B4.4 Fourier Analysis HT22

Lecture 15: Fourier series for tempered distributions

1. Definition of Fourier series and examples
2. Characterisation of Fourier coefficients in two cases
3. Plancherel's theorem for Fourier series

The material corresponds to pp. 53–57 in the lecture notes and should be covered in Week 8.

Recap from lecture 14 and a definition

If $u \in \mathscr{D}'(\mathbb{R})$ is 2π periodic, then it is tempered and

$$\widehat{u} = \sum_{k \in \mathbb{Z}} 2\pi c_k \delta_k \ \text{ in } \ \mathscr{S}'(\mathbb{R}) \tag{1}$$

with

$$c_k = \frac{1}{2\pi} \langle u, \Psi \mathrm{e}^{-\mathrm{i}k(\cdot)} \rangle , \quad \Psi = \frac{\chi}{P\chi} , \quad \chi = \rho * \mathbf{1}_{(-1,2\pi+1]}.$$

By the Fourier inversion formula in $\mathscr{S}'(\mathbb{R})$ we then get

$$u = \sum_{k \in \mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}kx} \ \text{ in } \ \mathscr{S}'(\mathbb{R}). \tag{2}$$

Definition The series (2) is called the Fourier series for u and the numbers c_k are called the Fourier coefficients for u.

Convergence of Fourier series for a tempered distribution

In what sense does the Fourier series (2) converge?

Definition Let $v_k \in \mathscr{S}'(\mathbb{R})$ and $v \in \mathscr{S}'(\mathbb{R})$. Then we write

$$v = \sum_{k \in \mathbb{Z}} v_k \text{ in } \mathscr{S}'(\mathbb{R})$$

provided

$$\sum_{k=-l}^{k=m} v_k \to v \text{ in } \mathscr{S}'(\mathbb{R}) \text{ as } l, m \to \infty.$$

This is the same as saying that

$$\sum_{k=1}^{l} v_{-k} \to a \text{ in } \mathscr{S}'(\mathbb{R}) \text{ as } l \to \infty,$$

$$\sum_{k=0}^{m} v_k \to b \text{ in } \mathscr{S}'(\mathbb{R}) \text{ as } m \to \infty$$

and $v = a + b$.

Fourier series for regular distributions

Example Assume $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ is 2π periodic. Then for $k \in \mathbb{Z}$:

$$\begin{aligned}
2\pi\, c_k &= \left\langle u, \Psi \mathrm{e}^{-\mathrm{i}k(\cdot)} \right\rangle = \int_{-\infty}^{\infty} u(x)\Psi(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\
&= \sum_{j\in\mathbb{Z}} \int_{2\pi j}^{2\pi(j+1)} u(x)\Psi(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\
&= \sum_{j\in\mathbb{Z}} \int_{0}^{2\pi} u(x+2\pi j)\Psi(x+2\pi j)\mathrm{e}^{-\mathrm{i}k(x+2\pi j)}\,\mathrm{d}x \\
&= \sum_{j\in\mathbb{Z}} \int_{0}^{2\pi} u(x)\Psi(x+2\pi j)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\
&= \int_{0}^{2\pi} u(x)\mathrm{e}^{-\mathrm{i}kx} P\Psi(x)\,\mathrm{d}x = \int_{0}^{2\pi} u(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x
\end{aligned}$$

Thus c_k are in this case the usual Fourier coefficients that some of you have seen in prelims.

Characterization of Fourier coefficients in two cases

Proposition Let $(c_k)_{k\in\mathbb{Z}}$ be a doubly infinite sequence of complex numbers.
(1) Then $(c_k)_{k\in\mathbb{Z}}$ are the Fourier coefficients for a 2π periodic C^∞ function if and only if, for each $m \in \mathbb{N}_0$,

$$k^m c_k \to 0 \text{ as } |k| \to \infty.$$

In this case the Fourier series converges in the C^∞ sense: the series, together with all its term-by-term differentiated series, converge uniformly.
(2) Then $(c_k)_{k\in\mathbb{Z}}$ are the Fourier coefficients for a 2π periodic distribution if and only if the sequence has moderate growth: there exist constants $C \geq 0$ and $M \in \mathbb{N}_0$ such that

$$|c_k| \leq C\left(1 + k^2\right)^{\frac{M}{2}}$$

holds for all $k \in \mathbb{Z}$.

The proof of (1) is left as an exercise and we proved (2) in lecture 14.

Example Recall that we have shown that the periodisation of a test function

$$P\phi(x) = \sum_{k\in\mathbb{Z}} \phi(x + 2\pi k)$$

gives rise to a linear map $P\colon \mathscr{S}(\mathbb{R}) \to \mathrm{C}^{\infty}_{2\pi}(\mathbb{R})$, the space of 2π periodic C^{∞} functions. By the Poisson summation formula we have

$$P\phi(x) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \widehat{\phi}(k)\mathrm{e}^{\mathrm{i}kx}.$$

Given a 2π periodic C^{∞} function f, its Fourier coefficients c_k satisfy $k^m c_k \to 0$ as $|k| \to \infty$ for any $m \in \mathbb{N}_0$. The function

$$\psi(x) = \sum_{k\in\mathbb{Z}} c_k \left(\rho_\varepsilon * \mathbf{1}_{(k-2\varepsilon,k+2\varepsilon)}\right)(x)$$

is therefore for $\varepsilon \in (0, \frac{1}{10})$ a Schwartz test function, so by the Fourier inversion formula its inverse Fourier transform is also a Schwatz test function, say ϕ. It follows that $P\phi(x) = f(x)$, so that the map P is onto.
Exercise What is the kernel of $P\colon \mathscr{S}(\mathbb{R}) \to \mathrm{C}^{\infty}_{2\pi}(\mathbb{R})$?

Plancherel's theorem for Fourier series

Theorem If $u\colon \mathbb{R} \to \mathbb{C}$ is a 2π periodic $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R})$ function with Fourier coefficients c_k, then

$$u = \sum_{k \in \mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}kx} \ \text{ in } \ \mathrm{L}^2(0, 2\pi]$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} \left| u(x) \right|^2 \mathrm{d}x = \sum_{k \in \mathbb{Z}} \left| c_k \right|^2 \tag{3}$$

The identity (3) is called Parseval's identity and can also be expressed as $\|u\|_2^2 = 2\pi \|(c_k)_{k \in \mathbb{Z}}\|_{\ell_2}^2$.
Conversely, if $\left(C_k\right)_{k \in \mathbb{Z}} \in \ell_2(\mathbb{Z})$, then

$$u = \sum_{k \in \mathbb{Z}} C_k \mathrm{e}^{\mathrm{i}kx}$$

with convergence in $\mathrm{L}^2(0, 2\pi]$ (and u is a 2π periodic $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R})$ function with Fourier coefficients C_k).

Plancherel's theorem for Fourier series–proof

Proof. Assume first that u is a 2π periodic C^∞ function. Then we have in particular that its Fourier series converges uniformly:

$$u(x) = \sum_{k \in \mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}kx} \quad \text{holds uniformly in } x \in \mathbb{R}.$$

In particular it therefore also converges in $L^2(0, 2\pi]$ and

$$\begin{aligned}
\int_0^{2\pi} \left| u(x) \right|^2 \mathrm{d}x &= \int_0^{2\pi} \sum_{k,l \in \mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}kx} \overline{c_l \mathrm{e}^{\mathrm{i}lx}} \, \mathrm{d}x \\
&= \sum_{k,l \in \mathbb{Z}} c_k \overline{c_l} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(k-l)x} \, \mathrm{d}x \\
&= 2\pi \sum_{k \in \mathbb{Z}} \left| c_k \right|^2.
\end{aligned}$$

Plancherel's theorem for Fourier series–proof

Next, we consider the general case where $u\colon \mathbb{R} \to \mathbb{C}$ is 2π periodic and $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R})$. Put $u_t = \rho_t * u$, where $\big(\rho_t\big)_{t>0}$ is the standard mollifier on $\mathbb{R}$. Then u_t is a 2π periodic C^∞ function and

$$\int_0^{2\pi} \big| u - u_t \big|^2 \, \mathrm{d}x \to 0 \text{ as } t \searrow 0.$$

Now for each $t > 0$ the Fourier series of u_t converges uniformly, say

$$u_t(x) = \sum_{k \in \mathbb{Z}} c_k(t) \mathrm{e}^{\mathrm{i}kx} \text{ uniformly in } x \in \mathbb{R}.$$

It is not difficult to see that $c_k(t) \to c_k$ as $t \searrow 0$ for each $k \in \mathbb{Z}$.
We clearly also have for $s, t > 0$ that $u_s - u_t$ is a 2π periodic C^∞ function with Fourier coefficients $c_k(s) - c_k(t)$ and according to what we just proved,

$$\int_0^{2\pi} \big| u_s - u_t \big|^2 \, \mathrm{d}x = 2\pi \sum_{k \in \mathbb{Z}} \big| c_k(s) - c_k(t) \big|^2.$$

Plancherel's theorem for Fourier series–proof

Because $\left(u_t\right)_{t>0}$ is Cauchy in $\mathrm{L}^2(0,2\pi]$ as $t \searrow 0$, also $\left(c_k(t)\right)_{k\in\mathbb{Z}}$ is Cauchy in $\ell_2(\mathbb{Z})$ as $t \searrow 0$. But the latter is complete by the Riesz-Fischer theorem so for some $\left(a_k\right)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ we have

$$\left\|\left(c_k(t)\right) - \left(a_k\right)\right\|_{\ell_2} \to 0 \text{ as } t \searrow 0.$$

It follows that $c_k = a_k$ for all $k \in \mathbb{Z}$, hence that $\left(c_k\right)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ and that

$$\begin{aligned}
\int_0^{2\pi} \left|u\right|^2 \mathrm{d}x &= \lim_{t\searrow 0} \int_0^{2\pi} \left|u_t\right|^2 \mathrm{d}x \\
&= \lim_{t\searrow 0} 2\pi \sum_{k\in\mathbb{Z}} \left|c_k(t)\right|^2 \\
&= 2\pi \sum_{k\in\mathbb{Z}} \left|c_k\right|^2.
\end{aligned}$$

Plancherel's theorem for Fourier series–proof

Finally in order to see that we also have convergence in $\mathrm{L}^2(0,2\pi]$ we consider for m, $n \in \mathbb{N}$:

$$\int_0^{2\pi} \left| u(x) - \sum_{k=-m}^{k=n} c_k \mathrm{e}^{\mathrm{i}kx} \right|^2 \mathrm{d}x = \int_0^{2\pi} |u(x)|^2 \, \mathrm{d}x - 2\pi \sum_{k=-m}^{k=n} \left| c_k \right|^2 \to 0$$

as m, $n \to \infty$. This concludes the proof in one direction.
Concersely suppose $\left(C_k \right)_{k \in \mathbb{Z}} \in \ell_2(\mathbb{Z})$, that is, $C_k \in \mathbb{C}$ and $\sum_{k \in \mathbb{Z}} \left| C_k \right|^2 < \infty$. Define

$$u(x) = \sum_{k \in \mathbb{Z}} C_k \mathrm{e}^{\mathrm{i}kx}.$$

Clearly the sequence $\left(C_k \right)_{k \in \mathbb{Z}}$ is in particular of moderate growth, so by an earlier result $u \in \mathscr{S}'(\mathbb{R})$ is a 2π periodic distribution with Fourier coefficients C_k. By the previous part of the proof it follows that the convergence is in $\mathrm{L}^2(0,2\pi]$ and so that $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R})$. $\square$