# B4.4 Fourier Analysis HT22

Lecture 9: $L^2$ based Sobolev spaces and the general convolution rule

1. $L^2$ based Sobolev spaces
2. A special case of the Sobolev embedding theorem
3. The Fourier transform of a compactly supported distribution
4. The general convolution rule
5. Representation in terms of Bessel kernel

The material corresponds to pp. 33–38 in the lecture notes and should be covered in Week 5.

## Another look at Sobolev spaces

An often used approach when solving PDEs and related equations is to first find a distributional or weak (meaning *generalized*) solution by use of general principles. For instance, when it is a linear PDE with constant coefficients we could attempt to do it using the Fourier transform. But often we expect the solution is more *regular* and not just a general distribution. A convenient way to quantify regularity of distributions is by use of Sobolev spaces. Recall that we defined for $k \in \mathbb{N}_0$ and $p \in [1, \infty]$,

$$\mathsf{W}^{k,p}(\mathbb{R}^n) = \Bigl\{ u \in \mathsf{L}^p(\mathbb{R}^n) : \ \partial^\alpha u \in \mathsf{L}^p(\mathbb{R}^n) \ \text{ for each } \ |\alpha| \leq k \Bigr\},$$

and the local variant

$$\mathsf{W}^{k,p}_{\mathrm{loc}}(\mathbb{R}^n) = \Bigl\{ u \in \mathsf{L}^p_{\mathrm{loc}}(\mathbb{R}^n) : \ \partial^\alpha u \in \mathsf{L}^p_{\mathrm{loc}}(\mathbb{R}^n) \ \text{ for each } \ |\alpha| \leq k \Bigr\}.$$

In fact, we defined these spaces on each open subset $\Omega$ of $\mathbb{R}^n$.

## $\mathrm{L}^2$ based Sobolev spaces

These correspond to taking the exponent $p = 2$:

$$\mathrm{W}^{k,2}(\mathbb{R}^n) = \Big\{ u \in \mathrm{L}^2(\mathbb{R}^n) : \, \partial^\alpha u \in \mathrm{L}^2(\mathbb{R}^n) \text{ for each } |\alpha| \leq k \Big\},$$

and

$$\mathrm{W}^{k,2}_{\mathrm{loc}}(\mathbb{R}^n) = \Big\{ u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^n) : \, \partial^\alpha u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^n) \text{ for each } |\alpha| \leq k \Big\}.$$

The former is equipped with the inner product

$$\big(u, v\big)_{\mathrm{W}^{k,2}} := \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} \big(\partial^\alpha u\big)(x) \overline{\big(\partial^\alpha v\big)(x)} \, \mathrm{d}x$$

and corresponding norm $\|u\|_{\mathrm{W}^{k,2}} = \sqrt{(u, u)_{\mathrm{W}^{k,2}}}$. Because it is complete in the corresponding metric it is an example of a Hilbert space. If you follow Functional Analysis 1 & 2 you will have seen some of their general theory already. Here we will not use this abstract viewpoint.

## $\mathrm{L}^2$ based Sobolev spaces

We can characterize the Sobolev space $\mathrm{W}^{k,2}(\mathbb{R}^n)$ by use of the Fourier transform. Using the Plancherel theorem and then the differentiation rule we calculate for $\phi \in \mathscr{S}(\mathbb{R}^n)$:

$$
\begin{aligned}
\|\phi\|^2_{\mathrm{W}^{k,2}} &= \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} \left| \partial^\alpha \phi \right|^2 \mathrm{d}x \\
&= (2\pi)^n \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} \left| \widehat{\partial^\alpha \phi} \right|^2 \mathrm{d}\xi \\
&= (2\pi)^n \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} \left| (\mathrm{i}\xi)^\alpha \widehat{\phi}(\xi) \right|^2 \mathrm{d}\xi \\
&= (2\pi)^n \int_{\mathbb{R}^n} \left( \sum_{|\alpha| \leq k} \left| \xi^\alpha \right|^2 \right) |\widehat{\phi}|^2 \, \mathrm{d}\xi.
\end{aligned}
$$

## $L^2$ based Sobolev spaces

Here we record the inequality

$$n^{1-k}|\xi|^{2k} \leq \sum_{|\alpha|=k} |\xi^\alpha|^2 \leq |\xi|^{2k} \quad (k \in \mathbb{N})$$

and consequently

$$(2n)^{1-k}\left(1+|\xi|^2\right)^k \leq \sum_{|\alpha|\leq k} |\xi^\alpha|^2 \leq \left(1+|\xi|^2\right)^k \tag{1}$$

It follows that the Sobolev norm $\| \cdot \|_{W^{k,2}}$ is equivalent to the norm

$$\|\phi\|_{\mathrm{H}^k} = \left\|\left(1+|\xi|^2\right)^{\frac{k}{2}}\widehat{\phi}\right\|_2.$$

The norm $\| \cdot \|_{\mathrm{H}^k}$ also derives from an inner product, namely

$$\left(\phi,\psi\right)_{\mathrm{H}^k} = \int_{\mathbb{R}^n} \widehat{\phi}(\xi)\overline{\widehat{\psi}(\xi)}\left(1+|\xi|^2\right)^k \mathrm{d}\xi$$

## $\mathrm{L}^2$ based Sobolev spaces

As indicated in the notation for the norm and inner product one often denotes

$$\mathrm{H}^k(\mathbb{R}^n) := \left\{ u \in \mathscr{S}'(\mathbb{R}^n) : \left(1+|\xi|^2\right)^{\frac{k}{2}} \widehat{u} \in \mathrm{L}^2(\mathbb{R}^n) \right\}$$

By the equivalence of the norms $\|\cdot\|_{\mathrm{W}^{k,2}}$ and $\|\cdot\|_{\mathrm{H}^k}$ on $\mathscr{S}(\mathbb{R}^n)$ it follows that

$$\mathrm{H}^k(\mathbb{R}^n) = \mathrm{W}^{k,2}(\mathbb{R}^n)$$

and that the norms remain equivalent in this wider context. *Exercise: Prove it using mollification.*

**Example** If $u$, $\Delta u \in \mathrm{L}^2(\mathbb{R}^n)$, then $u \in \mathrm{H}^2(\mathbb{R}^n)$. The proof is another application of the Plancherel theorem and the diffeentiation rule (see lecture notes for details).

## $L^2$ based Sobolev spaces

**Definition** Sobolev spaces of order $s \in \mathbb{R}$ are defined as

$$\mathrm{H}^s(\mathbb{R}^n) := \left\{ u \in \mathscr{S}'(\mathbb{R}^n) : \left(1 + |\xi|^2\right)^{\frac{s}{2}} \widehat{u} \in \mathsf{L}^2(\mathbb{R}^n) \right\}$$

and equipped with the inner product

$$\left(u, v\right)_{\mathrm{H}^s} = \int_{\mathbb{R}^n} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\left(1 + |\xi|^2\right)^s \mathrm{d}\xi$$

and corresponding norm $\|u\|_{\mathrm{H}^s} := \sqrt{(u, u)_{\mathrm{H}^s}}$.

**Remark** Note that $\mathrm{H}^0(\mathbb{R}^n) = \mathsf{L}^2(\mathbb{R}^n)$ and that the scale is nested: when $s < t$, then

$$\mathrm{H}^t(\mathbb{R}^n) < \mathrm{H}^s(\mathbb{R}^n).$$

To see the latter observe that $\left(1 + |\xi|^2\right)^s \leq \left(1 + |\xi|^2\right)^t$ holds for all $\xi \in \mathbb{R}^n$ and therefore that

$$\|u\|_{\mathrm{H}^s} \leq \|u\|_{\mathrm{H}^t}$$

when $u \in \mathrm{H}^t(\mathbb{R}^n)$.

## A special case of the Sobolev embedding theorem

The *regularity* of tempered distributions in $\mathrm{H}^s(\mathbb{R}^n)$ increases with $s \in \mathbb{R}$. An instance of this is documented in the following

**Proposition** Let $u \in \mathrm{H}^s(\mathbb{R}^n)$ and assume $s > \frac{n}{2}$. Then $u \in \mathsf{C}^k(\mathbb{R}^n)$ for each $k \in \mathbb{N}_0$ with $k < s - \frac{n}{2}$. In fact, $\partial^\alpha u \in \mathsf{C}_0(\mathbb{R}^n)$ for each $|\alpha| < s - \frac{n}{2}$.

*Proof.* The proof goes via the Plancherel theorem, the differentiation rule, the Fourier inversion formula and the Riemann-Lebesgue lemma. The assumption $u \in \mathrm{H}^s(\mathbb{R}^n)$ amounts to

$$u \in \mathscr{S}'(\mathbb{R}^n) \text{ and } \left(1+|\xi|^2\right)^{\frac{s}{2}} \widehat{u} \in \mathsf{L}^2(\mathbb{R}^n).$$

Because $s > 0$ this implies in particular that $\widehat{u} \in \mathsf{L}^2(\mathbb{R}^n)$: note that $\left(1+|\xi|^2\right)^{-\frac{s}{2}}$ is a bounded $\mathsf{C}^\infty$ function and so

$$\widehat{u} = \left(1+|\xi|^2\right)^{-\frac{s}{2}} \left( \left(1+|\xi|^2\right)^{\frac{s}{2}} \widehat{u} \right) \in \mathsf{L}^2(\mathbb{R}^n).$$

## A special case of the Sobolev embedding theorem–proof continued...

Fix a multi-index $\alpha$ of length $|\alpha| < s - \frac{n}{2}$. By the differentiation rule $\widehat{\partial^\alpha u} = \left(\mathrm{i}\xi\right)^\alpha \widehat{u}$. Now

$$|\xi^\alpha| = \prod_{j=1}^{n} |\xi_j|^{\alpha_j} \leq \prod_{j=1}^{n} |\xi|^{\alpha_j} = |\xi|^{|\alpha|} \leq \left(1+|\xi|^2\right)^{\frac{|\alpha|}{2}}$$

for all $\xi \in \mathbb{R}^n$. Consequently

$$\begin{aligned} |\xi^\alpha \widehat{u}(\xi)| &\leq \left(1+|\xi|^2\right)^{\frac{|\alpha|}{2}} |\widehat{u}(\xi)| \\ &= \left(1+|\xi|^2\right)^{\frac{|\alpha|-s}{2}} \left(\left(1+|\xi|^2\right)^{\frac{s}{2}} |\widehat{u}(\xi)|\right) \end{aligned}$$

Integrate over $\xi \in \mathbb{R}^n$ and use the Cauchy-Schwarz inequality to estimate.

## A special case of the Sobolev embedding theorem–proof continued...

$$\begin{aligned}\int_{\mathbb{R}^n} \left|\xi^\alpha \widehat{u}(\xi)\right| \mathrm{d}\xi &\leq \left\|\left(1+|\cdot|^2\right)^{\frac{|\alpha|-s}{2}}\right\|_2 \left\|\left(1+|\cdot|^2\right)^{\frac{s}{2}} |\widehat{u}|\right\|_2 \\ &= \left\|\left(1+|\cdot|^2\right)^{\frac{|\alpha|-s}{2}}\right\|_2 \|u\|_{\mathrm{H}^s}\end{aligned}$$

The right-hand side is finite since by integration in polar coordinates:

$$\left\|\left(1+|\cdot|^2\right)^{\frac{|\alpha|-s}{2}}\right\|_2^2 = \omega_{n-1} \int_0^\infty \frac{r^{n-1}}{\left(1+r^2\right)^{s-|\alpha|}} \,\mathrm{d}r.$$

Here the exponent $n-1-2(s-|\alpha|) = 2(|\alpha|-(s-\frac{n}{2}))-1 < -1$ so the integral converges, and thus $\widehat{\partial^\alpha u} \in \mathrm{L}^1(\mathbb{R}^n)$. Using the Fourier inversion formula in $\mathscr{S}'$ we get

$$\partial^\alpha u = \left(2\pi\right)^{-n} \widetilde{\mathcal{F}}\left(\widehat{\partial^\alpha u}\right),$$

and so $\partial^\alpha u \in \mathrm{C}_0(\mathbb{R}^n)$ by Riemann-Lebesgue. $\square$

## The Fourier transform of a compactly supported distribution

**Proposition** Let $v \in \mathscr{E}'(\mathbb{R}^n)$. Then $\widehat{v}$ is a moderate $C^\infty$ function and

$$\widehat{v}(\xi) = \left\langle v, \mathrm{e}^{-\mathrm{i}\xi\cdot(\cdot)} \right\rangle.$$

*Proof.* Take $\chi \in \mathscr{D}(\mathbb{R}^n)$ so $\chi = 1$ near $\operatorname{supp}(v)$. Then we have $\chi v = v$. To check this simply calculate for $\phi \in \mathscr{S}(\mathbb{R}^n)$,

$$\langle v, \phi \rangle = \langle v, \chi\phi + (1-\chi)\phi \rangle = \langle v, \chi\phi \rangle = \langle \chi v, \phi \rangle$$

since $(1-\chi)\phi = 0$ near $\operatorname{supp}(v)$. But then from previous results we get that $\widehat{v} = (2\pi)^{-n}\widehat{v} * \widehat{\chi}$ is a moderate $C^\infty$ function and

$$\widehat{v}(\xi) = (2\pi)^{-n}\big(\widehat{v} * \widehat{\chi}\big)(\xi) = (2\pi)^{-n}\big\langle \widehat{v}, \widehat{\chi}(\xi - \cdot) \big\rangle.$$

## The Fourier transform of a compactly supported distribution–proof continued..

Here

$$
\begin{aligned}
\widehat{v}(\xi) = (2\pi)^{-n}\big\langle \widehat{v}, \widehat{\chi}(\xi - \cdot)\big\rangle &= (2\pi)^{-n}\big\langle \widehat{v}, \widetilde{\widehat{\chi}}(\cdot - \xi)\big\rangle \\
&\overset{\text{FIF}}{=} (2\pi)^{-2n}\big\langle \widehat{v}, \mathcal{F}^3\chi(\cdot - \xi)\big\rangle \\
&= (2\pi)^{-2n}\big\langle \widehat{v}, \tau_{-\xi}\mathcal{F}^3\chi\big\rangle \\
&= (2\pi)^{-2n}\big\langle v, \mathrm{e}^{-\mathrm{i}\xi\cdot(\cdot)}\mathcal{F}^4\chi\big\rangle \\
&\overset{\text{FIF}}{=} \big\langle v, \mathrm{e}^{-\mathrm{i}\xi\cdot(\cdot)}\chi\big\rangle \\
&= \big\langle v, \mathrm{e}^{-\mathrm{i}\xi\cdot(\cdot)}\big\rangle
\end{aligned}
$$

concluding the proof. $\square$

FIF = Fourier inversion formula in $\mathscr{S}'$

## The general convolution rule

**Theorem** Let $u \in \mathscr{S}'(\mathbb{R}^n)$, $v \in \mathscr{E}'(\mathbb{R}^n)$. Then $u * v \in \mathscr{S}'(\mathbb{R}^n)$ and

$$\widehat{u * v} = \widehat{u}\widehat{v}. \tag{2}$$

*Proof.* We prove the result by use of mollification and the basic convolution rule. First recall from B4.3 that $u * v$ is defined by the rule

$$\langle u * v, \phi \rangle = \langle u, \widetilde{v} * \phi \rangle \,, \quad \phi \in \mathscr{D}(\mathbb{R}^n).$$

For the standard mollifier $\left(\rho_\varepsilon\right)_{\varepsilon > 0}$ we have that $v_\varepsilon := \rho_\varepsilon * v \in \mathscr{D}(\mathbb{R}^n)$ and by the basic convolution and the dilation rules

$$\widehat{v_\varepsilon} = \widehat{\rho_\varepsilon}\widehat{v} = d_\varepsilon \widehat{\rho} \widehat{v}.$$

Here we have for $\phi \in \mathscr{S}(\mathbb{R}^n)$ that $d_\varepsilon \widehat{\rho} \phi \to \phi$ in $\mathscr{S}(\mathbb{R}^n)$ as $\varepsilon \searrow 0$. [*Exercise: Check it.*]

## The general convolution rule – proof continued...

Another use of the basic convolution rule yields

$$\widehat{u * v_\varepsilon} = \widehat{u}\widehat{v_\varepsilon} = \widehat{u}\widehat{v} d_\varepsilon \widehat{\rho}$$

and we note that $\widehat{u}\widehat{v} \in \mathscr{S}'(\mathbb{R}^n)$ since $\widehat{v}$ is a moderate $\mathrm{C}^\infty$ function. Next, for $\phi \in \mathscr{S}(\mathbb{R}^n)$ we get as $\varepsilon \searrow 0$,

$$\left\langle \widehat{u * v_\varepsilon}, \phi \right\rangle = \left\langle \widehat{u}\widehat{v}, d_\varepsilon \widehat{\rho} \phi \right\rangle \to \left\langle \widehat{u}\widehat{v}, \phi \right\rangle.$$

From the Fourier inversion formula in $\mathscr{S}'$ and $\mathscr{S}'$ continuity of the Fourier transform, $u * v_\varepsilon \to \mathcal{F}^{-1}\left(\widehat{u}\widehat{v}\right)$ in $\mathscr{S}'(\mathbb{R}^n)$ as $\varepsilon \searrow 0$. But we also have from B4.3 that $u * v_\varepsilon \to u * v$ in $\mathscr{D}'(\mathbb{R}^n)$ as $\varepsilon \searrow 0$, so $\langle u * v, \phi \rangle = \langle \mathcal{F}^{-1}\left(\widehat{u}\widehat{v}\right), \phi \rangle$ for $\phi \in \mathscr{D}(\mathbb{R}^n)$. Because the right-hand side is a tempered distribution and because $\mathscr{D}(\mathbb{R}^n)$ is $\mathscr{S}$ dense in $\mathscr{S}(\mathbb{R}^n)$ it follows that $u * v \in \mathscr{S}'(\mathbb{R}^n)$ and (2) holds. This concludes the proof. $\square$

**Exercise:** Show that $v * \phi \in \mathscr{S}(\mathbb{R}^n)$ when $v \in \mathscr{E}'(\mathbb{R}^n)$ and $\phi \in \mathscr{S}(\mathbb{R}^n)$. Use this to give another proof of the general convolution rule.

## An extension of the convolution product

**Definition** Let $u$, $v \in \mathscr{S}'(\mathbb{R}^n)$ and assume that $\widehat{v}$ is a moderate $\mathrm{C}^\infty$ function. We then *define* the convolution $u * v \in \mathscr{S}'(\mathbb{R}^n)$ by the rule

$$u * v := \mathcal{F}^{-1}\bigl(\widehat{u}\widehat{v}\bigr).$$

**Remarks** It is an extension because $\widehat{v}$ can be a moderate $\mathrm{C}^\infty$ function also when the support of $v$ is not compact. Also note that the general convolution rule ensures it is a consistent extension of the convolution product defined for $u \in \mathscr{S}'(\mathbb{R}^n)$ and $v \in \mathscr{E}'(\mathbb{R}^n)$.
With the obvious definition of $v * u$, we have $u * v = v * u$. Furthermore using the rules for the Fourier transform we can also show

$$\partial^\alpha\bigl(u * v\bigr) = \bigl(\partial^\alpha u\bigr) * v = u * \bigl(\partial^\alpha v\bigr)$$

remains true for any multi-index $\alpha \in \mathbb{N}_0$.

## $\mathrm{L}^2$ based Sobolev spaces: representation using Bessel kernel

We defined for each $s \in \mathbb{R}$ the Sobolev space of order $s$ by

$$\mathrm{H}^s(\mathbb{R}^n) = \left\{ u \in \mathscr{S}'(\mathbb{R}^n) : \, \left(1+|\xi|^2\right)^{\frac{s}{2}} \widehat{u} \in \mathrm{L}^2(\mathbb{R}^n) \right\}$$

The function $\xi \mapsto \left(1+|\xi|^2\right)^{-\frac{s}{2}}$ is a moderate $\mathrm{C}^\infty$ function and its inverse Fourier transform

$$g_s := \mathcal{F}^{-1}_{\xi \to x}\left( \left(1+|\xi|^2\right)^{-\frac{s}{2}} \right)$$

is called the *Bessel kernel of order s*. By the extended convolution rule and the Fourier inversion formula we have for $u \in \mathrm{H}^s(\mathbb{R}^n)$ that

$$u = \mathcal{F}^{-1}\left( \left(1+|\xi|^2\right)^{-\frac{s}{2}} \left(1+|\xi|^2\right)^{\frac{s}{2}} \widehat{u} \right) = g_s * \mathcal{F}^{-1}\left( \left(1+|\xi|^2\right)^{\frac{s}{2}} \widehat{u} \right)$$

Therefore

$$\mathrm{H}^s(\mathbb{R}^n) = \left\{ g_s * f : \, f \in \mathrm{L}^2(\mathbb{R}^n) \right\}$$

and this is why these spaces are also known as *Bessel potential spaces*.

## Bessel potential spaces with exponent $p$ [Not examinable]

Let $s \in \mathbb{R}$ and $p \in (1, \infty)$. The corresponding Bessel potential space is then defined as

$$\mathrm{H}^{s,p}(\mathbb{R}^n) := \left\{ g_s * f : \ f \in \mathsf{L}^p(\mathbb{R}^n) \right\}$$

equipped with the norm $\|u\|_{\mathrm{H}^{s,p}} := \|f\|_p$.

**Theorem on Bessel potentials:** When $k \in \mathbb{N}_0$ and $p \in (1, \infty)$ we have $\mathrm{H}^{k,p}(\mathbb{R}^n) = \mathsf{W}^{k,p}(\mathbb{R}^n)$.

**The Trace Theorem:** The *trace operator* $\mathrm{Tr}\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^{n-1} \times \{0\})$ is defined by $\mathrm{Tr}(\phi) := \phi(x', 0)$, $x' \in \mathbb{R}^{n-1}$. If $k \in \mathbb{N}$, $p \in (1, \infty)$ and $k > \frac{1}{p}$, then the trace operator *extends by continuity* to a continuous linear and surjective map

$$\mathrm{Tr}\colon \mathrm{H}^{k,p}(\mathbb{R}^n) \to \mathrm{H}^{k-\frac{1}{p},p}(\mathbb{R}^{n-1})$$