B4.4 Fourier Analysis HT22

Lecture 12: Elliptic PDEs and fundamental solutions

1. Characterization of ellipticity
2. Mapping properties in Sobolev spaces
3. Gårding inequalities
4. Existence of fundamental solutions
5. Hypoellipticity
6. Non-examinable proofs FYI

The material corresponds to pp. 44–45 and 57–59 in the lecture notes and should be covered in Week 6.

Elliptic differential operators

Let $p(\partial)$ be a linear differential operator with constant coefficients

$$p(\partial) = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha \quad c_\alpha \in \mathbb{C}$$

of order $m \in \mathbb{N}_0$ (so $c_\alpha \neq 0$ for at least one multi-index α of length m). The *symbol* of $p(\partial)$ is the complex polynomial $p(\mathrm{i}\xi)$ and its m-homogeneous part

$$p_m(\mathrm{i}\xi) = \sum_{|\alpha| = m} c_\alpha \left(\mathrm{i}\xi\right)^\alpha$$

is called the *principal symbol* for $p(\partial)$.

Definition The differential operator $p(\partial)$ of order m is elliptic if its principal symbol has no real zero except $\xi = 0$, that is,

$$p_m(\mathrm{i}\xi) \neq 0 \text{ for all } \xi \in \mathbb{R}^n \setminus \{0\}.$$

Examples

- A 0 order differential operator is simply multiplication by a complex constant. It is elliptic precisely when the constant is non-zero.
- The Laplacian Δ on $\mathbb{R}^n$ is elliptic. Furthermore it is *homogeneous of order* 2 because its symbol equals its principal symbol which is $-|\xi|^2$.
- Powers of the Laplacian on $\mathbb{R}^n$, Δ^k, are elliptic for each $k \in \mathbb{N}$. Δ^k is homogeneous of order $2k$ and has symbol $\left(-|\xi|^2\right)^k$.
- $(1-\Delta)^k$ is elliptic of order $2k$ for each $k \in \mathbb{N}$. Its symbol is $(1+|\xi|^2)^k$ and has *no* real zero. *Note* it is related to the Bessel kernel.
- If $p(\partial)$ is elliptic of order $m \in \mathbb{N}$ and $q(\partial)$ is a differential operator of order at most $m-1$, then $p(\partial)+q(\partial)$ is also elliptic.
- The Cauchy-Riemann differential operators $\frac{\partial}{\partial \bar{z}}$ and $\frac{\partial}{\partial z}$ on $\mathbb{C}$ are both elliptic of order 1.
- The heat operator $\partial_t - \Delta$ and the wave operator $\partial_t^2 - \Delta$ are not elliptic (on $\mathbb{R}^n$ for any $n \in \mathbb{N}$).

Characterization of ellipticity

Lemma Let $p(\partial)$ be a linear differential operator with constant coefficients of order $m \in \mathbb{N}$:

$$p(\partial) = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha$$

Then $p(\partial)$ is elliptic if and only if there exist constants $c > 0$, $R > 0$ such that

$$\left| p(\mathrm{i}\xi) \right| \geq c|\xi|^m$$

holds for all $\xi \in \mathbb{R}^n$ with $|\xi| \geq R$.

Proof. '*Only if*' Assume that $p(\partial)$ is elliptic of order m. The function $\xi \mapsto \left| p_m(\mathrm{i}\xi) \right|$ is continuous and hence assumes its minimum value on the unit sphere $\mathbb{S}^{n-1}$. Say $\xi_0 \in \mathbb{S}^{n-1}$ and $\left| p_m(\mathrm{i}\xi) \right| \geq \left| p_m(\mathrm{i}\xi_0) \right|$ for all $\xi \in \mathbb{S}^{n-1}$. Because $\xi_0 \neq 0$ we have

$$a := \left| p_m(\mathrm{i}\xi_0) \right| > 0.$$

Characterization of ellipticity–proof

For $\xi \in \mathbb{R}^n \setminus \{0\}$ we have $a \leq \left|p_m(\mathrm{i}\frac{\xi}{|\xi|})\right|$, so by homogeneity

$$\left|p_m\big(\mathrm{i}\xi\big)\right| \geq a|\xi|^m$$

holds for all $\xi \in \mathbb{R}^n$. Because the polynomial $p - p_m$ has degree at most $m-1$ we have with

$$b := \sum_{|\alpha| \leq m-1} |c_\alpha|$$

that $\left|p(\mathrm{i}\xi) - p_m(\mathrm{i}\xi)\right| \leq b|\xi|^{m-1}$ holds for all $\xi \in \mathbb{R}^n$ with $|\xi| \geq 1$. We therefore get for $\xi \in \mathbb{R}^n$ with $|\xi| \geq 1$:

$$\begin{aligned}
\left|p(\mathrm{i}\xi)\right| &\geq \left|p_m(\mathrm{i}\xi)\right| - \left|p(\mathrm{i}\xi) - p_m(\mathrm{i}\xi)\right| \\
&\geq a|\xi|^m - b|\xi|^{m-1} \\
&= \big(a - \frac{b}{|\xi|}\big)|\xi|^m
\end{aligned}$$

Characterization of ellipticity–proof

If therefore $R \geq 1$ and $|\xi| \geq R$, then

$$\left|p(\mathrm{i}\xi)\right| \geq \left(a - \frac{b}{R}\right)|\xi|^m$$

and so we conclude this part of the proof with $c = a - \frac{b}{R}$ and $R > \max\left\{1, \frac{b}{a}\right\}$.

'*if*' Conversely if $p(\partial)$ is not elliptic, then for some $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ we have $p_m(\mathrm{i}\xi_0) = 0$. Consequently

$$\mathbb{R} \ni t \mapsto p(\mathrm{i}t\xi_0) = \left(p - p_m\right)(\mathrm{i}\xi_0 t)$$

is a polynomial of degree at most $m-1$, which is not compatible with the bound in the lemma. $\square$

Often we prefer to estimate the symbols with the quantity $\left(1+|\xi|^2\right)^{\frac{s}{2}}$ that appears in the definition of the Sobolev spaces $\mathrm{H}^s(\mathbb{R}^n)$.

Characterization of ellipticity–another variant

Corollary The differential operator $p(\partial)$ of order $m \in \mathbb{N}$ is elliptic if and only if there exist constants $C > 0$, $r > 0$ such that

$$\big|p(\mathrm{i}\xi)\big| \geq C\big(1 + |\xi|^2\big)^{\frac{m}{2}}$$

holds for all $\xi \in \mathbb{R}^n$ with $|\xi| \geq r$.

Proof. The key to the proof is the elementary inequality

$$\frac{t^m}{\big(1+t^2\big)^{\frac{m}{2}}}\big(1+|\xi|^2\big)^{\frac{m}{2}} \leq |\xi|^m \leq \big(1+|\xi|^2\big)^{\frac{m}{2}}$$

that holds for all $\xi \in \mathbb{R}^n$, $t \geq 0$ with $|\xi| \geq t$. [Prove it as an exercise]
The corollary now follows from the previous lemma where we note that the relation between the constants can be taken as

$$C = c\frac{R^m}{\big(1+R^2\big)^{\frac{m}{2}}} \text{ and } r = R.$$

We leave the details as an exercise. $\square$

A useful convention about H^s norms

When dealing with tempered distributions and the Sobolev spaces $\mathrm{H}^s(\mathbb{R}^n)$ that we defined for each $s \in \mathbb{R}$ as

$$\mathrm{H}^s(\mathbb{R}^n) := \left\{ u \in \mathscr{S}'(\mathbb{R}^n) : \ \left(1+|\xi|^2\right)^{\frac{s}{2}} \widehat{u} \in \mathsf{L}^2(\mathbb{R}^n) \right\}$$

it is often useful to *define* for $v \in \mathscr{S}'(\mathbb{R}^n)$ that

$$\|v\|_2 := \begin{cases} \|v\|_2 & \text{when } v \in \mathsf{L}^2(\mathbb{R}^n) \\ +\infty & \text{when } v \notin \mathsf{L}^2(\mathbb{R}^n) \end{cases}$$

and correspondingly for the H^s-norms we *define* for $u \in \mathscr{S}'(\mathbb{R}^n)$:

$$\|u\|_{\mathrm{H}^s} := \left\| \left(1+|\xi|^2\right)^{\frac{s}{2}} \widehat{u} \right\|_2 .$$

Thus for $u \in \mathscr{S}'(\mathbb{R}^n)$ we have that $u \in \mathrm{H}^s(\mathbb{R}^n)$ if and only if

$$\|u\|_{\mathrm{H}^s} < +\infty$$

Mapping properties on the Sobolev scale H^s

Proposition Let $p(\partial)$ be a differential operator of order $m \in \mathbb{N}$. Then for each $s \in \mathbb{R}$ we have $p(\partial)\colon \mathrm{H}^s(\mathbb{R}^n) \to \mathrm{H}^{s-m}(\mathbb{R}^n)$ is linear and

$$\|p(\partial)u\|_{\mathrm{H}^{s-m}} \leq c\|u\|_{\mathrm{H}^s}$$

for all $u \in \mathrm{H}^s(\mathbb{R}^n)$, where

$$c = \sum_{|\alpha| \leq m} |c_\alpha|.$$

Proof. First note that $\bigl|p(\mathrm{i}\xi)\bigr| \leq c\bigl(1+|\xi|^2\bigr)^{\frac{m}{2}}$ holds for all $\xi \in \mathbb{R}^n$. We therefore have for $u \in \mathrm{H}^s(\mathbb{R}^n)$,

$$\begin{aligned}
\|p(\partial)u\|_{\mathrm{H}^{s-m}} &= \|\bigl(1+|\xi|^2\bigr)^{\frac{s-m}{2}} p(\mathrm{i}\xi)\widehat{u}\|_2 \\
&\leq c\|\bigl(1+|\xi|^2\bigr)^{\frac{s-m}{2}}\bigl(1+|\xi|^2\bigr)^{\frac{m}{2}}\widehat{u}\|_2 \\
&= c\|\bigl(1+|\xi|^2\bigr)^{\frac{s}{2}}\widehat{u}\|_2 \\
&= c\|u\|_{\mathrm{H}^s},
\end{aligned}$$

as required. $\square$

Gårding inequalities for elliptic operators

Theorem Assume that $p(\partial)$ is an elliptic differential operator of order $m \in \mathbb{N}$ and let $s > 0$. Then we have for $u \in \mathsf{L}^2(\mathbb{R}^n)$ that

$$\|u\|_{\mathrm{H}^s} \leq c\big(\|u\|_2 + \|p(\partial)u\|_{\mathrm{H}^{s-m}}\big), \tag{1}$$

where $c > 0$ is a constant.

Proof. By the characterization of ellipticity we can find constants $C > 0$, $r > 0$ such that

$$\big|p(\mathrm{i}\xi)\big| \geq C\big(1 + |\xi|^2\big)^{\frac{m}{2}}$$

holds for all $\xi \in \mathbb{R}^n$ with $|\xi| \geq r$. We claim the constant

$$c = \max\Big\{(2\pi)^{\frac{n}{2}}(1 + r^2)^s, \frac{1}{C}\Big\}.$$

works in Gårding's inequality (1). We can assume that $p(\partial)u \in \mathrm{H}^{s-m}(\mathbb{R}^n)$ as otherwise there is nothing to prove.

Gårding inequalities–proof

By Plancherel's theorem, $\widehat{u} \in \mathrm{L}^2(\mathbb{R}^n)$, hence we can estimate

$$\begin{aligned}\int_{B_r(0)} \big(1+|\xi|^2\big)^s |\widehat{u}(\xi)|^2 \,\mathrm{d}\xi &\leq \big(1+r^2\big)^s \int_{B_r(0)} |\widehat{u}(\xi)|^2 \,\mathrm{d}\xi \\ &\leq \big(1+r^2\big)^s \|\widehat{u}\|_2^2 \\ &\leq c^2 \|u\|_2^2\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_r(0)} \big(1+|\xi|^2\big)^s |\widehat{u}(\xi)|^2 \,\mathrm{d}\xi &= \int_{\mathbb{R}^n\setminus B_r(0)} \frac{\big(1+|\xi|^2\big)^s}{|p(\mathrm{i}\xi)|^2} |p(\mathrm{i}\xi)\widehat{u}(\xi)|^2 \,\mathrm{d}\xi \\ &\leq \frac{1}{C^2} \int_{\mathbb{R}^n\setminus B_r(0)} \big(1+|\xi|^2\big)^{s-m} |p(\mathrm{i}\xi)\widehat{u}(\xi)|^2 \,\mathrm{d}\xi \\ &\leq c^2 \|p(\partial)u\|_{\mathrm{H}^{s-m}}^2 .\end{aligned}$$

Addition of the two inequalities concludes the proof. □

Example 1 Let

$$q(\partial) = \sum_{|\alpha|\leq 1} q_\alpha \partial^\alpha, \quad q_\alpha \in \mathbb{C}$$

be a first order differential operator and $f \in \mathrm{L}^2(\mathbb{R}^n)$. If $u \in \mathrm{L}^2(\mathbb{R}^n)$ satisfies the PDE

$$\Delta u + q(\partial)u = f \text{ in } \mathscr{S}'(\mathbb{R}^n).$$

Then $u \in \mathrm{H}^2(\mathbb{R}^n)$ and $\|u\|_{\mathrm{H}^2} \leq c\big(\|u\|_2 + \|f\|_2\big)$ holds by Gårding's inequality because the Laplacian, and hence also $\Delta + q(\partial)$, is elliptic.

Example 2 The L^2 norm cannot be removed on the right-hand side in the Gårding inequality: The function $u = \mathrm{e}^{x^2-y^2}\cos\big(2xy\big)$, $(x, y) \in \mathbb{R}^2$, is C^∞ but does not belong to $\mathrm{H}^s(\mathbb{R}^2)$ for any $s \in \mathbb{R}$. Because

$$u = \mathrm{Re}\big(\mathrm{e}^{z^2}\big), \quad z = x + \mathrm{i}y \in \mathbb{C},$$

it is harmonic and so it is clear that we need the term $c\|u\|_2$ on the right-hand side in the Gårding inequality.

Fundamental solutions were defined for linear differential operators $p(\partial)$ in B4.3 as follows: Any $E \in \mathscr{D}'(\mathbb{R}^n)$ satisfying

$$p(\partial)E = \delta_0 \text{ in } \mathscr{D}'(\mathbb{R}^n)$$

is a fundamental solution for $p(\partial)$.

In B4.3 we found fundamental solutions for the Laplacian on $\mathbb{R}^n$ and the Cauchy-Riemann operators $\frac{\partial}{\partial \bar{z}}$, $\frac{\partial}{\partial z}$ on $\mathbb{C}$. In each case they were tempered distributions with singular support in $\{0\}$.

Example Find a fundamental solution for $\left(1 - \Delta\right)^k$ on $\mathbb{R}^n$ for n, $k \in \mathbb{N}$. Assume $E \in \mathscr{S}'(\mathbb{R}^n)$ satisfies $\left(1 - \Delta\right)^k E = \delta_0$ in $\mathscr{S}'(\mathbb{R}^n)$. Then we get by use of the differentiation rule,

$$\left(1 + |\xi|^2\right)^k \widehat{E} = 1 \text{ in } \mathscr{S}'(\mathbb{R}^n).$$

Fundamental solutions

The function

$$\xi \mapsto \left(1 + |\xi|^2\right)^{-k}$$

is a moderate C^∞ function, hence especially a tempered distribution, so by the Fourier inversion formula

$$E = E_k := \mathcal{F}^{-1}\left(\left(1 + |\xi|^2\right)^{-k}\right) \in \mathscr{S}'(\mathbb{R}^n).$$

By inspection we see that this *is* a fundamental solution.

We record that $E_k = g_{2k}$ is the Bessel kernel of order $2k$ that we encountered in lecture 9. We claim that it has singular support contained in $\{0\}$. We will prove a slightly more general result.

Bessel kernels again

Theorem The Bessel kernel of order $s \in \mathbb{R}$, $g_s = \mathcal{F}^{-1}\left(1+|\xi|^2\right)^{-\frac{s}{2}}$, is a tempered distribution, whose Fourier transform, $\widehat{g_s} = \left(1+|\xi|^2\right)^{-\frac{s}{2}}$, is a moderate C^∞ function. Furthermore, $\mathrm{sing.supp}(g_s) = \{0\}$.

Proof. Only the statement about the singular support needs proof. Fix a direction $1 \leq j \leq n$. Then for $m \in \mathbb{N}$ we calculate by use of Leibniz

$$\partial_j^m \widehat{g}_s = -s\xi_j \partial_j^{m-1} \widehat{g}_{s+2} - s(m-1)\partial_j^{m-2}\widehat{g}_{s+2}. \tag{2}$$

We next claim that for all $s \in \mathbb{R}$, $m \in \mathbb{N}_0$ there exists a constant $c(m,s) > 0$ such that

$$\left|\partial_j^m \widehat{g}_s\right| \leq c(m,s)\widehat{g}_{s+m} \tag{3}$$

We prove this by induction on $m \in \mathbb{N}_0$. It is trivially true for $m = 0$, and for $m = 1$ and any $s \in \mathbb{R}$ we have

$$\begin{aligned} \left|\partial_j \widehat{g}_s\right| &= \left|s\xi_j\right|\left(1+|\xi|^2\right)^{-\frac{s+2}{2}} \\ &\leq |s|\left(1+|\xi|^2\right)^{-\frac{s+1}{2}} = |s|\widehat{g}_{s+1}. \end{aligned}$$

Bessel kernels again–proof

Let $k \in \mathbb{N}$ and assume that (3) holds for $m \in \{0\,, \dots\,, k\}$ and all $s \in \mathbb{R}$. Then

$$\begin{aligned}
\left|\partial_j^{k+1}\widehat{g}_s\right| &\overset{(2)}{=} \left|-s\xi_j\partial_j^k\widehat{g}_{s+2} - sk\partial_j^{k-1}\widehat{g}_{s+2}\right| \\
&\leq |s\xi_j|c(k,s+2)\widehat{g}_{s+k+2} + |s|kc(k-1,s+2)\widehat{g}_{s+k+1} \\
&\leq |s|\big(c(k,s+2) + kc(k-1,s+2)\big)\widehat{g}_{s+k+1}
\end{aligned}$$

and the claim follows by induction.
Note that $\widehat{g}_t \in \mathrm{L}^1(\mathbb{R}^n)$ for $t > n$, hence for given $s \in \mathbb{R}$ we have by virtue of (3) that $\partial_j^m\widehat{g}_s \in \mathrm{L}^1(\mathbb{R}^n)$ for $m + s > n$, and so by the differentiation rule and Riemann-Lebesgue

$$\left(-\mathrm{i}x_j\right)^m g_s = \mathcal{F}^{-1}_{\xi\to x}\left(\partial_j^m\widehat{g}_s\right) \in \mathrm{C}_0(\mathbb{R}^n).$$

We generalize this as follows.

Bessel kernels again–proof

Let $\alpha \in \mathbb{N}_0^n$ be a multi-index and note that by the differentiation rule

$$\partial^\alpha\big(\big(-\mathrm{i}x_j\big)^m g_s\big) = \mathcal{F}^{-1}_{\xi \to x}\big(\big(\mathrm{i}\xi\big)^\alpha \partial_j^m \widehat{g}_s\big)$$

and if $m > n - s + |\alpha|$, then the right-hand side is in $\mathrm{C}_0(\mathbb{R}^n)$ by Riemann-Lebesgue. We deduce that $x_j^m g_s \in \mathrm{C}^{[m-n+s-1]}(\mathbb{R}^n)$ for all $m \in \mathbb{N}$ with $m > n - s - 1$, and therefore that $g_s \in \mathrm{C}^{[m-n+s-1]}(\mathbb{R}^n \setminus \{x : \, x_j = 0\})$ for all $m > n - s - 1$, that is, $g_s \in \mathrm{C}^\infty(\mathbb{R}^n \setminus \{x : \, x_j = 0\})$. But this is true for each direction j and since

$$\bigcup_{j=1}^n \big(\mathbb{R}^n \setminus \{x : \, x_j = 0\}\big) = \mathbb{R}^n \setminus \{0\}$$

we conclude that $g_s \in \mathrm{C}^\infty(\mathbb{R}^n \setminus \{0\})$, and so $\operatorname{sing.supp}(g_s) \subseteq \{0\}$.

Bessel kernels again–proof

Finally, note that $\partial^\alpha \widehat{g}_s \in \mathsf{L}^1(\mathbb{R}^n)$ for each multi-index α with $|\alpha| > n - s$, and so by the differentiation rule and Riemann-Lebesgue

$$\big(\mathrm{i}x\big)^\alpha g_s = \mathcal{F}^{-1}\big(\partial^\alpha \widehat{g}_s\big) \in \mathsf{C}_0(\mathbb{R}^n).$$

If g_s was C^∞ on $\mathbb{R}^n$ it would follow that g_s is rapidly decreasing. But the same argument applies to the derivatives $\partial^\beta g_s$, and consequently it would follow that $g_s \in \mathscr{S}(\mathbb{R}^n)$. But then also $\widehat{g}_s \in \mathscr{S}(\mathbb{R}^n)$ which is false. Therefore $0 \in \mathrm{sing.supp}(g_s)$ and the proof is concluded. $\square$

Existence of fundamental solutions

Theorem Every linear elliptic differential operator of order $m \in \mathbb{N}$ with constant coefficients,

$$p(\partial) = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha$$

has a fundamental solution $E \in \mathscr{D}'(\mathbb{R}^n)$.

A famous result by Ehrenpreiss (1954) and Malgrange (1955-56) states that any linear differential operator with constant coefficients, not identically zero, has a fundamental solution.

Hypoellipticity

Definition The differential operator $p(\partial)$ is hypoelliptic if $u \in \mathscr{D}'(\Omega)$ and $p(\partial)u \in C^\infty(\Omega)$ imply that $u \in C^\infty(\Omega)$.

Proposition If $E \in \mathscr{D}'(\mathbb{R}^n)$ is a fundamental solution for $p(\partial)$ and $\mathrm{sing.supp}(E) \subseteq \{0\}$, then $p(\partial)$ is hypoelliptic.

This follows from B4.3 Theorem 6.7.

Theorem An elliptic linear differential operator with constant coefficients admits a fundamental solution that is singularly supported in $\{0\}$, hence is hypoelliptic.

Existence of fundamental solutions: How to prove it?

A *formal* calculation using the Fourier transform indicates that

$$\text{“}E = \mathcal{F}^{-1}\left(\tfrac{1}{p(\mathrm{i}\xi)}\right)\text{”}.$$

For the approach to be feasible we would need to know that $\frac{1}{p(\mathrm{i}\xi)}$ is a tempered distribution, and at this stage this is far from clear. The problem is that the zero set for the symbol,

$$Z_p = \{\xi \in \mathbb{R}^n : \, p(\mathrm{i}\xi) = 0\}$$

could be large. It is at this point where it is useful to know the differential operator is elliptic, because it implies that Z_p is contained in a ball centered at 0 and that outside this ball the symbol $p(\mathrm{i}\xi)$ is bounded away from 0. In fact, it is bounded below by a positive multiple of $|\xi|^m$ when the order of the differential operator is m.

Existence of fundamental solutions: How to prove it?

However, even in the elliptic case we still cannot use the brute force direct approach with the Fourier transform and the theory we have developed in this course. Instead we must approach it in a more circumventional manner where the calculation with the Fourier transform is merely a guiding principle.

Using arguments from algebraic geometry it is possible to show that a *general* linear differential operators with constant coefficients (not identically zero) admits a fundamental solution that is a tempered distribution. This was done, in increasing levels of generality, in works of Hörmander in 1958, Bernstein and Gel'fand in 1969 and Atiyah in 1970.

Existence of fundamental solutions–proof [Not examinable]

Proof. Let c, $R > 0$ be the constants in the lemma characterizing elliptic operators:

$$\left|p(\mathrm{i}\xi)\right| \geq c|\xi|^m \ \text{ for } \ \xi \in \mathbb{R}^n \ \text{ with } \ |\xi| \geq R. \tag{4}$$

Consequently if $\chi = \rho * \mathbf{1}_{B_{R+1}}$, then $\chi \in \mathscr{D}(\mathbb{R}^n)$ has support in $\overline{B_{R+2}(0)}$ and $\chi = 1$ on $\overline{B_R(0)}$, so

$$\xi \mapsto \frac{1-\chi(\xi)}{p(\mathrm{i}\xi)}$$

is a tempered distribution and we can define

$$F := \mathcal{F}^{-1}\left(\frac{1-\chi(\xi)}{p(\mathrm{i}\xi)}\right) \in \mathscr{S}'(\mathbb{R}^n).$$

Note that $\mathcal{F}\big(p(\partial)F\big) = p(\mathrm{i}\xi)\widehat{F} = 1-\chi$, so

$$p(\partial)F = \mathcal{F}^{-1}\big(1-\chi\big).$$

We now need to find a feasible replacement for "$\mathcal{F}^{-1}\big(\frac{\chi(\xi)}{p(\mathrm{i}\xi)}\big)$".

Existence of fundamental solutions–proof [Not examinable]

Fix $\xi_0 \in \mathbb{S}^{n-1}$. For $\xi \in \mathbb{R}^n$ with $|\xi| \leq R+2$ and $z \in \mathbb{C}$ we have

$$|\xi + \xi_0 z| \geq |z| - R - 2.$$

Hence if we take $r \geq 2R+2$, then we infer from (4) that

$$\left| p\bigl(\mathrm{i}(\xi + \xi_0 z)\bigr) \right| \geq cR^m \tag{5}$$

for all $\xi \in \mathbb{R}^n$, $z \in \mathbb{C}$ with $|\xi| \leq R+2$, $|z| \geq r$.
Guided by the Fourier inversion formula and Cauchy's integral formula we put

$$G(\zeta) = (2\pi)^{-n} \int_{\mathbb{R}^n} \chi(\xi) \frac{1}{2\pi\mathrm{i}} \int_{|z|=r} \frac{\mathrm{e}^{\mathrm{i}(\xi + z\xi_0)\cdot\zeta}}{p(\mathrm{i}\xi + \mathrm{i}\xi_0 z)} \frac{\mathrm{d}z}{z} \, \mathrm{d}\xi \, , \; \zeta \in \mathbb{C}^n.$$

Here we used the shorthand $\mathrm{i}(\xi + \xi_0 z)\cdot\zeta := \mathrm{i}\sum_{j=1}^n (\xi_j + \xi_{0,j} z)\zeta_j$ in the argument of the exponential (note: no complex conjugation in second factor here).

Existence of fundamental solutions–proof [Not examinable]

We claim that $G\colon \mathbb{C}^n \to \mathbb{C}$ is a well-defined entire function whose derivatives can be calculated by differentiating behind the integral signs. This is an easy consequence of Lebesgue's dominated convergence theorem and the following bound that follows from (5) and $0 \leq \chi(\xi) \leq \mathbf{1}_{B_{R+2}(0)}$:

$$\left|\chi(\xi)\frac{\mathrm{e}^{\mathrm{i}(\xi+\xi_0 z)\cdot\zeta}}{p\big(\mathrm{i}(\xi+\xi_0 z)\big)z}\big(\mathrm{i}(\xi+\xi_0 z)\big)^{\alpha}\right| \leq \frac{\mathrm{e}^{(R+2)|\mathrm{Im}(\zeta)|+r|\zeta|}}{cR^m r}(R+2+r)^{|\alpha|}$$

valid for any $\xi \in \mathbb{R}^n$, $z \in \mathbb{C}$ with $|z| = r$ and $\zeta \in \mathbb{C}^n$, $\alpha \in \mathbb{N}_0^n$. Taking $\alpha = 0$ shows that $G(\zeta)$ is well-defined, taking $\alpha = e_j$ we see that G is complex differentiable with respect to the variable ζ_j, and with a general multi-index α we see that any differentiation of G can be done by differentiating behind the integral signs.

In particular the restriction of G to $\mathbb{R}^n$ is therefore a C^∞ function that can be differentiated by differentiation behind the integral signs. (Note that because of the complex variable z in the definition of G we cannot assert that G is a tempered distribution.)

Existence of fundamental solutions–proof [Not examinable]

Therefore by differentiation behind the integral signs we find

$$\begin{aligned}\big(p(\partial)G\big)(x) &= (2\pi)^{-n}\int_{\mathbb{R}^n}\chi(\xi)\frac{1}{2\pi\mathrm{i}}\int_{|z|=r}\mathrm{e}^{\mathrm{i}(\xi+\xi_0 z)\cdot x}\,\frac{\mathrm{d}z}{z}\,\mathrm{d}\xi\\ &= (2\pi)^{-n}\int_{\mathbb{R}^n}\chi(\xi)\mathrm{e}^{\mathrm{i}\xi\cdot x}\,\mathrm{d}\xi\\ &= \mathcal{F}^{-1}_{\xi\to x}\big(\chi(\xi)\big),\end{aligned}$$

where we used Cauchy's integral formula and then Fourier's inversion formula. Consequently if we define

$$E = F + G \in \mathscr{D}'(\mathbb{R}^n),$$

then

$$p(\partial)E = \mathcal{F}^{-1}\big(1-\chi\big) + \mathcal{F}^{-1}\big(\chi\big) = \mathcal{F}^{-1}\big(1\big) = \delta_0$$

and the proof is complete. □

Hypoellipticity–proof [Not examinable]

The proof is similar to the proof that the Bessel kernels are singularly supported in $\{0\}$. We have from the previous proof that $E = G + F$ and since $G \in \mathrm{C}^{\infty}(\mathbb{R}^n)$ we only need to consider

$$F = \mathcal{F}^{-1}\Big(\frac{1-\chi}{p(\mathrm{i}\xi)}\Big),$$

where we recall that $\chi = \rho * \mathbf{1}_{B_{R+1}(0)}$ and $\big|p(\mathrm{i}\xi)\big| \geq c|\xi|^m$ for $|\xi| \geq R$. The key to the proof is now to observe that if $P \in \mathbb{C}[\xi]$ is a polynomial, not identically 0, and $1 \leq j \leq n$ a direction, then for each $k \in \mathbb{N}_0$ we have

$$\partial_j^k\Big(\frac{1}{P}\Big) = \frac{Q_k}{P^{k+1}}$$

valid at all $\xi \in \mathbb{R}^n$ with $P(\xi) \neq 0$, where Q_k is a polynomial recursively determined by

$$Q_0 = 1 \ \text{ and } \ Q_k = P\partial_j Q_{k-1} - kQ_{k-1}\partial_j P$$

This is achieved by induction on $k \in \mathbb{N}_0$.

Hypoellipticity–proof [Not examinable]

The recursion for the polynomials Q_k shows that if P has degree $m \in \mathbb{N}$, then Q_k has degree at most $k(m-1)$. Consequently we have for a multi-index $\alpha \in \mathbb{N}_0^n$ by the ellipticity bound

$$\xi^\alpha \partial_j^k \widehat{F} = \mathcal{O}(|\xi|^{|\alpha|-m-k}) \text{ as } |\xi| \to \infty.$$

Given $l \in \mathbb{N}$ take $k > n+l-m$. Since then $l-m-k < -n$ we have that $\xi^\alpha \partial_j^k \widehat{F} \in \mathrm{L}^1(\mathbb{R}^n)$ for all multi-indices α of length at most l. By the differentiation rule and Riemann-Lebesgue we therefore have

$$\partial_x^\alpha \Big((-\mathrm{i}x_j)^k F \Big) = \mathcal{F}^{-1}_{\xi \to x} \Big(\xi^\alpha \partial_j^k \widehat{F} \Big) \in \mathrm{C}_0(\mathbb{R}^n),$$

so $x_j^k F \in \mathrm{C}^l(\mathbb{R}^n)$, and hence $F \in \mathrm{C}^l(\mathbb{R}^n \setminus \{x : x_j = 0\})$. Consequently, $F \in \mathrm{C}^l(\mathbb{R}^n \setminus \{0\})$ for all $l \in \mathbb{N}$ and so we have proved that $\mathrm{sing.supp}(F) \subseteq \{0\}$. As for the Bessel kernels we see that F cannot be C^∞ on $\mathbb{R}^n$ because it would mean that F is a Schwartz test function and therefore also $\widehat{F} = \frac{1-\chi}{p(\mathrm{i}\xi)}$ would have to be, which it is not. $\square$