B4.4 Fourier Analysis HT22

Lecture 8: The Fourier transform on L^2 and Plancherel's theorem

1. Plancherel's theorem
2. The Fourier transform on L^p and the Hausdorff-Young inequality
3. The Hilbert transform on L^2

The material corresponds to pp. 31–32 in the lecture notes and should be covered in Week 4.

The Fourier transform so far...

We have defined the Fourier transform on tempered distributions and have so far recorded the following mapping properties:

- $\mathcal{F}\colon \mathrm{L}^1(\mathbb{R}^n) \to \mathrm{C}_0(\mathbb{R}^n) \subset \mathrm{L}^\infty(\mathbb{R}^n)$,
- $\mathcal{F}\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n) \quad$ (bijection),
- $\mathcal{F}\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n) \quad$ (bijection),

and we have seen that the Fourier transform of the Heaviside function $H \in \mathrm{L}^\infty(\mathbb{R})$ is a distribution of order 1.

We mentioned that the Wiener algebra $\mathcal{F}\big(\mathrm{L}^1(\mathbb{R}^n)\big)$ is strictly smaller than $\mathrm{C}_0(\mathbb{R}^n)$ (you will show this on a problem sheet). Can we say something about the Fourier transform on other L^p spaces? –For instance, what kind of tempered distribution is $\mathcal{F}(f)$ when $f \in \mathrm{L}^2(\mathbb{R}^n)$?

The Plancherel theorem

Theorem The Fourier transform $\mathcal{F}\colon \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$ is bijective and $(2\pi)^{-\frac{n}{2}}\mathcal{F}$ is unitary (isometric and onto):

$$\|\widehat{f}\|_2 = (2\pi)^{\frac{n}{2}}\|f\|_2 \tag{1}$$

and

$$\mathcal{F}\big(\mathrm{L}^2(\mathbb{R}^n)\big) = \mathrm{L}^2(\mathbb{R}^n).$$

Remark The identity (1) is called *Plancherel's formula* as is the closely related formula

$$\int_{\mathbb{R}^n} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,\mathrm{d}\xi = (2\pi)^n \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,\mathrm{d}x. \tag{2}$$

Proof of Plancherel's theorem

Proof. For ϕ, $\psi \in \mathscr{S}(\mathbb{R}^n)$ we get by the Fourier inversion formula and the product rule

$$\int_{\mathbb{R}^n} \phi\overline{\psi}\,\mathrm{d}x = \int_{\mathbb{R}^n} \phi\mathcal{F}\big(\mathcal{F}^{-1}\overline{\psi}\big)\,\mathrm{d}x = \int_{\mathbb{R}^n} \widehat{\phi}\mathcal{F}^{-1}\overline{\psi}\,\mathrm{d}x.$$

Here we have that

$$\begin{aligned}\big(\mathcal{F}^{-1}\overline{\psi}\big)(x) &= (2\pi)^{-n}\int_{\mathbb{R}^n}\overline{\psi}(y)\mathrm{e}^{\mathrm{i}x\cdot y}\,\mathrm{d}y\\ &= (2\pi)^{-n}\overline{\int_{\mathbb{R}^n}\psi(y)\mathrm{e}^{-\mathrm{i}x\cdot y}\,\mathrm{d}y} = (2\pi)^{-n}\overline{\widehat{\psi}(x)},\end{aligned}$$

hence we arrive at

$$\int_{\mathbb{R}^n}\widehat{\phi}(\xi)\overline{\widehat{\psi}(\xi)}\,\mathrm{d}\xi = (2\pi)^n\int_{\mathbb{R}^n}\phi(x)\overline{\psi(x)}\,\mathrm{d}x.$$

Proof of Plancherel's theorem continued...

If we take $\psi = \phi$ in the previous formula we get

$$\|\widehat{\phi}\|_2 = (2\pi)^{\frac{n}{2}} \|\phi\|_2.$$

Note that we have now established Plancehrel's formulas (1) and (2) for test functions. In order to extend the formulas and conclude the proof of the theorem we use approximation and that $\mathrm{L}^2(\mathbb{R}^n)$ is complete. Fix $f \in \mathrm{L}^2(\mathbb{R}^n)$. By a result from B4.3 we can find a sequence (ϕ_j) in $\mathscr{D}(\mathbb{R}^n)$ such that $\|f - \phi_j\|_2 \to 0$. In particular we have then that $\phi_j \to f$ in $\mathscr{S}'(\mathbb{R}^n)$ and so, by $\mathscr{S}'$ continuity of the Fourier transform, also $\widehat{\phi_j} \to \widehat{f}$ in $\mathscr{S}'(\mathbb{R}^n)$. We now take $\phi = \phi_k - \phi_j$ in the formula above:

$$\|\widehat{\phi_k} - \widehat{\phi_j}\|_2 = (2\pi)^{\frac{n}{2}} \|\phi_k - \phi_j\|_2.$$

The sequence (ϕ_j) is Cauchy in $\mathrm{L}^2(\mathbb{R}^n)$ because it is convergent there, and consequently we see that also the sequence $(\widehat{\phi_j})$ is Cauchy in $\mathrm{L}^2(\mathbb{R}^n)$.

Proof of Plancherel's theorem continued...

But then it is convergent by completeness, that is, there exists $g \in \mathrm{L}^2(\mathbb{R}^n)$ such that $\|g - \widehat{\phi}_j\|_2 \to 0$. In particular we must then have $\widehat{\phi}_j \to g$ in $\mathscr{S}'(\mathbb{R}^n)$, and then necessarily $\widehat{f} = g \in \mathrm{L}^2(\mathbb{R}^n)$ and we conclude that (1) holds for f. An entirely similar argument can be used to establish (2).
We have shown that $(2\pi)^{-\frac{n}{2}}\mathcal{F}\colon \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$ is isometric (that is (1) holds). To see that it is onto we just recall the Fourier inversion formula in $\mathscr{S}'(\mathbb{R}^n)$. Accordingly we have

$$f = (2\pi)^{-n}\widetilde{\widehat{f}} = \mathcal{F}\Big((2\pi)^{-n}\widetilde{f}\Big) \in \mathcal{F}\big(\mathrm{L}^2(\mathbb{R}^n)\big).$$

This completes the proof. $\square$

The Fourier transform of on L^2 function

Let $f \in \mathrm{L}^2(\mathbb{R}^n)$. Then f is in particular a tempered distribution and so we may Fourier transform it:

$$\left\langle \widehat{f}, \phi \right\rangle = \left\langle f, \widehat{\phi} \right\rangle \quad \phi \in \mathscr{S}(\mathbb{R}^n).$$

This is how we define $\widehat{f}$ and it is important to note that it *cannot be defined* as a Lebesgue integral as we did for L^1 functions simply because f need not be integrable. However, Plancherel's theorem tells us that the tempered distribution $\widehat{f}$ is then actually an L^2 function.

The Fourier transform of on L^2 function

You might recall that we observed before that we could represent the Fourier transform of an L^p function as an $\mathscr{S}'$ limit:

$$\widehat{f}(\xi) = \lim_{j\to\infty} \int_{B_j(0)} f(x) \mathrm{e}^{-\mathrm{i}\xi\cdot x}\, \mathrm{d}x \ \text{ in } \ \mathscr{S}'(\mathbb{R}^n) \tag{3}$$

This of course remains true in particular for L^2 functions. Now in view of Plancherel's formula (1) we can improve the convergence: when $f \in \mathrm{L}^2(\mathbb{R}^n)$, then the convergence in (3) takes place in $\mathrm{L}^2(\mathbb{R}^n)$.

This means convergence in the L^2 norm and *does not automatically* entail convergence almost everywhere in $\xi \in \mathbb{R}^n$. General results from integration theory however say that there exists a *subsequence* (j_k) along which the convergence holds almost everywhere.

The Fourier transform on L^p

We have shown that $\mathcal{F}\colon \mathrm{L}^1(\mathbb{R}^n) \to \mathrm{L}^\infty(\mathbb{R}^n)$ and $\mathcal{F}\colon \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$. We can use this to say something about the Fourier transform of an L^p function for exponents $p \in (1,2)$: Let $f \in \mathrm{L}^p(\mathbb{R}^n)$ and write $f = f_1 + f_2$, where

$$f_1 = \begin{cases} f & \text{if } |f| \geq 1, \\ 0 & \text{if } |f| < 1, \end{cases} \qquad \text{and} \qquad f_2 = \begin{cases} 0 & \text{if } |f| \geq 1, \\ f & \text{if } |f| < 1. \end{cases}$$

Because $\|f_1\|_1 \leq \|f\|_p$ and $\|f_2\|_2 \leq \|f\|_p$ we get

$$\widehat{f} = \widehat{f_1} + \widehat{f_2} \in \mathrm{C}_0(\mathbb{R}^n) + \mathrm{L}^2(\mathbb{R}^n).$$

But in fact a much more precise result holds true!

The Fourier transform on L^p

The Hausdorff-Young inequality For $p \in (1,2)$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have for $f \in \mathrm{L}^p(\mathbb{R}^n)$ that $\widehat{f} \in \mathrm{L}^q(\mathbb{R}^n)$ and

$$\|\widehat{f}\|_q \leq \bigl(2\pi\bigr)^{\frac{n}{q}} \|f\|_p$$

holds. [*Not examinable*]

For $p > 2$ the Fourier transform of an L^p function can be a distribution of positive order. We saw already an example of this with the Heaviside function where the Fourier transform had order 1.

Example: The Hilbert transform again

Recall that the Hilbert transform was defined for each $\phi \in \mathscr{S}(\mathbb{R})$ as

$$\mathcal{H}(\phi) := \frac{1}{\pi}\Big(\mathrm{pv}\big(\tfrac{1}{y}\big) * \phi\Big)(x) = \lim_{\varepsilon \searrow 0}\Big(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\Big)\frac{\phi(x-y)}{\pi y}\,\mathrm{d}y$$

and that hereby $\mathcal{H}\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ as a linear map. We claim that $\mathcal{H}$ extends by continuity to $\mathrm{L}^2(\mathbb{R})$ and that hereby the extended map

$$\mathcal{H}\colon\ \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$$

is unitary (isometric and onto).

Example: The Hilbert transform again

To prove that $\mathcal{H}$ extends by continuity to $\mathsf{L}^2(\mathbb{R})$ we use the Fourier transform and Plancherel's theorem.
Let $\phi \in \mathscr{S}(\mathbb{R}^n)$. Then $\widehat{\mathcal{H}(\phi)}(\xi) = -\mathrm{i}\,\mathrm{sgn}(\xi)\widehat{\phi}(\xi)$ and so by Plancherel's formula (1) we get

$$
\begin{aligned}
\|\mathcal{H}(\phi)\|_2 &= (2\pi)^{-\frac{n}{2}}\|\widehat{\mathcal{H}(\phi)}\|_2 \\
&= (2\pi)^{-\frac{n}{2}}\|\widehat{\phi}\|_2 \\
&= \|\phi\|_2.
\end{aligned}
$$

In particular we have that $\mathcal{H}\colon \mathscr{S}(\mathbb{R}) \to \mathsf{L}^2(\mathbb{R})$ is uniformly continuous if on the domain $\mathscr{S}(\mathbb{R})$ we use the metric $d(\phi,\psi) = \|\phi - \psi\|_2$. Because $\mathscr{S}(\mathbb{R})$ is dense in $\mathsf{L}^2(\mathbb{R})$ we can now appeal to the an abstract result to conclude that $\mathcal{H}$ admits a unique continuous extension to $\mathsf{L}^2(\mathbb{R})$.

Example: The Hilbert transform again

Abstract extension theorem Let (M_1, d_1) be a metric space and (M_2, d_2) be a *complete* metric space. Assume that S is a dense subset of M_1 and that $T\colon S \to M_2$ is *uniformly continuous*. Then there exists a unique continuous map $\overline{T}\colon M_1 \to M_2$ such that $\overline{T}|_S = T$.
Under these circumstances we say that the T extends to M_1 by continuity and because the extension is unique we just denote it by T again.

The proof of the abstract extension theorem is left as an exercise.

In order to see that the extension of $\mathcal{H}$ to $\mathrm{L}^2(\mathbb{R})$ is unitary we use that it is isometric on $\mathscr{S}(\mathbb{R})$ and that $\mathscr{S}(\mathbb{R})$ is dense in $\mathrm{L}^2(\mathbb{R})$. For $f \in \mathrm{L}^2(\mathbb{R})$ find a sequence (ϕ_j) in $\mathscr{S}(\mathbb{R})$ with $\phi_j \to f$ in $\mathrm{L}^2(\mathbb{R})$. Then we get

$$\|\mathcal{H}(f)\|_2 = \lim_{j\to\infty} \|\mathcal{H}(\phi_j)\|_2 = \lim_{j\to\infty} \|\phi_j\|_2 = \|f\|_2$$

so $\mathcal{H}$ is isometric on $\mathrm{L}^2(\mathbb{R})$.

Example: The Hilbert transform again

Next, we use Plancherel's theorem to see that it is onto: Let $g \in \mathsf{L}^2(\mathbb{R})$. Define $f(x) = \mathcal{F}^{-1}_{\xi \to x}\big(\mathrm{i}\,\mathrm{sgn}(\xi)\widehat{g}(\xi)\big)$. Then $f \in \mathsf{L}^2(\mathbb{R})$ and taking a sequence (ϕ_j) in $\mathscr{S}(\mathbb{R})$ such that $\phi_j \to f$ in $\mathsf{L}^2(\mathbb{R})$ we calculate:

$$
\begin{aligned}
\mathcal{H}(f) &= \lim_{j\to\infty} \mathcal{H}(\phi_j) \\
&= \lim_{j\to\infty} \mathcal{F}^{-1}\big(-\mathrm{i}\,\mathrm{sgn}(\xi)\widehat{\phi_j}(\xi)\big) \\
&= \mathrm{F}^{-1}\big(-\mathrm{i}\,\mathrm{sgn}(\xi)\widehat{f}(\xi)\big) \\
&= \mathrm{F}^{-1}\big(\widehat{g}\big) = g.
\end{aligned}
$$

This concludes the proof. $\square$