polynomial growth: for each multi-indicex $\alpha \in \mathbb{N}_0^n$ there exist constants $c_\alpha \geq 0$, $m_\alpha \in \mathbb{N}_0$ so

$$\left|(\partial^\alpha a)(x)\right| \leq c_\alpha \left(1 + |x|^2\right)^{\frac{m_\alpha}{2}}$$

holds for all $x \in \mathbb{R}^n$.

Example Schwartz test functions, polynomials and functions such as $\cos p(x)$, $\sin p(x)$, where $p(x) \in \mathbb{C}[x]$, are moderate C^∞ functions. The functions

$$\mathbb{R} \ni x \mapsto \mathrm{e}^x \text{ and } \mathbb{R}^n \ni x \mapsto \mathrm{e}^{|x|^2}$$

are not.
It is clear that a moderate C^∞ function $a\colon \mathbb{R}^n \to \mathbb{C}$ in particular is a tempered L^∞ function and so defines a tempered distribution:

$$\phi \mapsto \int_{\mathbb{R}^n} \phi a \, \mathrm{d}x.$$

Properties of the set of moderate C^∞ functions

If a, $b\colon \mathbb{R}^n \to \mathbb{C}$ are moderate C^∞ functions, $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{N}_0^n$, then

- $a + \lambda b$ (it is a vector space)
- ab (it is an algebra)
- $\partial^\alpha a$ (it is closed under differentiation)

are moderate C^∞ functions.

The proof is straight forward and left as an exercise.

The key bound for moderate C^∞ functions

Proposition Let $a\colon \mathbb{R}^n \to \mathbb{C}$ be a moderate C^∞ function. Then the map

$$\mathscr{S}(\mathbb{R}^n) \ni \phi \mapsto a\phi \in \mathscr{S}(\mathbb{R}^n)$$

is linear and $\mathscr{S}$ continuous. More precisely we have the following bound: for all $k,\ l \in \mathbb{N}_0$ we have that

$$\overline{S}_{k,l}(a\phi) \leq 2^l \overline{c}_l (n+1)^{\overline{m}_l} \overline{S}_{k+\overline{m}_l,l}(\phi)$$

holds for all $\phi \in \mathscr{S}(\mathbb{R}^n)$, where

$$\overline{c}_l := \max_{|\beta| \leq l} c_\beta\,, \quad \overline{m}_l := \max_{|\beta| \leq l} m_\beta$$

and the numbers $c_\beta \geq 0$, $m_\beta \in \mathbb{N}_0$ are the numbers in the polynomial growth condition satisfied by $\partial^\beta a$.

Proof of key bound

Let α, $\beta \in \mathbb{N}_0^n$ be multi-indices with $|\alpha| \leq k$, $|\beta| \leq l$. Then for $\phi \in \mathscr{S}(\mathbb{R}^n)$:

$$\begin{aligned}
\left|x^\alpha \partial^\beta (a\phi)\right| &= \left|x^\alpha \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \partial^\gamma a \partial^{\beta-\gamma}\phi\right| \leq \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \left|\partial^\gamma a\right| \left|x^\alpha \partial^{\beta-\gamma}\phi\right| \\
&\leq \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} c_\gamma \left(1+|x|^2\right)^{\frac{m_\gamma}{2}} \left|x^\alpha \partial^{\beta-\gamma}\phi\right| \\
&\leq \overline{c}_l \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \left(1+|x_1|+\cdots+|x_n|\right)^{\overline{m}_l} \left|x^\alpha \partial^{\beta-\gamma}\phi\right| \\
&\leq \overline{c}_l \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} (n+1)^{\overline{m}_l - 1} \left(1+\sum_{j=1}^n |x_j|^{\overline{m}_l}\right) \left|x^\alpha \partial^{\beta-\gamma}\phi\right| \\
&\leq \overline{c}_l \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} (n+1)^{\overline{m}_l - 1} \left(\overline{S}_{k,l}(\phi) + n\overline{S}_{k+\overline{m}_l,l}(\phi)\right)
\end{aligned}$$

Proof of key bound and multiplication with moderate C^∞ functions

hence we continue with

$$\begin{aligned} \left| x^\alpha \partial^\beta (a\phi) \right| &\leq \overline{c}_l \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} (n+1)^{\overline{m}_l} \overline{S}_{k+\overline{m}_l, l}(\phi) \\ &\leq \overline{c}_l (n+1)^{\overline{m}_l} 2^l \overline{S}_{k+\overline{m}_l, l}(\phi) \end{aligned}$$

where we in the last inequality used that $\sum_{\gamma \leq \beta} \binom{\beta}{\gamma} = 2^{|\beta|} \leq 2^l$. This is the required bound and the rest is then clear. □

We then have the obvious adjoint identity:

$$\int_{\mathbb{R}^n} (a\phi)\psi \, \mathrm{d}x = \int_{\mathbb{R}^n} \phi(a\psi) \, \mathrm{d}x$$

holds for all ϕ, $\psi \in \mathscr{S}(\mathbb{R}^n)$ that allows us to define $au \in \mathscr{S}'(\mathbb{R}^n)$ for each $u \in \mathscr{S}'(\mathbb{R}^n)$ by the rule

$$\langle au, \phi \rangle := \langle u, a\phi \rangle, \quad \phi \in \mathscr{S}(\mathbb{R}^n).$$

It is clear how to define ua and that we have $au = ua$.

Multiplication with moderate C^∞ functions

As usual because the product is defined by the adjoint identity scheme it defines a map

$$\mathscr{S}'(\mathbb{R}^n) \ni u \mapsto au \in \mathscr{S}'(\mathbb{R}^n)$$

that is linear and $\mathscr{S}'$ continuous. Furthermore, the Leibniz rule holds:

$$\partial_j(au) = (\partial_j a)u + a\partial_j u$$

for each direction $1 \leq j \leq n$. The proof is straight forward from the definitions and left as an exercise.

The consistency extends beyond $\mathscr{S}$: when u is a tempered L^1 function, then

$$T_{au} = aT_u$$

holds. In fact, when u is a tempered measure we have consistency.

Convolution of a tempered distribution and a Schwartz test function

We defined $u * \theta$ for each $u \in \mathscr{S}'(\mathbb{R}^n)$ and $\theta \in \mathscr{S}(\mathbb{R}^n)$ by the adjoint identity scheme:

$$\langle u * \theta, \phi \rangle := \langle u, \widetilde{\theta} * \phi \rangle$$

for $\phi \in \mathscr{S}(\mathbb{R}^n)$. Hereby the map

$$\mathscr{S}'(\mathbb{R}^n) \ni u \mapsto u * \theta \in \mathscr{S}'(\mathbb{R}^n)$$

is linear and $\mathscr{S}'$ continuous. Furthermore, with the natural definitions we have $u * \theta = \theta * u$. But we can say more:

Proposition If $u \in \mathscr{S}'(\mathbb{R}^n)$, $\theta \in \mathscr{S}(\mathbb{R}^n)$, then $u * \theta$ is a moderate C^∞ function and $(u * \theta)(x) = \langle u, \theta(x - \cdot) \rangle$ for $x \in \mathbb{R}^n$. Furthermore, for each multi-index $\alpha \in \mathbb{N}_0^n$:

$$\partial^\alpha (u * \theta) = (\partial^\alpha u) * \theta = u * (\partial^\alpha \theta). \tag{1}$$

Convolution of a tempered distribution and a Schwartz test function

Proof. In order to show that $u * \theta \in \mathrm{C}^{\infty}(\mathbb{R}^n)$, that we have the formula $(u * \theta)(x) = \langle u, \theta(x - \cdot) \rangle$ and the differentiation rule (1) we can proceed as we did in B4.3. We leave that as an exercise and we then only have to show that $u * \theta$ is a moderate C^{∞} function. In view of (1) it suffices to show that $u * \theta$ has polynomial growth. To do that we invoke the boundedness property of u. Accordingly we find constants $c \geq 0$, k, $l \in \mathbb{N}_0$, so

$$\bigl| \langle u, \phi \rangle \bigr| \leq c \overline{S}_{k,l}(\phi)$$

holds for all $\phi \in \mathscr{S}(\mathbb{R}^n)$.

For each fixed $x \in \mathbb{R}^n$ we take $\phi = \theta(x - \cdot) = \widetilde{\bigl(\tau_x \theta\bigr)}$ in the bound for u whereby, by virture of the formula for $u * \theta$, we get

$$\bigl| u * \theta(x) \bigr| \leq c \overline{S}_{k,l}\bigl(\theta(x - \cdot)\bigr).$$

To see that this bound implies polynomial growth we let α, $\beta \in \mathbb{N}_0^n$ be multi-indices with $|\alpha| \leq k$, $|\beta| \leq l$.

Convolution of a tempered distribution and a Schwartz test function

For $x,\ y \in \mathbb{R}^n$ we estimate as follows using the binomial formula:

$$
\begin{aligned}
\left|y^\alpha \partial_y^\beta \theta(x-y)\right| &= \left|(y-x+x)^\alpha \bigl(\partial^\beta \theta\bigr)(x-y)\right| \\
&\leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \left|(x-y)^\gamma \bigl(\partial^\beta \theta\bigr)(x-y)\right| \left|x^{\alpha-\gamma}\right| \\
&\leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} S_{\gamma,\beta}(\theta)\left|x^{\alpha-\gamma}\right| \leq \overline{S}_{k,l}(\theta) \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \left|x^{\alpha-\gamma}\right| \\
&= \overline{S}_{k,l}(\theta) \prod_{j=1}^{n} \bigl(1+|x_j|\bigr)^{\alpha_j} \leq \overline{S}_{k,l}(\theta)\bigl(1+|x|\bigr)^{|\alpha|} \\
&\leq \overline{S}_{k,l}(\theta)\bigl(1+|x|\bigr)^{k} \leq 2^{\frac{k}{2}} \overline{S}_{k,l}(\theta)\bigl(1+|x|^2\bigr)^{\frac{k}{2}}
\end{aligned}
$$

and consequently $\left|u * \theta(x)\right| \leq c 2^{\frac{k}{2}} \overline{S}_{k,l}(\theta)\bigl(1+|x|^2\bigr)^{\frac{k}{2}}$ for all $x \in \mathbb{R}^n$ as required. □

Approximation and mollification in the tempered context

We saw in B4.3 that many results about distributions could be established by first proving them for C^∞ functions and then use mollification to transfer them to distributions. We can also use this technique for tempered distributions. Recall the standard mollifier $\left(\rho_\varepsilon\right)_{\varepsilon>0}$ on $\mathbb{R}^n$. We then have

Proposition If $u \in \mathscr{S}'(\mathbb{R}^n)$, then $\rho_\varepsilon * u$ is a moderate C^∞ function and

$$\rho_\varepsilon * u \to u \text{ in } \mathscr{S}'(\mathbb{R}^n)$$

as $\varepsilon \searrow 0$.

Proof. We have more or less already proved it. That $\rho_\varepsilon * u$ is a moderate C^∞ function follows from the previous result and to prove the convergence we just need to observe that, because u is $\mathscr{S}$ continuous, for $\phi \in \mathscr{S}(\mathbb{R}^n)$,

$$\rho_\varepsilon * \phi \to \phi \text{ in } \mathscr{S}(\mathbb{R}^n)$$

as $\varepsilon \searrow 0$. But this was established in example 3 of lecture 3. □

Approximation and mollification in the tempered context

As in B4.3 we can go one step further and approximate a tempered distribution by test functions from $\mathscr{D}(\mathbb{R}^n)$. For that we must combine mollification with truncation: simply multiply the mollified distribution by cut-off functions that equal 1 on increasingly large balls.

Proposition Let $u \in \mathscr{S}'(\mathbb{R}^n)$. Then there exists a sequence (u_j) in $\mathscr{D}(\mathbb{R}^n)$ such that

$$u_j \to u \text{ in } \mathscr{S}'(\mathbb{R}^n)$$

as $j \to \infty$.

We leave the proof as an exercise. Note that we in particular have that $u_j \in \mathscr{S}(\mathbb{R}^n)$, and so, just as in B4.3, we can think of the extension of a linear map $T \colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ to $\overline{T} \colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ by use of the adjoint identity scheme as an extension of T by $\mathscr{S}'$ continuity.

The convolution rule: the basic case

Proposition Let $u \in \mathscr{S}'(\mathbb{R}^n)$ and $\theta \in \mathscr{S}(\mathbb{R}^n)$. Then

$$\widehat{u * \theta} = \widehat{u}\widehat{\theta} \text{ and } \widehat{u\theta} = (2\pi)^{-n}\widehat{u} * \widehat{\theta}.$$

Proof. By definition we have for $\phi \in \mathscr{S}(\mathbb{R}^n)$: $\langle \widehat{u * \theta}, \phi \rangle = \langle u, \widetilde{\theta} * \widehat{\phi} \rangle$. We can now use results for Schwartz test functions (FIF = Fourier inversion formula on $\mathscr{S}$ and CR = convolution rule on $\mathscr{S}$):

$$\begin{aligned}
\langle \widehat{u * \theta}, \phi \rangle &\overset{\text{FIF}}{=} (2\pi)^{-n} \langle u, \widehat{\widehat{\theta}} * \widehat{\phi} \rangle \\
&\overset{\text{CR}}{=} \langle u, \widehat{\widehat{\theta}\phi} \rangle \\
&\overset{\text{defs}}{=} \langle \widehat{u}, \widehat{\theta}\phi \rangle \\
&\overset{\text{defs}}{=} \langle \widehat{u}\widehat{\theta}, \phi \rangle
\end{aligned}$$

The convolution rule: the basic case–proof continued...

For the second part we apply the just established result to $\widehat{u} \in \mathscr{S}'(\mathbb{R}^n)$, $\widehat{\theta} \in \mathscr{S}(\mathbb{R}^n)$ whereby we find (FIFs = Fourier inversion formulas in $\mathscr{S}$ and in $\mathscr{S}'$):

$$\begin{aligned} \widehat{\widehat{u} * \widehat{\theta}} &= \widehat{\widehat{u}}\widehat{\widehat{\theta}} \\ &\overset{\text{FIFs}}{=} (2\pi)^{2n}\widetilde{u}\widetilde{\theta} \\ &= (2\pi)^{2n}\widetilde{u\theta} \\ &\overset{\text{FIFs}}{=} (2\pi)^{n}\widehat{\widehat{u\theta}} \end{aligned}$$

and so by FIFs again we arrive at $\widehat{u} * \widehat{\theta} = (2\pi)^n \widehat{u\theta}$. The proof is finished. □

Example The Hilbert transform is defined for each $\phi \in \mathscr{S}(\mathbb{R})$ as

$$\mathcal{H}(\phi) := \frac{1}{\pi}\Big(\mathrm{pv}\big(\tfrac{1}{y}\big) * \phi\Big)(x) = \lim_{\varepsilon \searrow 0}\Big(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\Big)\frac{\phi(x-y)}{\pi y}\,\mathrm{d}y.$$

We know that hereby $\mathcal{H}(\phi)$ is a moderate C^{∞} function, so that in particular $\mathcal{H} \colon \mathscr{S}(\mathbb{R}) \to \mathscr{S}'(\mathbb{R})$ is linear. It is the most basic example of a *singular integral operator*. What can we say about the decay of $\mathcal{H}(\phi)$ at infinity and is it integrable?

We can use the convolution rule and Example 1 from lecture 6 to find its Fourier transform:

$$\widehat{\mathcal{H}(\phi)} = -\mathrm{i}\,\mathrm{sgn}(\xi)\widehat{\phi}(\xi).$$

When $\widehat{\phi}(0) = \int_{\mathbb{R}} \phi \,\mathrm{d}x \neq 0$, then it is discontinuous at $\xi = 0$ and so in that case $\mathcal{H}(\phi) \notin \mathrm{L}^1(\mathbb{R})$ by the Riemann-Lebesgue lemma.

But can we get positive results?

The Hilbert transform

To get positive results we can use the principle about smoothness versus decay at infinity together with the Fourier inversion formula. Assume

$$\phi \in \mathscr{S}(\mathbb{R}) \text{ and } \int_{\mathbb{R}} x^j \phi(x)\,\mathrm{d}x = 0 \text{ for } j \in \{0,1,2\}. \tag{2}$$

Then $\mathcal{H}(\phi) \in \mathrm{L}^1(\mathbb{R})$. Indeed, note that, by the differentiation rule, (2) amounts to $\widehat{\phi}(0) = \widehat{\phi}'(0) = \widehat{\phi}''(0) = 0$, so $\widehat{\mathcal{H}(\phi)} = -\mathrm{i}\,\mathrm{sgn}(\xi)\widehat{\phi}(\xi) \in \mathrm{C}^2(\mathbb{R})$ and then because $\widehat{\phi} \in \mathscr{S}(\mathbb{R})$ it is clear that also $\widehat{\mathcal{H}(\phi)} \in \mathrm{W}^{2,1}(\mathbb{R})$. Now by the Fourier inversion formula in $\mathscr{S}'$ and the differentiation rule,

$$(-\mathrm{i}x)^j \mathcal{H}(\phi)(x) = \frac{1}{2\pi}\mathcal{F}_{\xi\to -x}\left(\frac{\mathrm{d}^j}{\mathrm{d}\xi^j}\left(-\mathrm{i}\,\mathrm{sgn}(\xi)\widehat{\phi}(\xi)\right)\right)$$

for $j = 0,\ 1,\ 2$, and so $x^j\mathcal{H}(\phi)(x) \in \mathrm{C}_0(\mathbb{R})$ by the Riemann-Lebesgue lemma. Consequently we have for a constant $c > 0$ that $\left|\mathcal{H}(\phi)(x)\right| \leq \frac{c}{1+x^2}$ for all $x \in \mathbb{R}$ and so $\mathcal{H}(\phi) \in \mathrm{L}^1(\mathbb{R})$ when (2) holds.