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C7.5: General Relativity 1

University of Oxford: Part C

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Disclaimer: There are almost certainly typos in the notes, if something does not look correct or needs further explanation please let me know.

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Contents

Recommended books and resources

There are a large variety of good textbooks and lecture notes on general relativity. This course borrows from a number of them, in various different places. An assortment of textbooks that have been used in writing these notes are:

- Wald, General Relativity

A very thorough introduction to the subject.

- Weinberg, Gravitation and cosmology

- Carroll, An introduction to general relativity, spacetime and geometry.

Aimed more at particle physicists. We will follow this in the cosmology section and borrow bits for elsewhere.

- Hartle, Gravity, an introduction to Einstein's general relativity

- Misner, Thorne and Wheeler, Gravitation

It is a very big book.

- Nakahara, Geometry, Topology and Physics

An excellent book for learning about geometry and topology and will be useful for the differential geometry section of the notes.

There are also a number of useful lecture notes online. In particular:

- Joe Kier's lecture notes from 2020

- David Tong's lecture notes

- Sean Carroll's lecture notes

- Harvey Reall's lecture notes

Conventions

- We will use the god-given signature convention of mostly plus $(-,+,+,+)$. This may differ with the convention you have used in other courses, especially field theory courses. This convention is preferable when thinking about geometry as it gives positive spatial distances. For quantum field theory the other convention is preferable since it ensures that energies and frequencies are positive. You may map between the two conventions through *Wick rotation*, essentially allowing the coordinates to become complex.

- Spacetime indices will be taken to be greek letters from the middle of the alphabet: $\mu, \nu, \rho, ...$ and run over $0, 1, 2, 3$. Latin indices $i, j, k, ..$ run over the spatial directions and take values $1, 2, 3$.

- We employ Einstein summation convention, repeated indices are summed over, unless otherwise stated.

- We work in units where the speed of light c is set to 1. Occasionally it is instructive to reintroduce c which can be done by dimensional analysis.

- The Minkowski metric will be denoted by $\eta_{\mu\nu} = \text{diagonal}(-1, 1, 1, 1)_{\mu\nu}$.

- After introducing curvature we will take the metric to be $g_{\mu\nu}$ and the determinant will be $\det(g_{\mu\nu}) \equiv g$.

Useful formulae

- The Lagrangian for the geodesic equation of a massive test particle is

$$\mathcal{L}\Big(\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}, x^\mu\Big) = \sqrt{-g_{\mu\nu}(x)\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\,,$$

  with λ an arbitrary parameter along the worldline.

- The geodesic equation for a massive particle is

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = 0\,, \qquad g_{\mu\nu}(x)\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = -1\,,$$

  where τ is the proper time. For light, the first equation takes the same form just replacing τ with an affine parameter. The second is modified by $-1 \to 0$.

- The Christoffel symbols (Levi–Civita connection) are

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \Big(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho} \Big) \, .$$

- The Riemann tensor is

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\rho\lambda} \Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\sigma\lambda} \Gamma^{\lambda}{}_{\nu\rho} \, .$$

  - Symmetries

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} \, ,$$
$$R_{\mu\nu\rho\sigma} = R_{\sigma\rho\mu\nu} \, .$$

  - Bianchi identity 1

$$R^{\mu}{}_{\nu\rho\sigma} + R^{\mu}{}_{\rho\sigma\nu} + R^{\mu}{}_{\sigma\nu\rho} = 0 \, .$$

  - Bianchi Identity 2

$$\nabla_\mu R^{\sigma}{}_{\lambda\nu\rho} + \nabla_\nu R^{\sigma}{}_{\lambda\rho\mu} + \nabla_\rho R^{\sigma}{}_{\lambda\mu\nu} = 0 \, .$$

- Ricci tensor

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$$

- Ricci scalar

$$R = R_{\mu\nu} g^{\mu\nu} \, .$$

- Einstein tensor

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \, .$$

- Einstein–Hilbert action plus cosmological constant,

$$S = \frac{1}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g} \Big(R + \Lambda \Big) \, .$$

- Under a variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ we have

$$\delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma} \, ,$$
$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} \, ,$$
$$\delta R_{\mu\nu} = \nabla_\rho \delta \Gamma^{\rho}{}_{\mu\nu} - \nabla_\mu \delta \Gamma^{\rho}{}_{\rho\nu} \, .$$

1 Introduction

Gravity is one of the four[1] fundamental forces alongside electromagnetism, the strong nuclear force and the weak nuclear force. Of these forces gravity is by far the weakest force, the ratio of the gravitational force to electric force acting on an electron is 10^{-36}.[2] Despite this gravity plays a dominant role in shaping the large scale structure of the universe, this is because the strong and weak forces have a very short range, while, though electromagnetism is a long range force it is both attractive and repulsive and for bodies of macroscopic dimensions the repulsion of like charges is approximately balanced by the attraction of oppositely charges. On the other hand, gravity is only an attractive force, thus for sufficiently large bodies the gravitational field of the sum of all its constituents adds up to become the dominant force.

The leading candidate for a theory of gravity for some time was Newton's theory of gravitation. This however, is a non-relativistic theory of gravity and therefore is incompatible with special relativity: it is not invariant under Lorentz transformations. One can see this by thinking about what would happen if the sun suddenly disappeared. For 8 minutes, the time it takes for light to travel from the sun to Earth, we would be completely oblivious. This is because special relativity tells us that no signal can travel faster than light: the Earth must continue on its orbit for these 8 minutes, after which, it is flung out of the solar system leading to almost certain death for all life on Earth. However, Newton's theory of gravity acts instantaneously, we would be flung out of the solar system immediately. In Newton's theory, the force on one mass depends on the location of the other mass at the same time.

Einstein's breakthrough lead to a conceptual revolution in the way that we view spacetime. The fact that objects with the same initial conditions travel along the same curve, independent of their mass, hints that the curve that is followed is a property of the geometry of spacetime rather than a force acting on the body. General relativity (GR) understands gravity as the curvature of spacetime and the trajectories within spacetime as geodesics on this curved space. Or as John Wheeler once said, *"Mass tells space how to curve, while curved space tells matter how to move".*

The aim of this course is to introduce you to General relativity and by the end of it to allow you to perform calculations. Among other topics we will see how gravity bends light, the corrections to the motion of the planets and some cosmology. This is a large topic and

[1] One should probably add *currently known to physics* at this point.

[2] You can see this very clearly by holding two magnets together, gravity is not strong enough to pull one magnet to the floor. A current research direction conjectures that in any quantum theory of gravity, the strength of gravity is weaker relative to any other gauge force [1].

we will therefore omit many interesting directions, but this will lay the foundation for further study and for the follow up course general relativity II.

The notes are organised as follows. We begin by reviewing special relativity and Newtonian gravity in section 2. To understand general relativity properly we need to understand the underlying geometry of spacetime. This requires knowledge of the sophisticated tools of *differential geometry* to describe curved spacetime, which we will study in sections 3 and 4. With these new tools we are finally in a position to introduce Einstein's equations and physics in curved spacetime in section 5. The Schwarzschild solution is the go to solution of general relativity and we will use it as a testing ground for studying many interesting topics in GR including black holes, the motion of the planets and the bending of light in section 6. Our final chapter will be on the large scale structure of the universe with a trek through the world of cosmology.

2 Special relativity and Newtonian gravity

We begin with a whirlwind exploration of special relativity. This section is by no means meant to be an introduction to special relativity, more a refresher on the subject and to emphasise the pertinent points. For example we will immediately assume that the reader is familiar with the Einstein summation convention, that is repeated indices are summed over and one should be an 'up' index and the other a 'lower' index. For readers in need of a more thorough introduction there are a number of excellent texts to consult. For example the notes by Joe Minahan, the book 'Special Relativity: An Introduction with 200 Problems and Solutions' by Michael Tsamparlis and Bernard Schutz's book, 'A First Course in General Relativity'.

By the end of the 18th century two areas of physics that were in conflict had emerged: Newtonian mechanics and Electromagnetism. Newtonian mechanics has a notion of absolute time and the equations of motion are invariant under Galilean coordinate transformations. The transformation law between two coordinate systems, the latter moving at a uniform speed v in the x direction of the former, is

$$(t', x', y', z') = (t, x - vt, y, z)\,. \tag{2.1}$$

Galilean transformations imply that the speed of light should change in different reference frames moving with respect to each other. This is incompatible with Maxwell's equations describing electromagnetism where the speed of light is fixed. A resolution to this problem was proposed by conjecturing a preferred frame, the frame of the physical medium in which light propagates, called the *Ether*. The speed of light in any other rest frame would then be modified by the Newtonian addition of velocities. An experiment by Michelson and Morley in 1887 to detect the Ether failed, the speed of light does not satisfy the Newtonian law of addition of velocities. Either Newtonian mechanics or Maxwell's equations required modification.

2.1 Special relativity

Einstein gave the resolution to this problem in 1905 with the introduction of special relativity. In special relativity a key role is played by the so-called *inertial reference frame*. Such a frame satisfies three key properties:

1. There is a universal time coordinate which can be synchronised everywhere in the inertial frame.

2. The spatial slices are Euclidean space, satisfying the usual Euclidean axioms.

3. A body with no external forces will move with constant velocity within the frame with respect to the clocks and measuring sticks of the frame.

We can construct an infinite number of such frames and we need to understand how to map between them. This is where Einstein's two postulates for special relativity are needed:

1. The laws of physics in any inertial reference frame are identical.

2. The speed of light in a vacuum in two different reference frames does not change and is given by [3] $c = 299782458\text{m s}^{-1}$.

> For the second postulate it is important to note the word vacuum. The speed of light through different mediums is less than c, this is why we have refraction when light travels through glass for example. In [2] they even managed to stop light propagating temporarily before letting it propagate again.

The group of spacetime coordinate transformations which map inertial reference frames to inertial reference frames are the Lorentz transformations. One may then rephrase the principle of special relativity to be that the laws of nature are invariant under Lorentz transformations. This requires the abandonment of the Newtonian idea of absolute time. Events which are simultaneous in one inertial reference frame need not be simultaneous in another frame (see problem sheets 0 and 1 for examples to work through).

2.1.1 Lorentz transformations

A Lorentz transformation is a linear transformation from one spacetime coordinate system $x^\mu = (c\,t, x, y, z)$ to another x'^μ, of the form

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \,, \tag{2.2}$$

where Λ is a constant matrix, i.e. spacetime independent, and satisfies

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma} \,. \tag{2.3}$$

The matrix η is the famed Minkowski metric, which in our signature, is given by

$$\eta_{\mu\nu} = \text{diag}(-1,1,1,1)_{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\mu\nu} \,. \tag{2.4}$$

[3] There is a nice mnemonic (the number of letters in each word is the value here) to remember this: *We guarantee certainty clearly referring to this light mnemonic.*

The set of matrices satisfying (2.3) is the group $\mathrm{O}(1,3)$.[4] We could also add in constant shifts of the coordinates, $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$, with a^μ a constant four-vector. This would enhance the Lorentz group to the Poincaré group. For our purposes we will only need to consider the Lorentz group though and so we set $a^\mu = 0$ from now on.

From the definition of the Lorentz group it follows that it leaves the line element, (sometimes also called length element, invariant interval)

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \,, \tag{2.5}$$

invariant.[5] This is merely the proper distance between the two spacetime events: (t, x, y, z) and $(t+\mathrm{d}t, x+\mathrm{d}x, y+\mathrm{d}y, z+\mathrm{d}z)$. Here, d stands for an infinitesimal displacement, you also see δ and Δ to mean the same thing.

Aside: The group $\mathrm{O}(1,3)$ described above is sometimes called the *homogeneous Lorentz group*. It admits a proper subgroup defined by imposing

$$\Lambda^0{}_0 \geq 1 \,, \quad \det\Lambda = 1 \,. \tag{2.6}$$

The proper subgroup restricts to all transformations which can be smoothly joined to the identity. The *improper* Lorentz transformations involve either space inversion $\det\Lambda = -1\,,\ \Lambda^0{}_0 \geq 1$, or time reversal $\det\Lambda = 1\,,\ \Lambda^0{}_0 \leq 1$. Space and time inversions are known not to be exact symmetries of nature and therefore when we say Lorentz transformation what we really mean is the proper Lorentz transformations.

We now want to understand what types of transformations the proper Lorentz group admits. There is a further subgroup consisting of spatial rotations taking the form:

$$\Lambda^0{}_0 = 1 \,, \quad \Lambda^0{}_i = \Lambda^i{}_0 = 0 \,, \quad \Lambda^i{}_j = R_{ij} \,, \tag{2.7}$$

with R an $\mathrm{SO}(3)$ matrix: $RR^T = 1\,,\ \det R = 1$. The remaining transformations are known as *boosts* which mix the space and time directions. Examples of the two types of transformation are[6]

$$\Lambda^{\text{Rotation}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad \Lambda^{\text{Boost}} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.8}$$

[4] More generally one could consider $\mathrm{O}(p,q)$ which satisfy (2.3) but now with η having p -1's and q $+1$'s.

[5] One can show that the Lorentz transformations are the only non-singular coordinate transformations that leave $\mathrm{d}s^2$ invariant. Here non-singular means that both $x'(x)$ and $x(x')$ are well behaved differential functions and thus $\frac{\partial x^\mu}{\partial x'^\nu}$ has an inverse. When we consider $\mathrm{d}s^2 = 0$ there is an enhancement of the symmetry group. You will show this in problem sheet 1.

[6] Note that we only work with the proper Lorentz group.

The first is a rotation in the x, y directions and the second is a boost in the x direction. The rotation parameter is compact $\theta \in [0, 2\pi)$ while the boost parameter, known as the *rapidity* is non-compact $\phi \in (-\infty, \infty)$. Altogether the Lorentz group has six parameters, split evenly between boost and rotations. Rotations commute amongst themselves but do not commute with boosts, thus the Lorentz group is non-abelian.

Rather than considering the matrices Λ in (2.8) it is useful to consider their generators.[7] These are matrices T^a which satisfy

$$\Lambda = \mathrm{e}^{\mathrm{i}\theta_a T^a}\,, \tag{2.9}$$

where θ_a are constant parameters. The generators for the two matrices appearing in (2.8) are

$$T^{\text{Rotation}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad T^{\text{Boost}} = \begin{pmatrix} 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.10}$$

with similar expressions for the other generators.

Exercise 2.1: Addition of rapidity

1. Compute the addition of the rapidity under two successive boosts along the x axis.
2. Show that the generators in (2.10) give the matrices in equation (2.8)
3. Compute the commutator of the generators ((2.10)) of a boost along x and y.
4. Compute the commutator of the generators ((2.10)) for a boost along x and rotation in the x-y-plane.
5. Compute the commutator of the generators ((2.10))of a boost along x and rotation in the y-z-plane.

The interpretation of the rotations is clear from our understanding of Galilean symmetries but what is the interpretation of the boosts? You may not be surprised but this corresponds to changing coordinates to that of a frame moving with a constant velocity with respect to the first.

Under a boost in the x-direction the transformed coordinates are

$$t' = t\cosh\phi - x\sinh\phi\,, \quad x' = -t\sinh\phi + x\cosh\phi\,. \tag{2.11}$$

[7] We are considering the Lie Algebra of the Lie Group.

The point $x' = 0$ is then moving, as viewed from the original frame, with velocity

$$v = \frac{x}{t} = \tanh\phi\,. \tag{2.12}$$

Motivated by this it is useful to perform the replacement $\phi = \operatorname{arctanh} v$ in the transformations to obtain

$$\begin{aligned} t' &= \gamma(t - vx)\,, \\ x' &= \gamma(x - vt)\,, \end{aligned} \qquad \text{with} \quad \gamma = (1 - v^2)^{-1/2}\,. \tag{2.13}$$

Exercise 2.2: Addition of velocity

Compute the addition of the velocity under two successive boosts along the x axis. You should contrast this with the addition of rapidities.

Understanding these transformation lead to a whole range of interesting phenomena from time dilation to length contraction. *In problem sheet 0 and 1 you will review some of these problems. There will also be some worked examples given in these notes in section 2.4.*

2.1.2 Causal structure and worldlines

The invariance of the proper distance between spacetime events allows us to make the following definition:

Definition 1 (Timelike, Null, Spacelike) *The interval between two spacetime events x^μ and $x^\mu + \mathrm{d}x^\mu$ is*

$$\begin{aligned} \textit{Timelike separated if} \quad & \mathrm{d}s^2 < 0\,, \\ \textit{Null or Lightlike separated if} \quad & \mathrm{d}s^2 = 0\,, \\ \textit{Spacelike separated if} \quad & \mathrm{d}s^2 > 0\,. \end{aligned} \tag{2.14}$$

This can be visualised by studying a spacetime diagram. Focussing only on the t, x coordinates and suppressing the y, z directions we may associate to every point p in spacetime a light cone, see figure 1. For simplicity let the point p be the origin, this is of course just a shift in the coordinate, remember our a^μ's that we removed earlier. Then the path light takes passing through the point p is given by $x = \pm t$. These straight lines moving at the speed of light define the light cone. Points lying on the cone are *null/lightlike separated* from our point p. We can divide the light cone into two cones, the future and past light cones. Those points residing in the light cones are *timelike separated* while those outside the light cones are *spacelike separated*. To fully complete the diagram into a cone we should add in the y, z

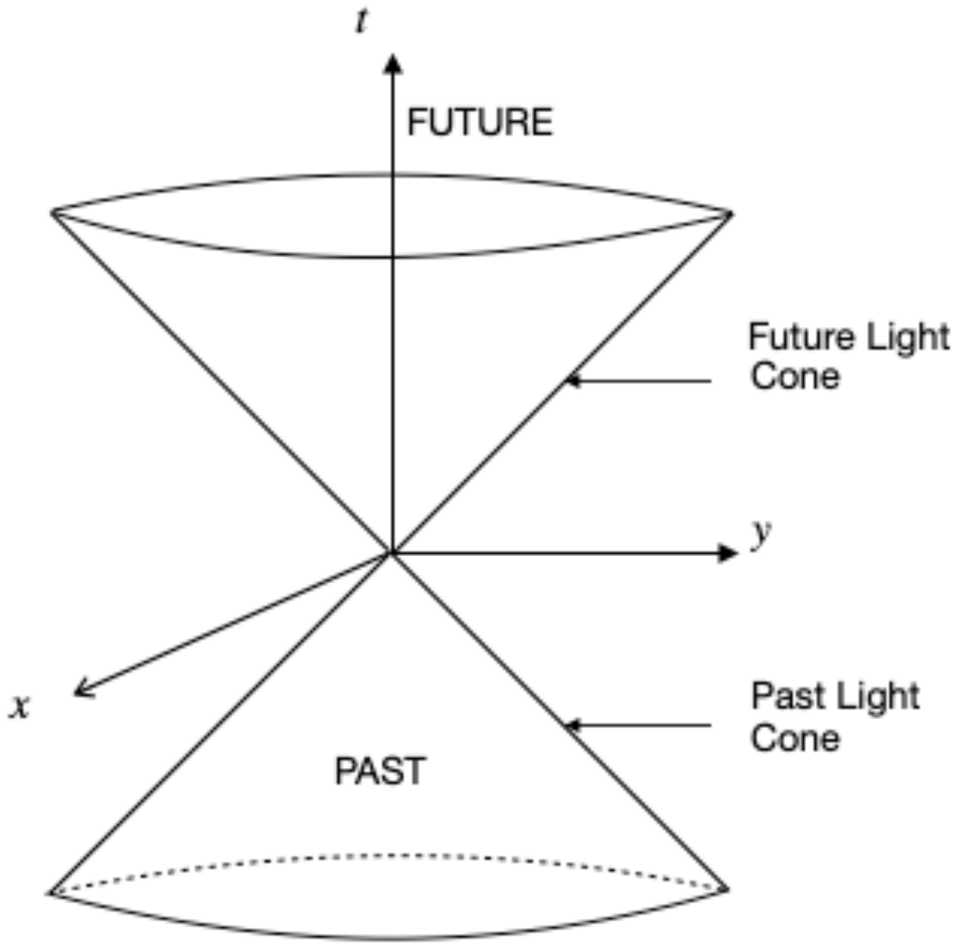


Figure 1: The lightcone diagram. The pink areas are time-like separated from the point at the centre, while points in the blue area are space-like separated. The dotted lines are light-like separated.

directions, from which we can see the conical structure. In figure 1 we have sketched the light cone in the t, x-plane

There is a slight subtlety about this diagram which we should point out. If we perform a boost of out coordinates in the x-direction and map this onto the light-cone it seems that the light-cone begins to close up, see figure 1. This is a Euclidean viewpoint however, whereas this properly resides in Lorentzian space. From a Lorentzian viewpoint the axes remain orthogonal and it is not difficult to see that the lines $t = \pm x$ get mapped precisely to the lines $t' = \pm x'$. Thus, we see that our classification in terms of timelike, null and spacelike separated points is also preserved under Lorentz transformations from this viewpoint.

Note that when we are looking at the distance between points we are drawing straight lines between the points and computing the distance of this line. This works in Minkowski space but when we introduce curved spacetime this is no longer correct since the notion of a straight line in this sense ceases to exist. We should really think about computing the tangent of a path between the two points, for straight lines the tangent lies along the curve but for more general curves this is no longer true. Let us define a path γ in our spacetime and parametrise it by $\lambda \in [\lambda_1, \lambda_2]$. We may choose local inertial coordinates so that we may define our curve as $x^\mu(\lambda)$, note that λ is arbitrary and need not be identified with the time coordinate. We can then compute the tangent to the curve, $\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$. For a spacelike curve, that is one which for every infinitesimal interval it is spacelike, we define the (change of) *proper length* to be

$$\Delta s = \int_{\lambda_1}^{\lambda_2} \sqrt{\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \mathrm{d}\lambda \, . \tag{2.15}$$

There is no such analogue along a null curve since $\mathrm{d}s^2 = 0$. For a timelike curve, one in which the infinitesimal intervals are all time like, we define the *proper time* τ via:

$$\Delta \tau = \int_{\lambda_1}^{\lambda_2} \sqrt{-\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \mathrm{d}\lambda \, , \tag{2.16}$$

where this is understood to be the change in proper time after following the curve from $x^\mu(\lambda_1)$ to $x^\mu(\lambda_2)$. One may worry that this depends on the parametrisation of the curve, however it is a simple exercise (exercise 2.3) to show that the above is independent of the parametrisation of the curve.

Exercise 2.3: Reparamterisation invariance

Show that the definition of the change in proper time in equation 2.16 is independent of the parametrisation of the curve.

The proper time is useful because of the *clock postulate*.

Clock Postulate *An accurate clock moving along a timelike worldline measures the proper time along the worldline.*

This point of view makes the "twin paradox" and similar puzzles clear. Two worldlines which have two intersections at different events will have proper times which measure their respective proper times, however these numbers in general will be different since the paths are different.

It is often convenient to parametrise a timelike curve by the proper time since for a timelike path we have:

$$\eta_{\mu\nu}\frac{\mathrm{d}x^\mu(\tau)}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu(\tau)}{\mathrm{d}\tau} = -1\,. \tag{2.17}$$

Exercise 2.4: Parametrising a timelike curve

Show that it is always possible to find a parametrisation of a timelike curve so that it satisfies (2.17), and moreover that it is unique up to constant shifts.

Massive paths Let us now consider the worldlines of massive particles, these follow timelike paths. From our earlier discussion we will use the proper time as the parameter along the path with the path starting at $\tau = 0$ for simplicity. The tangent vector is known as the *four-velocity* U^μ:

$$U^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\,. \tag{2.18}$$

This is automatically normalised, $\eta_{\mu\nu}U^\mu U^\nu = -1$ since we parametrised the curve using the proper time! We may define the *energy-momentum four-vector* as

$$p^\mu = mU^\mu\,, \tag{2.19}$$

with m the mass of the particle. The mass is a fixed quantity independent of inertial frame, this is what you may have been used to calling the rest mass. The energy is defined simply to be p^0, and as one component of a four-vector is not invariant under Lorentz transformations. Note that in the particle's rest frame we have $p^0 = m$ (recall $c = 1$) and so this is the celebrated $E = mc^2$. Note that the energy in the rest frame is the norm of the energy momentum four vector. In a general frame we have

$$E^2 - p^i p_i = m^2\,, \tag{2.20}$$

which is the general version of Einstein's famous formula.

2.1.3 Some more formal aspects: vectors, one-forms and tensors

Vectors and Vector fields To probe the structure of Minkowski space it is necessary to introduce the concepts of vectors and tensors. We will give a full treatment of this subject later in section 3 introducing only the necessary notation for the moment. You may be used to thinking of a vector as something stretching from one point to another and which can be freely moved around. In relativity this is no longer true and so we must be more careful by what we mean by a vector.

To each point p in spacetime we associate the set of all possible vectors located at that point. A useful class of vectors to consider are the tangent vectors to curves going through the point p. In a n-dimensional spacetime there are n independent such vectors and they span a *vector space* called the *tangent space* at p, and denoted by T_p.

Definition 2 (Vector space) *A vector space, V is a set of elements, $v \in V$, which we call vectors, that may be added together or multiplied by elements of a field, $\mathbb{F}$ (e.g real or complex numbers). The operations of the vector addition and multiplication must satisfy the following axioms:*

1. *Associativity*
$$u + (v + w) = (u + v) + w\,, \qquad \forall u, v, w, \in V\,, \tag{2.21}$$

2. *Commutativity*
$$u + v = v + u \quad \forall u, v, \in V\,, \tag{2.22}$$

3. *Identity element. There exists a $0 \in V$ such that*
$$v + 0 = v\,, \quad \forall v \in V\,, \tag{2.23}$$

4. *Inverse elements. For every $v \in V$ there exists a $-v \in V$ such that*
$$v + (-v) = 0 = (-v) + v\,, \tag{2.24}$$

5. *Compatability of scalar multiplication with field multiplication.*
$$a(bv) = (ab)v\,, \quad \forall\ a, b \in \mathbb{F}\,,\ \textit{and}\ v \in V \tag{2.25}$$

6. *Identity element of scalar multiplication. There exists a $1 \in \mathbb{F}$ such that*
$$1v = v\,, \quad \forall v \in V\,, \tag{2.26}$$

7. *Distributivity of scalar multiplication.*
$$a(u + v) = au + av\,, \quad \forall a \in \mathbb{F}\,,\ \textit{and}\ u, v \in V\,, \tag{2.27}$$

8. *Distributivity or scalar multiplication.*
$$(a + b)u = av + bv\,, \quad \forall a, b \in \mathbb{F}\ \ \textit{and}\ v \in V \tag{2.28}$$

A vector is a perfectly well-defined geometric object defined at the point p. We may also define a *vector field* to be a set of vectors with exactly one defined at each point in spacetime. The set of all the tangent spaces T_p of a manifold[8] M is known as the *tangent bundle* $T(M)$. This is a $2n$-dimensional manifold which is an example of a fiber bundle. It is important to emphasise that neither the vector nor the vector field transform under Lorentz transformations.

It is often useful to decompose vectors into components in terms of some basis of the tangent space. Recall that a *basis* is a set of vectors which both spans the vector space and is linearly independent. There are an infinite number of possible bases, but each will have the same number of basis elements, the dimension of the manifold here. Let us imagine that at every point we set up a basis with four vectors $\hat{e}_\mu$. The any vector V can be expanded in terms of this basis as $V = V^\mu \hat{e}_\mu$. Here V^μ are known as the *components of the vector.* This is sometimes sloppily called a vector or contravariant vector, however this is not correct, V is the vector and V^μ are the components of the vector.

A standard example of a vector in spacetime, and one that will appear frequently, is the tangent to a curve. We can specify a curve by specifying coordinates in terms of a parameter, $x^\mu(\lambda)$. The tangent vector has components

$$V^\mu = \frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda}\,, \tag{2.29}$$

and the vector is

$$V = V^\mu \hat{e}_\mu\,. \tag{2.30}$$

Under a Lorentz transformation the coordinates transform according to (2.2), and from this we may deduce the transformation of the components of the four-vector V^μ, when the coordinate system is transformed as in (2.2), to be

$$V^\mu \to V'^\mu = \Lambda^\mu{}_\nu V^\nu\,. \tag{2.31}$$

Since the vector itself does not change under Lorentz transformations, and the parametrisation with λ is unaltered it follows that the basis vectors transform according to

$$\hat{e}_\mu = \Lambda^\nu{}_\mu \hat{e}'_\nu\,. \tag{2.32}$$

This is just multiplication by the inverse of the Lorentz transformation which transforms the coordinates, therefore

$$\hat{e}'_\mu = \Lambda_\mu{}^\nu \hat{e}_\nu\,. \tag{2.33}$$

[8] We will define a manifold later in section 3.

To summarise, we have introduced a set of coordinates labelled by upper indices which transform in a certain way under Lorentz transformations. We then considered vector components with upper indices which transformed in the same way as the coordinates. The basis vectors associated with the coordinate system transformed via the inverse matrix and were labelled by a lower index. These transformations leave invariant the vector, that is summing over the vector components with the basis vectors.

Co-vectors and one-forms Once we have a vector space we can define an associated vector space known as the *dual vector space*. It is usually denoted with an asterisk, so that the dual vector space of the Tangent space T_p is T_p^*. The dual space is the space of all linear maps from the original vector space to the real numbers, so that if $\omega \in T_p^*$ then by definition of a linear map[9]

$$\omega(aV + bW) = a\omega(V) + b\omega(W) \in \mathbb{R}\,, \tag{2.34}$$

for all $V, W \in T_p$ and $a, b \in \mathbb{R}$. It follows that T_p^* is a vector space itself and since it is finite dimensional its dual vector space is T_p. We may introduce a basis of dual vectors $\hat{\theta}^\mu$ by fixing

$$\hat{\theta}^\mu(\hat{e}_\nu) = \delta^\mu_\nu\,. \tag{2.35}$$

Every dual vector can be written in components in terms of this basis as

$$\omega = \omega_\mu \hat{\theta}^\mu\,. \tag{2.36}$$

Typically one refers to the elements of T_p as *contravariant* four vectors and elements of T_p^* as *covariant vectors*, or even *one-forms*, (a name that will make more sense after we have introduced differential geometry in section 3). The set of all cotangent spaces over M is called the *cotangent bundle* $T^*(M)$. The action of a dual vector field on a vector field is no longer a single number but a *scalar*, depending on the spacetime position. A scalar has no indices and is left invariant under Lorentz transformations.

The component notation is useful when considering the action of a dual vector on a vector:

$$\omega(V) = \omega_\mu V^\nu \hat{\theta}^\mu(\hat{e}_\nu) = \omega_\mu V^\nu \delta^\mu_\nu = \omega_\mu V^\mu\,. \tag{2.37}$$

Since the action of a co-vector on a vector is a constant it is invariant under Lorentz transformations and we must have

$$\omega'_\mu V'^\mu = \omega'_\mu \Lambda^\mu{}_\sigma V^\sigma \equiv \omega_\mu V^\mu\,. \tag{2.38}$$

[9] We require a linear map, $f : V \to W$ to be additive $f(u+v) = f(u)+f(v)$ for all $u, v \in V$ and homogeneous of degree 1 so that $f(cu) = cf(u)$ for all $u \in V$ and $c \in \mathbb{F}$.

If we were dealing with vector and co-vector fields it would be a *scalar*. It is from here that we can obtain the transformation of the dual vector: a *covariant* four-vector is a quantity which transforms as

$$\omega_\mu \to \omega'_\mu = \Lambda_\mu{}^\nu \omega_\nu \,, \tag{2.39}$$

where

$$\Lambda_\mu{}^\nu \equiv \eta_{\mu\rho}\eta^{\nu\sigma}\Lambda^\rho{}_\sigma \,, \tag{2.40}$$

with $\eta^{\mu\nu}$ the inverse of $\eta_{\mu\nu}$, which are numerically the same.

Exercise 2.5: The inverse of the Lorentz transformation
Using the properties of the Lorentz transformation show that $\Lambda_\mu{}^\nu$ is the inverse of $\Lambda^\mu{}_\nu$.

The simplest example of a co-vector is the *gradient* of a scalar function,

$$\mathrm{d}\phi = \frac{\partial\phi}{\partial x^\mu}\hat{\theta}^\mu \,. \tag{2.41}$$

You can see that the components indeed transform in the correct way.

Observe that the metric η can be used to raise and lower indices. This allows us to transform a contravariant vector into a co-vector and vice-versa. The metric then acts as a map from $T_p(M) \to T_p^*(M)$, while the inverse metric acts as a map from $T_p^*(M) \to T_p(M)$.

Exercise 2.6: Raising and lowering indices

- Show that $\eta_{\mu\nu}V^\nu$ transforms as the components of a co-vector if V^ν transforms as the components of a vector.
- Show that $\eta^{\mu\nu}V_\nu$ transforms as the components of a vector if V_ν transforms as the components of a co-vector.

Note that because of the map between contravariant and covariant vectors via the Minkowski metric we can define an *inner product* on two vectors as

$$\eta(V, W) = \eta_{\mu\nu}V^\mu W^\mu \,. \tag{2.42}$$

Two vectors whose inner product vanishes are called *orthogonal*. Since it is a scalar the dot product is left invariant under Lorentz transformations and therefore orthogonality is basis and frame independent. We can define the *norm* of a vector to be the inner product with itself. Unlike in Euclidean geometry this is not positive definite, instead

$$\text{if } \eta_{\mu\nu}V^\mu V^\nu \text{ is } \begin{cases} < 0\,, & V^\mu \text{ is timelike}\,, \\ = 0\,, & V^\mu \text{ is lightlike or null}\,, \\ > 0\,, & V^\mu \text{ is spacelike}\,, \end{cases} \tag{2.43}$$

This is the more mathematical definition of these concepts from our earlier discussion.

Tensors One may extend the notion of a vector and co-vector to a *tensor*. A tensor of type (rank) (k,l), is a multilinear map from a collection of dual vectors and vectors to $\mathbb{R}$:

$$T: \quad T_p^* \times ... \times T_p^* \times T_p \times .. \times T_p \to \mathbb{R}\,, \tag{2.44}$$

for example a scalar is a tensor of rank (0,0), a co-vector is a rank $(0,1)$ tensor and a contravariant vector is a tensor of rank $(1,0)$. The space of all tensors of a fixed rank (k,l) forms a vector space. To construct a basis for this space it is useful to define the *tensor product* $\otimes$. If T is a (k,l)-tensor and S a (m,n)-tensor then $T\otimes S$ is a $(k+m,l+n)$ tensor defined to be

$$\begin{aligned} &T\otimes S\big(\omega^{(1)},...\omega^{(k)},...,\omega^{(k+m)},V^{(1)},...,V^{(l)},...,V^{(l+n)}\big) \\ &\qquad = T\big(\omega^{(1)},...\omega^{(k)},V^{(1)},...,V^{(l)}\big)S\big(\omega^{(k+1)},...,\omega^{(k+m)},V^{(l+1)},...,V^{(l+n)}\big)\,. \end{aligned} \tag{2.45}$$

Let T be a tensor of rank (k,l), then under a Lorentz transformation its components transform as

$$T'^{\mu_1...\mu_k}{}_{\nu_1...\nu_l} = \Lambda^{\mu_1}{}_{\mu_1'}...\Lambda^{\mu_k}{}_{\mu_k'}\Lambda_{\nu_1}{}^{\nu_1'}...\Lambda_{\nu_l}{}^{\nu_l'}T^{\mu_1'...\mu_k'}{}_{\nu_1'...\nu_l'}\,. \tag{2.46}$$

One can uses tensors to construct additional tensors either by taking linear combinations of tensors with the same upper and lower indices, direct products, contraction, or differentiation. Note that the order of the indices of a tensor matters.

Some tensors that will appear regularly are: the metric which is a $(0,2)$ tensor, with the inverse being a $(2,0)$ tensor, the *Kronecker delta* δ^μ_ν which is a $(1,1)$ tensor, and finally the Levi–Civita tensor which is a $(0,4)$ tensor. Not only can the metric be used to raise and lower indices of a tensor, it can also be used to contract indices. *Contraction* takes a (k,l) tensor to a $(k-1,l-1)$ tensor by

$$T^{\mu\nu\rho}{}_{\mu\sigma} \equiv S^{\nu\rho}{}_{\sigma}\,. \tag{2.47}$$

2.1.4 Newton's law in special relativity and energy momentum

We now want the special relativity version of Newton's second law. The requirement that it be tensorial puts some stringent constraints on the possible form, we must introduce a force four-vector f^μ satisfying

$$f^\mu = m\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}x^\mu(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}p^\mu(\tau)\,. \tag{2.48}$$

For electromagnetism and the Lorentz force law ($f = q(E + v\times B)$) we find

$$f^\mu = qU^\nu F_\nu{}^\mu\,, \tag{2.49}$$

where F is the field strength of the electromagnetism gauge field and q is the charge of the particle and U^μ its four-velocity.

Although p^μ provides a complete description of the energy and momentum of a particle for extended systems it is necessary to go further and define the *energy-momentum tensor*, or *stress tensor*, $T^{\mu\nu}$. This is a symmetric $(2,0)$ tensor which tells us all we need to know about the energy like aspects of a system: energy density, pressure, stress etc.. Consider a *fluid*. This is a continuum of matter described macroscopic quantities such as temperature, pressure, entropy, viscosity, etc. We will work with *perfect fluids* which are completely characterised by their pressure and density. This in particular means that they are isotropic (same in every direction) in the rest frame.

To understand this let us first consider *dust*. This is a collection of particles which are at rest with respect to each other, as a perfect fluid they have zero pressure. Since all the particles have an equal velocity in any fixed inertial frame we can imagine a four-velocity field $U^\mu(x)$ defined over all spacetime. We can define the *number-flux four-vector*

$$N^\mu = nU^\mu \,, \tag{2.50}$$

where n is the number density of the particles as measured in their rest frame. Then N^0 is the number density of particles as measured in any other frame, while N^i is the flux of particles in the i'th direction. Let us imagine each of the particles have the same mass m. Then in the rest frame the energy density of the dust is given by

$$\rho = nm \,. \tag{2.51}$$

This completely specifies the dust, however this only measures the energy density in the rest frame, how do we measure it in other frames? Notice that both n and m are 0-components of four-vectors in their rest frame: $N^\mu = (n, 0, 0, 0)$ and $p^\mu = (m, 0, 0, 0)$. Therefore ρ is the $\mu = 0, \nu = 0$ component of the tensor $p \otimes N$ as measured in the rest frame. We are therefore lead to define the energy momentum tensor for dust

$$T^{\mu\nu}_{\text{dust}} = p^\mu N^\nu = nmU^\mu U^\nu = \rho U^\mu U^\nu \,, \tag{2.52}$$

where ρ is the energy density as measured in the rest frame.

We can now consider other perfect fluids. The key point is the isotropic in the rest frame property which implies that the energy momentum tensor must take a diagonal form in the rest frame, since there cannot be a net flux of momentum in an orthogonal direction. Moreover the spacelike components must all be equal $T^{11} = T^{22} = T^{33}$, there are only two independent components. We will take the two independent parameters to be the energy density ρ and the pressure p (note that p is also used for momentum but will always come with a superscript or subscript). In the rest frame the energy momentum tensor takes the form

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \mathrm{p} & 0 & 0 \\ 0 & 0 & \mathrm{p} & 0 \\ 0 & 0 & 0 & \mathrm{p} \end{pmatrix} . \tag{2.53}$$

We want a formula which is good in any frame and therefore we want to write this in terms of tensors. For dust we had $T^{\mu\nu} = \rho U^\mu U^\mu$, so we may guess that there should be $(\rho + \mathrm{p})U^\mu U^\mu$, which gives $\rho + \mathrm{p}$ in the 00 component and zero elsewhere in the rest frame. To include the remainder we should find something which is of the form $\mathrm{p}\mathrm{diag}(-1, 1, 1, 1)$ this is of course given by the Minkowski metric! The general form of the energy momentum tensor for a perfect fluid is

$$T^{\mu\nu} = (\rho + \mathrm{p})U^\mu U^\nu + \mathrm{p}\eta^{\mu\nu} . \tag{2.54}$$

This will be important when we consider the cosmology section of the course.

2.2 Newtonian gravity

Above we have reviewed special relativity, now we want to understand why special relativity is incompatible with Newtonian gravity, in particular we will see that it is not Lorentz invariant. To do this we can cast Newtonian gravity in terms of a field theory. The force acting on a particle of mass m is

$$F = -m\nabla\Phi(t, \vec{x}) , \tag{2.55}$$

where the gravitational field $\Phi(t, \vec{x})$ is determined by the surrounding matter distribution $\rho(t, \vec{x})$, through

$$\nabla^2\Phi(t, \vec{x}) = 4\pi G_N \rho(t, \vec{x}) , \tag{2.56}$$

where G_N is Newton's constant with approximate value

$$G_N \sim 6.67 \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2} . \tag{2.57}$$

This is simply a rewriting into field theory language of the inverse square law of Newton. For example if there is a mass M concentrated at a single point at $(t, \vec{0})$, then the mass density is

$$\rho(t, \vec{x}) = M\delta^{(3)}(\vec{x}) , \tag{2.58}$$

which gives the gravitational field,

$$\Phi(\vec{x}) = -\frac{GM}{r}\,, \qquad r^2 = \vec{x}\cdot\vec{x}\,. \tag{2.59}$$

This can be extended to more complicated matter distributions, either summing up contributions from the location of point-like particles or more generally by using the Greens function for the Laplacian and the mass density

$$\Phi(\vec{x}) = -\int \mathrm{d}^3x'\,\frac{G\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,. \tag{2.60}$$

Exercise 2.7: Newton's theorem
Newton's theorem states that the gravitational field outside of a spherically symmetric mass distribution depends only on its total mass. Show this by using (2.55), (2.56) and Gauss' theorem.

We can now insert the gravitational force law into Newton's second law of motion $F = ma$. At this point one should ask oneself whether the inertial mass appearing in Newton's second law is the same as the one appearing in the gravitational force law (2.55), there is no reason that they need to be the same. Application of Newton's second law gives

$$\vec{a} = -\frac{m_G}{m_i}\nabla\Phi\,, \tag{2.61}$$

with $\vec{a}$ the acceleration Starting with Galileo, Christaan Huygens all the way to more recent experimental data has shown that $m_i = m_G$ to an accuracy of 10^{-13}. This is known as the *weak equivalence principle.* In the Newtonian theory this appears as an isolated unexplained fact, however it is this experimental fact that underlies general relativity. Since all bodies with the same initial conditions fall along the same curve regardless of their composition, we can interpret that curve to be a property of the geometry of the spacetime not of a force acting on the body.

2.2.1 Equivalence Principles

The *Weak equivalence principle* was one of the starting points for the development of GR. It is motivated by thought experiments using Newtonian gravity. The exact equality of $m_i = m_g$ is one version of the weak equivalence principle. Newtonian gravity gives no explanation for why this should be true. A theory of gravity should be able to explain this. Another way to formulate the weak equivalence principle is

The trajectory of a freely falling test body depends only on its initial position and initial *velocity and is independent of the composition of the body.*

A consequence of the weak equivalence principle is that it is not possible to tell the difference between constant acceleration and a constant gravitational field. Suppose that you are in a closed box and consider the two situations 1) you are on earth, 2) you are in a spaceship undergoing constant acceleration. Within Newtonian mechanics there is no local experiment that you can perform which distinguishes the two.[10] Another version of this is 1) the box is in free fall 2) you are floating in deep space. Again there is not local experiment that you can conduct to tell the difference. If the two situations can't be distinguished why do we describe them so differently?

This motivated Einstein's equivalence principle:

1. The weak equivalence principle is valid and
2. In a local inertial frame the results of all non-gravitational experiments will be indistinguishable from the results of the same experiments in an inertial frame in Minkowski spacetime.

The weak equivalence principle implies that 2) is valid for test bodies. The fact that test bodies which include ordinary matter which is held together by the three other forces, gives evidence that the electromagnetic and nuclear forces also obey 2).

Implications The equivalence principle has many implications.

One is that it implies that light is bent in a gravitational field. Consider a uniform gravitational field and a freely falling lab. Inside the lab the Einstein equivalence principle says that light rays must move on straight lines. But a straight line with respect to the lab corresponds to a curved path with respect to a frame at rest relative to the Earth. The effect is small so we cannot demonstrate this in the classroom but with more sensitive equipment in a lab this has been experimentally verified.

2.2.2 Planetary orbits in Newtonian mechanics

We now have all the technology we need to study the orbits of the planets within Newtonian mechanics. Let us set up a coordinate system where the massive body of mass M is at $r = 0$

[10] One of the important words is *local*. You can use tidal forces to distinguish between the two. Roughly if you drop two masses on Earth they will ever so slightly come together because the direction gravity acts on them is slightly different, they are pulled to the centre of the Earth. On a spaceship this is not the case and they fall down never getting closer together. This however is a non-local experiment, you need to watch the masses fall for a while and for a distance.

and the planet of mass m is a distance r from that point. The Lagrangian describing the system is

$$\mathcal{L} = \frac{m}{2}\big(\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2\big) - V(r)\,, \qquad V(r) = -\frac{mMG}{r}\,. \tag{2.62}$$

To make this more tractable it is useful to change coordinates to polar coordinates rather than Cartesian coordinates:

$$x = r\sin\theta\cos\phi\,, \quad y = r\sin\theta\sin\phi\,, \quad z = r\cos\theta\,. \tag{2.63}$$

The Lagrangian becomes

$$\mathcal{L} = \frac{m}{2}\Big(\dot{r}^2 + r^2\big(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\big)\Big) + \frac{mMG}{r}\,. \tag{2.64}$$

We can now compute the equations of motion via the Euler–Lagrange equations: we find

$$\begin{aligned} \ddot{r} - r\big(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\big) + \frac{MG}{r^2} &= 0\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) - r^2\sin\theta\cos\theta\dot{\phi}^2 &= 0\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}(r^2\sin^2\theta\dot{\phi}) &= 0\,. \end{aligned} \tag{2.65}$$

First let us consider the $\dot{\theta}$ equation. If we kick the particle off in the $\theta = \frac{\pi}{2}$ plane with $\dot{\theta} = 0$ then it will remain in that plane. We will make this choice from now on. The coordinate ϕ is an ignorable coordinate since it does not appear explicitly in the Lagrangian. Recall that for every ignorable coordinate there is an associated conserved charge, in this case it will be the angular momentum. We may define

$$l = r^2\dot{\phi}\,, \tag{2.66}$$

which is conserved. We have now solved the last two equations of (2.65) and only the first remains. Then we have

$$\ddot{r} - \frac{l^2}{r^3} + \frac{MG}{r^2} = 0\,. \tag{2.67}$$

To proceed further it is useful to note that there is one more conserved quantity, the Energy of the system. This follows since the Lagrangian is explicitly time independent, thus

$$Em = \frac{m}{2}\Big(\dot{r}^2 + r^2\big(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\big)\Big) + V(r)\,, \tag{2.68}$$

is conserved. We can now substitute $\dot{\theta}$ and $\dot{\phi}$ into this final condition to obtain an equation for $\dot{r}$ only:

$$E = \frac{1}{2}\dot{r}^2 + \frac{l^2}{2r^2} - \frac{MG}{r} \equiv \frac{1}{2}\dot{r}^2 + V_N(r)\,. \tag{2.69}$$

We can now study the orbits by looking at the Newtonian potential. At large distances the attractive $-r^{-1}$ dominates, while the angular momentum prohibits the particle from getting too close to the origin, see figure 2.

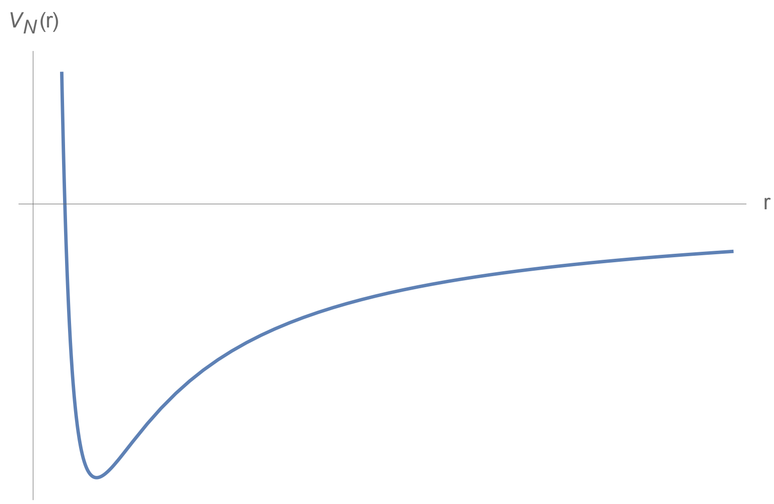


Figure 2: A representative example of the Newtonian potential. There are three interesting behaviours to consider. When $E > 0$ and the motion describes a fly-by coming from infinity and heading back to infinity. For $E = V_N(r_*) < 0$ the motion is a circular orbit. Finally for $0 > E > V_N(r_*)$ we obtain elliptic orbits.

The potential has a minimum when

$$V'(r_*) - \frac{MG}{r_*^2} - \frac{l^2}{r_*^3} = 0 \qquad r_* = \frac{l^2}{MG}\,. \tag{2.70}$$

The planet can happily sit at $r = r_*$ for all time on a circular orbit, note that $E < 0$ in this case. The planet could also oscillate back and forth around the minima. This happens when $E < 0$ so that the planet cannot escape off to infinity. This describes an orbit where the distance to the massive body varies, as you may expect this is the usual elliptic orbit. For $E > 0$ the motion describes a flyby, the planet gets close to the massive body, never reaching it, before being flung off to infinity. Clearly such a planet would be dead and inhospitable for life.

So far we have discussed the radial motion of the planet, this does not tell us about the full motion however. We consider the orbit trajectory, the flyby motion is not so interesting for us. We now need to solve the angular momentum equations (2.66). To solve the coupled equations we start by employing a change of coordinates

$$u = r^{-1}\,, \tag{2.71}$$

and then view this as a function of ϕ. This works nicely because

$$\dot{u} = \frac{\mathrm{d}u}{\mathrm{d}\phi}\dot{\phi} = lu^2\frac{\mathrm{d}u}{\mathrm{d}\phi}\,, \tag{2.72}$$

where we have used (2.66). We have

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -l\frac{\mathrm{d}u}{\mathrm{d}\phi}\,. \tag{2.73}$$

The conservation of energy equation (2.69) becomes

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + \left(u - \frac{GM}{l^2}\right)^2 = \frac{2E}{l^2} + \frac{G^2M^2}{l^4}\,. \tag{2.74}$$

This turns out to be straightforward to solve, the solution is

$$u(\phi) = \frac{GM}{l^2}(1 + e\cos\phi)\,. \tag{2.75}$$

In the original radial coordinate we have

$$r(\phi) = \frac{l^2}{GM}\frac{1}{1 + e\cos\phi}\,. \tag{2.76}$$

This is the equation for a conic section with the eccentricity given by

$$e = \sqrt{1 + \frac{2El^2}{G^2M^2}}\,. \tag{2.77}$$

The shape of the orbit depends on the eccentricity. Motion with $E > 0$ is not in a bounded orbit, tracing out a hyperbola for $e > 1$ and a parabola for $e = 1$. Objects in orbit have $e < 1$ with elliptical orbits. An important thing to note about this solution is that the orbit does not *precess*, its closest approach to the origin, known as the *perihelion*[11] is always at the same point it never moves and nor does the furthest point of the orbit, the *aphelion*. This disagrees with observations of Mercury's orbit and is the first observational discrepancy of Newtonian gravity.[12]

[11] Strictly this is for the closest approach to the sun. *Helios* is the word for the sun in greek, while *peri* means around.

[12] To perform a more accurate computation one should also take into account the effect of the gravitation fields of the other planets. This is notoriously difficult since one has to study a multi-body problem. Instead, what one can do is imagine that the other planets form a shell of mass along their orbit, acting equally. One can then evaluate the force due to this mass shell. This approximation works if one considers the problem over a long enough time, but is poor if taken for a short time scale. Since planets closer to the sun have quicker orbits over a long enough time this approximation will give a reasonable result and we can avoid trying to solve this multi-body computation.

2.3 Problems with Newtonian gravity and why we need GR

Newton's theory of gravitation is successful in explaining the motions of the moon and planets. Some irregularities in the orbit of Uranus remained unexplained until the irregularities were used independently by John Couch Adams and Jean Joseph Le Verrier in 1846, to predict the existence and position of Neptune. There were still issues with predictions from Newtonian gravity and experimental data however. The precession of the perihelion of Mercury was one such problem. It was shown to be out by 43"/century[13], recall that in the section above we showed that the perihelion does not precess in Newtonian gravity. We will see later how GR corrects this. A more obvious (and mathematical) problem arose after Einstein's work on special relativity in 1905. Newtonian gravity is incompatible with special relativity. A body can, in principle, be accelerated to a speed greater than the speed of light. Moreover, effects are instantaneous in Newtonian gravity clearly this is not allowed in special relativity where the speed of light gives an upper bound on the transfer of information.

Despite Newtonian gravities' failings it is sufficient for studying a large range of phenomena. To understand when a relativistic theory is needed let us consider a circular orbit around a star of mass M. The speed of the planet is easily computed by equating the centripetal force with the gravitational force giving,

$$\frac{v^2}{r} = \frac{GM}{r^2}\,. \tag{2.78}$$

Relativistic effects become important when $v \sim c$ and therefore the dimensionless parameter which governs corrections to Newtonian gravity is

$$\frac{GM}{rc^2}\,. \tag{2.79}$$

There is a convenient length scale which one can construct from a mass and the fundamental constants known as the *Schwarzschild radius*,[14]

$$r_s = \frac{2GM}{c^2}\,. \tag{2.80}$$

Relativistic corrections to gravity are then necessary when $r_S \sim 2r$. By this measure the earth is not a relativistic system $r_S \sim 10^{-2}m$ and the corrections on the surface of the Earth are of the order 10^{-8}. For satellites in orbit this is even smaller $\sim 10^{-9}$ however for GPS satellites clocks with such high precision are needed that this effect can be seen and if GR

[13] The " stands for arcseconds, with 3600 arcseconds(=3600") in a degree.

[14] We will see this appear later when we consider the Schwarzschild solution in section 6.

was not taken into account would stop working very soon. The sun has $r_S \sim 3$km and for Mercury the corrections are of order 10^{-7}, clearly very small but over a century the precession of Mercury's perihelion adds up to the previously quoted 43".

General relativity is the theory that replaces both Newtonian gravity and special relativity. However, general relativity is not the final theory of gravity, one eventually needs a theory of quantum gravity. General relativity breaks down for very extreme phenomena where quantum effects become important, e.g. the Big Bang and inside black holes. If one views gravity as a classical field theory and attempts to quantise it one finds that it is perturbatively non-renormalizable (if you do the QFT courses these words will reappear). Essentially this means that to obtain sensible observable results we must absorb infinities in computations by introducing new parameters. For a renormalizable theory we need to introduce only a finite number of these new parameters but for a non-renormalizable theory we need to introduce an infinite number, rendering the theory unable to give meaningful predictions. A candidate theory for quantum gravity, but no means the only candidate, is string theory. We still do not know what quantum gravity really is! We should emphasise that a theory of quantum gravity is only needed for these extreme phenomena and so for the large part general relativity is sufficient.

2.4 Some worked examples

We have now completed our review of special relativity and Newtonian gravity. We present some worked examples on special relativity below.

2.4.1 Proper time along an accelerated worldline

We treat the planets as being at rest relative to each other in this question.

Leia begins at rest on the planet Polis Massa and sets off in a spaceship to visit a distant planet called Alderaan. Alderaan is at rest relative to Polis Massa and is a proper distance D away. Leia's spaceship accelerates during the journey at a constant rate α,

$$\eta_{\mu\nu} a^\mu a^\nu = \alpha^2 \,, \tag{2.81}$$

where a^μ is the four-acceleration of Leia. We want to answer two questions: 1) what path does Leia take in terms of coordinates centred on Polis Massa? 2) How much time passes, from Leia's point of view until she reaches Alderaan?

We can choose coordinates (t, x, y, z) where the worldline of Polis Massa is simply $(t, 0, 0, 0)$ and the worldline of Alderaan is $(t, D, 0, 0)$ (recall that the two planets are at rest relative to each other). Leia's world line is then of the form

$$(t(\tau), x(\tau), 0, 0)\,, \tag{2.82}$$

where τ is the proper time along Leia's worldline. Since we have parametrised Leia's worldline by the proper time we have

$$-\dot{t}(\tau)^2 + \dot{x}(\tau)^2 = -1 \qquad \dot{\bullet} \equiv \frac{\mathrm{d}\bullet}{\mathrm{d}\tau}\,. \tag{2.83}$$

Leia's acceleration is therefore,

$$a = \big(\ddot{t}(\tau), \ddot{x}(\tau), 0, 0\big) = \left(\frac{\dot{x}(\tau)\ddot{x}(\tau)}{\sqrt{1+\dot{x}(\tau)^2}}, \ddot{x}(\tau), 0, 0\right), \tag{2.84}$$

where for the second equality we have used (2.83) to eliminate $\ddot{t}(\tau)$. Since Leia's acceleration is constant, (2.81), we have

$$\alpha^2 = \frac{\ddot{x}(\tau)^2}{1+\dot{x}(\tau)^2}\,. \tag{2.85}$$

We have that $\dot{x}(\tau) > 0$ and therefore the solution for $\dot{x}(\tau)$ is

$$\dot{x}(\tau) = \sinh(\alpha\tau + \beta)\,, \tag{2.86}$$

with β a constant of integration. Since Leia began at rest on Polis Massa, we take $\beta = 0$. Integrating again and using that Leia begins at Polis Massa at $\tau = 0$, i.e. $x(0) = 0$, we have

$$x(\tau) = \frac{1}{\alpha}\big(\cosh(\alpha\tau) - 1\big)\,. \tag{2.87}$$

Inserting this into (2.83), solving for $t(\tau)$ and imposing $t(0) = 0$ we find

$$t(\tau) = \frac{1}{\alpha}\sinh(\alpha\tau)\,. \tag{2.88}$$

Leia reaches Alderaan when

$$\tau = \operatorname{arccosh}(1 + \alpha D)\,, \tag{2.89}$$

If αD is large then $\tau \sim \frac{1}{\alpha}\log(\alpha D)$ and therefore no matter how large D is, for a sufficiently large acceleration Leia can reach Alderaan in a "reasonable" proper time. On the other hand, when Leia reaches Alderaan

$$t = \sqrt{D^2 + \frac{2D}{\alpha}}\,, \tag{2.90}$$

and therefore no matter how large α is it always takes at least a time of D (recall $c = 1$) to reach Alderaan as viewed from Polis Massa.

2.4.2 Null curves in Minkowski space

By now we have all seen that a straight line is a null curve in Minkowski space but are there more? Note that we are not asking about geodesics. Consider the curve, given in inertial coordinates, by

$$x^\mu = (\lambda, \sin\lambda, \cos\lambda, 0)\,. \tag{2.91}$$

The tangent to the vector is

$$v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = (1, \cos\lambda, -\sin\lambda, 0)\,, \tag{2.92}$$

and has norm

$$v^\mu v^\nu \eta_{\mu\nu} = -1 + \cos^2\lambda + \sin^2\lambda = 0\,. \tag{2.93}$$

This is a null curve that is not straight, it is not a geodesic however.

2.4.3 Ladders and barns

Barry and Paul Chuckle have been employed by Albert E. to put a ladder in a barn, a simple feat you would imagine but these are the Chuckle brothers and nothing is simple with them. Albert E. stands outside the barn, and tells Barry and Paul to run very quickly at a constant speed in a straight line through the barn carrying the ladder. The barn has doors at the front and back, and two apprentices (Jimmy and Brian) stand at either door ready to close or open them. Initially the front door is open and the back door is closed. The proper length of the ladder is l, while the proper length of the barn is b with $b < l$.

Albert E. claims that if Barry and Paul run fast enough, and that there is no slacking, then both doors of the barn can be temporarily closed with both Barry, Paul and the ladder inside the barn. One of the apprentices can then open the back door again so that Barry, Paul and the ladder can pass through the barn safely. The brothers are stumped, "oh dear, oh dear" says Barry, "the ladder is bigger than the barn, it will never work". To put their minds at rest show that the ladder will fit in a chosen reference frame.

Let us work in inertial coordinates where the barn is at rest, which corresponds to Albert E.'s point of view. In these coordinates the front of the barn is at $x^\mu = (\lambda, 0, 0, 0)$ while the back of the barn is at $x^\mu = (\lambda, b, 0, 0)$.

The worldline of the front of the ladder in this reference frame is $x^\mu = (\lambda, v\lambda, 0, 0)$, where v is the velocity of the ladder. We have chosen coordinates so that the front of the ladder enters the barn at $\lambda = 0$. The back of the ladder follows the worldline $x^\mu = (\lambda, \lambda v - L, 0, 0)$ for some L which is not l!

First we must work out what L is in terms of l. We could of course perform a Lorentz transformation to switch to the rest frame of the ladder, the proper length of the ladder is then the coordinate length in this frame. We will use an alternative approach, staying in the original coordinate frame. How can we measure a length? Well we can define it to be half the proper time along the worldline at one end of the ladder between the emission and reception of a light signal which bounces off the other end of the body. The worldlines of the points making up the ladder are given by $x^\mu = (\lambda, \lambda v - \beta, 0, 0)$ where $\beta \in [0, L]$ and their tangent vectors are

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = (1, v, 0, 0)\,. \tag{2.94}$$

We now want to find a spacelike straight line orthogonal to this tangent vector. Such a worldline is given by $n^\mu = (-v\tilde{\lambda}, -\tilde{\lambda}, 0, 0)$. This curve meets the front of the ladder at $\tilde{\lambda} = 0$ and the back of the ladder at $\tilde{\lambda} = L(1 - v^2)^{-1}$. We want to calculate the proper length of this curve with $\tilde{\lambda} \in [0, \frac{L}{1-v^2}]$. To do so we should parametrise the curve by the proper length. The norm of the tangent of the above vector is $\eta_{\mu\nu}\dot{n}^\mu(\tilde{\lambda})\dot{n}^\nu(\tilde{\lambda}) = 1 - v^2$. Then the proper length is

$$s = \tilde{\lambda}\sqrt{1 - v^2}\,. \tag{2.95}$$

The ladder then has proper length

$$l = s\Big|_{\tilde{\lambda} = \frac{L}{1-v^2}} = \frac{L}{\sqrt{1 - v^2}}\,. \tag{2.96}$$

The entire ladder can fit into the barn from Albert E.'s perspective if $b \geq L$ and therefore the Chuckle brothers must run at a speed of

$$v \geq \sqrt{1 - \frac{b^2}{l^2}}\,. \tag{2.97}$$

Both doors of the barn can be closed if: the front of the ladder is still in the barn, $b > v\lambda$ and the back of the ladder is in the barn $\lambda v - L > 0$. Since $t = \lambda$ the ladder is in the barn for

$$\frac{l\sqrt{1 - v^2}}{v} \leq t \leq \frac{b}{v}\,. \tag{2.98}$$

We see that Albert E. sees the ladder fully inside the barn with the doors closed.

Now consider what happens from the Chuckle brother's perspective. We can do a Lorentz transformation to coordinates in which they are at rest:

$$(t', x', y', z') = (\gamma t - \gamma v x, \gamma x - \gamma v t, y, z)\,, \qquad \gamma = \frac{1}{\sqrt{1 - v^2}}\,. \tag{2.99}$$

In the Chuckle brother's coordinates the barn follows the worldline $(\lambda, -v\lambda, 0, 0)$, while the back of the barn follows the worldline $(\lambda, -v\lambda + b\sqrt{1-v^2}, 0, 0)$.
The front door can be closed when the front of the barn passes the back of the ladder, so $-v\lambda < -l$ and therefore the front door of the barn is closed for $t' > \frac{l}{v}$.
The back door must open when the front of the ladder is about to go through it. So it is closed until $b\sqrt{1-v^2} - vt' = 0$ and therefore the back door is closed for $t' \in [0, \frac{b}{v\gamma}]$. In summary we have

$$\begin{cases} \text{Front door closed} & \frac{l}{v} \leq t' \, , \\ \text{Back door closed} & 0 \leq t' \leq \frac{b}{v\gamma} \, . \end{cases} \tag{2.100}$$

Since the ladder is longer than the barn $l > b$ and $\gamma \geq 1$ it follows that there is no time for which both doors are closed from the point of view of the Chuckle brothers. The entire ladder never fits into the barn from their perspective. The two view-points are depicted in figure 3.

Having been convinced by your arguments the brothers were off with a "to me, to you".[15]

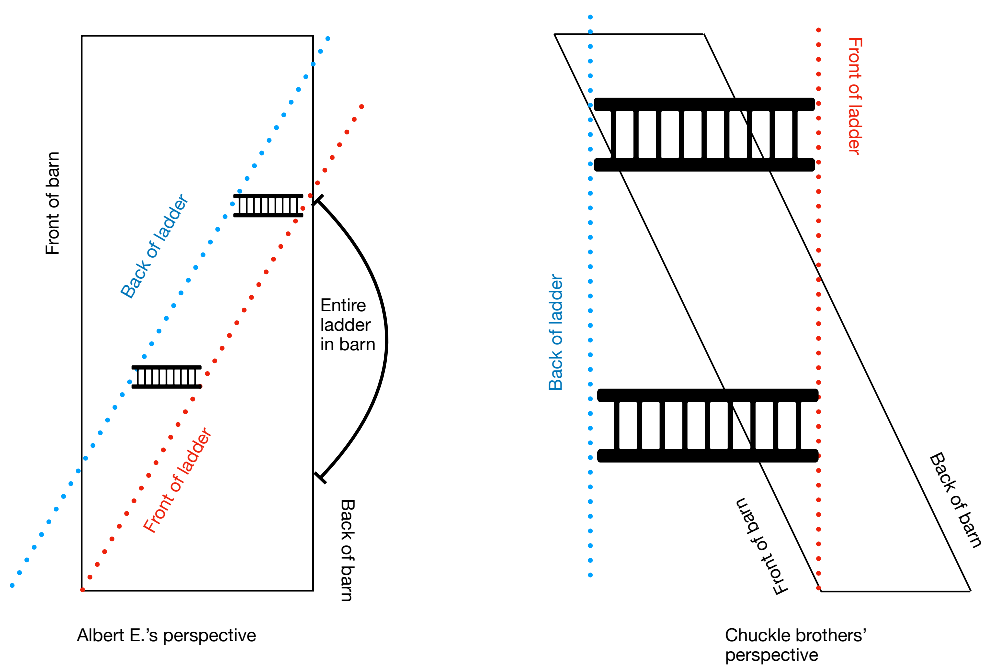


Figure 3: The two different perspectives of the ladder and barn. On the left from the perspective of Albert E., a stationary observer in the rest frame of the barn. On the right from the perspective of the Chuckle brothers carrying the ladder.

[15]ChuckleVision was a British children's comedy tv show following the antics of the Chuckle brothers Barry and Paul. Carrying a ladder was a common theme.

3 Differential Geometry

The study of general relativity is the study of curved spacetime and so to make progress we need to learn differential geometry. Our discussion will not be all encompassing, there will be both topics and proofs that we omit. Instead we will build up all the necessary mathematical structure, in a logical order, that we will need. As we proceed many of the objects that we will introduce may already be familiar to you, they will however take a different guise in places. This section will have different levels of mathematical rigour, in some sections we will be good mathematicians defining everything while in other places we will give more physics inspired definitions.

This section closely follows the excellent book by Nakahara [3].

3.1 Manifolds

Before we begin to define a manifold we need to define a topological space.

Definition 3 (Topological space) *Let X be any set and $\mathcal{T} = \{U_i | i \in I\}$ denote a certain collection of subsets of X. The pair $(X, \mathcal{T})$ is called a* topological space *if $\mathcal{T}$ satisfies*

1. *Both the set X and the empty set $\emptyset$ are open subsets: $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.*

2. *If J is any, possibly infinite, sub-collection of I, then the family $\{U_j | j \in J\}$ satisfies $\cup_{j\in J} U_j \in \mathcal{T}$.*

3. *If K is any finite sub-collection of I then the set $\{U_k | k \in K\}$ satisfies $\cap_{k\in K} U_k \in \mathcal{T}$.*

What is the intuition behind this definition? We have a set of objects X, for example the people in the lecture room. We now group people into subsets U_i, for example one subset could be the set of all people who were born on a Thursday another could be those who woke up before 7am this morning and so forth. Now we can define a topology $\mathcal{T}$ by taking a choice of these different subsets, this is the $i \in I$. This space needs to satisfy some properties though. 1) Both the empty set and the set X itself must be in $\mathcal{T}$. 2) If we take any subset J of the open sets U_i then the combination (union) of them all is one of the open sets U_i in our topology. 3) If we take a subset K and intersect them all then the intersection (elements common to all sets) is also an open set U_i in our topology.

Sometimes X alone is called a topological space, i.e. without associating to it a topology, here we mean that a topology is associated. The sets U_i are called *open sets* (we may sometimes refer to them as coordinate patches, the reason why will become obvious later) and $\mathcal{T}$ gives a *topology* to X. So when we talk about the open sets, what we really mean is *does this set appear in the topology?*. Therefore what we call an open set depends on what topology we have given the space.

Example 3.1: Topologies

a) If X is a set and $\mathcal{T}$ a collection of all subsets of X then this is a topological space, and is known as the *discrete topology*.

b) Let X be a set and take $\mathcal{T} = \{\emptyset, X\}$. This is then a topological space and the topology is known as the *trivial topology*. While the discrete topology is too stringent, this topology is too trivial.

c) Take $X = \mathbb{R}$. All open subsets (a, b) (a, b may be $\mp\infty$ respectively) and their unions define a topology known as the *usual topology*.

Exercise 3.1: Usual topology vs discrete topology
Consider the usual topology on $\mathbb{R}$ and show that if we allow for an infinite number of open sets in condition 3 for the definition of a topological space, then the usual topology reduces to the discrete topology.

Definition 4 (Metric) *A* metric $d : X \times X \to \mathbb{R}$ *is a function that for any* $x, y, z \in X$ *satisfies:*

1. $d(x, y) = d(y, x)$,

2. $d(x, y) \geq 0$ *with equality iff* $x = y$,

3. $d(x, y) + d(y, z) \geq d(x, z)$.

If X is endowed with a metric then X is made a topological space whose open sets are given by open discs

$$U_\epsilon(x) = \{y \in X | d(x, y) < \epsilon\}, \tag{3.1}$$

and all possible unions. The topology $\mathcal{T}$ is called the *metric topology* determined by d.

Definition 5 (Neighbourhood) *Suppose $\mathcal{T}$ gives a topology to X. Then N is a* neighbourhood *of the point $x \in X$ if N is a subset of X and N contains at least one open set U_i which contains x. Note that there is no requirement for N to be open, in the case where it is open it is called an* open neighbourhood.

The intuition behind this definition is that for a set N of the open sets U_i to be in the neighbourhood of a point x, then at least one of the sets in N must contain x.

Definition 6 (Hausdorff space) *A topological space* $(X, \mathcal{T})$ *is a* Hausdorff space *if for an arbitrary pair of distinct points* $x, y \in X$*, there always exists neighbourhoods* U_x *and* U_y *such that* $U_x \cap U_y = \emptyset$.

Example 3.2: A non-Hausdorff example
Let $X = \{A, B, C, D\}$ define the sets

$$U_0 = \emptyset\,, \quad U_1 = \{A\}\,, \quad U_2 = \{A, B\}\,, \quad U_3 = \{A, B, C, D\}\,. \tag{3.2}$$

Then the topology $\mathcal{T} = \{U_0, U_1, U_2, U_3\}$ makes X a topological space but it is not Hausdorff. First note that both the empty set and X are in the topology $\mathcal{T}$, satisfying point (1) of the definition of a topological space. Note that the union of these sets is within $\mathcal{T}$ thereby satisfying point (2). Finally the intersection of any of the sets is within $\mathcal{T}$ and therefore it is a topological space. To see why it is not Hausdorff it suffices to show that we can pick two points which have no open sets in which one of the points is in and that the intersection of these open sets is not the empty set. There are a few choices we could make but an obvious one is C, D. They both appear in only one open set and therefore the space cannot be Hausdorff.

Most examples in physics that one encounters are Hausdorff spaces. We will *assume* this is the case throughout in this course since it is a form of apotropaic magic: it protects us from evil (=bad things occuring).

Definition 7 (Continuous) *Let* X *and* Y *be topological spaces. A map* $f : X \to Y$ *is* continuous *if the inverse image of an open set in* Y *is an open set in* X*. Note that a continuous function does* not *need to map an open set in* X *to an open set in* Y*,* $f(x) = x^2$ *is an example of a continuous function that would fail this requirement.*

Definition 8 (Closed, closure, interior, boundary) *Let* $(X, \mathcal{T})$ *be a topological space.*

- *A subset* A *of* X *is* closed *if its complement* $X - A \in \mathcal{T}$ *in* X *is an open set.*
- *The* closure *of the subset* A *is the smallest closed set that contains* A *and is denoted by* $\bar{A}$.
- *The* interior *of* A *is the largest open subset of* A *and is denoted by* A°.

- *The* boundary $b(A)$ *of* A *is the complement of* A° *in* $\bar{A}$*:* $b(A) = \bar{A} - A^\circ$.

An open set is always disjoint from its boundary while a closed set always contains its boundary. To make this a little more clear let us consider a concrete example.

Example 3.3: Open and closed sets

- Let us consider $\mathbb{R}^2$, the two-dimensional plane, with the metric topology and let A be the following open set, $A = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$. Then the closure of A is

$$\bar{A} = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}\,. \tag{3.3}$$

  The interior of A is itself $A^\circ = A$. The boundary is then the complement of A° in $\bar{A}$, thus

$$b(A) = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}\,. \tag{3.4}$$

  It therefore agrees with our usual understanding of these concepts.

- Consider the set $X = \{a, b, c, d, e, f\}$ with topology

$$\mathcal{T} = \{X, \emptyset, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e,f\}\}\,. \tag{3.5}$$

  By definition the sets, $\{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e,f\}$ are all open in $(X, \mathcal{T})$.

  Now let us consider the following sets and whether they are open or closed (or both). First it is useful to work out the closed sets in our topology, they are $X - \mathcal{T}$ and thus we have $\{X, \emptyset, \{b,c,d,e,f\}, \{a,b,e,f\}, \{b,e,f\}, \{a\}\}$.

  1. $\{a\}$. This is both open, since it appears in $\mathcal{T}$ and closed since $X - \{a\} = \{b,c,d,e,f\} \in \mathcal{T}$. We can see that its closure is itself since it is closed and its interior is itself since it is open.
  2. $\{b,c\}$. This is not open since it does not appear in $\mathcal{T}$. We have $X - \{b,c\} = \{a,d,e,f\}$ which does not appear in $\mathcal{T}$ either and is therefore this set is not closed either. The closure of $\{b,c\}$ is the smallest closed sets that contains $\{b,c\}$, and from the set of closed sets above we see that it is $\{b,c,d,e,f\}$. The interior is the largest open subset of A, we see that this is necessarily the empty set $\emptyset$. The boundary is therefore $\{b,c,d,e,f\}$.
  3. $\{c,d\}$. This is open since it appears in $\mathcal{T}$, but is not closed since $\{a,b,e,f\}$ does not appear in $\mathcal{T}$. The closure is $\{b,c,d,e,f\}$, while the interior is itself. The boundary is therefore $\{b,e,f\}$.

4. $\{a, b, e, f\}$. This is not open since it does not appear in $\mathcal{T}$, it is closed however since $X - \{a, b, e, f\} = \{c, d\}$ does appear in $\mathcal{T}$. The closure is itself since it is a closed set while the interior is $\{a\}$. Therefore the boundary is $\{b, e, f\}$.

For completeness we give the following two definitions, though we will not call on them much:

Definition 9 (Covering) *Let $(X, \mathcal{T})$ be a topological space. A family $\{A_i\}$ of subsets of X is called a* covering *of X if*

$$\bigcup_{i \in I} A_i = X\,. \tag{3.6}$$

If all the A_i happen to be the open sets of the topology $\mathcal{T}$ then the covering is called an open covering.

Definition 10 (Compact) *Consider a set X and all possible coverings of X. The set X is* compact *if for every open covering $\{U_i | i \in I\}$ there exists a* finite *subset J of I such that $\{U_j | j \in J\}$ is also a covering of X.*

Theorem 1 *Let X be a subset of $\mathbb{R}^n$, then X is compact if and only if it is closed and bounded.*

Definition 11 (Connected, Arcwise connected, Simply connected) *Let X be a topological space.*

i) X is connected *if it cannot be written as $X = X_1 \cup X_2$ where X_1 and X_2 are both open and $X_1 \cap X_2 = \emptyset$. Otherwise X is called* disconnected.

ii) A topological space is called arcwise connected *if for any points $x, y \in X$ there exists a continuous map $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. Only in a few pathological cases is arcwise connectedness not equivalent to connectedness.*

iii) A loop *in a topological space X is a continuous map $f : [0, 1] \to X$ such that $f(0) = f(1)$. If every loop in X can be continuously shrunk to a point, X is called* simply connected.

Some simple examples are:

Example 3.4: Connectedness

- $\mathbb{R}^2 - \mathbb{R}$ is not arcwise connected.
- $\mathbb{R}^2 - \{0\}$ is arcwise connected but not simply connected.

- $\mathbb{R}^3 - \{0\}$ is arcwise connected and simply connected.
- The n-dimensional torus is arcwise connected but not simply connected.

The main purpose of topology is to classify spaces. Suppose we have several figures, we want to be able to say which are equal and which are different, and probably more fundamentally what does being equal or different mean. In topology two figures are equivalent if it is possible to deform them continuously into each other. We therefore construct an equivalence relation under which geometrical objects are classified according to whether it is possible to deform one into the other. Of course these are just words and we should define this more mathematically. To wit let us define a *homeomorphism*

Definition 12 (Homeomorphism) *Let X_1 and X_2 be two topological spaces. A map $f : X_1 \to X_2$ is a* homeomorphism *if it is continuous and has an inverse $f^{-1} : X_2 \to X_1$ which is also continuous. If there exists a homeomorphism between X_1 and X_2 we say that X_1 and X_2 are* homeomorphic *to each other.*

The classic example of two homeomorphic spaces are a donut and a coffee mug, see here for a gif of this classic example courtesy of wikipedia.

One would like a quick way to understand whether two spaces are homeomorphic to each other. Even today we cannot fully characterise the equivalence classes between spaces. One modest statement that we can make is that if two spaces have different *topological invariants* then they are not homeomorphic to each other. A topological invariant is conserved under homeomorphisms. It may be a number such as the number of connected components of the space, an algebraic structure such as a group or a ring which can be constructed from the space, or something like connectedness, compactness or the Hausdorff property. If we knew the complete class of topological invariants we could specify the equivalence classes easily, however so far we only know a partial list. As such even if all the known topological invariants of two spaces coincide these spaces may still not be homeomorphic.

We are now finally in a position to define a *manifold*. An n-dimensional manifold is a space which looks locally like $\mathbb{R}^n$. Globally it need not be $\mathbb{R}^n$ but we may glue local patches, each of which look like $\mathbb{R}^n$ together to get the full global space. A manifold is then homeomorphic to $\mathbb{R}^n$ locally. The local homeomorphism allows us to give each point on the manifold a set of n numbers called local *coordinates*. If the manifold is not homeomorphic to $\mathbb{R}^n$ then we need to cover it in more than one patch, and so we need to introduce several local coordinates. We will require that the transition functions between these coordinates on the overlapping region are *smooth*. In this way we can develop the usual notion of calculus on a manifold. Topology is based on continuity, while manifolds is based on smoothness. With that let us begin with our definitions again.

Definition 13 (Differentiable manifold) *M is an n-dimensional* differentiable manifold *if it satisfies:*

1. *M is a Hausdorff topological space,*
2. *M is provided with a family of pairs $\{(U_i, \varphi_i)\}$;*
3. *$\{U_i\}$ is a family of open sets which covers M: $\cup_i U_i = M$.*
4. *φ_i is a homeomorphism from U_i onto an open subset U_i' of $\mathbb{R}^n$,*
5. *Given U_i and U_j such that $U_i \cap U_j \neq \emptyset$, then the map $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$ from $\varphi_j(U_i \cap U_j)$ to $\varphi_i(U_i \cap U_j)$ is infinitely differentiable. ψ_{ij} is known as a* transition function.

In figure 4 we have represented (well copied the image from Nakahara) the ideas above.

The pair (U_i, φ_i) are called a *chart* and the collection of charts is called an *atlas*. The subsets U_i are called the *coordinate neighbourhood* while the φ_i is called the *coordinate function*, or simply the *coordinate*. The homeomorphism φ_i is represented by n functions $\{x^1(p), ..., x^n(p)\}$, with this set $\{x^\mu(p)\}$ also called the *coordinate*. A point $p \in M$ exists independently of its coordinates, however we will often be sloppy and denote the point p through its coordinates.

If U_i and U_j overlap, two coordinate systems are assigned to the same point in $U_i \cap U_j$. Axiom 5 asserts that the transition function from one coordinate system to another be smooth $\mathbb{C}^\infty$. One may be alarmed by this but there is not reason for trepidation, it is analogous to labelling a point by Euclidean coordinates and polar coordinates. The map φ_i assigns n coordinates values $x^\mu, (1 \leq \mu \leq n)$ to a point $p \in U_i \cap U_j$, while φ_j assigns coordinates y^μ to the same point. The transition function from y to x, $x^\mu = x^\mu(y)$ is given by n functions of n variables, and is the explicit form of the map $\psi_{ji} = \varphi_j \circ \varphi_i^{-1}$. The differentiablility in the definition is then in the usual sense we are familiar from calculus. All this leads to us being able to move over M so long as we choose coordinates which vary in a smooth way over the manifold.

If the union of two atlases $\{(U_i \varphi_i)\}$ and $\{(V_j, \psi_j)\}$ is again an atlas, then these two atlases are said to be *compatible*. The compatibility is an equivalence relation. This equiva-

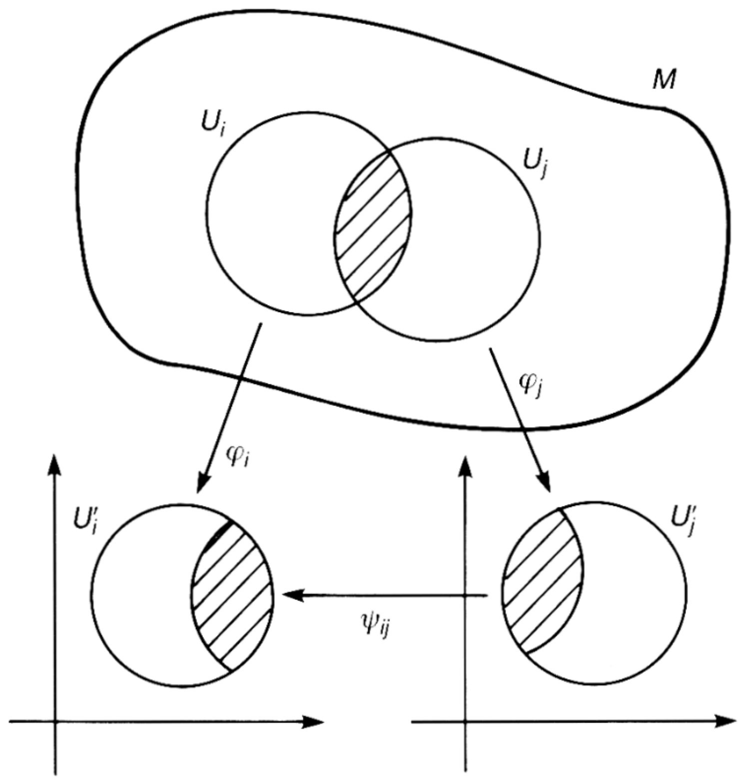


Figure 4: Here we see the manifold M and two coordinate charts. The homeomorphisms φ_i maps U_i onto an open set of $U_i' \subset \mathbb{R}^n$ providing coordinates for the point $p \in M$. If $U_i \cap U_j \neq \emptyset$ the transition functions from one coordinate system to another is smooth.

lence class is called the *differentiable structure*. Mutually compatible atlases define the same differentiable structure on M.

Let us briefly comment on manifolds with a boundary. We have assumed that the coordinate neighbourhood U_i is homeomorphic to an open set of $\mathbb{R}^n$. In some cases this is too restrictive. If a topological space M is covered by a family of open sets $\{U_i\}$ each of which is homeomorphic to an open set $H^n \equiv \{(x^1, ..., x^n) \in \mathbb{R}^n | x^n \geq 0\}$, M is said to be a manifold with boundary. The analogous plot of figure 4 for the manifold with a boundary is given in figure 5.

The set of points which are mapped to points with $x^n = 0$ is called the *boundary* of M and is denoted by ∂M. The coordinates on ∂M are given by $n-1$ numbers $(x^1, ..., x^{n-1}, 0)$. We now need to be careful when we define smoothness on the overlaps. The map ψ_{ij} : $\varphi_j(U_i \cap U_j) \to \varphi(U_i \cap U_j)$ is defined on an open set of H^n in general, and ψ_{ij} is said to be

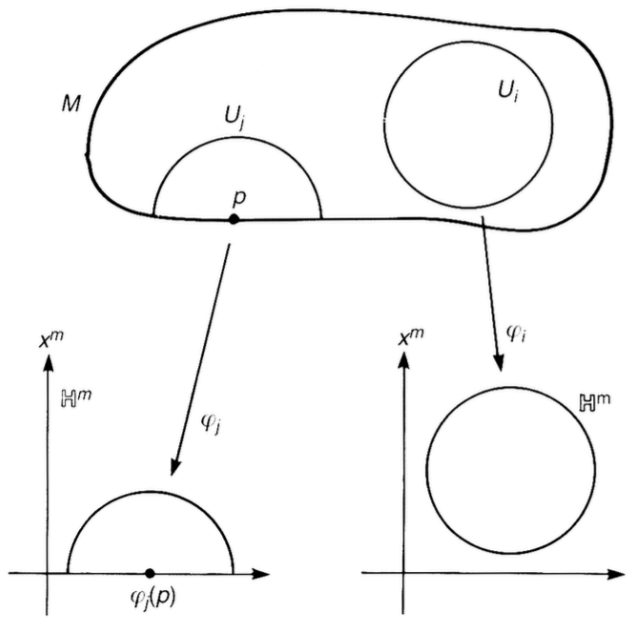


Figure 5: A manifold with a boundary. The point p is on the boundary. Note the subtle difference, for a manifold without a boundary the left figure would be extended below x^n=0.

smooth if it is C^∞ in an open set of $\mathbb{R}^n$ which contains $\varphi_j(U_i \cap U_j)$.

Example 3.5: Charts on some manifolds

- $\mathbb{R}^n$ is a differentiable manifold trivially. A single chart covers the whole space and we take φ to be the identity map.

- Let $n = 1$ and let us impose connectedness. Then there are two choices, either $\mathbb{R}$ or the circle S^1. Let us work out an atlas for S^1. For concreteness let us embed the circle in $\mathbb{R}^2$ via $x^2 + y^2 = 1$. We will need at least two charts. We can take them as in figure 6. Define $\varphi_1^{-1} : (0, 2\pi) \to S^1$ by[a]

$$\varphi_1^{-1} : \theta \to (\cos\theta, \sin\theta)\,, \tag{3.7}$$

whose image is $S^1 - \{(1, 0)\}$. Similarly define $\varphi_2^{-1} : (-\pi, \pi) \to S^1$ by

$$\varphi_2^{-1} : \theta \to (\cos\theta, \sin\theta)\,, \tag{3.8}$$

whose image is $S^1 - \{(-1, 0)\}$. Clearly both φ_i^{-1} are invertible and all the maps are continuous, thus the φ_i's are homeomorphisms. The transition functions seem trivial for this example but one must be careful to end up in the correct domain. The two charts overlap on the upper and lower hemispheres and therefore we have

$$\varphi_2(\varphi_1^{-1}(\theta)) = \begin{cases} \theta & \text{if } \theta \in (0,\pi) \\ \theta - 2\pi & \text{if } \theta \in (\pi, 2\pi) \end{cases} . \tag{3.9}$$

The transition function isn't defined at $\theta = 0$ or $\theta = \pi$, nonetheless it is smooth on each of the two overlapping open sets as required.

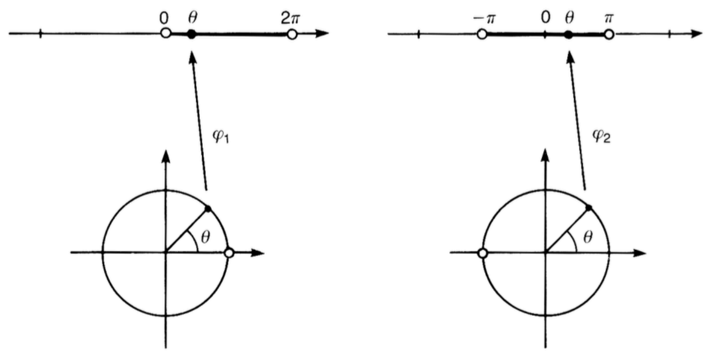


Figure 6: Two charts on S^1.

- Let us consider a slightly less trivial example, the n-dimensional sphere S^n. We may realise it by embedding it in $\mathbb{R}^{n+1}$. (Note that embedding it in a higher-dimensional space is just for convenience and not a necessary requirement for being a manifold, in fact some n-dimensional spaces cannot be embedded in $\mathbb{R}^{n+1}$, for example hyperbolic space.)

  We can realise the n-dimensional sphere S^n in $\mathbb{R}^{n+1}$ as

$$\sum_{i=0}^{n} (x^i)^2 = 1 . \tag{3.10}$$

  We can introduce coordinate neighbourhoods

$$\begin{aligned} U_{i+} \equiv & \{(x^0, x^1, ..., x^n) \in S^n | x^i > 0\} , \\ U_{i-} \equiv & \{(x^0, x^1, ..., x^n) \in S^n | x^i < 0\} . \end{aligned} \tag{3.11}$$

  Next define the coordinate map $\varphi_{i+} : U_{i+} \to \mathbb{R}^n$ to be

$$\varphi_{i+}(x^0, ..., x^n) = (x^0, ..., x^{i-1}, x^{i+1}, ..., x^n) , \tag{3.12}$$

  and $\varphi_{i-} : U_{i-} \to \mathbb{R}^n$ to be

$$\varphi_{i-}(x^0, ...x^{i-1}, x^{i+1}, ..., x^n) . \tag{3.13}$$

Note that the domains of φ_{i+} and φ_{i-} are different and they have no overlap. Instead they are the projections of the hemispheres $U_{i\pm}$ to the plane $x^i = 0$. The transition functions can be obtained simply from the above maps. As an example let us take S^2, then we have six coordinate neighbourhoods: $U_{x\pm}, U_{y\pm}, U_{z\pm}$. The transition function $\psi_{(y-)(x+)} \equiv \varphi_{y-} \circ \varphi_{x+}^{-1}$ is given by

$$\psi_{(y-)(x+)} : (y, z) \to \left(\sqrt{1 - y^2 - z^2}, z\right) . \tag{3.14}$$

This is infinitely differentiable on $U_{x+} \cap U_{y-}$.

[a]Until now we would just have taken the range to be $\theta \in [0, 2\pi)$ and been happy with this. However this does not meet our requirement of being a chart since it is not an open set. This would present problems later when we try to differentiate anything at $\theta = 0$. Recall that the derivative requires us to be able to take limits from both sides, and since there is nothing smaller than 0 we are stuck.

We have seen that to describe n-dimensional spheres we need more than one chart. The need to deal with multiple charts arises when we consider manifolds of non-trivial topology. When we come to discuss general relativity we will care a lot about changing coordinates and the limitations of the coordinate systems. In almost all situations that we will consider a single set of coordinates generally covers enough of the space to tell us everything we need to know. However as one progresses in physics, topology becomes more important. We will not see much of this but you may see this in some of your other physics/mathematics courses.

3.2 Calculus on manifolds

The reason why differentiable manifolds are useful is because it allows us to use the usual calculus we have developed on $\mathbb{R}^n$ for curved backgrounds. One of the key requirements is smoothness of the transition functions, which implies that the calculus is independent of the chosen coordinates.

3.2.1 Differentiable maps

Let $f : M \to N$ be a map from an m-dimensional manifold M to an n-dimensional manifold N. A point $p \in M$ is mapped to a point $f(p) \in N$. We may take a chart (U, φ) on M and a chart (V, ψ) in N where for all $p \in U$, $f(p) \in V$. Moreover let $\varphi(U) = U' \subset \mathbb{R}^m$ and $\psi(V) = V' \subset \mathbb{R}^n$. Then f has the following coordinate presentation:

$$\psi \circ f \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n . \tag{3.15}$$

If we write $\varphi(p) = \{x^\mu\}$ and $\psi\big(f(p)\big) = \{y^\alpha\}$ then, $\psi \circ f \circ \varphi^{-1}$ is just the usual vector-valued function $y = \psi \circ f \circ \varphi^{-1}(x)$ of m variables. Sometimes it is useful to abuse notation and write $y = f(x)$ or $y^\alpha = f^\alpha(x^\mu)$ when we know the coordinate systems on M and N that are in use.

Definition 14 (Smooth function) *We say that a function $\hat{f} : M \to \mathbb{R}$ is* smooth *if the map $\hat{f} \circ \varphi^{-1} : U' \to \mathbb{R}$ is smooth for all charts. We let the set of all smooth functions on M be denoted by $\mathcal{F}(M)$.*

Definition 15 (Smooth map, Differentiable map) *We say that a map $f : M \to N$ between two manifolds is smooth if the map $\psi \circ f \circ \varphi^{-1} : U' \to V'$ is smooth for all charts $\varphi : M \to \mathbb{R}^m$ and $\psi : N \to \mathbb{R}^n$. If $y = \psi \circ f \circ \varphi^{-1}(x)$ is C^∞ then we say that f is* differentiable *at p. This is actually independent of the coordinate system.*

Note that our definition of a smooth function is a particular case of a smooth map as defined directly above. There we have taken $N = \mathbb{R}$ and therefore we do not need the second coordinate map ψ.

Definition 16 (Diffeomorphism) Definition *Let $f : M \to N$ be a homeomorphism and ψ and φ coordinate functions. If $\psi \circ f \circ \varphi^{-1}$ is invertible and both are C^∞, then f is called a* diffeomorphism *and M is said to be* diffeomorphic *to N and vice-versa. This is denoted by $M \equiv N$.*

Since the map is invertible it follows that if $M \equiv N$ then $\dim M = \dim N$. Homeomorphisms classify spaces according to whether it is possible to deform one space into another *continuously*. *Diffeomorphisms* classify spaces into equivalence classes according to whether it is possible to deform one space into the other *smoothly*. As such a diffeomorphism is stronger than a homeomorphism, it requires that both the map and its inverse are smooth. Two diffeomorphic manifolds are viewed as the same manifold.

The question as to whether a homeomorphism is a diffeomorphism is quite subtle and far beyond the scope of this course, but let us give a small taste. Given a differentiable structure on your manifold, this is defined as an equivalence of atlases, one must then ask is it possible that a manifold admits different differentiable structures. This would mean that we can pick two atlases on a manifold which are *not compatible*. Finding such examples is non-trivial and it is known that it is only possible if $\dim(M) \geq 4$. In 1956 Milnor showed that the seven-sphere S^7 admits 28 differentiable structures. More bizarrely it has been shown that $\mathbb{R}^4$ has uncountably many pairwise non-diffeomorphic open subsets each of which is homeomorphic to $\mathbb{R}^4$. We will assume that our manifold admits a unique differentiable structure for simplicity.

3.2.2 Tangent Vectors

Having defined maps on a manifold we can define other objects on the manifold. The elementary notion of a vector no longer works: where is the origin? what is a straight line? etc.. On a manifold a vector is defined to be a *tangent vector* to a curve in M.

To define a tangent vector we need a curve $\gamma : (a, b) \to M$ and a function $f : M \to \mathbb{R}$. For simplicity let $0 \in (a, b)$ and let us parametrise our curve by t. We define the *tangent vector at* $\gamma(0)$ to be the directional derivative of a function $f(\gamma(t))$ along the curve $\gamma(t)$ at $t = 0$. The tangent vector to γ at $\gamma(0)$ is the linear map X_p from the space of smooth functions of M to $\mathbb{R}$ defined by

$$X_p[f] = \frac{\mathrm{d}f(\gamma(t))|}{\mathrm{d}t}\bigg|_{t=0} . \tag{3.16}$$

Note that this satisfies two important properties.

1. It is linear. For any smooth functions f, g and constant α we have $X_p[f + g] = X_p[f] + X_p[g]$ and $X_p[\alpha f] = \alpha X_p[f]$.

2. It satisfies the Leibniz rule: $X_p[fg] = fX_p[g] + X_p[f]g$.

In terms of local coordinates we have

$$X_p[f] = \frac{\partial f}{\partial x^\mu}\frac{\mathrm{d}x^\mu(\gamma(t))}{\mathrm{d}t}\bigg|_{t=0} . \tag{3.17}$$

Notice the abuse of notation, the first term should really be

$$\frac{\partial\big(f \circ \varphi^{-1}(x)\big)}{\partial x^\mu} , \tag{3.18}$$

we will often employ this abuse of notation.

> **Exercise 3.2: Tangent vectors at p**
> Show that the set of all tangent vectors at p forms an n-dimensional vector space $T_p(M)$.

We have just found that the tangent vector can be written as

$$X_p = X_p^\mu\left(\frac{\partial}{\partial x^\mu}\right), \qquad X_p^\mu = \frac{\mathrm{d}x^\mu(\gamma(t))}{\mathrm{d}t}\bigg|_{t=0} \tag{3.19}$$

then

$$\frac{\mathrm{d}f(\gamma(t))|}{\mathrm{d}t}\bigg|_{t=0} = X^\mu\frac{\partial f}{\partial x^\mu} \equiv X_p[f] . \tag{3.20}$$

We define X_p to be the tangent vector to M at $p = \gamma(0)$ along the direction given by the curve $\gamma(t)$. There is then a natural basis of $T_p(M)$ for us to take given by

$$\left\{\frac{\partial}{\partial x^\mu}\bigg|_p, \mu = 1, ..., \dim(M)\right\} . \tag{3.21}$$

This is chart dependent since we chose a chart in the neighbourhood of p. Choosing a different chart would give a different basis of $T_p(M)$. The basis defined in this way is called the *coordinate basis*.

Let us see how the coordinate basis changes when we choose different coordinates. Let $p \in U_i \cap U_j$ and let $x = \varphi_i(p)$ and $x' = \varphi_j(p)$ be two charts defined in the neighbourhood of the point p. For a smooth function f we have:

$$\begin{aligned} \left(\frac{\partial}{\partial x^\mu}\right)\bigg|_p [f] &= \frac{\partial}{\partial x^\mu}(f \circ \varphi^{-1})\bigg|_{\varphi(p)} \\ &= \frac{\partial}{\partial x^\mu}[(f \circ \varphi'^{-1}) \circ (\varphi' \circ \varphi^{-1})]\bigg|_{\varphi(p)} . \end{aligned} \tag{3.22}$$

Let $F' = f \circ \varphi'^{-1}$, this is a function in the x' coordinates. Moreover note that $\varphi' \circ \varphi^{-1}$ is simplify the functions $x'^\mu(x)$, that is the primed coordinates in terms of the unprimed ones. Hence it follows that we have:

$$\begin{aligned} \left(\frac{\partial}{\partial x^\mu}\right)\bigg|_p [f] &= \frac{\partial}{\partial x^\mu}\left[F'(x'(x))\right]\bigg|_{\varphi(p)} \\ &= \frac{\partial x'^\nu}{\partial x^\mu}\bigg|_{\varphi(p)} \frac{\partial F'(x')}{\partial x'^\nu}\bigg|_{\varphi'(p)} \\ &= \frac{\partial x'^\nu}{\partial x^\mu}\bigg|_{\varphi(p)} \frac{\partial}{\partial x'^\nu}\bigg|_p [f] , \end{aligned} \tag{3.23}$$

and therefore we have

$$\frac{\partial}{\partial x^\mu}\bigg|_p = \frac{\partial x'^\nu}{\partial x^\mu}\bigg|_{\varphi(p)} \frac{\partial}{\partial x'^\nu}\bigg|_p . \tag{3.24}$$

This then defines the transformation of one basis into another. It is now straightforward to work out the transformation of the components of a vector X^μ into the components of another basis X'^μ by using that the vector X is invariant. It follows that:

$$X'^\nu = X^\mu \frac{\partial x'^\nu}{\partial x^\mu}\bigg|_{\varphi(p)} . \tag{3.25}$$

Components of vectors that transform in this way are known as *contravariant*.

We have defined a tangent vector X as a differential operator acting on functions along a curve passing through the point p, but there is some redundancy in this since two curves passing can give the same tangent vector at p. This leads use to define an equivalence class of curves on M.

Definition 17 (Equivalence class of curves) *If two curves $\gamma_1(t)$ and $\gamma_2(t)$ satisfy* *(i)* $\gamma_1(0) = \gamma_2(0) = p$,

(ii) $\frac{\mathrm{d}x^\mu(\gamma_1(t))}{\mathrm{d}t}\Big|_{t=0} = \frac{\mathrm{d}x^\mu(\gamma_2(t))}{\mathrm{d}t}\Big|_{t=0}$,

then $\gamma_1(t)$ and $\gamma_2(t)$ yield the same differential operator X at p. This allows us to define the equivalence relation between curves at the point p, $\gamma_1(t) \sim \gamma_2(t)$. We identify the tangent vector *X with the* equivalence class of curves

$$[\gamma(t)] = \left\{ \tilde{\gamma}(t) \Big| \gamma(0) = \tilde{\gamma}(0) \text{ and } \frac{\mathrm{d}x^\mu(\gamma(t))}{\mathrm{d}t}\Big|_{t=0} = \frac{\mathrm{d}x^\mu(\tilde{\gamma}(t))}{\mathrm{d}t}\Big|_{t=0} \right\} \tag{3.26}$$

rather than a particular representative of the curve.

All the equivalence classes of curves at a point $p \in M$, i.e. all the tangent vectors at p, form a vector space called the *tangent space* of M at p, $T_p(M)$. From our discussion above we can take the basis of vectors for $T_p(M)$ to be $e_\mu = \frac{\partial}{\partial x^\mu}$. It follows that $\dim T_p(M) = \dim(M)$.

It is often convenient to use the coordinate basis, however the basis clearly depends on the coordinates that we are using. At times it is convenient to use other bases, one such basis which is a 'non-coordinate bases' is known as vielbeins. These are necessary when defining spinors on curved backgrounds and can often make computations simpler. We will introduce these later in the problem sheets.

Note that for two distinct points p and q the tangent spaces $T_p(M)$ and $T_q(M)$ are different. We cannot add vectors from one to a vector in the other. In fact even to compare the vectors in $T_p(M)$ with the vectors in $T_q(M)$ we need to introduce the notion of *parallel transport*, which will appear shortly.

3.2.3 Vector fields

So far we have defined tangent vectors at a single point, but what we really want is an object with a choice of tangent vector for every point in our manifold. These objects are called fields.

A vector field X is defined to be a smooth assignment of tangent vectors X_p to each point $p \in M$. Here we mean that if we feed a function into the vector field then we obtain another function which is the differentiation of the first. The vector field is then smooth if, starting with our smooth function f we obtain another smooth function $X[f]$. Therefore a vector field defines a map $X : C^\infty(M) \to C^\infty(M)$. To evaluate $X[f]$ at a point p we have

$$X[f]\Big|_p = X_p[f]\,. \tag{3.27}$$

We denote the space of all vector fields on M to be $\mathcal{X}(M)$.

3.2.4 One-forms

Since $T_p(M)$ is a vector space, there exists a dual vector space to $T_p(M)$ whose element is a linear function from $T_p(M) \to \mathbb{R}$. The dual space is called the *cotangent space* at p, and is denoted by $T_p^*(M)$. An element ω_p of $T_p^*(M)$ is a map $\omega_p : T_p(M) \to \mathbb{R}$ and is called a *dual vector/cotangent vector* or in the context of differential forms a *one-form*. The simplest example of a one-form is the differential $\mathrm{d}f$ for a smooth function f on M. The action of a vector V_p on f is $V_p[f] = V_p^\mu \frac{\partial f}{\partial x^\mu} \in \mathbb{R}$. The action of $\mathrm{d}f \in T_p^*(M)$ on $V_p \in T_p(M)$ is defined by

$$\langle \mathrm{d}f, V_p \rangle \equiv V_p[f] = V_p^\mu \frac{\partial f}{\partial_{x^\mu}} \in \mathbb{R}\,. \tag{3.28}$$

This is then $\mathbb{R}$-linear in both V_p and f. In terms of the coordinate basis we have

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu\,, \tag{3.29}$$

and it is natural to regard $\{\mathrm{d}x^\mu\}$ as a basis of $T_p^*(M)$. This is a dual basis since

$$\left\langle \mathrm{d}x^\mu, \frac{\partial}{\partial x^\nu} \right\rangle = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu\,. \tag{3.30}$$

We can then write an arbitrary one-form as

$$\omega_p = \omega_{p,\mu}\mathrm{d}x^\mu\,. \tag{3.31}$$

If we take a vector V_p and a one-form ω_p we may define the *inner product* between one-forms and vectors $\langle\ ,\ \rangle : T_p^*(M) \times T_p(M) \to \mathbb{R}$ to be

$$\langle \omega_p, V_p \rangle = \omega_{p,\mu} V^\nu \left\langle \mathrm{d}x^\mu, \frac{\partial}{\partial x^\nu} \right\rangle = \omega_{p,\mu} V_p^\nu \delta^\mu_\nu = \omega_{p,\mu} V^\mu\,. \tag{3.32}$$

The inner product is defined between a vector and a covector. Since ω_p is defined without reference to any coordinate system for a point $p \in U_i \cap U_j$ we have

$$\omega_p = \omega_{p,\mu}\mathrm{d}x^\mu = \omega'_{p,\mu}\mathrm{d}x'^\mu\,, \tag{3.33}$$

with x and x' as before. Then we have

$$\omega'_\nu = \omega_\mu \frac{\partial x^\mu}{\partial x'^\nu}\,, \tag{3.34}$$

which is the transformation of the components of a co-vector.

We will often denote the set of one-forms to be $\Omega^{(1)}(M)$.

3.2.5 Tensors

Definition 18 (Tensor) *A* tensor *at the point p of type (q,r) is a multilinear object which maps q elements of $T_p^*(M)$ and r elements of $T_p(M)$ to $\mathbb{R}$:*

$$T : \otimes^q T_p^*(M) \otimes^r T_p(M) \to \mathbb{R}\,. \tag{3.35}$$

We define $\mathcal{T}_p^{(q,r)}(M)$ to be the set of (q,r) tensors at $p \in M$.

An element of $\mathcal{T}^{(q,r)}(M)$ can be written in terms of the coordinate bases described above as

$$T = T^{\mu_1 \ldots \mu_q}{}_{\nu_1 \ldots \nu_r} \frac{\partial}{\partial x^{\mu_1}} \ldots \frac{\partial}{\partial x^{\mu_q}} \mathrm{d}x^{\nu_1} \ldots \mathrm{d}x^{\nu_r}\,. \tag{3.36}$$

Let $V_i = V_i^\mu \frac{\partial}{\partial x^\mu}$ with $1 \le i \le r$ and $\omega_j = \omega_{j\mu} \mathrm{d}x^\mu$ with $1 \le j \le q$ then the action of T is

$$T(\omega_1, \ldots, \omega_q; V_1, \ldots . V_r) = T^{\mu_1 \ldots \mu_q}{}_{\nu_1 \ldots \nu_r}\, \omega_{1\mu_1} \ldots . \omega_{q\mu_q} V_1^{\nu_1} \ldots . V_r^{\nu_r}\,. \tag{3.37}$$

The transformation of the components of a tensor under a change of coordinates follows simply from the transformation of the components of a contravariant vector and a covariant vector. We have

$$\boxed{T'^{\mu'_1 .. \mu'_q}{}_{\nu'_1 .. \nu'_r} = T^{\mu_1 .. \mu_q}{}_{\nu_1 .. \nu_r} \frac{\partial x'^{\mu'_1}}{\partial x^{\mu_1}} \ldots \frac{\partial x'^{\mu'_q}}{\partial x^{\mu_q}} \frac{\partial x^{\nu_1}}{\partial x'^{\nu'_1}} \ldots \frac{\partial x^{\nu_r}}{\partial x'^{\nu'_r}}\,.} \tag{3.38}$$

Note that in all cases given the placement of indices this is really the only type of transformation that we can really have.

As before we can define a *Tensor field* of type (q,r) to be a smooth assignment of an element of $\mathcal{T}_p^{q,r}(M)$ at each point $p \in M$. The set of tensor fields of type (q,r) on M is denoted by $\mathcal{T}^{q,r}(M)$.

Operations on Tensor fields There are a variety of different operations we can do on tensor fields to generate new tensor fields. We can add or subtract them, multiply by smooth functions. This is just the statement that tensors at a point p form a vector space. These operations preserve the rank of the tensor but there are operations we can do which do not preserve the rank.

Given a tensor field S of rank (q,r) and a tensor field T of rank (s,t) we may take the tensor product of the two tensor fields together to form a tensor field $S \otimes T$ of rank $(q+s, r+t)$. The *tensor product* $S \otimes T$ is defined by

$$S \otimes T(\omega_1, .., \omega_q, \eta_1, .., \eta_s, X_1, \ldots, X_r, Y_1, \ldots, Y_t) = S(\omega_1, .., \omega_q, X_1, \ldots, X_r) T(\eta_1, .., \eta_s, Y_1, \ldots, Y_t)\,. \tag{3.39}$$

In terms of components we have:

$$(S\otimes T)^{\mu_1...\mu_q\nu_1..\nu_s}{}_{\rho_1...\rho_r\sigma_1...\sigma_t} = S^{\mu_1...\mu_q}{}_{\rho_1...\rho_r}T^{\nu_1..\nu_s}{}_{\sigma_1...\sigma_t} \ . \tag{3.40}$$

Given an (q,r) tensor we may define a $(q-1,r-1)$ tensor through *contraction*. To do this we replace one of the $T^*_p(M)$ entries with a basis vector $\hat{\theta}^\mu$ and the corresponding $T_p(M)$ entry with the dual vector $\hat{e}_\mu$ and then sum over the indices. For example we have

$$S(\omega_1, ..\omega_{q-1}, X_1, ..., X_{r-1}) = \sum_{\mu=1}^{n} T(\omega_1, ..\omega_{q-1}, \hat{\theta}^\mu, X_1, ..., X_{r-1}, \hat{e}_\mu)\,. \tag{3.41}$$

The placement of either of the basis vector/co-vector is arbitrary and moving it around we obtain a typically different tensor, again of rank $(q-1,r-1)$. In terms of components we have

$$S^{\mu_1\cdots\mu_{q-1}}{}_{\nu_1..\nu_{r-1}} = T^{\mu_1\cdots\mu_{q-1}\sigma}{}_{\nu_1..\nu_{r-1}\sigma} \ . \tag{3.42}$$

Another operation we can do is to symmetrise and anti-symmetrise the tensor. For example given a rank $(0,r)$ tensor T we can define the symmetrisation of T to be

$$S[T(X_1, ..., X_r)] = \frac{1}{r!}\sum_{\sigma\in S_r} T(X_{\sigma(1)}, ..., X_{\sigma(r)})\,, \tag{3.43}$$

and the anti-symmetrisation to be

$$A[T(X_1, .., X_r)] = \frac{1}{r!}\sum_{\sigma\in S_r} \mathrm{sign}(\sigma)T(X_{\sigma(1)}, ..., X_{\sigma(r)})\,, \tag{3.44}$$

where S_r is the permutation group of r objects. In components we have

$$S[T]_{\mu_1...\mu_r} = \frac{1}{r!}\sum_{\sigma\in S_r} T_{\mu_{\sigma(1)}\cdots\mu_{\sigma(r)}} \equiv T_{(\mu_1...\mu_r)}\,, \tag{3.45}$$

and

$$A[T]_{\mu_1...,\mu_r} = \frac{1}{r!}\sum_{\sigma\in S_r} \mathrm{sign}(\sigma)T_{\mu_{\sigma(1)}\cdots\mu_{\sigma(r)}} \equiv T_{[\mu_1...\mu_r]} \tag{3.46}$$

Note that it only makes sense to symmetrise and anti-symmetrise over objects of the same type, or the same placement of indices. We also divide by the symmetry factor $r!$ which is the number of permutations, this ensures that if an object is (anti-)symmetric then (anti-)symmetrisation acts as the identity.

3.2.6 Induced maps

Definition 19 (Push-forward/differential map) *A smooth map $f : M \to N$ naturally induces a map f_* called the* differential map *or* push-forward*,*

$$f_* : T_p(M) \to T_{f(p)}(N)\,. \tag{3.47}$$

The explicit form of f_* is obtained by the definition of the directional derivative along a curve. Let $g \in \mathcal{F}(N)$ then $g \circ f \in \mathcal{F}(M)$. A vector $V \in T_p(M)$ acts on $g \circ f$ to give a number $V[g \circ f]$. We can now define $f_* V \in T_{f(p)}(N)$ by

$$(f_* V)[g] \equiv V[g \circ f]\,. \tag{3.48}$$

See figure 7 for a pictorial representation.

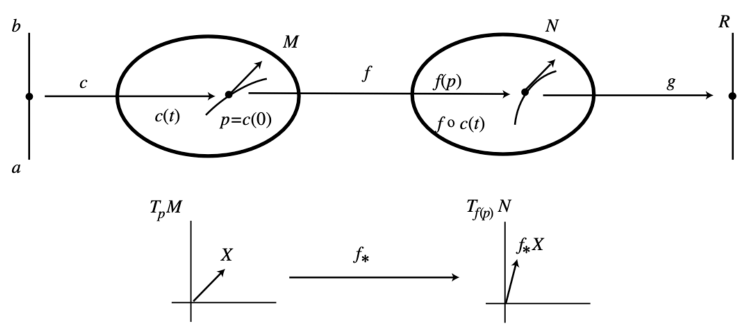


Figure 7: A map $f : M \to N$ induces the differential map $f_* : T_p(M) \to T_{f(p)}(N)$. Note that the mapping is performed by mapping the curve $c(t)$ between the two manifolds using the map f.

[**Aside:** Note that we do not require the map f to be invertible here, just smooth, so it need not be a diffeomorphism. We will however require this later so let us just assume it from now on.]

We can write this more explicitly by introducing coordinates. Let us introduce the charts (U, φ) on M and (V, ψ) on N, then

$$(f_* V)[g \circ \psi^{-1}(y)] = V[g \circ f \circ \varphi^{-1}(x)]\,, \tag{3.49}$$

where $x = \varphi(p)$ and $y = \psi(f(p))$. Let $V = V^\mu \frac{\partial}{\partial x^\mu}$ and $f_* V = W^\alpha \frac{\partial}{\partial y^\alpha}$, then in components it reads

$$W^\alpha = V^\mu \frac{\partial}{\partial x^\mu} y^\alpha(x)\,. \tag{3.50}$$

This is nothing but the Jacobian of the map $f : M \to N$ that we are all familiar with. So far we have done this for vector fields, but this can easily be extended to tensors of type $(q, 0)$ without additional thought. Let us consider an example to put this into practice:

Example 3.6: Push forward

Let (x^1, x^2) and (y^1, y^2, y^3) be coordinates on M and N respectively, and let $V = a\frac{\partial}{\partial x^1} + b\frac{\partial}{\partial x^2}$. Take the map $f : M \to N$ whose coordinate representation is

$$f(x^1, x^2) = \left(x^1, x^2, \sqrt{1-(x^1)^2-(x^2)^2}\right). \tag{3.51}$$

Then

$$f_*V = V^\mu \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial}{\partial y^\alpha} = a\frac{\partial}{\partial y^1} + b\frac{\partial}{\partial y^2} - \left(a\frac{y^1}{y^3} + b\frac{y^2}{y^3}\right)\frac{\partial}{\partial y^3}. \tag{3.52}$$

Definition 20 (Pull-back) *A smooth map f also induces a map between cotangent space*

$$f^* : T^*_{f(p)}(N) \to T^*_p(M), \tag{3.53}$$

which is called the pull-back. *If we take $V \in T_p(M)$ and $\omega \in T^*_{f(p)}(N)$ then the pull-back of ω by f^* is defined to be*

$$\langle f^*\omega, V\rangle = \langle \omega, f_*V\rangle. \tag{3.54}$$

In components we have

$$(f^*\omega)_\mu = \omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu}. \tag{3.55}$$

The pull-back can be extended to tensors of type $(0, r)$.

We have defined the push-forward and pull-back for tensors of type $(q, 0)$ and $(0, r)$ respectively. Note that the push-forward induces a map in the same direction as the map f while the pull-back induces a map in the opposite direction to the original map. If f is a diffeomorphism $f : M \to N$, then we also have the inverse $f^{-1} : N \to M$ and therefore we can transport any object from M to N and back to our hearts content.

3.3 Flows and Lie derivatives

Definition 21 (Integral Curve) *Let X be a vector field on M. An* integral curve $x(t)$ *of X is a curve in M whose tangent vector at $x(t)$ is $X|_x$. Given a chart (U, φ), this means that*

$$\frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t} = X^\mu(x(t)), \tag{3.56}$$

where $x^\mu(t)$ is the μ'th component of $\varphi(x(t))$ and $X = X^\mu\frac{\partial}{\partial x^\mu}$.

As always we have very much abused notation, using x to denote a point in M as well as its coordinates. Finding an Integral curve is equivalent to solving the ODE with initial conditions $x^\mu(0) = x_0^\mu$. The existence and uniqueness theorems for ODEs implies that there is always a unique solution, at least locally, with the given initial data.

Definition 22 (Flow) *Let $\sigma(t, x_0)$ be an integral curve of X which passes through the point x_0 at $t = 0$, and denote the coordinate by $\sigma^\mu(t, x_0)$. The flow equation becomes*

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma^\mu(t, x_0) = X^\mu(\sigma(t, x_0))\,, \tag{3.57}$$

with the initial condition

$$\sigma^\mu(0, x_0) = x_0^\mu\,. \tag{3.58}$$

Then the map $\sigma : \mathbb{R} \times M \to M$ is called a flow *generated by $X \in \mathcal{X}(M)$.*

From the definition one can show that a flow satisfies the rule

$$\sigma(t, \sigma^\mu(s, x_0)) = \sigma(t + s, x_0)\,, \tag{3.59}$$

for any $s, t \in \mathbb{R}$, such that both sides make sense. This follows from the uniqueness of the ODE with fixed initial condition.

Theorem 2 *For any point $x \in M$, there exists a differentiable map $\sigma : \mathbb{R} \times M \to M$ such that*

(i) $\sigma(0, x) = x$,

(ii) $t \mapsto \sigma(t, x)$ *is a solution of* (3.57) *and* (3.58),

(iii) $\sigma(t, \sigma^\mu(s, x)) = \sigma(t + s, x)$.

Note that the initial point is denoted by x to emphasise that σ is a map $\mathbb{R} \times M \to M$.

We may imagine a flow as a the flow of a steady stream. If a particle is observed at a point x at $t = 0$ it will be found at $\sigma(t, x)$ at later time t.

see exmaple 3.3

Example 3.7: Integral Curves

- Let $M = \mathbb{R}^2$ and let $X((x, y)) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ be a vector field in M. Then

$$\sigma(t, (x, y)) = (x\cos t - y\sin t, x\sin t + y\cos t)\,, \tag{3.60}$$

is a flow generated by X. The flow through (x,y) is a circle whose centre is at the origin. Clearly $\sigma(t,(x,y)) = (x,y)$ if $t = 2\pi n, n \in \mathbb{Z}$. If $(x,y) = (0,0)$, the flow stays at $(0,0)$.

- Consider the sphere S^2 in polar coordinates with the vector field $X = \partial_\phi$. The integral curves are:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = 1\,, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = 0\,, \tag{3.61}$$

and this has solution:

$$\theta = \theta_0\,, \quad \phi = \phi_0 + t\,. \tag{3.62}$$

The associated diffeomorphism is $\sigma_t : (\theta,\phi) \to (\theta, \phi + t)$, and the flow lines are simply the lines of constant latitude on the sphere.

3.3.1 One-parameter group of transformations

For fixed $t \in \mathbb{R}$ a flow $\sigma(t,x)$ is a diffeomorphism from M to M which we denoted by $\sigma_t : M \to M$. This map is made into a commutative group by the following rules:

1. $\sigma_t(\sigma_s(x)) = \sigma_{t+s}(x)$ i.e. $\sigma_t \circ \sigma_s = \sigma_{t+s}$, (Associative)
2. σ_0=identity map (Unit element),
3. $\sigma_{-t} = (\sigma_t)^{-1}$ (Inverse).

Exercise 3.3: Flow defines a commutative group
Show that a flow defines a commutative group.

This group is called the *one-parameter group of transformations*. Locally the group looks like the additive group $\mathbb{R}$, although they may not be isomorphic globally. For example in the example above (see equation (3.60)) we had that $\sigma_{2\pi n+t} = \sigma_t$ and we find that the one-parameter group is isomorphic to SO(2) the multiplicative group of 2×2 real matrices of the form;

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3.63}$$

or U(1) the multiplicative group of complex numbers of unit modulus $\mathrm{e}^{\mathrm{i}\theta}$.

We can consider an infinitesimal transformation and see where it maps the point x. Using (3.57) and (3.58) we find

$$\sigma^\mu_\epsilon(x) = \sigma^\mu(\epsilon, x) = x^\mu + \epsilon X^\mu(x)\,. \tag{3.64}$$

The vector field X in this context is called the *infinitesimal generator* of the transformation σ_t.

Given a vector field X the corresponding flow σ is often referred to as the *exponentiation* of X and is denoted by

$$\sigma^\mu(t,x) = \exp(tX)x^\mu \,. \tag{3.65}$$

To see why this is so, let us take a parameter t and evaluate the coordinate of a point which is separated from the initial point $x = \sigma(0,x)$ by the parameter distance t along the flow σ. The coordinate corresponding to the point $\sigma(t,x)$ is

$$\begin{aligned}\sigma^\mu(t,x) &= x^\mu + t\frac{\mathrm{d}}{\mathrm{d}s}\sigma^\mu(s,x)\Big|_{s=0} + \frac{t^2}{2!}\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^2\sigma^\mu(s,x)\Big|_{s=0} + \dots \\ &= \left[1 + t\frac{\mathrm{d}}{\mathrm{d}s} + \frac{t^2}{2!}\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^2 + \dots\right]\sigma^\mu(s,x)\Big|_{s=0} \\ &\equiv \exp\Big(t\frac{\mathrm{d}}{\mathrm{d}s}\Big)\sigma^\mu(s,x)\Big|_{s=0} \,.\end{aligned} \tag{3.66}$$

The last expression can also be written as $\sigma^\mu(t,x) = \exp(tX)x^\mu$ as in the definition above. Then the flow satisfies the following exponential properties:

1. $\sigma(0,x) = x = \exp(0X)x\,,$
2. $\frac{\sigma(t,x)}{\mathrm{d}t} = X\exp(tX)x = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\exp(tX)x\Big)\,,$
3. $\sigma(t,\sigma(s,x)) = \sigma(t,\exp(sX)x) = \exp(tX)\exp(sX)x = \exp\big((t+s)X\big)x = \sigma(t+s,x)\,.$

3.3.2 Lie Derivatives

We have now defined maps using flows, but what are these good for? One use is to construct the Lie derivative. This is a derivative which essentially tells us how something changes along the integral curve of a vector.

Let $\sigma(t,x)$ and $\tau(t,x)$ be two flows generated by the vector fields X and Y respectively:

$$\frac{\mathrm{d}\sigma^\mu(s,x)}{\mathrm{d}s} = X^\mu(\sigma(s,x))\,, \qquad \frac{\mathrm{d}\tau^\mu(t,x)}{\mathrm{d}t} = Y^\mu(\tau(t,x))\,. \tag{3.67}$$

Let us evaluate the change of the vector field Y along $\sigma(s,x)$. To do this we need to compare the vector Y at a point x with Y at a nearby point $x' = \sigma_\epsilon(x)$, see figure 8. We cannot simply take the difference between the components of Y at the two points since they belong to different tangent spaces: $T_x(M)$ and $T_{\sigma_\epsilon(x)}(M)$, and so the difference between the two vectors is ill-defined. To define a sensible derivative, we first map $Y|_{\sigma_\epsilon(x)}$ to $T_x(M)$ by using the push-forward $(\sigma_{-\epsilon})_* : T_{\sigma_\epsilon(x)}(M) \to T_x(M)$, after which the two vectors are in the same tangent space and we can take the difference between them, see figure 8.

Definition 23 (Lie Derivative) *The* Lie derivative *of a vector field Y along the flow σ of the vector field X is defined by*

$$\boxed{\mathcal{L}_X Y = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big[(\sigma_{-\epsilon})_* Y|_{\sigma_\epsilon(x)} - Y|_x\Big]\,. \qquad (3.68)}$$

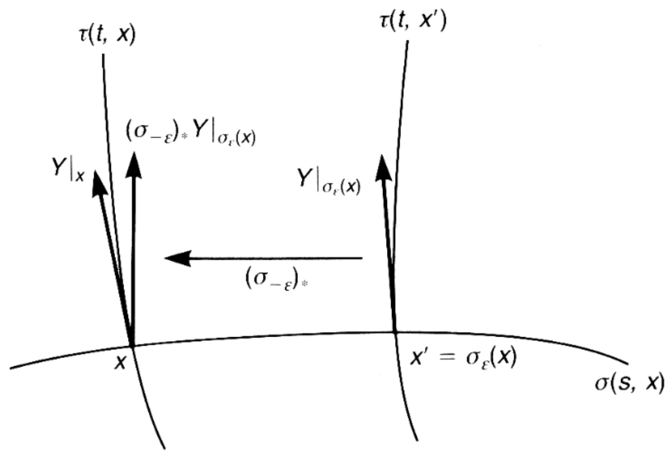


Figure 8: To compare a vector $Y|_x$ with the vector $Y|_{\sigma_\epsilon(x)}$ the latter must be transported back to x by the differential map $(\sigma_{-\epsilon})_*$, that is we use the push-forward.

This is still somewhat abstract, since it is a coordinate free expression, so let us write it in components. By writing this in components we obtain another expression for the Lie derivative of a vector field. Let (U,φ) be a chart with the coordinates x and let $X = X^\mu\frac{\partial}{\partial x^\mu}$ and $Y = Y^\mu\frac{\partial}{\partial x^\mu}$ be vector fields defined on U. Then $\sigma_\epsilon(x)$ has the coordinates $x^\mu + \epsilon X^\mu(x)$ and

$$\begin{aligned} Y|_{\sigma_\epsilon(x)} &= Y^\mu\big(x^\nu + \epsilon X^\nu(x)\big)e_\mu|_{x+\epsilon X} \\ &\simeq \Big[Y^\mu(x) + \epsilon X^\nu(x)\partial_\nu Y^\mu(x)\Big]e_\mu|_{x+\epsilon X}\,, \end{aligned} \qquad (3.69)$$

with $e_\mu = \frac{\partial}{\partial x^\mu} \equiv \partial_\mu$. Mapping this vector at $\sigma_\epsilon(x)$ to x using $(\sigma_{-\epsilon}(x))_*$ we obtain

$$\begin{aligned} (\sigma_{-\epsilon}(x))_* Y|_{\sigma_\epsilon(x)} = &\Big[Y^\mu(x) + \epsilon X^\lambda(x)\partial_\lambda Y^\mu(x)\Big]\partial_\mu\big(x^\nu - \epsilon X^\nu(x)\big)e_\nu|_x \\ =&\Big[Y^\mu(x) + \epsilon X^\lambda(x)\partial_\lambda Y^\mu(x)\Big]\Big[\delta^\nu_\mu - \epsilon\partial_\mu X^\nu(x)\Big]e_\nu|_x \\ =&Y^\mu(x)e_\mu|_x + \epsilon\big[X^\mu(x)\partial_\mu Y^\nu(x) - Y^\mu(x)\partial_\mu X^\nu(x)\big]e_\nu|_x + O(\epsilon^2)\,, \end{aligned} \qquad (3.70)$$

and therefore we find

$$\mathcal{L}_X Y = \big(X^\mu\partial_\mu Y^\nu - Y^\mu\partial_\mu X^\nu\big)e_\nu\,. \qquad (3.71)$$

This motivates the introduction of the Lie bracket.

Definition 24 (Lie bracket) *The* Lie bracket, $[\ ,\ ]$. *For vector fields X, Y on M we have*

$$[X,Y]f = X[Y[f]] - Y[X[f]]\,, \tag{3.72}$$

for all $f \in \mathcal{F}(M)$.

In components $[X, Y]$ reads

$$(X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu) e_\nu\,, \tag{3.73}$$

and in terms of the Lie bracket the Lie derivative of Y along X is

$$\mathcal{L}_X Y = [X,Y]\,. \tag{3.74}$$

Exercise 3.4: Some properties of the Lie bracket
Show that the Lie bracket defines a vector field. In addition show that it satisfies the following properties:

1. Bilinearity

$$\begin{aligned} [X, c_1 Y_1 + c_2 Y_2] &= c_1 [X, Y_1] + c_2 [X, Y_2]\,, \\ [c_1 X_1 + c_2 X_2, Y] &= c_1 [X_1, Y] + c_2 [X_2, Y]\,, \end{aligned} \tag{3.75}$$

   for any constants c_1 and c_2.

2. Skew symmetry

$$[X,Y] = -[Y,X]\,. \tag{3.76}$$

3. Jacobi Identity

$$[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0\,, \tag{3.77}$$

4. For X, Y vector fields and f a smooth function on M then

$$\begin{aligned} \mathcal{L}_{fX} Y &= f[X,Y] - Y[f]X\,, \\ \mathcal{L}_X (fY) &= f[X,Y] + X[f]Y \end{aligned} \tag{3.78}$$

5. For $f : M \to N$ then

$$f_*[X,Y] = [f_* X, f_* Y]\,. \tag{3.79}$$

Exercise 3.5: Algebra of Lie derivative
Show that for vector fields $X, Y, Z \in \mathcal{X}(M)$ we have

$$\mathcal{L}_X \mathcal{L}_Y Z - \mathcal{L}_Y \mathcal{L}_X Z = \mathcal{L}_{[X,Y]} Z\,. \tag{3.80}$$

Geometrically the Lie bracket shows the non-commutativity of two flows. Let us take the flows $\sigma(s,x)$ and $\tau(t,x)$ generated by X and Y respectively. If we first move a small parameter distance ϵ along the flow σ and then by δ along the second flow τ we end up at a point whose coordinates are

$$\begin{aligned}\tau^\mu\big(\delta,\sigma(\epsilon,x)\big) &\simeq \tau^\mu\big(\delta + x^\nu + \epsilon X^\nu(x)\big)\\ &\simeq x^\mu + \epsilon X^\mu(x) + \delta Y^\mu(x^\nu + \epsilon X^\nu)\\ &\simeq x^\mu + \epsilon X^\mu(x) + \delta Y^\mu(x) + \epsilon\delta X^\nu(x)\partial_\nu Y^\mu(x)\,.\end{aligned} \tag{3.81}$$

If we instead first move along τ and then move along σ we find

$$\sigma^\mu\big(\epsilon,\tau(\delta,x)\big) \simeq x^\mu + \delta Y^\mu(x) + \epsilon X^\mu(x) + \epsilon\delta Y^\nu(x)\partial_\nu X^\mu(x)\,. \tag{3.82}$$

The difference between the two points is proportional to the Lie bracket

$$\tau^\mu\big(\delta,\sigma(\epsilon,x)\big) - \sigma^\mu\big(\epsilon,\tau(\delta,x)\big) = \epsilon\delta[X,Y]^\mu\,. \tag{3.83}$$

The Lie bracket measures the failure of the parallelogram in figure 9 to close and therefore for the two flows to commute. It is easy to see that

$$\mathcal{L}_X Y = [X,Y] = 0 \qquad \Longleftrightarrow \qquad \sigma\big(s,\tau(t,x)\big) = \tau\big(t,\sigma(s,x)\big)\,. \tag{3.84}$$

Lie Derivative for one-forms: We now want to define the Lie derivative of a one-form $\omega \in \Omega^1(M)$ along X. This time we need to use the pull-back, and the Lie derivative of the one-form ω is

$$\boxed{\mathcal{L}_X\omega \equiv \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big[(\sigma_\epsilon)^*\omega|_{\sigma_\epsilon(x)} - \omega|_x\Big]\,,} \tag{3.85}$$

where $\omega|_x \in T_x^*(M)$ is ω at x. Introducing coordinates such that $\omega = \omega_\mu \mathrm{d}x^\mu$, we have

$$(\sigma_\epsilon)^*\omega|_{\sigma_\epsilon(x)} = \omega_\mu(x)\mathrm{d}x^\mu + \epsilon\big[X^\nu(x)\partial_\nu\omega_\mu(x) + \partial_\mu X^\nu(x)\omega_\nu(x)\big]\mathrm{d}x^\mu\,, \tag{3.86}$$

which leads to

$$\mathcal{L}_X\omega = \big(X^\nu\partial_\nu\omega_\mu + \partial_\mu X^\nu\omega_\nu\big)\mathrm{d}x^\mu\,. \tag{3.87}$$

This remains a one-form, that is $\mathcal{L}_X\omega \in T_x^*(M)$ since it is the difference of two one-forms at the same point.

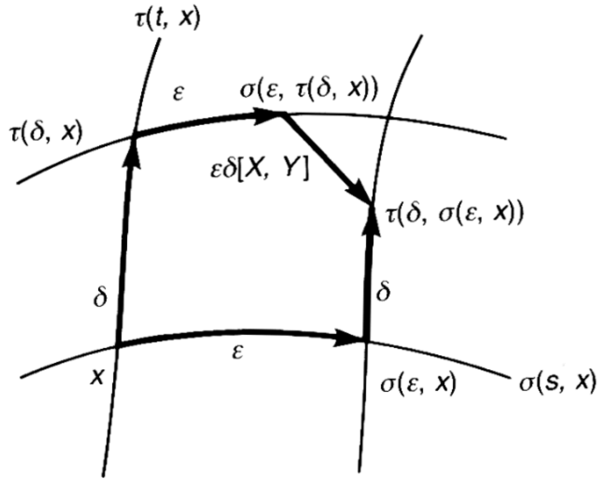


Figure 9: Moving first along the flow σ and then the flow τ or first along τ and then along σ we find that we may not end up at the same point. The difference is measured by the failure of the Lie bracket to vanish.

This may also be extended to functions f on M. Then

$$\begin{aligned}\mathcal{L}_X f &\equiv \lim_{\epsilon\to 0}\frac{1}{\epsilon}\big[f(\sigma_\epsilon(x)) - f(x)\big]\\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\big[f(x^\mu + \epsilon X^\mu(x)) - f(x^\mu)\big]\\ &= X^\mu(x)\frac{\partial f}{\partial x^\mu} = X[f]\,,\end{aligned} \tag{3.88}$$

which is just the usual directional derivative of f along X.

To extend this to more general tensors we need the following result:

Exercise 3.6: Properties of the Lie Derivative on tensors

1. Show that the Lie derivative satisfies:

$$\mathcal{L}_X(t_1 + t_2) = \mathcal{L}_X t_1 + \mathcal{L}_X t_2\,, \tag{3.89}$$

   where t_1 and t_2 are tensor fields of the same type.

2. Show that

$$\mathcal{L}_X(t_1\otimes t_2) = \big(\mathcal{L}_X t_1\big)\otimes t_2 + t_1\otimes\big(\mathcal{L}_X t_2\big)\,, \tag{3.90}$$

   with t_1 and t_2 tensors of arbitrary type.

3.4 Differential forms

Not all tensors are created equally, some will play a more prominent role than others. One class of interesting tensors are the p-forms, these are totally anti-symmetric $(0,p)$ tensor fields. To define them we must recall the definition of the anti-symmetrisation of a tensor. Consider $\omega \in \mathcal{T}_p^{(0,r)}(M)$, then its (total) anti-symmetrisation is given by:

$$A[\omega(X_1,...,X_r)] = \frac{1}{r!}\sum_{\sigma\in S_r} \text{sign}(\sigma)\omega(X_{\sigma(1)},..,X_{\sigma(r)})\,. \tag{3.91}$$

with $\text{sign}(\sigma)=+1$ for an even permutation and -1 for an odd permutation.

Definition 25 (Differential form) *A* differential form *of order r, or more succinctly an r-form, is a totally anti-symmetric tensor of type $(0,r)$.*

Definition 26 (Wedge product) *The* Wedge product $\wedge$ *of r one-forms is defined to be the totally anti-symmetric tensor product of the one-forms*

$$\mathrm{d}x^{\mu_1}\wedge\mathrm{d}x^{\mu_2}\wedge...\mathrm{d}x^{\mu_r} \equiv \sum_{\sigma\in S_r} sign(\sigma)\mathrm{d}x^{\mu_{\sigma(1)}}\otimes\mathrm{d}x^{\mu_{\sigma(2)}}\otimes....\otimes\mathrm{d}x^{\mu_{\sigma(r)}}\,. \tag{3.92}$$

Thus

$$\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu = \mathrm{d}x^\mu\otimes\mathrm{d}x^\nu - \mathrm{d}x^\nu\otimes\mathrm{d}x^\mu\,. \tag{3.93}$$

The wedge product satisfies the following conditions

- $\mathrm{d}x^{\mu_1}\wedge...\wedge\mathrm{d}x^{\mu_r}=0$ if some index is repeated.
- $\mathrm{d}x^{\mu_1}\wedge...\wedge\mathrm{d}x^{\mu_r} = \text{sign}(\sigma)\mathrm{d}x^{\mu_{\sigma(1)}}\wedge...\wedge\mathrm{d}x^{\mu_{\sigma(r)}}$.
- $\mathrm{d}x^{\mu_1}\wedge...\wedge\mathrm{d}x^{\mu_r}$ is linear in each $\mathrm{d}x^\mu$.

We will denote the vector space of r-forms at the point $p\in M$ by $\Omega_p^r(M)$, (another common notation is $\Lambda_p^{(r)}(M)$), a basis is provided by the set of all wedge products in (3.92). We can then expand an element of $\Omega_p^r(M)$ as

$$\omega = \frac{1}{r!}\omega_{\mu_1...\mu_r}\mathrm{d}x^{\mu_1}\wedge...\wedge\mathrm{d}x^{\mu_r}\,, \tag{3.94}$$

where $\omega_{\mu_1...\mu_r}$ are taken to be totally anti-symmetric. Since there are $\begin{pmatrix} m \\ r\end{pmatrix}$ choices of the set $\{\mu_1,....\mu_r\}$ out of $(1,2,...,m=\dim(M))$ the dimension of the vector space $\Omega_p^{(r)}(M)$ is

$$\begin{pmatrix} m \\ r\end{pmatrix} = \frac{m!}{r!(m-r)!}\,. \tag{3.95}$$

We take $\Omega^0_p(M) = \mathbb{R}$ and it follows that $\Omega^1_p(M) = T^*_p(M)$ from before. Moreover, since we are anti-symmetrising all the indices if r exceeds $m = \dim(M)$ then it vanishes identically. Furthermore we have the identity $\begin{pmatrix} m \\ r \end{pmatrix} = \begin{pmatrix} m \\ m-r \end{pmatrix}$ and it follows that $\dim \Omega^r_p(M) = \dim \Omega^{m-r}_p(M)$. Since $\Omega^r_p(M)$ is a vector space it is isomorphic to $\Omega^{(r-m)}_p(M)$.[16]

3.4.1 Exterior product

Definition 27 (Exterior product) *The* exterior product*, $\wedge$, is defined to be the following map $\wedge : \Omega^q_p(M) \times \Omega^r_p(M) \to \Omega^{q+r}_p(M)$. Its action follows by trivial extension of the wedge product defined above. Let $\omega \in \Omega^q_p(M)$ and $\xi \in \Omega^r_p(M)$ be an q-form and and r-form respectively. The action of the $(q+r)$-form $\omega \wedge \xi$ on $q+r$ vectors V_i is*

$$(\omega \wedge \xi)(V_1, ..., V_{q+r}) = \frac{1}{q!r!} \sum_{\sigma \in S_{q+r}} sign(\sigma) \omega\big(V_{\sigma(1)}, ..., V_{\sigma(q)}\big) \xi\big(V_{\sigma(q+1)}, ..., V_{\sigma(q+r)}\big) \,. \tag{3.96}$$

It follows that if $q + r > m = \dim(M)$ then $\omega \wedge \xi$ vanishes. With this product we can define and algebra

$$\Omega^*_p(M) \equiv \Omega^0_p(M) \oplus \Omega^1_p(M) \oplus ... \oplus \Omega^m_p(M) \,. \tag{3.97}$$

Example 3.8: Wedge product

Let us take $\mathbb{R}^3$ with coordinates (x, y, z) and consider the forms $\omega_1 = f(x)\mathrm{d}x + g(x)\mathrm{d}y$, $\omega_2 = \sin(x-z)\mathrm{d}y \wedge \mathrm{d}z$ and $\omega_3 = \mathrm{e}^{x+y+z}\mathrm{d}z$ then we have:

$$\begin{aligned} \omega_1 \wedge \omega_2 &= f(x)\sin(x-z)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \,, \\ \omega_1 \wedge \omega_3 &= \mathrm{e}^{x+y+z} f(x)\mathrm{d}x \wedge \mathrm{d}z + \mathrm{e}^{x+y+z} g(x)\mathrm{d}y \wedge \mathrm{d}z \,, \\ \omega_2 \wedge \omega_3 &= 0 \,. \end{aligned} \tag{3.98}$$

Exercise 3.7: Properties of the Wedge product

From the properties of the wedge product show that for $\xi \in \Omega^q_p(M)$, $\eta \in \Omega^r_p(M)$ and $\omega \in \Omega^s_p(M)$ that

$$\begin{aligned} \xi \wedge \eta &= (-1)^{qr} \eta \wedge \xi \,, \\ \xi \wedge \xi &= 0 \qquad \text{if } q \text{ odd} \,, \\ (\xi \wedge \eta) \wedge \omega &= \xi \wedge (\eta \wedge \omega) \,. \end{aligned} \tag{3.99}$$

[16] When the manifold is equipped with a metric the isomorphism is provided by the Hodge star operation $\star$. We will see the Hodge star later in section 4.1.3.

We may assign an r-form smoothly at each point on a manifold M. We denote the space of smooth r-forms on M by $\Omega^r(M)$, and take $\Omega^0(M) = \mathcal{F}(M)$ to be the space of smooth functions.

3.4.2 Exterior derivative

A useful map between p-forms and $p+1$-forms is the exterior derivative:

Definition 28 (Exterior Derivative) *The* exterior derivative d_r *is a map* $\Omega^r(M) \to \Omega^{r+1}(M)$, *whose action on an r-form is:*

$$\begin{aligned} \mathrm{d}\omega(X_1, ..., X_{r+1}) = &\sum_{i=1}^{r} (-1)^{i+1} X_i \omega(X_1, ..., \hat{X}_i, ..., X_{r+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{r+1}) \,, \end{aligned} \tag{3.100}$$

where the hats denote that this term should be removed.

We can write this in coordinates, consider the r-form

$$\omega = \frac{1}{r!} \omega_{\mu_1 \dots \mu_r} \mathrm{d}x^{\mu_1} \wedge ... \wedge \mathrm{d}x^{\mu_r} \,, \tag{3.101}$$

then the exterior derivative

$$\mathrm{d}_r \omega = \frac{1}{r!} \left(\frac{\partial}{\partial x^\nu} \omega_{\mu_1 \dots \mu_r} \right) \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge ... \wedge \mathrm{d}x^{\mu_r} \,. \tag{3.102}$$

It is common to drop the r subscript and simply write d, we will do this from now on. The wedge product automatically anti-symmetrises the coefficient so it is indeed a $(r+1)$-form that we obtain.

Exercise 3.8: Exterior derivative property
Show that for $\xi \in \Omega^q_p(M)$, $\eta \in \Omega^r_p(M)$ we have

$$\mathrm{d}(\xi \wedge \eta) = \mathrm{d}\xi \wedge \eta + (-1)^q \xi \wedge \mathrm{d}\eta \,. \tag{3.103}$$

Example 3.9: Exterior Derivative
Let us take $\mathbb{R}^3$ with coordinates (x, y, z). The generic r-forms are

$$\begin{aligned} \omega_0 &= f(x, y, z) \,, \\ \omega_1 &= \omega_x(x, y, z)\mathrm{d}x + \omega_y(x, y, z)\mathrm{d}y + \omega_z(x, y, z)\mathrm{d}z \,, \\ \omega_2 &= \omega_{xy}(x, y, z)\mathrm{d}x \wedge \mathrm{d}y + \omega_{yz}(x, y, z)\mathrm{d}y \wedge \mathrm{d}z + \omega_{zx}(x, y, z)\mathrm{d}z \wedge \mathrm{d}x \,, \\ \omega_3 &= \omega_{xyz}(x, y, z)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \,. \end{aligned} \tag{3.104}$$

The exterior derivative of these forms is

$$
\begin{aligned}
\mathrm{d}\omega_0 &= \frac{\partial}{\partial x}f(x,y,z)\mathrm{d}x + \frac{\partial}{\partial y}f(x,y,z)\mathrm{d}y + \frac{\partial}{\partial z}f(x,y,z)\mathrm{d}z\,,\\
\mathrm{d}\omega_1 &= \left(\frac{\partial}{\partial x}\omega_y - \frac{\partial}{\partial y}\omega_x\right)\mathrm{d}x\wedge\mathrm{d}y + \left(\frac{\partial}{\partial y}\omega_z - \frac{\partial}{\partial z}\omega_y\right)\mathrm{d}y\wedge\mathrm{d}z + \left(\frac{\partial}{\partial z}\omega_x - \frac{\partial}{\partial x}\omega_z\right)\mathrm{d}z\wedge\mathrm{d}x\,,\\
\mathrm{d}\omega_2 &= \left(\frac{\partial}{\partial x}\omega_{yz} + \frac{\partial}{\partial y}\omega_{zx} + \frac{\partial}{\partial z}\omega_{xy}\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z\,, \qquad (3.105)\\
\mathrm{d}\omega_3 &= 0\,.
\end{aligned}
$$

In the usual 3d vector calculus you may identify these as ‘grad’ for d acting on the scalar, ‘curl’ for the one-form and the ‘divergence’ for the two-form.

From either the coordinate free expression (3.100) or the one using the coordinates in (3.102), we can prove the important result that

$$
\mathrm{d}^2 = 0\,, \qquad (\mathrm{d}_{r+1}\mathrm{d}r = 0)\,. \tag{3.106}
$$

Using the coordinate form (3.102) we find

$$
\mathrm{d}^2\omega = \frac{1}{r!}\frac{\partial^2}{\partial x^\nu \partial x^\sigma}\omega_{\mu_1\ldots\mu_r}\mathrm{d}x^\nu\wedge\mathrm{d}x^\sigma\wedge\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_r}\,. \tag{3.107}
$$

Using that the derivative term is symmetric in $\nu\sigma$ while the wedge product is anti-symmetric in these indices it follows that this vanishes. Since $\mathrm{d}^2 = 0$ it follows that an exact form is always closed, though the converse need not be true. The failure of a closed form to be exact tells us interesting information about the topology of the underlying manifold.

Aside: Cohomology

The exterior derivative induces the sequence

$$
0 \xrightarrow{i} \Omega^0(M) \xrightarrow{\mathrm{d}_0} \Omega^1(M) \xrightarrow{\mathrm{d}_1} \ldots \xrightarrow{\mathrm{d}_{m-1}} \Omega^m(M) \xrightarrow{\mathrm{d}_m} 0\,, \tag{3.108}
$$

with i the inclusion map. This is known as the *de Rahm complex*. We let the set of all closed r-forms on M be denoted by $Z^r(M)$, so that for $\mathrm{d}_r : \Omega^r(M) \to \Omega^{r+1}(M)$, $\ker(\mathrm{d}_r) = Z^r(M)$, and denote the set of all exact r-forms to be $B^r(M)$, i.e. the B^r is the image of $\Omega^{r-1}(M)$ under $\mathrm{d}^{r-1} : \Omega^{r-1}(M) \to \Omega^r(M)$. Then the rth de-Rahm cohomology group is defined to be

$$
H^r(M) = Z^r(M)/B^r(M)\,. \tag{3.109}
$$

This is the dual space of the *homology group*, though we will not have time to consider either of these. The cohomology groups tell us important information about a manifold, their dimensions are topological invariants. Let $b^r = \dim(H^r(M))$, these are known as the *Betti numbers* of the manifold and are always finite. For a connected manifold one always has $b_0 = 1$, these are just the constant functions. The higher Betti numbers are non-zero when the manifold has some interesting topology, for example on a round two-sphere we have: $b^0(S^2) = 1 = b^2(S^2)$ and $b^1(S^2) = 0$. The Euler characteristic of a manifold can be given in terms of the Betti numbers by

$$\chi(M) = \sum_{r=0}^{m} (-1)^r b^r(M)\,. \tag{3.110}$$

For the S^2 we find $\chi(S^2) = 1 - 0 + 1 = 2$ which is the correct result!

We have seen that every exact form is closed, however not every closed form is exact, instead we have:

Theorem 3 (Poincaré's lemma) *If a coordinate neighbourhood U of a manifold M is contractible to a point $p \in M$, any closed form on U is also exact. In particular on $M = \mathbb{R}^m$, closed implies exact.*

Since we have been mapping our manifolds to $\mathbb{R}^m$ this says that for a general manifold any closed form is *locally exact*. That is if ω is a closed r-form, then in any neighbourhood $U \subset M$ it is always possible to find $\eta \in \Omega^{r-1}(M)$ such that $\omega = \mathrm{d}\eta$ on U. Since we cannot generally cover the manifold with a single coordinate patch, it may not be possible to find such an η everywhere on M. It is for this reason that we say the form is only *locally* exact rather than globally exact.

Example 3.10: Exact forms
Let us consider some examples.

- Consider $M = \mathbb{R}$. We can take a generic one-form to be $\omega = f(x)\mathrm{d}x$, with $f(x)$ some function. This is trivially closed since it is a top form, it is also exact since we can write

$$g(x) = \int_0^x \mathrm{d}x' f(x')\,, \tag{3.111}$$

  such that $\omega = \mathrm{d}g(x)$. This is all very boring because of the Poincaré's lemma.

- Now consider a circle, S^1. We can obtain a circle by looking at the phase $\mathrm{e}^{\mathrm{i}\phi} \in \mathbb{C}$. We can introduce the one-form $\omega = \mathrm{d}\phi$. Clearly this is once again closed since it is a top form, and from the way that it is written it seems that it must be exact once again, this however is not correct. The caveat is that ϕ is *not* a good coordinate everywhere on S^1, since it is not single valued, remember that we needed at least 2 patches on S^1. As such ϕ is not a good smooth function and so it is not a zero-form. Therefore $\mathrm{d}\phi$ is closed but not exact.

- Next consider $M = \mathbb{R}^2$. The Poincaré lemma ensures that all closed forms are exact. Wat happens if we remove a point? Consider instead $\mathbb{R}^2 - \{0,0\}$ and the one-form

$$\omega = -\frac{y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y\,. \tag{3.112}$$

This is not a smooth one-form on $\mathbb{R}^2$, since it blows up at the origin, however by restricting to $\mathbb{R}^2 - \{0,0\}$ we obtain a smooth one-form. A simple computation shows that ω is closed, but is it exact? If such a smooth function exists such that $\omega = \mathrm{d}f$ then the function f must satisfy:

$$\frac{\partial f}{\partial x} = -\frac{y}{x^2+y^2}\,, \quad \frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}\,. \tag{3.113}$$

The solution is

$$f(x,y) = \arctan\left(\frac{y}{x}\right) + \text{constant}\,, \tag{3.114}$$

so have we found an exact form? The answer is no, this is not a smooth function everywhere on $\mathbb{R}^2 - \{0,0\}$, along the line $x = 0$ it is ill-defined, and so ω is *not* exact. We see that removing a single point makes a big difference and closed no longer implies exact. A similar story holds for $\mathbb{R}^3$ and this is how magnetic monopoles sneak back into physics despite being forbidden by Maxwell's equations. See your favourite course on electromagnetism.

3.4.3 Interior product

We can now go from $\Omega^r(M) \to \Omega^{r+1}(M)$, what about the other way around? To do this we have to define the *Interior product*.

Definition 29 (Interior product) *Let Y be a vector field and $\omega \in \Omega^r(M)$ then*

$$i_Y\omega(X_1, ..., X_{r-1}) \equiv \omega(Y, X_1, ..., X_{r-1})\,. \tag{3.115}$$

If we introduce coordinates: $Y = Y^\mu \frac{\partial}{\partial x^\mu}$ then

$$i_Y\omega = \frac{1}{(r-1)!}Y^\nu \omega_{\nu\mu_1\ldots\mu_{r-1}}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{r-1}}\,. \tag{3.116}$$

Example 3.11:
Let us take $\mathbb{R}^3$ again with coordinates (x,y,z), and the usual coordinate basis, then we have

$$\mathrm{i}_{e_x}(\mathrm{d}x\wedge\mathrm{d}y) = \mathrm{d}y\,, \quad \mathrm{i}_{e_x}(\mathrm{d}y\wedge\mathrm{d}z) = 0\,, \quad \mathrm{i}_{e_x}(\mathrm{d}z\wedge\mathrm{d}x) = -\mathrm{d}z\,. \tag{3.117}$$

Using the interior product and exterior derivative gives a simple way of computing the Lie derivative of a form along the vector field X. We have for any r-form ω and vector field X that:

$$\mathcal{L}_X\omega = \big(\mathrm{d}\,\mathrm{i}_X + \mathrm{i}_X\,\mathrm{d}\big)\omega\,. \tag{3.118}$$

Exercise 3.9: Interior product identities

Show that the interior product satisfies the following:

$$\begin{aligned} \mathrm{i}_X^2 &= 0\,,\\ \mathrm{i}_X(\omega\wedge\eta) &= \mathrm{i}_X\omega\wedge\eta + (-1)^r\omega\wedge\mathrm{i}_X\eta\,,\\ \mathcal{L}_X\mathrm{i}_X\omega &= \mathrm{i}_X\mathcal{L}_X\omega\,. \end{aligned} \tag{3.119}$$

Hamiltonian mechanics in differential geometry We can now combine some of the differential geometry we have learnt so far to reformulate classical Hamiltonian mechanics. Recall that in classical mechanics the phase space is a manifold M parametrised by coordinates (q^i, p_j) where q^i are the positions of particles and p_j their momenta. Note that M must be even dimensional here. The Hamiltonian $H(q,p)$ is a function on M and Hamilton's equations are

$$\dot{q}^i = \frac{\partial H}{\partial p_i}\,, \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}\,. \tag{3.120}$$

Phase space comes equipped with the *Poisson bracket*, defined on a pair of functions f, g to act as

$$\{f,g\} = \frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q^j}\,, \tag{3.121}$$

from which the time evolution of a function is

$$\dot{f} = \{f, H\}\,, \tag{3.122}$$

with H the Hamiltonian. To obtain Hamilton's equations one should input $f = q^i$ and $f = p_i$ into the above.

Underlying this structure are forms. The key idea behind this is to convert the scalar function H into a vector field X_H on M. Particles will then follow trajectories which are the integral curves generated by X_H. To convert the scalar into a vector we introduce the *symplectic two-form* ω. This is a two-form which is closed $\mathrm{d}\omega = 0$ and is non-degenerate, $\omega\wedge\omega\wedge....\wedge\omega \neq 0$. A manifold equipped with such a two-form is called a *symplectic manifold*.

Any two-form provides a map $\omega : T_p(M) \to T_p^*(M)$, since given a vector field X we can simply take the inner product with ω to obtain a one-form, $\mathrm{i}_X\omega$. For our purposes we want to go in the opposite direction, we want to convert a scalar function into a vector field. This is possible if the map $\omega : T_p(M) \to T_p^*(M)$ is an isomorphism. This is equivalent to ω being non-degenerate. In this case we can define a vector field X_H via

$$\mathrm{i}_{X_H}\omega = -\mathrm{d}H\,. \tag{3.123}$$

In coordinate notation we have

$$X_H^\mu \omega_{\mu\nu} = -\partial_\nu H\,. \tag{3.124}$$

If we take the inverse to be $\omega^{\mu\nu}$ so that $\omega^{\mu\nu}\omega_{\nu\rho} = \delta^\mu_\rho$, then

$$X_H^\mu = \omega^{\mu\nu}\partial_\nu H\,. \tag{3.125}$$

The integral curves generated by X_H obey

$$\frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t} = X_H^\mu = \omega^{\mu\nu}\partial_\nu H\,. \tag{3.126}$$

These are the general form of Hamilton's equations, just written without reference to canonical coordinates. If we let $x^\mu = (q^i, p_j)$ and choose the symplectic form to have block diagonal form

$$\omega^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \Leftrightarrow \qquad \omega = \mathrm{d}p^i \wedge \mathrm{d}q_i \tag{3.127}$$

then the integral curves reduce precisely to Hamilton's equations (3.120).

To define the Poisson structure, we first note that we can repeat the map for obtaining a vector from a scalar for any function f, to obtain a vector field X_f. Then

$$\{f, g\} = \omega(X_f, X_g) = -\omega(X_g, X_f)\,. \tag{3.128}$$

This may be written in a multitude of different ways, we have

$$\{f, g\} = -\mathrm{i}_{X_f}\omega(X_g) = \mathrm{d}f(X_g) = X_g(f)\,. \tag{3.129}$$

It follows that the equation of motion in Poisson bracket structure is then

$$\dot{f} = \{f, H\} = X_H(f) = \mathcal{L}_{X_H} f\,. \tag{3.130}$$

We see that the Lie derivative along X_H generates time evolution!

So far we have not explained why the symplectic two-form was taken to be closed. This is required in order for the Poisson bracket to obey the Jacobi identity. It is also a necessary (and sufficient) condition for the symplectic form to be invariant under Hamiltonian flow.

3.4.4 Integration

We have learnt how to differentiate on a manifold using a vector field X, we now turn our attention to integration? The key question we want to ask is what can we integrate on a manifold M and how? The answer turns out to be our friends the differential forms.

Definition 30 (Orientation) *To begin we need to define an* orientation *on a manifold. Let M be a connected m-dimensional differentiable manifold. At a point $p \in M$ the tangent space $T_p(M)$ is spanned by the basis $\{e_\mu\} = \{\frac{\partial}{\partial x^\mu}\}$ where x^μ is the local coordinate on the chart U_i which contains p. Take U_j to be another chart such that $U_i \cap U_j \neq \emptyset$ and such that $p \in U_i \cap U_j$. Then the tangent space $T_p(M)$ is spanned by both $\{e_\mu\}$ or $\{\tilde{e}_\nu\} = \{\frac{\partial}{\partial y^\mu}\}$. The change of basis is*

$$\tilde{e}_\nu = \frac{\partial x^\mu}{\partial y^\nu} e_\mu \equiv \Lambda^\mu{}_\nu e_\mu \,. \tag{3.131}$$

If $\det(\Lambda) > 0$ *on $U_i \cap U_j$, the two bases $\{e_m\}$ and $\{\tilde{e}_\nu\}$ are said to defined the* same orientation *on $U_i \cap U_j$. If on the other hand* $\det(\Lambda) < 0$ *then they define the* opposite orientation*.*

Definition 31 (Orientable) *Let M be a connected manifold covered by $\{U_i\}$. The manifold M is* orientable *if for any overlapping charts U_i, U_j there exist local coordinates $\{x^\mu\}$ for U_i and $\{y^\nu\}$ for U_j such that* $\det \Lambda = \det\left(\frac{\partial x^\mu}{\partial y^\nu}\right) > 0$*. If M is* non-orientable*, Λ cannot be made positive in all intersections of charts.*

An example of a non-orientable manifold is the Möbius strip, see figure 10. To construct a Möbius strip take two rectangles and glue them together with a twist of π on one of the edges to glue.

If an m-dimensional manifold M is orientable there exists an m-form ω which is nowhere vanishing, called the *volume form* or *volume element*. It plays the role of the measure when we integrate a function $f \in \mathcal{F}(M)$ over M. Two volume elements are said to be *equivalent* if there exists a strictly positive function $h \in \mathcal{F}(M)$ such that $\omega = h\omega'$. A negative-definite function $h' \in \mathcal{F}(M)$ gives and inequivalent orientation to M. Therefore for any orientable manifold there are two inequivalent orientations, we may refer to one of them as right-handed and the other as left-handed.

Since the volume form is a top form locally it can be written as

$$\omega = h(x)\mathrm{d}x^1 \wedge ...\mathrm{d}x^m \,, \tag{3.132}$$

with the requirement that $h(x) \neq 0$. For the volume form to not flip orientation we must be able to patch this over the whole manifold without the handedness changing. Suppose that

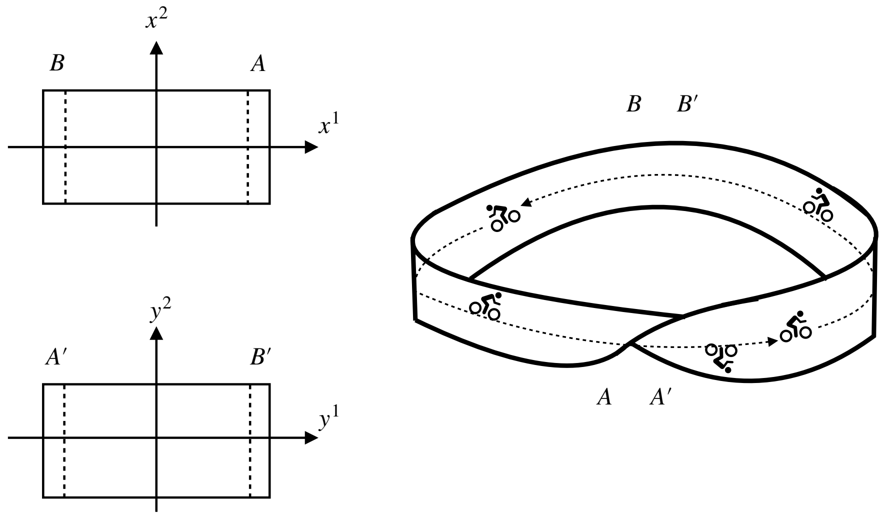


Figure 10: To construct the Möbius strip we glue two rectangles together: A with A' and B with B'. When joining A with A' we twist by π. The coordinate transformation on the A, A' intersection $y^1 = x^1$ and $y^2 = -x^2$, which has Jacobian -1 and is thus not orientable. We see that the cyclist going around the Möbius strip end up "up-side down" as they travel around the strip.

we have two sets of coordinates x^μ and y^ν on the charts U_i and U_j respectively, then in the new coordinates we have

$$\omega = h(x)\frac{\partial x^1}{\partial y^{\nu_1}}\mathrm{d}y^{\nu_1} \wedge .... \wedge \frac{\partial x^n}{\partial y^{\nu_m}}\mathrm{d}y^{\nu_m} = h(x)\det\left(\frac{\partial x^\mu}{\partial y^\nu}\right)\mathrm{d}y^1 \wedge .... \wedge \mathrm{d}y^m\,, \tag{3.133}$$

which makes clear that we may only define a volume form when the manifold is orientable, since the determinant appears. For the Möbius strip we see that we begin with volume form $\omega = \mathrm{d}x \wedge \mathrm{d}y$ but as we change charts this becomes $\omega = -\mathrm{d}x \wedge \mathrm{d}y$ and so ω is not of definite handedness on the Möbius strip.

With our volume form in tow we can now define integration of a function $f : M \to \mathbb{R}$ over an orientable manifold M. Let us take the volume form to be ω. Then in a coordinate neighbourhood U_i with coordinates x^μ we define the integration of an m-form $f\omega$ to be

$$\int_{U_i} f\omega \equiv \int_{\varphi(U_i)} f\big(\varphi_i^{-1}(x))\big)h\big(\varphi_i^{-1}(x)\big)\mathrm{d}x^1...\mathrm{d}x^m\,. \tag{3.134}$$

Notice that the right-hand side is just ordinary integration we are familiar with, albeit in m variables. Once the integral of f over U_i is defined it can be extended to an integration over all of M by making use of a *partition of unity*.

Definition 32 (Partition of unity) *Take an open covering $\{U_i\}$ on M such that each point of M is covered with a finite number of U_i. If this is always possible we call M* paracompact.[17] *If a family of differentiable functions $\epsilon_i(p)$ satisfies*

1. $0 \leq \epsilon_i(p) \leq 1$,
2. $\epsilon_i(p) = 0$ *if* $p \notin U_i$,
3. $\epsilon_1(p) + \epsilon_2(p) + ... = 1$ *for every point* $p \in M$.

The family $\{\epsilon_i(p)\}$ is called a partition of unity *for the covering $\{U_i\}$.*

From condition (3) it follows that

$$f(p) = \sum_i f(p)\epsilon_i(p) = \sum_i f_i(p)\,, \qquad f_i(p) \equiv \epsilon_i(p)f(p)\,. \tag{3.135}$$

Hence given a point $p \in M$ assumed paracompactness ensures that there are only a finite number of terms in the summation over i, this was one of the magical properties we imposed but forgot about. For each of the $f_i(p)$ we may define the integral over U_i via (3.134), and therefore we have

$$\int_M f\omega \equiv \sum_i \int_{U_i} f_i\omega\,. \tag{3.136}$$

Though a different choice of atlas gives a different set of coordinates and a different partition of unity the integral as defined above stays the same.

Example 3.12: Integrating on a circle

Let us consider integrating a function on the circle. Let us take the atlas as given in (3.7) and (3.8). Let $U_1 = S^1 - \{(1,0)\}$ and $U_2 = S^1 - \{(-1,0)\}$. Then we may give a partition of unity by fixing $\epsilon(\theta) = \sin^2\frac{\theta}{2}$ and $\epsilon_2(\theta) = \cos^2\frac{\theta}{2}$. Note that $\epsilon_1(0) = 0$ and $\epsilon_2(\pi) = 0$ and therefore they vanish at the removed points as required. Moreover $\epsilon_1(\theta) + \epsilon_2(\theta) = 1$ as required. Thus $\{\epsilon_i(\theta)\}$ furnishes us with a partition of unity for the atlas $\{U_i\}$. Let us integrate the function $f = \cos^2\theta$. Of course we know

$$\int_0^{2\pi} \mathrm{d}\theta \cos^2\theta = \pi\,, \tag{3.137}$$

but we should check with our partition of unity that we obtain the same result. We find

$$\int_{S^1} \mathrm{d}\theta\cos^2\theta = \int_0^{2\pi} \mathrm{d}\theta \sin^2\frac{\theta}{2}\cos^2\theta + \int_{-\pi}^{\pi} \mathrm{d}\theta\cos^2\frac{\theta}{2}\cos^2\theta = \frac{1}{2}\pi + \frac{1}{2}\pi = \pi\,. \tag{3.138}$$

[17] We will assume this is the case whenever we integrate something in this course.

So far we have left the function $h(x)$ appearing in the volume-form arbitrary. Since this gets multiplied by the Jacobian it changes between different coordinate patches and therefore there is no canonical way to pick this. Once we endow the manifold with a metric, as we are required to do in GR, there is a canonical choice that we can make.

We can also integrate forms over sub-manifolds of M, rather than the full manifold. A manifold Σ with dimension $k < n$ is a *sub-manifold* of M if we can find a map $\sigma : \Sigma \to M$ which is one-to-one and $\sigma_* : T_p(\Sigma) \to T_{\sigma(p)}(M)$ is also one-to-one. We can then integrate a k-form ω on M over a k-dimensional sub-manifold Σ by pulling the form back to Σ:

$$\int_{\sigma(\Sigma)} \omega = \int_\Sigma \sigma^* \omega \,. \tag{3.139}$$

For example consider a one-form A living on M and take C to be a one-dimensional manifold in M. We can introduce a map $\sigma : C \to M$ which defines a non-intersecting curve $\sigma(C)$ which is a sub-manifold of M. We can then pull-back A onto the curve and integrate to obtain,

$$\int_{\sigma(C)} A = \int_C \sigma^* A \,. \tag{3.140}$$

Let the curve trace out a path $x^\mu(\tau)$ in M then, in coordinates this reads

$$\int_C \sigma^* A = \int \mathrm{d}\tau A_\mu(x) \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \,, \tag{3.141}$$

which is precisely the way in which a worldline of a particle couples to the electromagnetic field.

Until now our focus has been on smooth manifolds without boundary. We saw that this can be extended to manifolds with a boundary in section 3.1. There we have charts $\varphi : M \to U_i$ where U_i is an open subset of $\mathbb{R}^m = \{(x^1, ..., x^m) | x^m \geq 0\}$. The boundary is denoted by ∂M, and is the sub-manifold fixed by $x^m = 0$.

Theorem 4 (Stokes Theorem) *For a manifold M with a boundary, for any $(m-1)$-form ω we have*

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega \,. \tag{3.142}$$

Stoke's theorem is the mother of all integral theorems. You may be familiar with the divergence theorem, Green's theorem, etc., this is the generalisation of those.

> **Exercise 3.10: Stoke's Theorem**
> Show that this reduces to Stoke's theorem on $\mathbb{R}^3$.

We will see integrals later in the course when we compute the mass of the Schwarzschild black hole.

4 Riemannian geometry

We now have all the necessary pre-requisites to introduce the most valuable player of general relativity: the metric. The introduction of a metric brings a whole slew of new objects that we can define. Here we will continue to talk about Riemannian geometry, this is spaces with Euclidean signature, whereas what we really want to consider is Lorentzian geometry.

4.1 The metric

Definition 33 (Riemannian metric) *Let M be a differentiable manifold.*
A Riemannian metric *g on M is a type $(0,2)$ tensor field on M which at each point $p \in M$ satisfies:*

- *Symmetric: $g_p(X,Y) = g_p(Y,X)$,*
- *$g_p(X,X) \geq 0$ with equality iff $X = 0$,*

with $X, Y \in T_p(M)$.

A tensor field g of type $(0,2)$ is a pseudo-Riemannian metric *if it satisfies the first condition and*

- *Non-degenerate. If for any $p \in M$ $g_p(X,Y) = 0$ for all $Y \in T_p(M)$ then $X|_p = 0$,*

We may extend the tensor g_p over the full manifold. With a choice of coordinates we can write the metric as

$$g = g_{\mu\nu}(x) \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \,. \tag{4.1}$$

This defines the metric to be a smooth tensor field over our whole manifold, that is a multi-linear map from $T(M) \times T(M) \to \mathcal{F}(M)$. We will often write this as the line element $\mathrm{d}s^2$,

$$\mathrm{d}s^2 = g_{\mu\nu}(x) \mathrm{d}x^\mu \mathrm{d}x^\nu \,, \tag{4.2}$$

in particular removing the tensor product. We can do this unambiguously because of the symmetry property of the metric. This also captures our intuitive understanding of the infinitesimal distance being measured by the infinitesimal coordinate separations $\mathrm{d}x^\mu$ weighted by the metric.

One can extract out the components by evaluating the metric on a pair of basis elements

$$g_{\mu\nu}(x) = g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) . \tag{4.3}$$

We may view $g_{\mu\nu}$ as a matrix, which by the symmetry property above is symmetric. This implies that the matrix is diagonalisable, with real eigenvalues. If there are i positive eigenvalues and j negative eigenvalues the pair (i, j) is called the *index* of the metric. If $j = 1$ the metric is called a *Lorentz metric*, for $j = 0$ we have a *Euclidean* metric. The number of negative entries is called the *signature* and by Sylvester's law of inertia[18], this is independent of the choice of basis. By an abuse of notation we will often call this symmetric matrix the metric when really this is just components of the metric tensor field in some coordinate basis.

For most applications of differential geometry, we are interested in manifolds with signature 0, i.e. a Riemannian manifold. The simplest example which you are probably familiar with, though maybe not in this language, is the metric on Euclidean space $\mathbb{R}^m$, which in Cartesian coordinates has the metric

$$g = \mathrm{d}x^1 \otimes \mathrm{d}x^1 + ... + \mathrm{d}x^m \otimes \mathrm{d}x^m \,, \tag{4.4}$$

which in components reads $g_{\mu\nu} = \delta_{\mu\nu}$.

4.1.1 Riemannian metric

A general Riemannian metric is a useful object to have in one's tool belt. It gives us a way of measuring the length of a vector X at each point

$$|X| = \sqrt{g(X, X)}\,. \tag{4.5}$$

Moreover we may measure the angle between two vectors

$$g(X, Y) = |X||Y| \cos\theta\,. \tag{4.6}$$

Furthermore, it can be used to measure the distance between two points p and q along a curve in M. For the curve $\sigma : [a, b] \to M$ with $\sigma(a) = p$ and $\sigma(b) = q$ the distance between the two points along the curve is

$$d(p, q) = \int_a^b \mathrm{d}t \sqrt{g(X, X)|_{\sigma(t)}}\,, \tag{4.7}$$

where X is the tangent vector field to the curve. If the curve has coordinates $x^\mu(t)$ then $X^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}$ and the distance is

$$d(p, q) = \int_a^b \mathrm{d}t \sqrt{g_{\mu\nu} \frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t} \frac{\mathrm{d}x^\nu(t)}{\mathrm{d}t}}\,. \tag{4.8}$$

Importantly this distance does not depend on the parametrisation of the curve, and only on the curve itself.

[18] This has nothing to do with inertia, Sylvester just wanted a law of inertia like Newton.

Example 4.1:
A less trivial example is the metric on a unit round two-sphere, denoted S^2,

$$\mathrm{d}s^2(S^2) = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \,. \tag{4.9}$$

Here $\theta \in (0, \pi)$ and $\phi \in [0, 2\pi)$. This is written in a chart which does not cover the full S^2, to show that this is a smooth tensor field one must define a second patch whose union with the above one covers S^2. To see this one can first realise the S^2 by embedding the two sphere in $\mathbb{R}^3$ via:

$$x^2 + y^2 + z^2 = 1 \,. \tag{4.10}$$

First introduce polar coordinates as

$$x = \sin\theta\cos\phi \,, \quad y = \sin\theta\sin\phi \,, \quad z = \cos\theta \,, \tag{4.11}$$

which define $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$ uniquely. This covers all of the S^2 but for the line of longitude $y = 0, x > 0$ and the points $(0, 0, \pm 1)$. We can define a second chart using a different set of polar coordinates:

$$x = -\sin\theta'\cos\phi' \,, \quad y = \cos\theta' \,, \quad z = \sin\theta'\sin\phi' \,, \tag{4.12}$$

where $\theta' \in (0, \pi)$ and $\phi' \in (0, 2\pi)$. The points $(0, \pm 1, 0)$ and the line $z = 0, x < 0$ is not covered however one can see that the union of these two charts covers the S^2. One can check that this makes S^2 a manifold. Now in the chart the metric we obtain is

$$\mathrm{d}s^2(S^2) = \mathrm{d}\theta'^2 + \sin^2\theta' \mathrm{d}\phi'^2 \,. \tag{4.13}$$

One can then check this defines a smooth tensor field.

We can use the metric to work out the circumference of the unit circle. Let us first compute the path along the equator. This is the integral curve of the vector field $X = \partial_\phi$ which we computed in example 3.7. The form of the integral curve is

$$\theta(\lambda) = \theta_0 \,, \quad \phi(\lambda) = \lambda \,, \tag{4.14}$$

and we can compute the length of the curve. We end up back where we are after $\lambda = 2\pi$ and so the path length is:

$$d = \int_0^{2\pi} \sqrt{\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2}\mathrm{d}\lambda = \int_0^{2\pi} \sin\theta_0 \mathrm{d}\lambda = 2\pi\sin\theta_0 \,. \tag{4.15}$$

The equator has $\theta = \frac{\pi}{2}$ and therefore we find that the circumference is 2π which is the correct value for a unit sphere. If we wanted to work with a non-unit sphere we should multiply the metric by a factor l^2 with l the radius. It is simple to see that the result in this case would then be $2\pi l$.

Exercise 4.1: Metric on S^2 from pull back

This metric in (4.9) is the pull back of the metric on Euclidean space to the S^2 defined by the polar coordinates. Show this.

The possible issues with the tensor field (4.9) came from the need to change charts since we cannot cover the S^2 with a single chart. The metric on S^2 will appear later in this course and we will simply ignore any subtleties of the regularity of the metric in (4.9) since, as we have just seen, these can be removed by properly considering the different patches of the S^2.

4.1.2 Lorentzian manifolds

For General Relativity we need to consider Lorentzian manifolds. The simplest example is Minkowski space. This is $\mathbb{R}^{1,m-1}$ equipped with the metric

$$\eta = -\mathrm{d}x^0 \otimes \mathrm{d}x^0 + \mathrm{d}x^1 \otimes \mathrm{d}x^1 + ... + \mathrm{d}x^{m-1} \otimes \mathrm{d}x^{m-1} , \tag{4.16}$$

which has components $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, ..., 1)$. Note that on a Lorentzian manifold we take the index to run over $0, 1, .., m-1$.

At any point p on a general Lorentzian manifold it is always possible to find an orthonormal basis $\{e_\mu\}$ of $T_p(M)$ such that locally the metric looks like the Minkowski metric

$$g_{\mu\nu}|_p = \eta_{\mu\nu} . \tag{4.17}$$

This is closely related to the equivalence principle we discussed in section 2.2.1, we will discuss the coordinates, known as *normal coordinates* that allow us to do this shortly. The fact that locally the metric looks like Minkowski space allows us to import some of the ideas of special relativity, namely we can classify the elements of $T_p(M)$ into three classes

- $g(X, X) > 0 \longrightarrow X$ is *spacelike* ,
- $g(X, X) = 0 \longrightarrow X$ is *lightlike* or *null* ,
- $g(X, X) < 0 \longrightarrow X$ is *timelike* .

At each point on M we can then draw light cones which are the null tangent vectors at that point. The novelty is that the directions of these light cones can vary smoothly as we move around the manifold. This specifies the causal structure of spacetime which determines which regions of spacetime can interact together.

As in the Riemannian case we can use the metric to determine the length of curves. The nature of a curve is inherited from the nature of its tangent vector. A curve is called *timelike* if its tangent vector is everywhere timelike. We then measure the proper time

$$\tau = \int_a^b \mathrm{d}\lambda \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\,. \tag{4.18}$$

4.1.3 Why is the metric useful?

The existence of a metric comes with a large number of benefits.

The metric as an isomorphism The metric gives a natural isomorphism between vectors and covectors, $g : T_p(M) \to T_p^*(M)$ for each p. In a coordinate basis we can write $X = X^\mu \partial_\mu$, and map it to a one-form $X = X_\mu \mathrm{d}x^\mu$, with the components given by

$$X_\mu = g_{\mu\nu} X^\nu\,. \tag{4.19}$$

We will usually say that we use the metric to lower (or raise) an index. What we really mean is that the metric provides an isomorphism between a vector space and its dual. Since g is non-degenerate and is thus invertible we also have the inverse map. We take the inverse of $g_{\mu\nu}$ to be $g^{\mu\nu}$ so that $g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho$. This can then be thought of as the components of a symmetric $(2,0)$ tensor

$$\hat{g} = g^{\mu\nu}\partial_\mu \otimes \partial_\nu\,. \tag{4.20}$$

Then

$$X^\mu = g^{\mu\nu} X_\nu\,. \tag{4.21}$$

The Volume form The metric also gives a natural volume form on the manifold M rather than the variety of volume forms we had previously. On a Riemannian manifold we take the volume form to be

$$\mathrm{vol}(M) = \sqrt{\det(g_{\mu\nu})}\mathrm{d}x^1 \wedge ...\mathrm{d}x^m\,, \tag{4.22}$$

and we use the shorthand $\sqrt{\det(g_{\mu\nu})} = \sqrt{g}$. On a Lorentzian manifold the determinant is negative and therefore we take the volume form to be

$$\mathrm{vol}(M) = \sqrt{-g}\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^{m-1}\,. \tag{4.23}$$

As it is written it looks coordinate dependent however it is not. To see this recall that if we change coordinates $y = y(x)$ we have (see (3.34))

$$\mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial y^\nu}\mathrm{d}y^\nu \equiv \Lambda^\mu{}_\nu \mathrm{d}y^\nu\,. \tag{4.24}$$

Then

$$\begin{aligned}
\mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^m &= \Lambda^1{}_{\nu_1}....\Lambda^m{}_{\nu_m}\mathrm{d}y^{\nu_1} \wedge ... \wedge \mathrm{d}y^{\nu_m} \\
&= \sum_{\sigma\in S_m} \Lambda^1{}_{\sigma(1)}...\Lambda^m{}_{\sigma(m)}\mathrm{d}y^{\sigma(1)} \wedge ... \wedge \mathrm{d}y^{\sigma(m)} \\
&= \sum_{\sigma\in S_m} \mathrm{sign}(\sigma)\Lambda^1{}_{\sigma(1)}...\Lambda^m{}_{\sigma(m)}\mathrm{d}y^1 \wedge ... \wedge \mathrm{d}y^m \\
&= \det(\Lambda)\mathrm{d}y^1 \wedge ...\mathrm{d}y^m \,,
\end{aligned} \tag{4.25}$$

where in the penultimate line we have used the properties of the wedge product and in the last line used the definition of the determinant. The metric components transform as

$$g_{\mu\nu} = \frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu}\tilde{g}_{\rho\sigma} \,, \tag{4.26}$$

and therefore

$$\det(g_{\mu\nu}) = \det(\tilde{g}_{\mu\nu})\big(\det(\Lambda)\big)^{-2} \,, \tag{4.27}$$

and therefore this cancels with the transformation of the wedge product leaving

$$\mathrm{vol}(M) = \sqrt{|g|}\mathrm{d}y^1 \wedge ...\mathrm{d}y^m \,, \tag{4.28}$$

which is therefore invariant.

We may rewrite the volume form as

$$\mathrm{vol}(M) = \frac{1}{m!}v_{\mu_1...\mu_m}\mathrm{d}x^{\mu_1} \wedge .... \wedge \mathrm{d}x^{\mu_m} \,, \qquad \text{where} \qquad v_{\mu_1...\mu_m} = \sqrt{|g|}\epsilon_{\mu_1...\mu_m} \,. \tag{4.29}$$

Here $\epsilon_{\mu_1...\mu_m}$ is the Levi–Civita symbol which is m-dimensional totally anti-symmetric tensor giving 1 for an even permutation of the indices, -1 for an odd permutation and 0 when an index is repeated. It follows that $v_{\mu_1....\mu_m}$ is a tensor, while $\epsilon_{\mu_1...\mu_m}$ is not, instead it is a tensor density (one needs to multiply by the square root of the determinant to obtain a tensor). Note that we define $\epsilon^{\mu_1...\mu_m}$ to again be the totally anti-symmetric tensor with $\epsilon^{1...m} = 1$, i.e. we do not raise the indices on ϵ with the metric. Instead we have

$$v^{\mu_1...\mu_m} = g^{\mu_1\nu_1}...g^{\mu_m\nu_m}v_{\nu_1...\nu_m} = \pm\frac{1}{\sqrt{|g|}}\epsilon^{\mu_1...\mu_m} \,. \tag{4.30}$$

Hodge dual On an oriented manifold M we can use the totally anti-symmetric tensor density to define a map which takes a p-form $\omega \in \Omega^p(M)$ to a $(m-p)$-form $\star\omega \in \Omega^{m-p}(M)$. We define this map to be

$$(\star\omega)_{\mu_1...\mu_{m-p}} = \frac{1}{p!}\sqrt{|g|}\epsilon_{\mu_1...\mu_{m-p}\nu_1...\nu_p}\omega^{\nu_1..\nu_p} \,. \tag{4.31}$$

This is called the *Hodge dual* and is independent of coordinates. One can see that it satisfies

$$\star(\star\omega) = \pm(-1)^{p(m-p)}\omega\,, \tag{4.32}$$

with $+$ for a Riemannian metric and $-$ for a Lorentzian.[19]

With the Hodge dual in tow we can define an inner product on each vector space $\Omega^r(M)$. If $\omega, \eta \in \Omega^r(M)$ then we define the inner product

$$\langle \eta, \omega \rangle \equiv \int_M \eta \wedge \star\omega\,. \tag{4.33}$$

With such an inner product one can look at operators on $\Omega^r(M)$ and their adjoints. The differential operator we have introduced on r-forms is the exterior derivative. For $\omega \in \Omega^r(M)$ and $\alpha \in \Omega^{r-1}(M)$ the adjoint is defined via

$$\langle \mathrm{d}\alpha, \omega \rangle = \langle \alpha, \mathrm{d}^\dagger\omega \rangle\,, \tag{4.34}$$

where the adjoint operator $\mathrm{d}^\dagger : \Omega^r(M) \to \Omega^{r-1}(M)$ is given by

$$\mathrm{d}^\dagger = \pm(-1)^{m(r+1)-1} \star \mathrm{d} \star\,. \tag{4.35}$$

One can then define a Laplacian $\Box : \Omega^r(M) \to \Omega^r(M)$ defined as[20]

$$\Box = (\mathrm{d} + \mathrm{d}^\dagger)^2 = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d}\,. \tag{4.36}$$

It can be defined on both Riemannian manifolds and Lorentzian, however it is only positive definite on Riemannian manifolds. On a function f the Laplacian acts as

$$\Box f = -\frac{1}{\sqrt{|g|}}\partial_\nu\Big(\sqrt{|g|}g^{\mu\nu}\partial_\mu f\Big)\,. \tag{4.37}$$

[19]One has actually seen the Hodge dual before, it was just hidden from view. Consider two vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$, We can take the cross product to obtain a third vector $\vec{c}$ as $\vec{a} \times \vec{b} = \vec{c}$. This however mixes a lot of different objects. This is equivalent in our new language to first use the metric to relate the vectors to one-forms. The cross product it really the wedge product of the two one-forms to give a two-form. We then take the Hodge dual of this two-form to obtain a one-form and then use the metric once again to extract out a vector. This more complicated route is hidden since the metric is just the Kronecker delta and so we can raise and lower indices with impunity. Going to curved space and a non-trivial metric these subtleties become relevant.

[20]You may also see the Laplacian denoted by $\triangle$ rather than $\Box$.

Aside: There is a beautiful interplay between the Eigenforms and Eigenvalues of the Laplacian and the topology of the space that we will not cover. If one defines a harmonic form to be one which is annihilated by the Laplacian $\Box\omega = 0$, then there is an isomorphism between the set of all harmonic forms and the cohomology group:

$$\mathrm{Harm}^r(M) \cong H^r(M)\,. \tag{4.38}$$

The Betti numbers which were the dimensions of the cohomology groups are then just the dimension of the group of harmonic r-forms on the manifold.

4.2 Connections and curvature

A vector field X is a directional derivative acting on a function $f \in \mathcal{F}(M)$. However so far we have not introduced such a derivative for tensors of type (q, r). The Lie derivative is not quite what we want since it also involves derivatives of the vector defining the direction and so we want to introduce something additional. This other derivative is more useful than the Lie derivative, but requires the introduction of a *connection* to map the vector spaces at one point to vector spaces at another. The resultant object is known as the *covariant derivative* and is distinct from the Lie derivative that we introduced previously.

Definition 34 (Affine connection) *An* affine connection*, which we denote by* ∇ *is a map* $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$*, that is* $(X, Y) \mapsto \nabla_X Y$ *which satisfies*

$$\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z\,, \tag{4.39}$$

$$\nabla_{(fX+gY)} Z = f\nabla_X Z + g\nabla_Y Z\,, \tag{4.40}$$

$$\nabla_X(fY) = X[f]Y + f\nabla_X Y\,, \tag{4.41}$$

for vector fields $X, Y, Z \in \mathcal{X}(M)$ *and functions* $f, g \in \mathcal{F}(M)$.

Let us take a chart (U, φ) with coordinate $x = \varphi(p)$ and define m^3 functions ${\Gamma^\mu}_{\nu\rho}$ called the *connection coefficients* by

$$\nabla_\nu e_\mu \equiv \nabla_{e_\nu} e_\mu = e_\lambda {\Gamma^\lambda}_{\nu\mu}\,, \tag{4.42}$$

where $\{e_\mu\} = \{\frac{\partial}{\partial x^\mu}\}$ is the coordinate basis in $T_p(M)$. The connection coefficients specify how the basis vectors change from point to point, i.e. how to map the tangent space $T_p(M)$ to $T_q(M)$. Using the properties of the connection we can work out the general covariant derivative of a vector field

$$\begin{aligned}\nabla_X Y &= \nabla_X(Y^\mu e_\mu) \\ &= X[Y^\mu]e_\mu + Y^\mu \nabla_X e_\mu \\ &= X^\nu \partial_\nu(Y^\mu)e_\mu + X^\nu Y^\mu \nabla_\nu e_\mu \\ &= X^\nu \Big(\partial_\nu Y^\mu + \Gamma^\mu{}_{\nu\rho} Y^\rho\Big) e_\mu \,.\end{aligned} \tag{4.43}$$

We can strip off the overall X^ν to write

$$\Big(\nabla_\nu Y\Big)^\mu = \frac{\partial Y^\mu}{\partial x^\nu} + \Gamma^\mu{}_{\nu\rho} Y^\rho \,, \tag{4.44}$$

so that

$$(\nabla_X Y)^\mu = X^\nu \nabla_\nu Y^\mu \tag{4.45}$$

On a function the covariant derivative coincides with both the Lie derivative and the regular partial derivative, however its action on vectors differs. While the Lie derivative $\mathcal{L}_X Y$ depends on both X and its first derivative, the covariant derivative depends only on X. This is the natural generalisation of the partial derivative on curved space.

We will often be sloppy and write

$$(\nabla_X Y)^\mu = \nabla_X Y^\mu \,. \tag{4.46}$$

Typically in older books, though some still like to use this stupid convention, one may see the semi-colon notation

$$\nabla_\nu Y^\mu = Y^\mu{}_{;\nu} \;. \tag{4.47}$$

We will refrain from using this convention to preserve our sanity.

At the moment the connection $\Gamma^\mu{}_{\nu\rho}$ is somewhat abstract. One may guess that it is a tensor however this is not correct. To see this let us consider how it transforms under a change of coordinates. Recall that the basis elements transform as

$$\tilde{e}_\nu = \Lambda^\mu{}_\nu e_\mu \,, \qquad \text{with} \qquad \Lambda^\mu{}_\nu = \frac{\partial x^\mu}{\partial y^\nu} \,. \tag{4.48}$$

Recall that a $(1,2)$ tensor $T^\mu{}_{\nu\rho}$ transforms as

$$\tilde{T}^{\mu_1}{}_{\nu_1\rho_1} = (\Lambda^{-1})^{\mu_1}{}_{\mu_2} \Lambda^{\nu_2}{}_{\nu_1} \Lambda^{\rho_2}{}_{\rho_1} T^{\mu_2}{}_{\nu_2\rho_2} \;. \tag{4.49}$$

We can compute the transformation of the connection. In the basis $\{\tilde{e}_\mu\}$ we have

$$\begin{aligned}
\nabla_{\tilde{e}_\rho}\tilde{e}_\nu &= \tilde{\Gamma}^\mu{}_{\nu\rho}\tilde{e}_\mu \\
&= \nabla_{\Lambda^\sigma{}_\rho e_\sigma}\Big(\Lambda^\tau{}_\nu e_\tau\Big) \\
&= \Lambda^\sigma{}_\rho\Big(\nabla_\sigma(\Lambda^\tau{}_\nu)e_\tau + \Lambda^\tau{}_\nu\nabla_\sigma e_\tau\Big) \\
&= \Lambda^\sigma{}_\rho\Big(\Lambda^\tau{}_\nu\Gamma^\kappa{}_{\sigma\tau} + \partial_\sigma\Lambda^\kappa{}_\nu\Big)e_\kappa \\
&= \Lambda^\sigma{}_\rho\Big(\Lambda^\tau{}_\nu\Gamma^\kappa{}_{\sigma\tau} + \partial_\sigma\Lambda^\kappa{}_\nu\Big)(\Lambda^{-1})^\mu{}_\kappa\tilde{e}_\mu\,.
\end{aligned} \tag{4.50}$$

From this we obtain

$$\tilde{\Gamma}^\mu{}_{\nu\rho} = (\Lambda^{-1})^\mu{}_\kappa\Lambda^\sigma{}_\rho\Lambda^\tau{}_\nu\Gamma^\kappa{}_{\sigma\tau} + (\Lambda^{-1})^\mu{}_\kappa\Lambda^\sigma{}_\rho\partial_\sigma\Lambda^\kappa{}_\nu\,. \tag{4.51}$$

The first term is the expected transformation term of a $(1,2)$ tensor, however there is an additional piece. This additional piece is independent of Γ and depends only on the $\partial\Lambda$. This is the characteristic transformation of a connection coefficient.

4.2.1 Differentiating other tensors

We can use the properties of the covariant derivative to extend its action to any tensor field. Consider a one-form ω, we want the covariant derivative to take the one-form and return another one-form, $\nabla_X\omega$, as such we should check its action on a vector field $Y \in \mathcal{X}(M)$. We impose that the connection obeys the (generalised) Leibniz identity, so that

$$\nabla_X(\omega(Y)) = (\nabla_X\omega)(Y) + \omega(\nabla_X Y)\,. \tag{4.52}$$

Since $\omega(Y)$ is a function we know that

$$\nabla_X(\omega(Y)) = X[\omega(Y)]\,. \tag{4.53}$$

Using the Leibniz condition we have

$$(\nabla_X\omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y)\,, \tag{4.54}$$

and reducing to coordinates we find

$$\begin{aligned}
X^\mu(\nabla_\mu\omega)_\nu Y^\nu &= X^\mu\partial_\mu(\omega_\nu Y^\nu) - \omega_\nu X^\mu\big(\partial_\mu Y^\nu + \Gamma^\nu{}_{\mu\rho}Y^\rho\big) \\
&= X^\mu\Big(\partial_\mu\omega_\rho - \Gamma^\nu{}_{\mu\rho}\omega_\nu\Big)Y^\rho\,.
\end{aligned} \tag{4.55}$$

We may then write

$$(\nabla_\mu\omega)_\rho \equiv \nabla_\mu\omega_\rho = \frac{\partial}{\partial x^\mu}\omega_\rho - \Gamma^\nu{}_{\mu\rho}\omega_\nu\,. \tag{4.56}$$

We can now extend this argument to an arbitrary tensor of rank (q, r), again imposing the generalised Leibniz identity, and we find

$$\begin{aligned}\nabla_\mu T^{\nu_1\ldots\nu_q}{}_{\rho_1\ldots\rho_r} =& \frac{\partial}{\partial x^\mu} T^{\nu_1\ldots\nu_q}{}_{\rho_1\ldots\rho_r} + \Gamma^{\nu_1}{}_{\mu\sigma} T^{\sigma\ldots\nu_q}{}_{\rho_1\ldots\rho_r} + \ldots + \Gamma^{\nu_q}{}_{\mu\sigma} T^{\nu_1\ldots\nu_{q-1}\sigma}{}_{\rho_1\ldots\rho_r} \\ & - \Gamma^\sigma{}_{\mu\rho_1} T^{\nu_1\ldots\nu_q}{}_{\sigma\ldots\rho_r} - \ldots - \Gamma^\sigma{}_{\mu\rho_r} T^{\nu_1\ldots\nu_q}{}_{\rho_1\ldots\rho_{r-1}\sigma} \,.\end{aligned} \tag{4.57}$$

In words, you first differentiate the tensor and then for each upper index you add in a $+\Gamma T$ and for every down index a $-\Gamma T$.

4.3 Torsion and curvature

Even though the connection is not a tensor we can use it to construct two tensors. The first is a rank $(1, 2)$ tensor T known as *Torsion*, the second is a rank $(1, 3)$ tensor known as *curvature* or the *Riemann tensor*.

Definition 35 (Torsion tensor) *The torsion tensor acts on $X, Y \in \mathcal{X}(M)$ and $\omega \in \Omega^1(M)$ by*

$$T(\omega : X, Y) = \omega\big(\nabla_X Y - \nabla_Y X - [X, Y]\big) \,. \tag{4.58}$$

We may equivalently think of this as a map $T : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \,. \tag{4.59}$$

Definition 36 (Curvature tensor) *The curvature acts on $X, Y, Z \in \mathcal{X}(M)$ and $\omega \in \Omega^1(M)$ as*

$$R(\omega : X, Y, Z) = \omega\big(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\big) \tag{4.60}$$

As for the torsion we may think of this as a map $\mathcal{X}(M) \times \mathcal{X}(M)$ to a differential operator acting on $\mathcal{X}(M)$ as

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \,. \tag{4.61}$$

Exercise 4.2: Torsion and Curvature are tensors

Check that both the Torsion and Curvature tensors are actually tensors. There are two ways of doing this.

1. Show that it is multi-linear in all arguments. For example show $T(\omega : fX, Y) = fT(\omega : X, Y)$ for all $f \in \mathcal{F}(M)$ and so forth.

2. Show that it transforms as a tensor should under a change of coordinates.

We can evaluate the tensors in a basis to obtain the component form. Let $\{\theta^\rho\} = \{\mathrm{d}x^\rho\}$ be a basis of the cotangent space and $\{e_\mu\} = \{\partial_\mu\}$ be our basis for the tangent space.

4.3.1 Component form of the torsion

In this basis the components of the torsion tensor are

$$\begin{aligned} T^{\rho}{}_{\mu\nu} &= T(\theta^{\rho} : e_{\mu}, e_{\nu}) \\ &= \theta^{\rho}\big(\nabla_{\mu}e_{\nu} - \nabla_{\nu}e_{\mu} - [e_{\mu}, e_{\nu}]\big) \\ &= \theta^{\rho}\big(\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\sigma}{}_{\nu\mu}\big)e_{\sigma} \\ &= \Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu}\,. \end{aligned} \tag{4.62}$$

We therefore end up with

$$\boxed{T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu}\,.} \tag{4.63}$$

So despite $\Gamma^{\sigma}{}_{\mu\nu}$ not being a tensor, the anti-symmetrised part is! The torsion tensor is clearly anti-symmetric in the two lowered indices.

Definition 37 (Torsion free) *We see that connections $\Gamma^{\sigma}{}_{\mu\nu}$ which are symmetric in the lowered indices have $T^{\rho}{}_{\mu\nu} = 0$ and are called* torsion-free*.*

4.3.2 Component form of the Riemann tensor

A similar computation for the Riemann tensor gives

$$\boxed{R^{\sigma}{}_{\rho\mu\nu} = \partial_{\mu}\Gamma^{\sigma}{}_{\nu\rho} - \partial_{\nu}\Gamma^{\sigma}{}_{\mu\rho} + \Gamma^{\lambda}{}_{\nu\rho}\Gamma^{\sigma}{}_{\mu\lambda} - \Gamma^{\lambda}{}_{\mu\rho}\Gamma^{\sigma}{}_{\nu\lambda}\,.} \tag{4.64}$$

4.3.3 The Ricci identity

Consider the commutator of covariant derivatives acting on a vector field, we have

$$\begin{aligned} \nabla_{[\mu}\nabla_{\nu]}X^{\sigma} =& \partial_{[\mu}\big(\nabla_{\nu]}X^{\sigma}\big) + \Gamma^{\sigma}{}_{[\mu|\lambda|}\nabla_{\nu]}Z^{\lambda} - \Gamma^{\rho}{}_{[\mu\nu]}\nabla_{\rho}X^{\sigma} \\ =& \partial_{[\mu}\partial_{\nu]}X^{\sigma} + \big(\partial_{[\mu}\Gamma^{\sigma}{}_{\nu]\rho}\big)X^{\sigma} + \big(\partial_{[\mu}X^{\rho}\big)\Gamma^{\sigma}{}_{\nu]\rho} + \Gamma^{\sigma}{}_{[\mu|\lambda|}\partial_{\nu]}X^{\lambda} \\ &+ \Gamma^{\sigma}{}_{[\mu|\lambda|}\Gamma^{\lambda}{}_{\nu]\rho}X^{\rho} - \Gamma^{\rho}{}_{[\mu\nu]}\nabla_{\rho}X^{\sigma}\,. \end{aligned} \tag{4.65}$$

The first term of the second line vanishes, while the third and fourth cancel. The second term on the second line and first on the third line combine to give the Riemann tensor while the last gives the torsion. Putting everything together we have the *Ricci identity*

$$\boxed{2\nabla_{[\mu}\nabla_{\nu]}X^{\sigma} = R^{\sigma}{}_{\rho\mu\nu}X^{\rho} - T^{\rho}{}_{\mu\nu}\nabla_{\rho}X^{\sigma}\,.} \tag{4.66}$$

Similar identities hold when acting on other tensors and can be shown following similar steps to the above.

4.3.4 Levi–Civita connection

So far the discussion has not required a metric. When a metric exists we have the following theorem.

Theorem 5 (Levi–Civita Connection) *There exists a unique, torsion free, connection that is compatible with the metric g:*

$$\nabla_X g = 0\,, \tag{4.67}$$

for all vector fields X. This connection is called the Levi–Civita connection.

Proof: To prove this we first show uniqueness before constructing the connection. Suppose that such a connection exists, then we have

$$X\Big(g(Y,Z)\Big) = \nabla_X\Big(g(Y,Z)\Big) = \Big(\nabla_X g\Big)(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\,. \tag{4.68}$$

Since $\nabla_X g = 0$ we have

$$X\Big(g(Y,Z)\Big) = g(Y,\nabla_X Z) + g(\nabla_X Y, Z)\,. \tag{4.69}$$

We may use our favourite trick and cyclically permute X, Y, Z to find

$$\begin{aligned} Y\Big(g(Z,X)\Big) &= g(Z,\nabla_Y X) + g(\nabla_Y Z, X)\,,\\ Z\Big(g(X,Y)\Big) &= g(X,\nabla_Z Y) + g(\nabla_Z X, Y)\,. \end{aligned} \tag{4.70}$$

By the no torsion condition we have

$$\nabla_X Y - \nabla_Y X = [X,Y]\,, \tag{4.71}$$

and therefore

$$\begin{aligned} X\big(g(Y,Z)\big) &= g(\nabla_Y Z, X) + g(\nabla_Y X, Z) + g\big([X,Y],Z\big)\,,\\ Y\big(g(Z,X)\big) &= g(\nabla_Z Y, X) + g(\nabla_Y X, Z) + g\big([Y,Z],X\big)\,,\\ Z\big(g(X,Y)\big) &= g(\nabla_Z Y, X) + g(\nabla_X Z, Y) + g\big([Z,X],Y\big)\,, \end{aligned} \tag{4.72}$$

Adding the first and second and subtracting the third we find

$$\begin{aligned} g(\nabla_Y X, Z) =&\frac{1}{2}\Big[X\big(g(Y,Z)\big) + Y\big(g(Z,X)\big) - Z\big(g(X,Y)\big)\\ &- g\big([X,Y],Z\big) - g\big([Y,Z],X\big) + g\big([Z,X],Y\big)\Big] \end{aligned} \tag{4.73}$$

With a non-degenerate metric this specifies the connection uniquely.

It remains to be seen that the connection as defined does satisfy the properties of a connection. We will present one of the terms to check. The most finicky one is $\nabla_{fX}Y = f\nabla_X Y$, so let us present that one

$$
\begin{aligned}
g(\nabla_{fY}X, Z) =& \frac{1}{2}\Big[X\big(g(fY,Z)\big) + fY\big(g(Z,X)\big) - Z\big(g(X,fY)\big) \\
& - g([X,fY],Z) - g([fY,Z],X) + g([Z,X],fY)\Big] \\
=& \frac{1}{2}\Big[fX\big(g(Y,Z)\big) + X(f)g(Y,Z) + fY\big(g(Z,X)\big) - fZ\big(g(X,Y)\big) - Z(f)g(X,Y) \\
& - fg([X,Y],Z) - X(f)g(Y,Z) - fg([Y,Z],X) + Z(f)g(Y,X) + fg([Z,X],Y)\Big] \\
=& g(f\nabla_Y X, Z)\,.
\end{aligned}
\tag{4.74}
$$

The coloured terms in the penultimate line cancel amongst themselves, leaving just the black terms as required. The other properties follow similarly. This then proves the uniqueness and has explicitly constructed such a connection.

In components we can evaluate

$$g(\nabla_\nu e_\mu, e_\rho) = \Gamma^\lambda{}_{\nu\mu} g_{\lambda\rho} = \frac{1}{2}\big(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\big)\,. \tag{4.75}$$

Multiplying by the inverse metric we have

$$\boxed{\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\big(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\big)\,.} \tag{4.76}$$

The connection compatible with the metric is called the *Levi–Civita connection* while the components of the Levi–Civita connection are called the *Christoffel symbols*.

There is a nice expression if you contract two indices of the Christoffel symbols, we have

$$\Gamma^\mu{}_{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\nu \sqrt{|g|} \tag{4.77}$$

To see this note

$$\Gamma^\mu{}_{\mu\nu} = \frac{1}{2} g^{\mu\rho}\partial_\nu g_{\mu\rho} = \frac{1}{2}\mathrm{tr}(g^{-1}\partial_\nu g) = \frac{1}{2}\mathrm{tr}(\partial_\nu \log g)\,, \tag{4.78}$$

for diagonalisable matrices we have $\mathrm{tr}\log A = \log\det(A)$ and therefore we find

$$\Gamma^\mu{}_{\mu\nu} = \frac{1}{2}\partial_\nu \log\det(g) = \frac{1}{\sqrt{\det(g)}}\partial_\nu \sqrt{\det(g)}\,. \tag{4.79}$$

This implies that

$$\sqrt{|g|}\nabla_\mu X^\mu = \sqrt{|g|}(\partial_\mu X^\mu + \Gamma^\mu{}_{\mu\nu}X^\nu) + \sqrt{|g|}\Big(\partial_\mu X^\mu + X^\nu \frac{1}{\sqrt{|g|}}\partial_\nu\sqrt{|g|}\Big) = \partial_\mu\big(\sqrt{|g|}X^\mu\big)\,. \tag{4.80}$$

Using this result we can prove the divergence theorem:

$$\int_M \mathrm{d}^m x\sqrt{|g|}\nabla_\mu X^\mu = \int_{\partial M}\mathrm{d}^{n-1}x\sqrt{\gamma}n_\mu X^\mu\,, \tag{4.81}$$

where γ_{ij} is the pull-back of the metric to ∂M, $\gamma = \det(\gamma_{ij})$ and n_μ is an outward pointing unit vector orthogonal to ∂M. On a Lorentzian manifold this holds provided that ∂M is either purely spacelike or purely timelike, which guarantees that $\gamma \neq 0$.

4.4 Parallel transport and geodesics

We have introduced the connection but we are yet to explain what it connects. It connects tangent spaces, or more generally any vector space at different points of the manifold. This map is called *parallel transport*. Take a vector field X with some associated integral curve γ with coordinates $x^\mu(\lambda)$ such that

$$X^\mu\big|_\gamma = \frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda}\,. \tag{4.82}$$

We say that a tensor field T is parallel transported along γ if

$$\nabla_X T = 0\,. \tag{4.83}$$

Let γ connect two points $p, q \in M$. The condition (4.83) provides a map from the vector space defined at p to the vector space defined at q. Consider a second vector field Y. In components (4.83) reads

$$X^\nu\big(\partial_\nu Y^\mu + \Gamma^\mu{}_{\nu\rho}Y^\rho\big) = 0\,. \tag{4.84}$$

If we evaluate it on the curve γ, we can write $Y^\mu = Y^\mu(x(\lambda))$ and therefore the condition is

$$\frac{\mathrm{d}Y^\mu}{\mathrm{d}\lambda} + X^\nu\Gamma^\mu{}_{\nu\rho}Y^\rho = 0\,. \tag{4.85}$$

This defines a set of coupled ordinary differential equations, given an initial condition at $p = \gamma(\lambda = 0)$ for example these can be solved to find a unique vector field at each point along the curve. This is path dependent and depends on the connection and the underlying path which was characterised by X here.

There is a subtle difference between what we are doing here and what we did with the push-forward and pull-back, which we used to define the Lie derivative. Here X only appears to define the map, there are no derivatives applied to X^μ as was for those maps. The connection does the work of relating the vector spaces along the curve and not the vector X.

4.4.1 Geodesics

Definition 38 (Affinely parametrised geodesic) *An* affinely parametrised geodesic *is an integral curve with tangent vector field X that obeys*

$$\nabla_X X = 0\,. \tag{4.86}$$

Along the curve γ with coordinates x^μ and tangent vector field X this implies

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu{}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} = 0\,. \tag{4.87}$$

We have thrown around the phrase affinely parametrised geodesic but what does this really mean? Consider a curve parametrised by λ and with tangent vector field X that satisfies (4.87). Let us parametrise it by some other parameter τ with $\lambda(\tau)$ and take $f = \frac{\mathrm{d}\lambda}{\mathrm{d}\tau} > 0$. The change in parametrisation leads to a different tangent vector to the curve since

$$Y^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = X^\mu \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\,. \tag{4.88}$$

We therefore have

$$\nabla_Y Y = \nabla_{fX} fX = f\nabla_X(fX) = fX[f]X + f^2\nabla_X X = X[f]Y\,. \tag{4.89}$$

We see that (4.89) defines the same geodesic however it is *not* affinely parametrised, the right-hand side of (4.87) no longer vanishes but is instead proportional to an arbitrary function multiplied by the vector field.

We see conversely that if we work in the opposite direction, with a non-affinely parametrised geodesic, that is a Y satisfying (4.89) then we can always perform a reparametrisation that gives us an affinely parametrised geodesic, therefore there is no loss of generality for us to consider affinely parametrised geodesics.

Observe that we still have some freedom to reparametrise our curve without ruining the affine parametrisation, this requires $X[f] = 0$. This is then equivalent to f being constant along the curve and therefore we may take $\lambda = a\tau + b$ with a, b constants without ruining the affine parametrisation. There is a two-parameter family of affine parameters for any affinely parametrised geodesic.

Exercise 4.3: Geodesic equation from Euler–Lagrange
Consider the action of a particle following the path $x^\mu(\lambda)$:

$$S = \int \mathrm{d}\lambda \sqrt{-g_{\mu\nu}(x)\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\,. \tag{4.90}$$

Using the Euler–Lagrange equations show that generically you obtain the non-affinely parametrised geodesics from the action with respect to the Levi–Civita connection.

What is the affine parameter for a time-like geodesic?

If we choose the Levi–Civita connection, since $\nabla_X g = 0$ it follows that for any vector field Y which is parallel transported along a geodesic defined by X we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} g(X, Y) = 0 \,. \tag{4.91}$$

The vector field Y makes the same angle with the tangent vector at each point along the geodesic. Further, this holds true if we replace Y by X in the expression above. Since the norm of the vector field X tangent to the geodesic classifies the character of the geodesic, (timelike/null/spacelike), if we define a geodesic using a metric compatible connection, then the nature of the geodesic does not change. This statement relies on us using a metric compatible connection though, in this course we will always take such a connection and therefore the nature of a geodesic is preserved throughout all spacetime.

Let us consider a timelike geodesic. When we vary the action (4.90) what are we extremising and is it a maximum of minimum? From our definition of the proper time, (4.18) we see that we are extremising the proper time, and geodesics maximise the proper time. Why is this true? Well given any time-like curve we can approximate it to arbitrary accuracy by a null curve. We should consider jagged null curves that follow the time-like one, see figure 11. As we increase the number of null curves the approximation gets better and better, while still having zero length. Timelike curves cannot therefore be curves with minimal proper time since they are infinitesimally close to curves of zero length (and therefore zero proper time). They must therefore maximise the proper time. This is why the twin who remains home in the twin paradox ages more, they are on a geodesic (for most of the journey). We should really say that this maximises the proper time locally. If we took a sphere, then there is more than one geodesic between two points, we can either go the short way around or the long way around. One is longer than the other (assuming the points are not opposite each other, i.e. picking the poles), but both maximise locally the length functional.

4.4.2 Normal coordinates

Geodesics allow for the construction of a particularly useful coordinate system. This holds independently of whether the Levi–Civita connection is employed or note, however it takes a particularly simple form when it is used. On a Riemannian manifold, in the neighbourhood of a point $p \in M$ we can always find coordinates such that

$$g_{\mu\nu}(p) = \delta_{\mu\nu} \,, \qquad \text{and} \qquad \partial_\rho g_{\mu\nu}(p) = 0 \,. \tag{4.92}$$

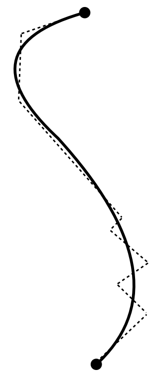


Figure 11: We approximate the time-like curve with null curves. As we increase the number of null curves the approximation gets better and better.

The same is true for Lorentzian manifolds with $\delta \to \eta$. These coordinates are known as *normal coordinates*. Since the first derivative of the metric vanishes at p it implies that the Christoffel symbols vanish there: $\Gamma^{\mu}{}_{\nu\rho}(p) = 0$. As we move away from p this does not need to continue to hold. It should be noted that one cannot generically make the second derivative of the metric vanish at p, it is only the first derivative. This means that it is not possible to pick the Riemann tensor to vanish at a given point.

We can brute force this. Start with a metric $\tilde{g}_{\mu\nu}$ in coordinates $\tilde{x}^{\mu}$ and try to find a new set of coordinates $x^{\mu}(\tilde{x})$ which satisfy the required conditions. In the new coordinates we have

$$\frac{\partial \tilde{x}^{\rho}}{\partial x^{\mu}} \frac{\partial \tilde{x}^{\sigma}}{\partial x^{\nu}} \tilde{g}_{\rho\sigma} = g_{\mu\nu} = \delta_{\mu\nu} \,. \tag{4.93}$$

We can take the point p to be the origin of both coordinate systems and Taylor expand around the point

$$\tilde{x}^{\rho} = 0 + \frac{\partial \tilde{x}^{\rho}}{\partial x^{\mu}}\Big|_{x=0} x^{\mu} + \frac{1}{2} \frac{\partial^2 \tilde{x}^{\rho}}{\partial x^{\mu} \partial x^{\nu}}\Big|_{x=0} x^{\mu} x^{\nu} + \dots \,. \tag{4.94}$$

Inserting the expansion into (4.93) together with the Taylor expansion of $\tilde{g}_{\mu\nu}$ we can try to solve the resulting PDEs. The first order variation implies

$$\frac{\partial \tilde{x}^{\rho}}{\partial x^{\mu}}\Big|_{x=0} \frac{\partial \tilde{x}^{\sigma}}{\partial x^{\nu}}\Big|_{x=0} \tilde{g}_{\rho\sigma}(p) = \delta_{\mu\nu} \,. \tag{4.95}$$

We can always find $\partial \tilde{x} / \partial x$ such that this is true, there are many choices. For $\dim M = m$ there are m^2 independent coefficients of $\partial \tilde{x}/\partial x$. The equation above contains $\frac{1}{2} m(m+1)$ conditions on these, since $\tilde{g}$ is symmetric. This leaves us with $\frac{1}{2} m(m-1)$ parameters which are un-fixed. Notice that this remainder is precisely the same number of components of the rotational group of $\mathrm{SO}(m)$ or $\mathrm{SO}(1, m-1)$, this is of course the group that leaves the flat metric unchanged and is therefore to be expected. Next consider the second order variations. There are $\frac{1}{2}m^2(m+1)$ independent components of $\partial^2\tilde{x}^\rho/\partial x^\mu \partial x^\nu$ which is the same number of components of $\partial_\rho g_{\mu\nu}$ and so we can always choose the first derivative of the metric at p to vanish. Consider now the second derivative term, requiring $\partial_\rho\partial_\sigma g_{\mu\nu} = 0$ imposes $\frac{1}{4}m^2(m+1)^2$ constraints. However the next term in the Taylor expansion is $\partial^3\tilde{x}^\rho/\partial x^\mu\partial x^\nu\partial x^\sigma$ and has only $\frac{1}{6}m^2(m+1)(m+2)$ independent coefficients: there are not enough independent coefficients to cancel all of the terms of the second derivative. The difference is the number of ways of characterising the second derivative of the metric that cannot be undone by coordinate transformations. This is precisely the number of independent components of the Riemann tensor, this is

$$\frac{1}{4}m^2(m+1)^2 - \frac{1}{6}m^2(m+1)(m+2) = \frac{1}{12}m^2(m+1)(m-1)\,. \tag{4.96}$$

One can explicitly construct the normal coordinates using the exponential map and geodesics flowing through the point p. One can consider all affinely parametrised geodesics through p and label the point q at a small fixed distance of the affine parameter by the coordinates of the geodesic flowing through q. One then essentially uses geodesics to construct your basis vectors. We will not consider this construction here.

The Equivalence principle Normal coordinates play an important role in GR. Any observer at a point p who parametrises their immediate surroundings using normal coordinates will experience a locally flat metric.

This is the mathematics underling the Einstein equivalence principle. Any freely falling observer, performing local experiments will not experience a gravitational field. Here free falling means following a geodesic and therefore they can use normal coordinates. The lack of gravitational field is the statement that $g_{\mu\nu}(p) = \eta_{\mu\nu}$.

There are limitations to the equivalence principle and the important word is **local**. There is a way to distinguish whether there is a gravitational field or at p. We simply compute the Riemann tensor. This depends on the second derivative of the metric and will in general be non-vanishing. However to measure the effects of the Riemann tensor one typically has to compare the result of an experiment at p with the result at a nearby point q, this is then a "non-local" observable, according to the equivalence principle.

4.4.3 Path dependence: Curvature and Torsion

We have introduced the curvature and torsion tensors but what are they really measuring? For the Riemann tensor let us consider a vector $Z_p \in T_p(M)$ and parallel transport it along a curve C to some point $r \in M$. In addition condition another curve C' along which we can parallel transport Z_p to r. We will see that the difference between the two vectors at r is determined by the Riemann tensor.

The meaning of curvature Let us construct our curves from two segments, generated by linearly independent vector fields X, Y and let us take $[X, Y] = 0$. (Recall that this implies that the parallelogram constructed from the vectors closes, see section 3.3.2). We take the points to be close and pick normal coordinates $x^\mu = (\tau, \sigma, 0, ..., 0)$ so that the starting point is at $x^\mu(p) = 0$, and the tangent vectors are aligned along the coordinates $X = \frac{\partial}{\partial \tau}$ and $Y = \frac{\partial}{\partial \sigma}$. The other corner points are $x^\mu(r) = (\delta\tau, 0, 0, ..)$, $x^\mu(s) = (0, \delta\sigma, 0, ...)$ and $x^\mu(r) = (\delta\tau, \delta\sigma, 0, ...)$, with $\delta\tau$ and $\delta\sigma$ small, see figure 12.

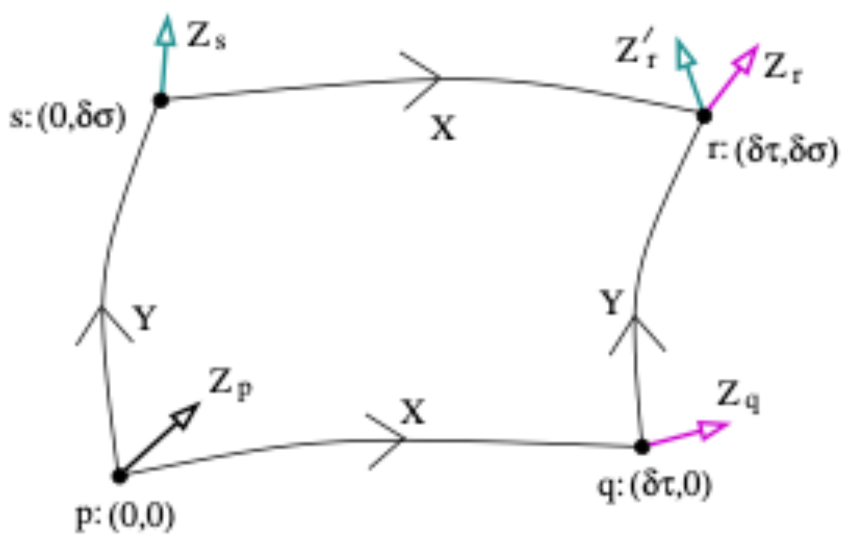


Figure 12: Parallel transporting a vector Z_p along two different paths does not give the same answer. From the lecture notes of Tong.

First parallel transport Z_p along X to obtain Z_q. Along the curve, by definition of the parallel transport Z^μ satisfies

$$\frac{\mathrm{d}Z^\mu}{\mathrm{d}\tau} + X^\nu \Gamma^\mu{}_{\rho\nu} Z^\rho = 0\,. \tag{4.97}$$

We can Taylor expand the solution as

$$Z_q^\mu = Z_p^\mu + \frac{\mathrm{d}Z^\mu}{\mathrm{d}\tau}\Big|_p \delta\tau + \frac{1}{2}\frac{\mathrm{d}^2 Z^\mu}{\mathrm{d}\tau^2}\Big|_p \delta\tau^2 + \mathcal{O}(\delta\tau^3)\,. \tag{4.98}$$

We can use normal coordinates at the point p which implies that ${\Gamma^\mu}_{\rho\nu}(p) = 0$ and therefore $\frac{\mathrm{d}Z^\mu}{\mathrm{d}\tau}\big|_p = 0$. To calculate the second derivative we differentiate (4.97), to obtain

$$\begin{aligned}
\frac{\mathrm{d}Z^\mu}{\mathrm{d}\tau^2}\Big|_{\tau=0} &= -\Big(X^\nu Z^\rho \frac{\mathrm{d}{\Gamma^\mu}_{\rho\nu}}{\mathrm{d}\tau} + \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau} Z^\rho {\Gamma^\mu}_{\rho\nu} + X^\nu \frac{\mathrm{d}Z^\rho}{\mathrm{d}\tau}{\Gamma^\mu}_{\rho\nu}\Big)\Big|_p \\
&= -X^\nu Z^\rho \frac{\mathrm{d}{\Gamma^\mu}_{\rho\nu}}{\mathrm{d}\tau}\Big|_p \\
&= -X^\nu X^\sigma Z^\rho \partial_\sigma {\Gamma^\mu}_{\rho\nu}\Big|_p
\end{aligned} \tag{4.99}$$

To get to the second line we have used that we are working in normal coordinates at p and the final line is because τ parametrises the integral curve of X. We find

$$Z^\mu_q = Z^\mu_p - \frac{1}{2} X^\nu X^\sigma Z^\rho \partial_\sigma {\Gamma^\mu}_{\rho\nu}\Big|_p \delta\tau^2 + \ldots. \tag{4.100}$$

Now we parallel transport again, this time along Y to the point r with resultant vector Z^μ_r. The Taylor expansion is

$$Z^\mu_r = Z^\mu_q + \frac{\mathrm{d}Z^\mu}{\mathrm{d}\sigma}\Big|_q \delta\sigma + \frac{1}{2}\frac{\mathrm{d}^2 Z^\mu}{\mathrm{d}\sigma^2}\Big|_q \delta\sigma^2 + \mathcal{O}(\delta\sigma^3)\,. \tag{4.101}$$

We can evaluate the first derivative $\frac{\mathrm{d}Z^\mu}{\mathrm{d}\sigma}\big|_q$ using the analogue of the parallel transport equation (4.97),

$$\frac{\mathrm{d}Z^\mu}{\mathrm{d}\sigma}\Big|_q = -Y^\nu Z^\rho {\Gamma^\mu}_{\rho\nu}\big|_q\,, \tag{4.102}$$

however since our normal coordinates are at p and not q we cannot argue that this term immediately vanishes, instead we can Taylor expand about p to get

$$Y^\nu Z^\rho {\Gamma^\mu}_{\rho\nu}\big|_q = Y^\nu Z^\rho X^\sigma \partial_\sigma {\Gamma^\mu}_{\rho\nu}\big|_p \delta\tau + \ldots. \tag{4.103}$$

One should also expand Y^ν and Z^ν however to leading order they multiply ${\Gamma^\mu}_{\rho\nu}(p) = 0$ ergo, only contribute at the next order. For the second order term in the Taylor expansion (4.101) there is a similar expression to before, we find

$$\begin{aligned}
\frac{\mathrm{d}^2 Z^\mu}{\mathrm{d}\sigma^2}\Big|_q &= -Y^\nu Y^\sigma Z^\rho \partial_\sigma {\Gamma^\mu}_{\rho\nu}\big|_q + \ldots \\
&= -Y^\nu Y^\sigma Z^\rho \partial_\sigma {\Gamma^\mu}_{\rho\nu}\big|_p + \ldots
\end{aligned} \tag{4.104}$$

After the dust settles we have

$$Z^\mu_r = Z^\mu_q - Y^\nu Z^\rho X^\sigma \partial_\sigma {\Gamma^\mu}_{\rho\nu}\big|_p \delta\tau\delta\sigma - \frac{1}{2} Y^\nu Y^\sigma Z^\rho \partial_\sigma {\Gamma^\mu}_{\rho\nu}\big|_p \delta\sigma^2 + \ldots. \tag{4.105}$$

and therefore

$$Z^\mu_r = Z^\mu_p - \frac{1}{2}\partial_\sigma {\Gamma^\mu}_{\rho\nu}\big|_p \Big[X^\nu X^\sigma Z^\rho \delta\tau^2 + 2Y^\nu Z^\rho X^\sigma \delta\sigma\delta\tau + Y^\nu Y^\sigma Z^\rho \delta\sigma^2\Big]\Big|_p + \ldots. \tag{4.106}$$

with ... cubic and higher terms. We can now consider the same computation for the path C'. We merely need to swap the role of $\tau \leftrightarrow \sigma$ and $X \leftrightarrow Y$, so that

$$Z'^{\mu}_{r} = Z^{\mu}_{p} - \frac{1}{2}\partial_\sigma \Gamma^{\mu}{}_{\rho\nu}\big|_p \Big[X^\nu X^\sigma Z^\rho \delta\tau^2 + 2X^\nu Z^\rho Y^\sigma \delta\sigma\delta\tau + Y^\nu Y^\sigma Z^\rho \delta\sigma^2\Big]\Big|_p + ... \tag{4.107}$$

and therefore

$$\begin{aligned} \Delta Z^{\mu}_{r} = Z^{\mu}_{r} - Z'^{\mu}_{r} &= -\Big(\partial_\sigma \Gamma^{\mu}{}_{\rho\nu} - \partial_\nu \Gamma^{\mu}{}_{\rho\sigma})|_p (Y^\nu Z^\rho X^\sigma)|_p \delta\sigma\delta\tau + ... \\ &= R^{\mu}{}_{\rho\sigma\nu} Y^\nu Z^\rho X^\sigma|_p \delta\sigma\delta\tau \,. \end{aligned} \tag{4.108}$$

The final expression follows from the Riemann tensor expression in normal coordinates. Although our calculation was performed in a certain choice of coordinates since the end result is an equality between tensors it must hold in any coordinate system. This is a common trick, normal coordinates generally simplify expressions.

The Riemann tensor tells us the path dependence of parallel transport. This is related to the concept of *holonomy*. If we transport a vector around a closed loop we can ask how it compares to the original vector. This is captured by the Riemann tensor. A particularly easy example is to consider a two-sphere. We can draw a loop by considering the intersection of three great circles. First go along the equator by $1/4$ of the circumference. Then make a $\pi/2$ turn and head to the north pole. At the north pole go south on another $\pi/2$ angle. You will end up with a triangle with angle $3\pi/2$. Now consider parallel transporting a vector along this loop. You will see that it changes direction when you get back to the start. Of course one could take any path and the direction you end up facing depends on the path. The set of all possible transformations of the vector at p along loops form a group known as the *holonomy group*. For a Riemannian manifold with a metric this is a subgroup of $\mathrm{SO}(m)$ while for a Lorentzian manifold it is a subgroup of $\mathrm{SO}(1, m-1)$.

The meaning of Torsion Torsion will not play a role in this course since we will excusively use the Levi–Civita connection which is torsion free. Before we completely rid ourselves of the torsion let us first understand its geometric meaning.

Take two vectors $X, Y \in T_p(M)$ and let us use coordinates x^μ. Starting at $p \in M$ we can use these vectors to construct two points infinitesimally close to p, let them be r and s respectively:

$$r : x^\mu + \epsilon X^\mu \qquad \text{and} \qquad s : x^\mu + \epsilon Y^\mu \,. \tag{4.109}$$

We can now parallel transport X along Y to give a new vector $X' \in T_s(M)$ and similarly parallel transport Y along X to get a new vector $Y' \in T_r(M)$. The new vectors have components

$$X' = (X^\mu - \epsilon\Gamma^\mu{}_{\nu\rho}Y^\nu X^\rho)\partial_\mu\,, \qquad Y' = (Y^\mu - \epsilon\Gamma^\mu{}_{\nu\rho}X^\nu Y^\rho)\partial_\mu\,. \tag{4.110}$$

Each now defines a new point. Starting from s and moving in the direction X' we get a new point

$$q : x^\mu + (X^\mu + Y^\mu)\epsilon - \epsilon^2\Gamma^\mu{}_{\nu\rho}Y^\nu X^\rho\,. \tag{4.111}$$

Similarly if we sit at r and move in the direction of Y' we get to a typically different point t with coordinates

$$t : x^\mu + (X^\mu + Y^\mu)\epsilon - \epsilon^2\Gamma^\mu{}_{\nu\rho}X^\nu Y^\rho\,. \tag{4.112}$$

The two points are not the same when $\Gamma^\mu{}_{\nu\rho} \neq \Gamma^\mu{}_{\rho\nu}$, i.e. when the connection has torsion. Torsion measures the failure of the parallelogram in figure 13 to close.

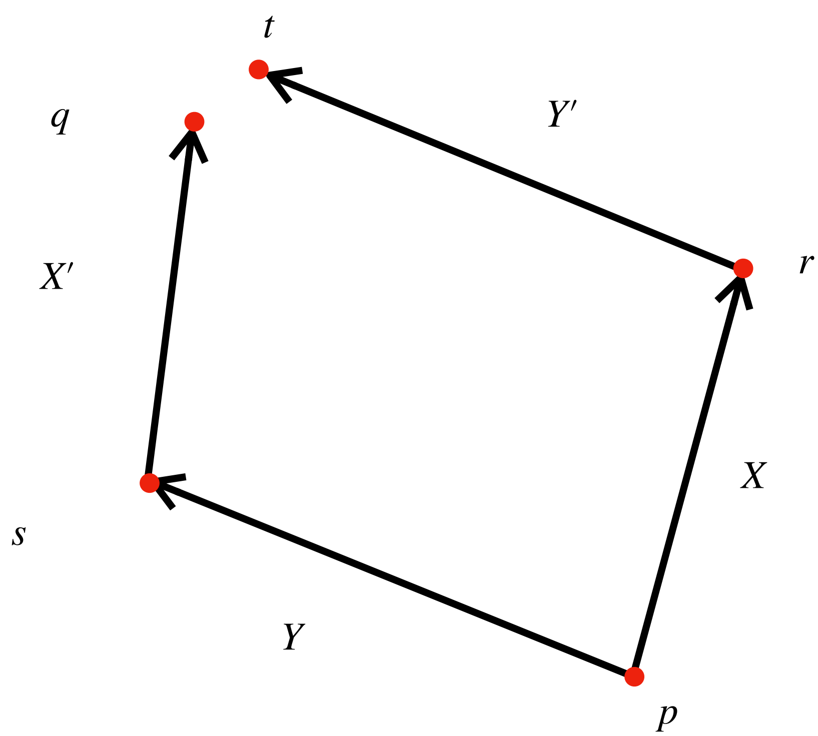


Figure 13: We transport the two vectors X and Y along each other. The failure for the parallelogram to close is measured by the torsion of the connection.

4.4.4 Geodesic deviation

In Euclidean space or in Minkowski spacetime, geodesics which are initially parallel will remain parallel forever. On a general curved manifold this notion of parallel is not possible, instead we can study whether nearby geodesics move together or apart, and characterise their relative acceleration.

Consider a one-parameter family of geodesics with coordinates $x^\mu(\tau : \sigma)$. Here τ is the affine parameter along the geodesics, all of which are tangent to the vector field X. Thus, along the surface spanned by $x^\mu(\tau : \sigma)$ we have

$$\frac{\partial x^\mu}{\partial \tau}\bigg|_\sigma = X^\mu \,. \tag{4.113}$$

The parameter σ labels the different geodesics, see figure 14. We can compute the tangent vector in the σ direction to be generated by a second vector field S so that

$$S^\mu = \frac{\partial x^\mu}{\partial \sigma}\bigg|_\tau \,. \tag{4.114}$$

This tangent vector is known as the *deviation vector*, its job is to take us from one geodesic to a nearby geodesic with the same affine parameter τ. The family of geodesics sweep out a 2d surface embedded in the manifold. We have freedom to choose coordinates so that on the surface $S = \frac{\partial}{\partial \sigma}$ and $X = \frac{\partial}{\partial \tau}$ consequently we have $[X, S] = 0$.

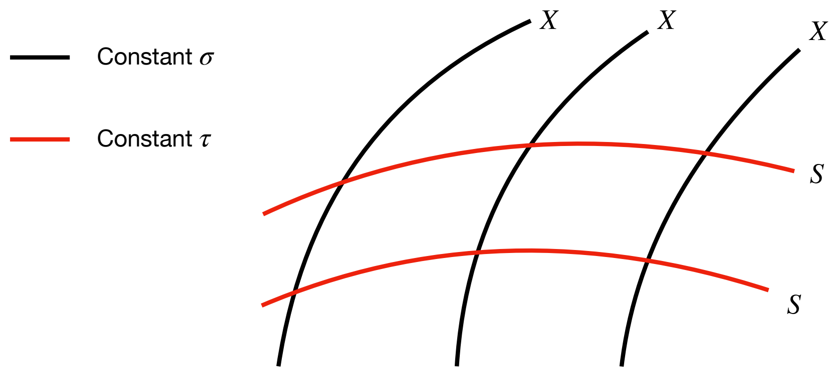


Figure 14: The black lines are geodesics generated by X while the red lines label constant τ and are generated by S with $[X, S] = 0$.

We can ask how neighbouring geodesics behave, do they converge, diverge, or remain the same distance apart? Consider a torsion free connection so that

$$\nabla_X S - \nabla_S X = [X, S] \,. \tag{4.115}$$

Since $[X, S] = 0$, we have

$$\nabla_X \nabla_X S = \nabla_X \nabla_S X = \nabla_S \nabla_X X + R(X, S)X \,, \tag{4.116}$$

where we have used the expression for the Riemann tensor in (4.61). Since X is tangent to geodesics we have $\nabla_X X = 0$ and therefore

$$\nabla_X \nabla_X S = R(X, S)X\,. \tag{4.117}$$

In index notation we have

$$X^\nu \nabla_\nu \big(X^\rho \nabla_\rho S^\mu\big) = R^\mu{}_{\nu\rho\sigma} X^\nu X^\rho S^\sigma\,. \tag{4.118}$$

If we take an integral curve γ associated to X as before we have

$$\frac{D^2 S^\mu}{D\tau^2} = R^\mu{}_{\nu\rho\sigma} X^\nu X^\rho S^\sigma\,, \tag{4.119}$$

with $D/D\tau$ the covariant derivative along the curve γ, $D/D\tau \equiv \frac{\partial x^\mu}{\partial \tau}\nabla_\mu$. The left hand side tells us how the deviation vector S changes as we move along the geodesic and it measures the relative acceleration of neighbouring geodesics. From (4.119) we see that the relative acceleration of neighbouring geodesics is measured by the Riemann tensor. This is nothing other than the *tidal forces* mentioned previously. Note that the relative acceleration vanishes for all families of geodesics if and only if the Riemann tensor vanishes.

4.5 Riemann tensor and its symmetries

We have just seen that the Riemann tensor is responsible for tidal forces, let us now study this tensor in more detail. The components of the Riemann tensor are given in (4.64). It is not hard to see that it is anti-symmetric in the final two indices:

$$R^\sigma{}_{\rho\mu\nu} = -R^\sigma{}_{\rho\nu\mu}\,. \tag{4.120}$$

This does not exhaust the symmetries however. If we lower an index then we have

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}\,, \tag{4.121}$$

$$R_{\mu[\nu\rho\sigma]} = 0\,, \tag{4.122}$$

$$\nabla_{[\mu} R_{\sigma\rho]\tau\nu} = 0\,. \tag{4.123}$$

These expressions can be proven by using normal coordinates.

4.5.1 Ricci and Einstein tensors

Given a rank $(1,3)$ tensor we can construct a rank $(0,2)$ tensor by contraction, for the Riemann tensor the resultant $(0,2)$-rank tensor is called the Ricci tensor.

Definition 39 (Ricci tensor) *The* Ricci *tensor is*

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}\,. \tag{4.124}$$

It inherits symmetry in its indices from the properties of the Riemann tensor and therefore

$$R_{\mu\nu} = R_{\nu\mu}\,. \tag{4.125}$$

We can play a similar game to create a scalar by contracting the indices again, the resultant scalar is known as the Ricci-scalar.

Definition 40 (Ricci Scalar) *The Ricci scalar is*

$$R = g^{\mu\nu} R_{\mu\nu}\,. \tag{4.126}$$

Using the metric compatible connection, the Bianchi identity implies that

$$\nabla^{\mu}\Big(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\Big) = 0\,. \tag{4.127}$$

Definition 41 (Einstein tensor) *The covariantly constant tensor*

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\,, \tag{4.128}$$

called the Einstein tensor*.*

This will appear in the next section when we consider GR and its conservation has physical consequences.

5 Einstein's equations

After defining all this mathematics we can now use it to introduce general relativity. Like the other forces, gravity is also mediated by some field, in this case it is the metric $g_{\mu\nu}$. It is a dynamical object, not something fixed and therefore there must be some rules as to how it can evolve. These are provided by the equations of motion following from the *Einstein–Hilbert* action.

5.1 The Einstein–Hilbert action

We want to write down an action for the gravity. Differential geometry places some rigid constraints on what this can be. We want the action to be diffeomorphism invariant, since physics should not depend on the choice of coordinates. It should therefore depend on the intrinsic properties of the metric.

Spacetime is a manifold M equipped with a metric of Lorentzian signature. The action is an integral over M and so we require a volume-form. Thankfully the metric provides us with a canonical volume-form, which is reparametrisation invariant with which we can integrate any scalar. Given that we only have a metric there is not really much that we can construct. The simplest non-trivial scalar we can construct is the Ricci scalar, and therefore we can guess the action

$$S_{\mathrm{EH}} = \int \mathrm{d}^4x\sqrt{-g}R\,. \tag{5.1}$$

As a quick check since the Ricci scalar takes the form $R \sim \partial\Gamma + \Gamma\Gamma$ and the Levi–Civita connection is $\Gamma \sim \partial g$ it follows that the action is second derivative in the metric. This is like all other actions that we have considered previously, they all had two derivatives of our fundamental field.

The equations of motion will follow from varying the action with respect to the metric. We start with a fixed metric and see how the action varies as we shift

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\,. \tag{5.2}$$

Writing the Ricci scalar as $R = g^{\mu\nu}R_{\mu\nu}$ the Einstein–Hilbert action changes as

$$\delta S = \int \mathrm{d}^4x\Big((\delta\sqrt{-g})g^{\mu\nu}R^{\mu\nu} + \sqrt{-g}(\delta g^{\mu\nu})R_{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\Big)\,. \tag{5.3}$$

It turns out that it is simpler to consider the variation with respect to the inverse metric, this is of course equivalent to considering the variation with the metric since

$$g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu\,, \quad \Rightarrow \quad (\delta g_{\mu\nu})g^{\nu\rho} + g_{\mu\nu}\delta g^{\nu\rho} = 0\,, \quad \Rightarrow \quad \delta g^{\nu\rho} = -g^{\nu\sigma}g^{\rho\mu}\delta g_{\sigma\mu}\,. \tag{5.4}$$

The second term in the variation of the Einstein–Hilbert action is already proportional to $\delta g^{\mu\nu}$, so want to consider the first and third terms only. First consider the variation of the determinant term. To do this we must remember a few properties of a diagonalisable matrix A, namely

$$\log\det A = \operatorname{tr}\log A\,. \tag{5.5}$$

(To prove this use that this is clearly true for a diagonal matrix since the determinant is the product of the eigenvalues while the trace is the sum of the eigenvalues. Since both the determinant and trace are invariant under conjugation it follows for any diagonalisable matrix.) Thus we have

$$\frac{1}{\det A}\delta\det A = \operatorname{tr}(A^{-1}\delta A)\,, \tag{5.6}$$

after recalling the properties of the (matrix) Logarithm. Applying this to the metric we have

$$\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}(-g)g^{\mu\nu}\delta g_{\mu\nu} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}\,. \tag{5.7}$$

Using the identity (5.4) we have

$$\boxed{\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\,,} \tag{5.8}$$

as claimed.

With this the variation of the Einstein–Hilbert action takes the form

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\Big(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\Big)\delta g^{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\,, \tag{5.9}$$

and it remains to consider the final term only. We claim that this term is a total derivative and can therefore be neglected by using Stoke's theorem under suitable assumptions of spacetime (no boundary). To confirm this we need to prove the following identity:

$$\delta R_{\mu\nu} = \nabla_\rho\delta\Gamma^\rho{}_{\mu\nu} - \nabla_\nu\delta\Gamma^\rho{}_{\mu\rho}\,, \tag{5.10}$$

where

$$\delta\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\big(\nabla_\mu\delta g_{\sigma\nu} + \nabla_\nu\delta g_{\mu\sigma} - \nabla_\sigma\delta g_{\mu\nu}\big)\,. \tag{5.11}$$

We start by looking at the variation of the Christoffel symbols $\Gamma^\rho{}_{\mu\nu}$. Though the Christoffel symbol is not a tensor the variation $\delta\Gamma^\rho{}_{\mu\nu}$ is a tensor. This is because it is the difference of two Christoffel symbols, one computed using the metric $g_{\mu\nu}$ and one with $g_{\mu\nu} + \delta g_{\mu\nu}$ and the term in the transformation of the Christoffel which shows that it is not a tensor is independent of the metric and therefore cancels in the difference. This observation makes our lives a lot simpler. It implies that at any point $p \in M$ we can work in normal coordinates such that $\partial_\rho g_{\mu\nu}|_p = 0$ and therefore $\Gamma^\rho{}_{\mu\nu}|_p = 0$. To linear order in the variation the change in the Christoffel symbol evaluated at p is

$$\begin{aligned}\delta\Gamma^\rho{}_{\mu\nu}|_p &= \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu\delta g_{\sigma\nu} + \partial_\nu\delta g_{\sigma\mu} - \partial_\sigma\delta g_{\mu\nu}\right)|_p \\ &= \frac{1}{2}g^{\rho\sigma}\left(\nabla_\mu\delta g_{\sigma\nu} + \nabla_\nu\delta g_{\sigma\mu} - \nabla_\sigma\delta g_{\mu\nu}\right)|_p\end{aligned} \tag{5.12}$$

where we have used that in normal coordinates we can replace partial derivatives with covariant derivatives. Both the left and right hand side are tensors and therefore this holds in any coordinate system, moreover the point p was arbitrary and therefore this holds in all coordinate systems at all points $p \in M$.

Next consider the variation of the Riemann tensor. In normal coordinates we have

$$R^\sigma{}_{\rho\mu\nu} = \partial_\mu\Gamma^\sigma{}_{\nu\rho} - \partial_\nu\Gamma^\sigma{}_{\mu\rho}\,, \tag{5.13}$$

and the variation is

$$\delta R^\sigma{}_{\rho\mu\nu} = \partial_\mu\delta\Gamma^\sigma{}_{\nu\rho} - \partial_\nu\delta\Gamma^\sigma{}_{\mu\rho} = \nabla_\mu\delta\Gamma^\sigma{}_{\nu\rho} - \nabla_\nu\delta\Gamma^\sigma{}_{\mu\rho}\,, \tag{5.14}$$

where we have once again used that in normal coordinates we can replace partial derivatives with covariant derivatives. As before we have a tensorial equation and therefore this must hold in any coordinate system not just normal coordinates. Contracting the σ and μ indices we find

$$\boxed{\delta R_{\rho\nu} = \nabla_\mu\delta\Gamma^\mu{}_{\nu\rho} - \nabla_\nu\delta\Gamma^\mu{}_{\rho\mu}\,.} \tag{5.15}$$

It follows that

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu\Big(g^{\rho\nu}\delta\Gamma^\mu{}_{\rho\nu} - g^{\mu\nu}\delta\Gamma^\rho{}_{\nu\rho}\Big) = \nabla_\mu X^\mu\,, \tag{5.16}$$

with X^μ the bracketed tensor. The variation of the Einstein–Hilbert action can then be written as

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\left[\Big(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\Big)\delta g^{\mu\nu} + \nabla_\mu X^\mu\right]. \tag{5.17}$$

The final term is a total derivative after using the identity (4.80) and with suitable assumptions on spacetime can be neglected. Requiring that the action is extremised, so that $\delta S = 0$, implies the equation of motion

$$\boxed{G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0\,.} \tag{5.18}$$

These are the *Einstein field equations* in the absence of matter. We may further simplify them by first contracting with the inverse metric to find $R = 0$ and therefore in the absence of matter Einstein's equations are simply

$$R_{\mu\nu} = 0\,. \tag{5.19}$$

A metric obeying this equation is known as Ricci flat. Though this looks deceptively simple this holds a very rich set of solutions, in fact not all solutions to this equation are known.

We threw away the boundary term in the usual cavalier way one does with such variational principles. We may make this more rigorous by introducing the Gibbons–Hawking boundary term to allow for M to admit a boundary. The addition of this term gives the same Einstein field equations as before even for a manifold with a boundary.

5.1.1 Newton's constant

As it stand the action we have given does not have the correct dimension, this will become a problem when we want to couple to matter, so let us remedy this. We take the coordinates to have dimension of length $[x] = L$ and therefore the metric is dimensionless. The Ricci scalar involves two derivatives and therefore it has dimension $[R] = L^{-2}$. Including the dimension of the integration measure the current action in (5.1) has dimension $[S] = L^2$. An action should have dimension of Energy $\times$ time and therefore we should multiply the action by an appropriate dimensionful constant. This constant is known as Newton's constant. The correct action is:

$$S_{\text{EH}} = \frac{c^3}{16\pi G_N} \int \mathrm{d}^4 x \sqrt{-g} R\,, \tag{5.20}$$

where c is of course the speed of light, and G_N is Newton's constant

$$G_N \sim 6.67 \times 10^{-11} m^3 \text{kg}^{-1} s^{-2}\,. \tag{5.21}$$

This will not change the equation of motion in the vacuum but once we couple to matter this will determine the strength of the coupling of the gravitational field to the matter.

If we are just interested in phenomena related to gravity it is sensible to set $G_N = 1$. Instead if we want to consider other phenomena other than gravity this is not so sensible since it defines the coupling of the forces. Instead the more useful convention is to pick $\hbar = 1$, which equates energy with time. With this convention Newton's constant has dimension $[G] = m^{-2}$. The corresponding energy scale is called the *Planck mass* and is given by

$$M_{pl}^2 = \frac{\hbar c}{8\pi G_N} \,. \tag{5.22}$$

It is around 10^{18} GeV which is a very high energy scale and far beyond anything we can probe experimentally. This is why the gravitational force is so weak, the coupling constant is much smaller than that of the other forces.

5.1.2 Cosmological constant

We motivated the Einstein–Hilbert action as the simplest action one can write down. There is in fact a simpler term we may write down other than the Einstein–Hilbert term we considered previously. We may simply add a constant to the volume form. The resulting action is

$$S = \frac{1}{16\pi G_N} \int \mathrm{d}^4x \sqrt{-g} (R - 2\Lambda) \,. \tag{5.23}$$

The constant Λ is known as the *cosmological constant* and has dimension $[\Lambda] = L^{-2}$. The minus sign in the action comes from thinking of the Lagrangian as $T - V$ with the cosmological constant playing the role of the potential energy V.

Varying the action as before yields the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\Lambda g_{\mu\nu} \,. \tag{5.24}$$

This time if we contract with the inverse metric we get $R = 4\Lambda$. Substituting this back into the vacuum Einstein equations, in the presence of a cosmological constant they become

$$\boxed{R_{\mu\nu} = \Lambda g_{\mu\nu} \,. \qquad (5.25)}$$

Metrics satisfying this property are known as Einstein metrics.

5.1.3 Higher derivative terms

The Einstein–Hilbert action with cosmological constant is the simplest thing we can write down. We may construct other scalars from the Riemann tensor, however they will introduce higher derivative terms. For example, there are three terms that we can add at four-derivative level in the metric

$$S_{4-\mathrm{deriv}} = \int \mathrm{d}^4x \sqrt{-g} \Big(c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \Big) \,, \tag{5.26}$$

with the c_i dimensionless constants. Generic choices of the constants will not give rise to higher derivative equations of motion with a well-defined initial value problem. Nonetheless there are certain combinations which conspire to keep the equations second order in derivatives. This goes by the name of *Lovelock's theorem* and says that in four-dimensions the combination

$$\frac{1}{8\pi^2}\int_M \mathrm{d}^4x\sqrt{g}(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) = \chi(M)\,, \tag{5.27}$$

where $\chi(M) \in \mathbb{Z}$, is the Euler character of M. Though not obvious, this is a total derivative, and therefore does not affect the classical equations of motion. Higher derivative terms only become relevant for fast varying fields. For us these terms will not be important and therefore we stick to the 2-derivative action.

5.1.4 Diffeomorphisms

A natural question to ask is how many degrees of freedom are there in the metric? Since it is a 4×4 symmetric matrix the naive guess is $\frac{1}{2}\times 4\times 5 = 10$ however this is not quite correct. Not all of these 10 components are physical. Two metrics which are related by a change of coordinates $x^\mu \to \tilde{x}^\mu(x)$ describe the same physical spacetime. This means that there is a redundancy in any given representation of the metric which removes precisely 4 of the 10 degrees of freedom, leaving just 6 actual degrees of freedom.

This redundancy is implemented by diffeomorphisms. Recall that a diffeomorphism is a map $\phi : M \to M$. We may use it to map all fields, including the metric on M to a new set of fields on M. The end result is physically indistinguishable from the original, it describes the same system just in a different set of coordinates. Such diffeomorphisms are analogous to the gauge transformations of a gauge theory such as elcetromagnetism.

Let us look at how diffeomorphisms modify the action. Consider a diffeomorphism which takes a point with coordinate x^μ to a nearby point with coordinates

$$x^\mu \to \tilde{x}^\mu = x^\mu + \delta x^\mu\,. \tag{5.28}$$

We can view this either as an active change in which one point with coordinates x^μ is mapped to another point with coordinates $x^\mu + \delta x^\mu$ or as a passive transformation in which we use two different coordinate patches to label the same point. Either viewpoint leads to the same conclusion, here we will take the passive viewpoint.

We can think of the change of coordinates as being generated by an infinitesimal vector field X,

$$\delta x^\mu = -X^\mu(x)\,. \tag{5.29}$$

Under the change of coordinates the metric transforms as

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}(x)\,. \tag{5.30}$$

We can invert the Jacobian matrix to find

$$\frac{\partial \tilde{x}^\mu}{\partial x^\rho} = \delta^\mu_\rho - \partial_\rho X^\mu \quad \Rightarrow \quad \frac{\partial x^\rho}{\partial \tilde{x}^\mu} = \delta^\rho_\mu + \partial_\mu X^\rho\,, \tag{5.31}$$

where the inverse holds to leading order in the variation X. Continuing to work infinitesimally we have

$$\begin{aligned}\tilde{g}_{\mu\nu}(\tilde{x}) =&(\delta^\rho_\mu + \partial_\mu X^\rho)(\delta^\sigma_\nu + \partial_\nu X^\sigma) g_{\rho\sigma}(x) \\ =&g_{\mu\nu}(x) + g_{\mu\rho}(x)\partial_\nu X^\rho + g_{\nu\rho}(x)\partial_\mu X^\rho\,.\end{aligned} \tag{5.32}$$

We can also Taylor expand the left-hand side to find

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \tilde{g}_{\mu\nu}(x+\delta x) = \tilde{g}_{\mu\nu}(x) - X^\lambda \partial_\lambda \tilde{g}_{\mu\nu}(x)\,. \tag{5.33}$$

Comparing the different metrics at the same point x we find that the metric undergoes the infinitesimal change

$$\delta g_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = X^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\rho}\partial_\nu X^\rho + g_{\nu\rho}\partial_\mu X^\rho\,. \tag{5.34}$$

This is precisely the Lie derivative of the metric. If we act with an infinitesimal diffeomorphism along X then the metric changes as

$$\delta g_{\mu\nu} = (\mathcal{L}_X g)_{\mu\nu}\,. \tag{5.35}$$

We may rewrite equation (5.34) by lowering the index on X^ρ to find

$$\delta g_{\mu\nu} = \partial_\mu X_\nu + \partial_\nu X_\mu + X^\rho(\partial_\rho g_{\mu\nu} - \partial_\mu g_{\rho\nu} - \partial_\nu g_{\mu\rho})\,, \tag{5.36}$$

the last term is just the Christoffel symbols and therefore we have

$$\delta g_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu\,. \tag{5.37}$$

We may put this together to see how the action changes. Under a general change of the metric the Einstein–Hilbert action changes as

$$\delta S = \int \mathrm{d}^4 x \sqrt{-g} G^{\mu\nu} \delta g_{\mu\nu}\,, \tag{5.38}$$

where we have discarded the boundary term. Insisting that $\delta S = 0$ for any variation $\delta g_{\mu\nu}$ gives the equations of motion $G^{\mu\nu} = 0$. In contrast, symmetries of the action are those variations

$\delta g_{\mu\nu}$ for which $\delta S = 0$ for any choice of metric. Since diffeomorphisms are symmetries we know that the action is invariant under changes of the form (5.37). Using the fact that $G_{\mu\nu}$ is symmetric we must have

$$\delta S = 2\int \mathrm{d}^4x\sqrt{-g}G^{\mu\nu}\nabla_\mu X_\nu = 0\,, \quad \text{for all} \quad X_\mu(x)\,. \tag{5.39}$$

After integrating by parts we find that this results in the Bianchi identity

$$\nabla_\mu G^{\mu\nu} = 0\,. \tag{5.40}$$

We learn that from the path integral perspective the Bianchi identity is a result of diffeomorphism invariance.

5.1.5 Coupling to matter

Until now the action has only involved gravity, and at most we can allow for test particles moving on geodesics. However matter is not just an actor doing what gravity says in spacetime, it also backreacts and affects the dynamics of spacetime. The first question to ask is how does matter couple to the metric? Consider matter which comes with a Lagrangian.

Scalar Field Consider first a scalar field $\phi(x)$. In flat spacetime the action takes the form

$$S_{\text{scalar}} = \int \mathrm{d}^4x\Big(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\Big)\,, \tag{5.41}$$

with $\eta^{\mu\nu}$ the inverse Minkowski metric.[21]

It is straightforward to generalise this to describe a field moving in curved spacetime, we simply need to replace the Minkowski metric with the curved metric, replace partial derivatives with covariant derivatives and introduce the volume form when we integrate in the action. This means that we take

$$S_{\text{scalar}} = \int \mathrm{d}^4x\sqrt{-g}\Big(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\Big)\,. \tag{5.42}$$

Despite upgrading the partial derivatives to covariant ones this is somewhat redundant here as they act the same on a scalar field: we keep it for later though.

Curved spacetime also introduces new possibilities for us to add to the action, for example we could add a term such as $\xi R\phi^2$ to the action which gives rise to extra couplings. We will not interest ourselves in such terms here however.

[21] Note that the minus sign is due to our mostly plus signature convention, you may be more used to the opposite convention when considering a field theory. The Lagrangian with take the form of kinetic energy minus potential energy.

Maxwell Theory The action of Maxwell theory from special relativity is

$$S_{\text{Maxwell}} = -\frac{1}{4}\int \mathrm{d}^4x \eta^{\mu\rho}\eta^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\,, \tag{5.43}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The electric and magnetic fields are encoded in F via

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}\,, \tag{5.44}$$

and the Bianchi identity $\mathrm{d}F = \mathrm{d}^2A = 0$ yields two of the four Maxwell equations

$$\nabla\cdot\vec{B} = 0\,, \qquad \nabla\times\vec{B} + \frac{\partial\vec{B}}{\partial t} = 0\,. \tag{5.45}$$

See problem sheet 1.

We may couple to curved space time through the minimal coupling outlined for the scalar theory. The action is

$$S_{\text{Maxwell}} = -\frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} = -\frac{1}{2}\int F\wedge\star F\,. \tag{5.46}$$

We again take $F = \mathrm{d}A$, which in components reads $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. Antisymmetry implies that we may replace the covariant derivatives with normal derivatives. The equations of motion are

$$\nabla^\mu F_{\mu\nu} = 0\,, \qquad \Leftrightarrow \qquad \mathrm{d}\star F = 0\,. \tag{5.47}$$

We have now seen the algorithm of how to couple matter to gravity for two examples, for generic matter we follow the exact same rules. It remains to be seen how coupling to matter change the Einstein equations of the previous section. We need to consider the combined action

$$S = \frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{-g}(R - 2\Lambda) + S_{\text{Matter}}\,, \tag{5.48}$$

where S_{Matter} is the action for any matter fields in the theory minimally coupled to gravity. When we vary the Einstein–Hilbert term we know that we will obtain the Einstein tensor, what about S_{Matter}?

Definition 42 (Energy-Momentum tensor) *We define the* Energy-Momentum tensor *to be*

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_{Matter}}{\delta g^{\mu\nu}}\,. \tag{5.49}$$

By construction $T_{\mu\nu}$ is symmetric. Varying the full action with respect to the metric we have

$$\delta S = \frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{-g}(G_{\mu\nu}+\Lambda g_{\mu\nu})\delta g^{\mu\nu} - \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}\,, \tag{5.50}$$

from which we may read the following equation of motion

$$G_{\mu\nu}+\Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}\,. \tag{5.51}$$

This is the Einstein equation describing gravity coupled to matter. Note that the presence of the energy-momentum tensor says that the matter distribution sources the curvature of the spacetime.

Example 5.1: Computing the energy momentum tensor

- For the scalar theory above the energy-momentum tensor is

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\Big(\frac{1}{2}\nabla^\rho\phi\nabla_\rho\phi + V(\phi)\Big)\,. \tag{5.52}$$

  To see this observe that

$$\mathcal{L}_{\text{scalar}} = \sqrt{-g}\Big(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\Big)\,, \tag{5.53}$$

  where we have used that the connection on a scalar field is equivalent to the usual derivative and therefore the variation with respect to the metric of the connection here is trivial. Therefore

$$\begin{aligned} T_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g}\Big(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\Big) \\ &= -\frac{2}{\sqrt{-g}}\Big[-\frac{1}{2}\sqrt{-g}\nabla_\mu\phi\nabla_\nu\phi + \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\Big(-\frac{1}{2}g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi - V(\phi)\Big)\Big] \\ &= \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\big(\tfrac{1}{2}\nabla^\rho\phi\nabla_\rho\phi - V(\phi)\big) \end{aligned} \tag{5.54}$$

  If we restrict to flat space then

$$T_{00} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\,, \tag{5.55}$$

  with ∇ the usual 3d spatial derivative. This is the energy density of a scalar field.

- For the Maxwell action we have

$$T_{\mu\nu} = g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\,, \tag{5.56}$$

  see problem sheet 1 and replace $\eta \to g$. In flat space we have

$$T_{00} = \frac{1}{2}\Big[\vec{E}^2 + \vec{B}^2\Big]\,. \tag{5.57}$$

  This is the energy density of the magnetic and electric fields.

6 Schwarzschild solution

Black holes are one of the most enigmatic objects and probably the reason why most of you are here. Well the moment is finally upon us and we will take our first steps to understanding black holes.

6.1 The Schwarzschild black hole

In 1915 Einstein had published his work on General relativity and made a comment saying that he was not optimistic that the equations he had found could be solved other than Minkowski space. Also in 1915, with the first world war raging in Europe, Karl Schwarzschild was in the German army on the Russian front performing ballistic calculations, and suffering from pemphigus a rare and painful autoimmune disease. Despite this, he worked on finding solutions to general relativity and found the first exact (non-trivial) solution to Einstein's field equations.[22] Schwarzschild's breakthrough was to use a convenient system of coordinates, taking a polar like coordinate system as opposed to Einstein's rectangular coordinate system. The metric that bears his name is

$$\mathrm{d}s^2 = -\left(1 - \frac{2G_N M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big)\,. \tag{6.1}$$

This solves Einstein's equations in a vacuum, $R_{\mu\nu} = 0$. The coordinate ranges are[23]

$$t \in \mathbb{R}\,, \quad 0 < \theta < \pi\,, \quad 0 < \phi < 2\pi\,. \tag{6.2}$$

The range of r is slightly more subtle. At $r = 2G_N M$ something funky is happening since the prefactor of $\mathrm{d}t^2$ and $\mathrm{d}r^2$ vanish and diverge respectively. For the moment we will keep $2G_N M < r < \infty$ and we are then safe. This value of the radial coordinate is called the *Schwarzschild radius* and will play a prominent role when we view the Schwarzschild solution as a black hole in section 6.3.

The depends on a single parameter M which is interpreted as the mass of the object. Indeed using our results from problem sheet 1(or the Linearised equations and their Newtonian limit we will see later in section **??**)

$$g_{00} = -(1 + 2\Phi)\,, \tag{6.3}$$

[22]Schwarzschild died in 1916 having left military service due to his illness.

[23]There are singularities at $\theta = 0, \pi$ however these are just the expected singularities from considering a two-sphere and attempting to use just one coordinate patch, as we studied in example 4.1. We should be careful about this but it is not a problem since we learnt earlier that this problem could be removed.

with Φ the Newtonian potential. For the Schwarzschild metric we have

$$\Phi = -\frac{G_N M}{r}\,, \tag{6.4}$$

which is the Newtonian potential for a point mass M at the origin.

We can compute the mass of the black hole by using Komar integrals, these will be explained in the GR2 course, but we will give a flavour of it here. The Schwarzschild solution admits a time-like Killing vector $K = \partial_t$: a Killing vector satisfies $\mathcal{L}_K g = 0$ which is equivalent to $\nabla_{(\mu}K_{\nu)} = 0$. To compute the Komar integral we must construct the dual one-form to the time-like Killing vector,

$$K = g_{00}\mathrm{d}t = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t\,. \tag{6.5}$$

The Komar integral is given by

$$M_{\text{Komar}} = -\frac{1}{8\pi G_N}\int_{S^2} \star\mathrm{d}K\,, \tag{6.6}$$

where the S^2 is any sphere with a radius larger than the horizon at $r = 2G_N M$ where the Killing vector has vanishing norm. Then

$$\mathrm{d}K = -\frac{2G_N M}{r^2}\mathrm{d}r\wedge\mathrm{d}t \quad \Rightarrow \quad \star\mathrm{d}K = -2G_N M\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\phi\,. \tag{6.7}$$

and therefore

$$M_{\text{Komar}} = M\,. \tag{6.8}$$

Note that $\mathrm{d}\star\mathrm{d}K = 0$ *and therefore it obeys an equation similar to Maxwell's equations* $\mathrm{d}\star F = 0$. *These are Maxwell's equations in the absence of any current and therefore one would expect the electric charge to vanish. Yet this electric charge is precisely the mass and this is non-zero. For the solution the mass is localised at the origin* $r = 0$ *where the field strength diverges. This allows for a non-trivial value.*

We may thus expect that this describes something physical only when $M > 0$. For $M = 0$ we find Minkowski space while for $M < 0$ the metric becomes unphysical.

6.1.1 Birkhoff's theorem

The Schwarzschild solution turns out the be the unique spherically symmetric asymptotically flat solution to the vacuum Einstein solutions, this fact is known as *Birkhoff's theorem.* This means that the Schwarzschild solution does not just describe the spacetime outside of a black hole but outside any non-rotating, spherically symmetric object such as a star or planet. We will sketch the proof of this fact since it allows us to get a feel for solving the Einstein equations.

The spherical symmetry of the metric means that it has an SO(3) isometry. If you hold up a round sphere and rotate it it looks the same no matter which way you rotate it. If instead you did the same with a golf ball, which has dimples then this rotational symmetry is broken. The distinction between these two situations should be captured by the metric. The metric on a round two-sphere will look the same wherever you sit on the sphere whereas the metric on the golf ball will depend on where you are.

To define this mathematically we need to use the concept of a flow that we introduce a number of lectures ago. A flow on a manifold M is a one-parameter family of diffeomorphisms $\sigma_t : M \to M$, and may be associated to a vector field $K \in \mathcal{X}(M)$ at each point along the flow which is tangent to the flow

$$K^\mu = \frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda}\,. \tag{6.9}$$

The flow is said to be an isometry if the metric looks the same at each point along a given flow line, mathematically this means that an isometry satisfies

$$\mathcal{L}_K g = 0\,, \quad \Leftrightarrow \quad \nabla_\mu K_\nu + \nabla_\nu K_\mu = 0\,. \tag{6.10}$$

A vector satisfying this equation is known as a *Killing vector field*. Sometimes it is simple to see that a vector generates an isometry, particularly when it is an ignorable coordinate, i.e. the metric does not depend on the coordinate. Sometimes, however, the Killing vectors are not so obvious.

There is a group structure underlying the symmetries, well technically a Lie algebra structure. This follows since the Lie derivative satisfies

$$\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X = \mathcal{L}_{[X,Y]}\,. \tag{6.11}$$

Killing vectors form a Lie algebra of the isometry group of the manifold. (See problem sheet 3 where we consider the Killing vectors on the round three-sphere).

One can then prove that the SO(3) isometry implies that the metric must take the form

$$\mathrm{d}s^2 = g_{\tau\tau}(\tau,\rho)\mathrm{d}\tau^2 + 2g_{\tau\rho}(\tau,\rho)\mathrm{d}\tau\mathrm{d}\rho + g_{\rho\rho}(\tau,\rho)\mathrm{d}\rho^2 + r^2(\tau,\rho)\mathrm{d}s^2(S^2)\,, \tag{6.12}$$

where

$$\mathrm{d}s^2(S^2) = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\,, \tag{6.13}$$

is the metric on a round two-sphere. The SO(3) isometry then acts on the two-sphere and leaves τ and ρ untouched. This is called a *foliation* of the space by S^2 leaves.

The size of the sphere is determined by $r(\tau,\rho)$ and it is convenient to redefine the coordinates such that r is a coordinate, we can then eliminate the ρ coordinate in favour of r. The metric becomes

$$\mathrm{d}s^2 = g_{\tau\tau}(\tau,r)\mathrm{d}\tau^2 + 2g_{\tau r}(\tau,r)\mathrm{d}\tau\mathrm{d}r + g_{rr}(\tau,r)\mathrm{d}r^2 + r^2\mathrm{d}s^2(S^2)\,. \tag{6.14}$$

The only subtlety we could encounter in doing this change of coordinates is if it is not possible to exchange ρ with r, for example r could have been independent of ρ. We can rule out these cases by imposing that asymptotically the spacetime looks like Minkowski space.

The logic now is to remove the cross term $\mathrm{d}\tau\mathrm{d}r$ by using a change of coordinates. If we define $\tilde{t}(\tau,r)$ then

$$\mathrm{d}\tilde{t} = \frac{\partial\tilde{t}}{\partial\tau}\mathrm{d}\tau + \frac{\partial\tilde{t}}{\partial r}\mathrm{d}r \tag{6.15}$$

and therefore, we can pick a change of coordinates such that we can remove the cross term. The resultant metric is

$$\mathrm{d}s^2 = -\mathrm{e}^{2\alpha(\tilde{t},r)}\mathrm{d}\tilde{t}^2 + \mathrm{e}^{2\beta(\tilde{t},r)}\mathrm{d}r^2 + r^2\mathrm{d}s^2(S^2)\,. \tag{6.16}$$

We have included a minus sign since we are looking for a Lorentzian metric and then we can introduce the exponential terms which are manifestly positive definite. This is the simplest form of the metric that we can achieve just through coordinate transformations and we now need to plug this into Einstein's equations. Observe that we have used symmetries to restrict the form of the metric and then used diffeomorphisms to write the metric in the simplest form possible. This makes solving Einstein's equations simpler, the correct ansatz and choice of coordinates simplifies

We can compute the Christoffel symbols for the metric, the non-trivial ones are[24]

$$\begin{aligned}
&\Gamma^{\tilde{t}}{}_{\tilde{t}\tilde{t}} = \partial_{\tilde{t}}\alpha\,, && \Gamma^{\tilde{t}}{}_{\tilde{t}r} = \partial_r\alpha\,, && \Gamma^{\tilde{t}}{}_{rr} = \mathrm{e}^{2\beta-2\alpha}\partial_{\tilde{t}}\beta\,,\\
&\Gamma^{r}{}_{\tilde{t}\tilde{t}} = \mathrm{e}^{2\alpha-2\beta}\partial_r\alpha\,, && \Gamma^{r}{}_{\tilde{t}r} = \partial_{\tilde{t}}\beta\,, && \Gamma^{r}{}_{rr} = \partial_r\beta\,,\\
&\Gamma^{\theta}{}_{r\theta} = \frac{1}{r}\,, && \Gamma^{r}{}_{\theta\theta} = -r\mathrm{e}^{-2\beta}\,, && \Gamma^{\phi}{}_{r\phi} = \frac{1}{r}\,,\\
&\Gamma^{r}{}_{\phi\phi} = -r\mathrm{e}^{-2\beta}\,, && \Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta\,, && \Gamma^{\phi}{}_{\theta\phi} = \frac{\cos\theta}{\sin\theta}\,.
\end{aligned} \tag{6.17}$$

[24]You should see that you can do this. There is also a mathematica file on the course webpage where this has been computed for you to check.

It follows that the non-vanishing components of the Riemann tensor are

$$\begin{aligned}
R^{\tilde{t}}{}_{r\tilde{t}r} &= \mathrm{e}^{2\beta-2\alpha}\Big(\partial_{\tilde{t}}^2\beta + (\partial_{\tilde{t}}\beta)^2 - \partial_{\tilde{t}}\alpha\partial_{\tilde{t}}\beta\Big) + \Big(\partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2\Big)\,, \\
R^{\tilde{t}}{}_{\theta\tilde{t}\theta} &= -r\mathrm{e}^{-2\beta}\partial_r\alpha\,, \\
R^{\tilde{t}}{}_{\phi\tilde{t}\phi} &= -r\mathrm{e}^{-2\beta}\sin^2\theta\partial_r\alpha\,, \\
R^{\tilde{t}}{}_{\theta r\theta} &= -r\mathrm{e}^{-2\alpha}\partial_{\tilde{t}}\beta\,, \\
R^{\tilde{t}}{}_{\phi r\phi} &= -r\mathrm{e}^{-2\alpha}\sin^2\theta\partial_{\tilde{t}}\beta\,, \\
R^{r}{}_{\theta r\theta} &= r\mathrm{e}^{-2\beta}\partial_r\beta\,, \\
R^{r}{}_{\phi r\phi} &= r\mathrm{e}^{-2\beta}\sin^2\theta\partial_r\beta\,, \\
R^{\theta}{}_{\phi\theta\phi} &= (1-\mathrm{e}^{-2\beta})\sin^2\theta\,.
\end{aligned} \tag{6.18}$$

From the Riemann tensor we can construct the Ricci tensor finding the non-trivial components

$$\begin{aligned}
R_{\tilde{t}\tilde{t}} &= \Big(\partial_{\tilde{t}}^2\beta + (\partial_{\tilde{t}}\beta)^2 - \partial_{\tilde{t}}\alpha\partial_{\tilde{t}}\beta\Big) + \mathrm{e}^{2\alpha-2\beta}\Big(\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\Big)\,, \\
R_{rr} &= -\Big(\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - \frac{2}{r}\partial_r\beta\Big) + \mathrm{e}^{2\beta-2\alpha}\Big(\partial_{\tilde{t}}^2\beta + (\partial_{\tilde{t}}\beta)^2 - \partial_{\tilde{t}}\alpha\partial_{\tilde{t}}\beta\Big) \\
R_{\tilde{t}r} &= \frac{2}{r}\partial_{\tilde{t}}\beta\,, \\
R_{\theta\theta} &= \mathrm{e}^{-2\beta}\Big(r\big(\partial_r\beta - \partial_r\alpha\big) - 1\Big) + 1\,, \\
R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta\,.
\end{aligned} \tag{6.19}$$

Our job is to now solve Einstein's equations in the vacuum, $R_{\mu\nu} = 0$. There is an obvious component to consider first, $R_{\tilde{t}r}$ which implies

$$\partial_{\tilde{t}}\beta = 0\,. \tag{6.20}$$

If we now take the $\tilde{t}$ derivative of $R_{\theta\theta}$ and use the above condition we find

$$\partial_{\tilde{t}}\partial_r\alpha = 0\,, \tag{6.21}$$

and therefore we have

$$\beta = \beta(r)\,, \qquad \alpha = f(r) + g(\tilde{t})\,. \tag{6.22}$$

The first term in the metric is then

$$-\mathrm{e}^{2f(r)+2g(\tilde{t})}\mathrm{d}\tilde{t}^2\,, \tag{6.23}$$

and by a redefinition of $\tilde{t}$ we can set

$$\mathrm{e}^{g(\tilde{t})}\mathrm{d}\tilde{t} = \mathrm{d}t\,, \tag{6.24}$$

and we end up with the metric

$$\mathrm{d}s^2 = -\mathrm{e}^{2f(r)}\mathrm{d}t^2 + \mathrm{e}^{2\beta(r)}\mathrm{d}r^2 + r^2\mathrm{d}s^2(S^2)\,. \tag{6.25}$$

We need to solve the remaining Einstein equations. Note that the metric is now independent of t, this naturally comes out of the Einstein equations, we did not impose this! This implies that *any spherically symmetric vacuum metric possesses a timelike Killing vector*. A metric with this property is called *stationary*, in fact the Schwarzschild metric is also *static* we will come back to this shortly.

We can now remove all $\tilde{t}$ derivatives and exchange $\alpha \to f$ in the Ricci tensor components and where we se $\tilde{t}$ replace with just t. We are free to add components and so we take the combination

$$0 = \mathrm{e}^{2\beta - 2f(r)}R_{tt} + R_{rr} = \frac{2}{r}\big(\partial_r f(r) + \partial_r \beta\big)\,. \tag{6.26}$$

We then have

$$f(r) = -\beta(r) + \mathrm{const}\,, \tag{6.27}$$

but we may rescale the time coordinate to set the constant to 0. Plugging this into $R_{\theta\theta}$ we find

$$\mathrm{e}^{2f(r)}\big(2r\partial_r f(r) + 1\big) = 1 \quad \Leftrightarrow \quad \partial_r\big(r\mathrm{e}^{2f(r)}\big) = 1\,, \tag{6.28}$$

which has solution

$$\mathrm{e}^{2f(r)} = 1 - \frac{r_S}{r}\,, \tag{6.29}$$

with r_S an undetermined constant which we will set to be $r_S = 2G_N M$. There is no remaining freedom except to set r_S to a certain value so the remaining components must vanish, and it turns out that they do, so we have solved Einstein's equations and derived the Schwarzschild solution.

Stationary vs Static There are two different meanings to time independence of a metric.

Definition 43 *Stationary A spacetime is* stationary *if it admits an everywhere timelike Killing vector field K. We typically normalise it so that asymptotically $K^2 \to -1$.*

Definition 44 *A spacetime is* static *if, in addition to being stationary, it is invariant under $t \to -t$, where t is the coordinate along the integral curves of K. This rules out* $\mathrm{d}t\mathrm{d}x$ *cross terms in the metric with x any other coordinate except t.*

We see that the Schwarzschild solution is static (and therefore also stationary).

Theorem 6 *Birkhoff's theorem*

The Schwarzschild solution is the unique spherically symmetric, asymptotically flat solution to Einstein's equations in a vacuum.

The beauty of Birkhoff's theorem lies in the fact that it must describe the spacetime around any spherically symmetric object: black holes, stars, footballs. Moreover since we did not assume time independence it equally applies to the spacetime around a collapsing star, provided the collapse is spherically symmetric.

6.2 Geodesics of Schwarzschild

We now want to consider the geodesics of the Schwarzschild metric. We have computed the Christoffel symbols above and could just substitute this into the geodesic equation (4.87) however if one did not already have the Christoffel symbols this is not necessarily the quickest method. Instead one should use the Euler–Lagrange equations for the Lagrangian

$$\mathcal{L} = \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\,, \tag{6.30}$$

and parametrise with an affine parameter. With the choice of an affine parameter we can then compute the Euler–Lagrange equations of $\mathcal{L}^2$ instead and obtain the same equations of motion. We take

$$\begin{aligned}\mathcal{L} &= g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\\ &= -\left(1-\frac{2G_NM}{r}\right)\dot{t}^2 + \left(1-\frac{2G_NM}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\,,\end{aligned} \tag{6.31}$$

with $\dot{\bullet} \equiv \frac{\mathrm{d}\bullet}{\mathrm{d}\lambda}$. Since we are using an affine parameter this is equal to a constant ϵ which we may take to be -1 for time-like geodesics, 0 for null and 1 for space-like geodesics.

Before embarking on a brute force approach we should take a step back to see how we can simplify the problem. The answer is to study the conserved quantities. Ignorable coordinates, ones which do not appear explicitly, give rise to conserved quantities since from the Euler–Lagrange equations[25] we find

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}x^\mu} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\dot{x}^\mu} = 0\,. \tag{6.32}$$

[25] More precisely Killing vectors define conserved quantities for geodesics. Let K be a Killing vector then, $K_\mu p^\mu$ is conserved along a geodesic with $p_\mu = \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}$. Here the Killing vectors are ∂_t and ∂_ϕ and so we obtain that the conserved quantities are simply p_t and p_ϕ.

The action has two such ignorable coordinates t and ϕ: giving

$$\begin{aligned}
2l &= \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\dot{\phi}} = 2r^2\sin^2\theta\dot{\phi}\,, \\
-2E &= \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\dot{t}} = -2\left(1-\frac{2G_NM}{r}\right)\dot{t}\,.
\end{aligned} \tag{6.33}$$

Of course these should be identified with the angular momentum and energy respectively. Next consider the equation for θ, we find

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(r^2\dot{\theta}) = r^2\sin\theta\cos\theta\dot{\phi}^2\,. \tag{6.34}$$

Recall that in computing the motion in Newtonian gravity we noted that if we started the particle at $\theta=\frac{\pi}{2}$ with $\dot{\theta}=0$ then it remained in the plane, the same is true here and so we can without loss of generality set $\theta=\frac{\pi}{2}$.

We can now plug this into (6.31) and equate with our constant parameter ϵ giving

$$\epsilon = -\left(1-\frac{2G_NM}{r}\right)^{-1}E^2 + \left(1-\frac{2G_NM}{r}\right)^{-1}\dot{r}^2 + r^{-2}l^2\,. \tag{6.35}$$

Rearranging we have

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = \frac{E^2}{2}\,, \tag{6.36}$$

with

$$V_{\text{eff}}(r) = -\frac{\epsilon}{2} + \frac{\epsilon G_NM}{r} + \frac{l^2}{2r^2} - \frac{l^2G_NM}{r^3}\,. \tag{6.37}$$

We should contrast this with the equivalent Newtonian expression in (2.69) for a massive particle which was

$$V_N(r) = -\frac{G_NM}{r} + \frac{l^2}{2r^2}\,. \tag{6.38}$$

We see that General relativity leads to additional corrections to the potential. The first term is simply a constant shift and so does not play much of a role since we can absorb it into a redefinition of the energy, the r^{-3} term is completely new and changes the Newtonian potential at small distances. Note that the effective potential vanishes at $r=2G_NM$ which is the Schwarzschild radius.

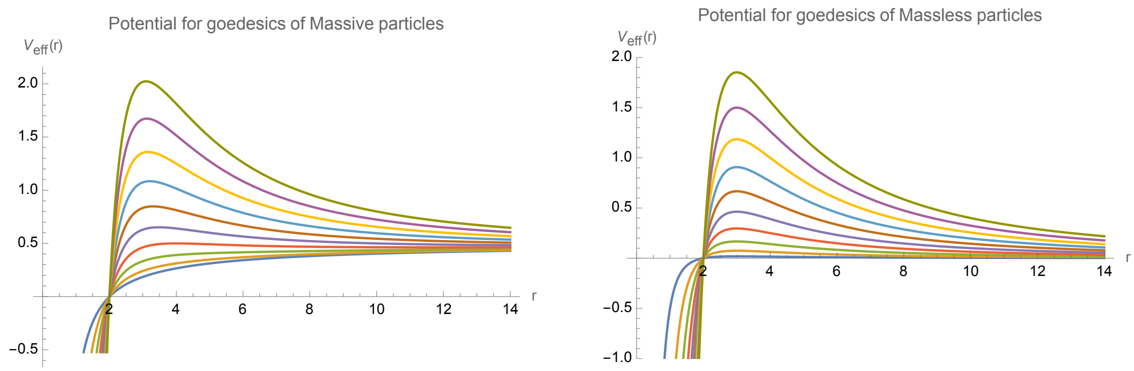
Let us reinstate the speed of light in the potential, we have

$$V_{\text{eff}}(r) = -\frac{\epsilon c^2}{2} + \frac{\epsilon G_NM}{r} + \frac{l^2}{2r^2} - \frac{l^2G_NM}{r^3c^2}\,, \tag{6.39}$$

then the equation for $\dot{r}$ is

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = \frac{1}{2}\frac{E^2}{c^2}\,. \tag{6.40}$$

We now want to analyse the different forms of trajectories that are possible. In figure 15 we have plotted the potential for various values of l, with fixed mass M, you should compare the left-hand side one with the plot in figure 2 for the Newtonian potential.



(a) Plots of the potential for massive particles, $\epsilon = -1$ and for $G_N M = 1$. The different plots correspond to increasing l.

(b) Plots of the potential for massless particles, $\epsilon = 0$ and for $G_N M = 1$. The different plots correspond to increasing l. Note the maximum at $3 \equiv 3G_N M$ in the massless case for all l.

Figure 15: Plots of the potential for massive and massless particles. Note that the plots tends to $-\frac{\epsilon}{2}$ as $r \to \infty$. Moreover the potentials both vanish at $2 = 2G_N M$ which is the Schwarzschild radius. This should be compared to the corresponding Newtonian plot in figure 2.

Circular orbits will be at points where the potential has a turning point. Then we are stuck in a circular orbit, which is stable if it corresponds to a minimum of the potential and unstable if it corresponds to a maximum. Differentiating the potential we have

$$V'_{\text{eff}}(r) = \frac{1}{r^4}\Big(3G_N l^2 M - l^2 r - G_N M \epsilon r^2\Big) \tag{6.41}$$

which potentially has two zeroes at

$$r_c = -\frac{l^2 \pm \sqrt{l^4 + 12 G_N l^2 M \epsilon}}{2 G_N M \epsilon}\,, \tag{6.42}$$

depending on the range of the parameters, for $\epsilon \neq 0$ and

$$r_c = 3 G_N M\,, \tag{6.43}$$

for $\epsilon = 0$.

For the massless photon the orbit is at a maximum and is therefore unstable. A photon can orbit in a circular orbit forever around the black hole, but any perturbation will send it flying off to either $r = 0$ or $r = \infty$. This is known as the *photon sphere*. The focussing effects mean that much of the light emitted from an accretion disc around a non-rotating black hole emerges from the photon sphere. In practice, it seems likely that the photographs by the Event Horizon Telescope do not have the required resolution to see this yet.

For massive particles there are different regimes depending on the angular momentum. For large l there will be two circular orbits, one stable and one unstable. In the $l \to \infty$ regime they are at

$$r_c = \left(3G_N M, \frac{l^2}{G_N M}\right). \tag{6.44}$$

The stable circular orbit gets further away while the unstable orbit approaches $3G_N M$ where the photon sphere is located. As we decrease l the two orbits come together and coincide when the discriminant of the quadratic in (6.41) vanishes. This is at

$$l = \sqrt{12} G_N M\,, \tag{6.45}$$

which gives

$$r_c = 6G_N M\,. \tag{6.46}$$

For smaller l there are no circular orbits and so $6G_N M$ is the smallest possible radius of a stable circular orbit of the Schwarzschild metric. We have derived that:

> The Schwarzschild solution possesses stable circular orbits for massive test masses for $r > 6G_N M$ and unstable circular orbits for $3G_N M < r < 6G_N M$.
>
> For massless particles/light there are unstable circular orbits at $r_c = 3G_N M$.

We should comment that these are the motions of geodesics. For an accelerating observer such as a rocket ship, there is nothing stopping them from dipping below $r = 3G_N M$ and then reemerging, so long as they stay away from $r = 2G_N M$.

Most experimental tests of general relativity involve the motion of test particles in the solar system. More recently, with the advancements in technology, using gravitational waves to test general relativity has also become possible. We will concentrate on three particular tests: the precession of perihelia, the bending of light and gravitational red-shift.

6.2.1 Perihelion precession

We saw when we consider the orbits in Newtonian gravity that the non-circular orbits were closed ellipses. Observation of the orbit of Mercury showed that the closed elliptic orbits of Newtonian gravity were not realised, instead the orbit precessed. A non-trivial check of General Relativity is then to show that the orbits of the planets precess. We can approximate the metric of the sun to be the Schwarzschild metric and take the planet to follow the geodesic of a massive particle.

The strategy is to describe the evolution of the radial coordinate r as a function of ϕ. If the orbit is a perfect ellipse $r(\phi)$ should be periodic with period 2π, then the perihelion occurs at the same point every orbit. Instead, for a non-closed ellipse the perihelion is shifted after every orbit. We will see that General Relativity gives a slight modification of the Newtonian result such that the orbit precesses.

First consider the radial equation of motion for a massive particle, (6.36), setting $\epsilon = -1$. To get an equation for $\frac{\mathrm{d}r}{\mathrm{d}\phi}$ we can use the chain rule and multiply the equation by

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^{-2} = \frac{r^4}{l^2}\,, \tag{6.47}$$

yielding

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 + \frac{r^4}{l^2} - \frac{2G_N M}{l^2}r^3 + r^2 - 2G_N M r = \frac{E^2 r^4}{l^2} \tag{6.48}$$

Define the new variable

$$x = \frac{l^2}{G_N M r}\,, \tag{6.49}$$

which gives rise to the Newtonian circular orbit when $x = 1$. The equation of motion becomes

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\phi}\right)^2 + \frac{l^2}{G_N^2 M^2} - 2x + x^2 - \frac{2G_N^2 M^2 x^3}{l^2} = \frac{E^2 l^2}{G_N^2 M^2}\,. \tag{6.50}$$

Next differentiate with respect to ϕ to obtain

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\phi^2} - 1 + x = \frac{3G_N^2 M^2 x^2}{l^2}\,. \tag{6.51}$$

In the Newtonian calculation the term on the right-hand-side would be absent and we could solve for x exactly. Here we will treat this as a perturbation around the Newtonian result.

We expand x into a Newtonian solution plus a small deviation

$$x = x_0 + x_1\,, \tag{6.52}$$

where the zeroth order part satisfies

$$\frac{\mathrm{d}^2 x_0}{\mathrm{d}\phi^2} - 1 + x_0 = 0\,. \tag{6.53}$$

The equation for the first order part becomes

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}\phi^2} + x_1 = \frac{3G_N^2 M^2}{l^2}x_0^2\,. \tag{6.54}$$

A solution to the zeroth order equation is (see (2.75))

$$x_0 = 1 + e\cos\phi\,, \tag{6.55}$$

which recall describes a perfect ellipse with eccentricity e, $e = 1 - \frac{b^2}{a^2}$ with a the semi-major axis, the distance from the centre to the farthest point on the ellipse and the semi-minor axis b the distance from the centre to the closest point. Plugging in the Newtonian solution into the first order equation of motion we find

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}\phi^2} + x_1 = \frac{3G_N^2 M^2}{l^2}(1 + e\cos\phi)^2\,. \tag{6.56}$$

A solution is given by

$$x_1 = \frac{3G_N^2 M^2}{l^2}\left[\left(1 + \frac{e^2}{2}\right) + e\phi\sin\phi - \frac{1}{6}e^2\cos 2\phi\right]. \tag{6.57}$$

The first term is just a constant displacement while the third oscillates around 0. The important effect is contained within the second term which accumulates over successive orbits. Combining only this term with the zeroth-order solution we have

$$x = 1 + e\cos\phi + \frac{3G_N^2 M^2 e}{l^2}\phi\sin\phi\,. \tag{6.58}$$

We should emphasise that this is not a full solution, it is an approximation but it encapsulates the part we are interested in. We may write

$$x = 1 + e\cos\Big((1-\alpha)\phi\Big)\,, \tag{6.59}$$

where

$$\alpha = \frac{3G_N^2 M^2}{l^2}\,. \tag{6.60}$$

where one should view this as a series expansion around $\alpha = 0$. It follows that during each orbit the perihelion advances by an angle

$$\Delta\phi = 2\pi\alpha = \frac{6\pi G_N^2 M^2}{l^2}\,. \tag{6.61}$$

We may replace the angular momentum in favour of the eccentricity by looking at the Newtonian solution. An ordinary ellipse satisfies

$$r = \frac{(1-e^2)a}{1 + e\cos\phi}\,, \tag{6.62}$$

with a the semi-major axis. This leads us to identify

$$l^2 \sim G_N M(1-e^2)a\,, \tag{6.63}$$

for the Newtonian orbit. Plugging this in and restoring the speed of light we find

$$\Delta\phi = \frac{6\pi G_N M}{c^2(1-e^2)a} \,. \tag{6.64}$$

Historically the precession of mercury was the first test of GR. The apparent discrepancy between observation and Newtonian gravity was known long before the advent of GR, and a number of solutions had been proposed including additional planets. For the motion of Mercury around the sun we have

$$\begin{aligned} \frac{G_N M_\odot}{c^2} &= 1.48 \times 10^3 m \,, \\ a &= 5.79 \times 10^{10} m \,, \\ e &= 0.2056 \,. \end{aligned} \tag{6.65}$$

This gives

$$\Delta\phi_{\text{Mercury}} = 5.01 \times 10^{-7} \text{radians/orbit} = 0.103\text{"/orbit} \tag{6.66}$$

with " denoting arcseconds. Mercury orbits once every 88 days and therefore

$$\Delta\phi_{\text{Mercury}} = 43.0\text{"/century} \,. \tag{6.67}$$

From our computation we conclude that the major axis of Mercury's orbit precesses at a rate of 43.0 arcseconds every 100 years. The observed value is 5601 arcseconds/100 years. Much of that is due to the precession of equinoxes in our geocentric coordinate system: 5025 arcseconds/100 years. The gravitational perturbations of the other planets contributes an additional 532 arcseconds/100 years leaving a 43 arcseconds/100 years to be explained by GR which is does quite well.

6.2.2 Bending of light

We can now extend these results for null geodesics. We have seen that there is an unstable circular orbit for light. What about other orbits? The fate of other light rays depends on the relative value of their energy E to their angular momentum l. The maximum value of the potential is

$$V_{\text{null}}(r_*) = \frac{l^2}{54 G_N^2 M^2} \,, \tag{6.68}$$

and therefore the physics depends on how this compares with the right-hand side of (6.36). There are two possibilities we need to consider

- $E < \frac{l}{\sqrt{27} G_N M}$. The energy of the light is lower than the angular momentum barrier. This means that light emitted from $r < r_*$ cannot escape to infinity; it will orbit the star before falling back towards the origin. For light coming from infinity it will not fall into the star but will instead bounce off the angular momentum barrier and return to infinity: the light will be scattered.

- $E > \frac{l}{\sqrt{27}G_N M}$. The energy of light is greater than the angular momentum barrier. Light can be emitted from $r < r_*$ and escape to infinity (this is only true for $r_S < r$). Meanwhile light coming from infinity is captured by the star/black hole.

To see this more clearly let us once again use the inverse parameter $u = \frac{1}{r}$. The equation of motion becomes

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + u^2\big(1 - 2G_N M u\big) = \frac{E^2}{l^2}\,. \tag{6.69}$$

Differentiating again we find

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = 3G_N M u^2\,. \tag{6.70}$$

We may once again work perturbatively. At zeroth order we can ignore the $G_N M$ term on the right-hand-side. Then to leading order we have

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = 0\,, \qquad \Rightarrow \qquad u = \frac{1}{b}\sin\phi\,, \tag{6.71}$$

for b a constant. Reinstating r we have $r\sin\phi = b$: which is the equation of a horizontal straight line, a distance b above the origin, see 16. The distance b is known as the *impact parameter*.

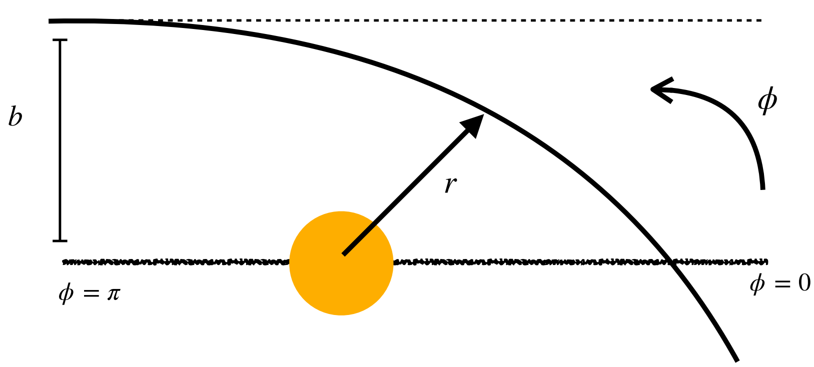


Figure 16: Light bending in the Schwarzschild metric. The dashed line at the top is the constant line $r\sin\phi = b$. The curved line is the geodesic.

With the zeroth order solution we can now solve (6.70) in an expansion around $\frac{G_N M}{b} = \beta$. We have

$$u = u_0 + \beta u_1 + ... \tag{6.72}$$

At first order we need to solve

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + u_1 = \frac{3\sin^2\phi}{b} = \frac{3(1-\cos 2\phi)}{2b}\,. \tag{6.73}$$

The general solution is

$$u_1 = A\cos\phi + B\sin\phi + \frac{1}{2b}(3+\cos 2\phi)\,, \tag{6.74}$$

where the first two parts are the solutions of the homogeneous part and A, B two integration constants. We should choose them so that the initial trajectory at $\phi = \pi$ agrees with the straight line u_0. For this to hold we must take $A = \frac{2}{b}$ and $B = 0$ so that $u_1 \to 0$ as $\phi \to \pi$. To leading order in β the solution is

$$u = \frac{1}{b}\sin\phi + \frac{G_N M}{2b^2}(3 + 4\cos\phi + \cos 2\phi)\,. \tag{6.75}$$

What angle does the particle escape to $r = \infty \Leftrightarrow u = 0$? Before the correction this was at $\phi = 0$, within our perturbative approach we can approximate $\sin\phi \sim \phi$ and $\cos\phi \sim 1$ to find that the particle escapes at

$$\phi \sim -\frac{4G_N M}{b}\,. \tag{6.76}$$

This means that the light is bent by gravity, this bending of light is known as *gravitational lensing*.

For the sun, $\frac{G_N M_\odot}{c^2} \sim 1.48$ km. If the light rays just graze the surface of the sun, then the impact parameter is the radius of the sun $R_\odot \sim 7 \times 10^5$ km. This gives a scattering angle of $\phi \sim 8.6 \times 10^{-5}$ radians or $\phi \sim 1.8$". The Newtonian prediction gives only half of this contribution.

There is a difficulty in testing this prediction since things behind the sun are rarely visible. By sheer coincide, the size of the moon in the sky is about the same size of the Sun which leads to total solar eclipses. This means that during a solar eclipse the light from the sun is blocked allowing for the measurement of stars whose light passes nearby the Sun. This can then be compared with the positions of these stars 6 months later when the Sun is behind the Earth and so the light from the star sources is not lensed by the Sun.

The first measurement was carried out in 1919 by two expeditions lead by Arthur Eddington and Frank Watson Dyson (we will see Eddington's name again shortly).[26] Since then

[26] This result was considered spectacular news and made the front page of most major newspapers.

our evidence of the bending of light is more impressive. Clusters of galaxies have been seen to distort the light from a background source often revealing a distinct ring-like pattern of multiple copies of the light source. See figure 17.

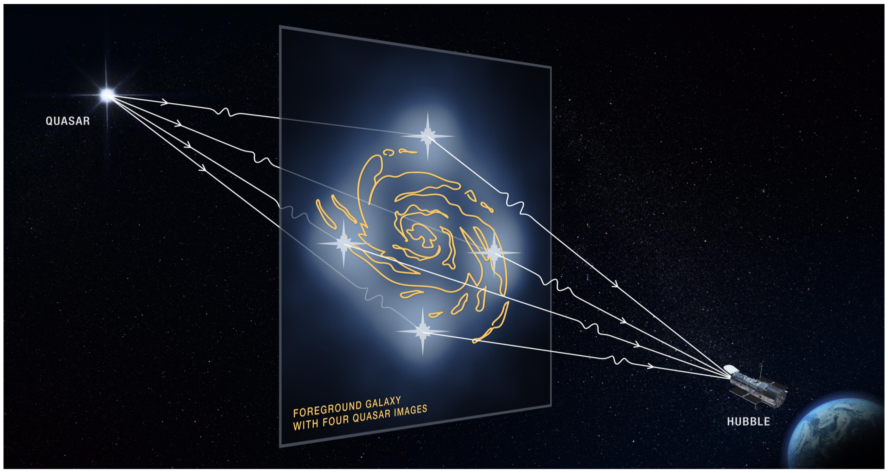


Figure 17: A diagram of light lensing picked up by the Hubble telescope. Notice that there are four copies of the distant quasar in the picture obtained by Hubble. Image credited to NASA, ESA and STScl.

6.2.3 Gravitational red shift

We have seen two tests of general relativity, next we will now look at time dilation due to strong gravitational fields.

Let us consider an observer with four velocity U^μ who is stationary in Schwarzschild coordinates, i.e. $U^i = 0$.[27] The four-velocity is normalised so that $U_\mu U^\mu = 1$, which for our stationary observer in a Schwarzschild background implies

$$U^0 = \left(1 - \frac{2G_N M}{r}\right)^{-1/2} . \tag{6.77}$$

Such an observer measures the frequency of a photon following a null geodesic $x^\mu(\lambda)$ to be

$$\omega = -g_{\mu\nu} U^\mu \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} . \tag{6.78}$$

We have

$$\begin{aligned} \omega =& \left(1 - \frac{2G_N M}{r}\right)^{1/2} \frac{\mathrm{d}t}{\mathrm{d}\lambda} \\ =& \left(1 - \frac{2G_N M}{r}\right)^{-1/2} E \, , \end{aligned} \tag{6.79}$$

[27] We could allow for the observer to be moving, however the difference is just to superimpose the usual Doppler shift on top of the gravitational effect and therefore we consider the simpler example.

where E was defined to be the conserved quantity associated to time translations when we worked out the geodesics. Since E is conserved it follows that ω will have different values when measured at different radial distances. For a photon emitted at r_1 and an observer at r_2, the observed frequencies will be related by

$$\frac{\omega_2}{\omega_1} = \sqrt{\frac{1 - 2G_N M/r_1}{1 - 2G_N M/r_2}}\,. \tag{6.80}$$

This is the exact result for the frequency shift, in the limit $r \gg 2G_N M$ we have

$$\begin{aligned}\frac{\omega_2}{\omega_1} =& 1 - \frac{G_N M}{r_1} + \frac{G_N M}{r_2} \\ &= 1 + \Phi(r_1) - \Phi(r_2)\,,\end{aligned} \tag{6.81}$$

with $\Phi = -G_N M/r$ the Newtonian potential.

We see that the frequency goes down as Φ increases which happens as we climb out of a gravitational field, leading to a red-shift. On the other hand photons which fall towards the gravitating body are blue shifted. Gravitational red-shift was first detected in 1960 by Pound and Rebka using gamma rays travelling a distance of 72-feet (about 22m) which was the height of the physics building at Harvard. Increasingly precise tests have found excellent agreement with GR. There is a cosmological counterpart to this, where light is red-shifted in an expanding universe.

Time delay Since the temporal component of the metric is

$$g_{00}(x) = 1 + 2\Phi(x)\,, \tag{6.82}$$

we see that there is a connection between time and gravity. Let us once again use the Schwarzschild solution. An observer sitting at a fixed distance r from the origin will measure a time interval

$$\mathrm{d}\tau^2 = -g_{00}\mathrm{d}t^2 = \left(1 - \frac{2G_N M}{r}\right)\mathrm{d}t^2\,. \tag{6.83}$$

For an asymptotic observer at $r \to \infty$ who measures a time t, an observer at r will measure the time T

$$T(r) = t\sqrt{1 - \frac{2G_N M}{r}}\,. \tag{6.84}$$

It follows that time goes slower in the presence of a massive gravitating object. Notice that at $r = r_S$ that time seems to stop for the observer at r_S. We will come back to this later.

We can make this more quantitive by considering two observers: Alice and Bob. Bob has gone up in a hot air balloon while Alice is on the surface of the earth at r_A. Bob is at a distance $r_B = r_A + \Delta r$. The time measured by Bob is

$$
\begin{aligned}
T_B = t\sqrt{1 - \frac{2G_NM}{(r_A + \Delta r)}} &\sim t\sqrt{1 - \frac{2G_NM}{r_A} + \frac{2G_NM\Delta r}{r_A^2}} \\
&\sim t\sqrt{1 - \frac{2G_NM}{r_A}}\left(1 + \frac{G_NM\Delta r}{r_A^2}\right) = T_A\left(1 + \frac{G_NM\Delta r}{r_A^2}\right).
\end{aligned}
\tag{6.85}
$$

A double expansion has been utilised where we assume $\Delta r \ll r_A$ and $\frac{2G_NM}{r_A} \ll 1$. If the hot air balloon flies a distance $\Delta r = 1000m$ above Alice then taking the radius of the Earth to be $r_A \approx 6000$km the difference in times is about 10^{-12} and therefore over the whole day Bob ages by an extra 10^{-18} seconds or so. Clearly this is a small amount, in the vicinity of a black hole this can be more pronounced. Recall that the smallest stable orbit was at $r = 3G_NM$ and such a person experiences time at a rate of $T = 3^{-1/2}t \approx 0.6t$ compared to an asymptotic observer at $r \to \infty$. For more dramatic results one would need to fly closer to the horizon and then return to asymptotic infinity.

This also gives a different perspective on the gravitational redshift. Bob doesn't like Alice and wants to ruin her day so he hovers above Alice and chucks peanuts at her. He throws peanuts at time intervals ΔT_B. Alice, wise to Bob's antics, opens up an umbrella. The peanuts hit the umbrella at time intervals ΔT_A where as above

$$
\Delta T_A = \Delta T_B\sqrt{\frac{1 + 2\Phi(r_A)}{1 + 2\Phi(r_B)}} \approx \Big(1 + \Phi(r_A) - \Phi(r_B)\Big)\Delta T_B\,. \tag{6.86}
$$

We have that $r_A < r_B$ and therefore $\Phi(r_A) < \Phi(r_B) < 0$ and hence $\Delta T_A < \Delta T_B$. Alice receives the peanuts at a higher frequency than Bob threw them.

Having seen the peanuts hitting the umbrella Bob decides to instead shine a light down at Alice with a frequency $\omega_B \sim \Delta T_B^{-1}$. Alice will then receive the light at a frequency ω_A where

$$
\omega_A \approx \Big(1 + \Phi(r_A) - \Phi(r_B)\Big)^{-1}\omega_B\,. \tag{6.87}
$$

This is a higher frequency $\omega_A > \omega_B$ and therefore a shorter wave-length. The light is therefore blue-shifted. In contrast if Alice retaliates and shines a light up to Bob then the frequency decreases and the light is redshifted.

6.3 Schwarzschild solution as a black hole

We have now studied some geodesics for the Schwarzschild solution and some phenomena. Each time we have carefully avoided the Schwarzschild radius $r_S = 2G_NM$ and also $r = 0$. At both of these points something funky happens with the metric, at least one of the components of the metric diverges or vanishes. The interpretation of the singularities is different for the two cases. The divergence at $r = 0$ is a genuine *singularity*. General relativity breaks down here and we need a theory of quantum gravity. GR predicts its own death! In contrast the divergence at $r = 2G_N M$ is a result of our choice of coordinates. This surface is referred to as the *event horizon* or simply the *horizon* of the black hole. Many of the surprising properties of a black hole happen here.

There is a simple way to check whether a divergence is due to a singularity or a poor choice of coordinates. We can build scalar quantities, these are then independent of coordinates, if they diverge in one coordinate system they diverge in all and the spacetime is sick at this point. One the other hand if it does not diverge we cannot say much, one would have to consider all possible scalar quantities to concretely say it is just a coordinate singularity. Since the Einstein equations in a vacuum set $R_{\mu\nu} = 0$ it follows that the simplest scalar quantities one can construct R and $R_{\mu\nu}R^{\mu\nu}$ both vanish. The next simplest is the *Kretschmann scalar* $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$. For the Schwarzschild metric we find

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G_N^2M^2}{r^6}\,. \tag{6.88}$$

There is no pathology at $r = 2G_N M$ while there is at $r = 0$ where it diverges.

One way to understand the geometry of spacetime is to explore its causal structure as defined by light cones. We therefore consider radial null curves, i.e. those with constant θ, ϕ and $\mathrm{d}s^2 = 0$, such that they satisfy

$$\mathrm{d}s^2 = 0 = -\left(1 - \frac{2G_N M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1}\mathrm{d}r^2\,, \tag{6.89}$$

which gives

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\left(1 - \frac{2G_N M}{r}\right)^{-1}. \tag{6.90}$$

This measures the slope of the light cones on a spacetime diagram of the t-r plane. For large r the slope is ± 1 as it would be for flat spacetime. On the other hand as we approach $r = 2G_N M$ we get $\frac{\mathrm{d}t}{\mathrm{d}r} \to \pm\infty$ and the light cones close up, see figure 18. Thus a light ray which approaches $r = 2G_N M$ never seems to get there, at least in this coordinate system. This apparent inability to get to $r = 2G_N M$ is actually an illusion and an artefact of a bad choice of coordinates. An in-falling light ray or massive particle has no trouble reaching this radius. On the other hand an observer far away would never be able to tell. If we all hovered outside a black hole and one of your class mates jumped in the black hole sending back signals the whole way down we would simply see the signals reach us less frequently, see figure 19.

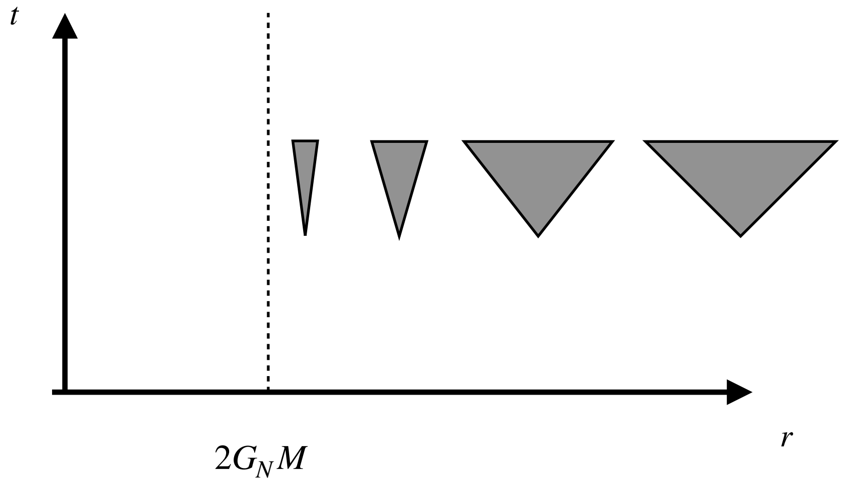


Figure 18: In Schwarzschild coordinates the light cones appear to close up as we approach the horizon. We will see that this is not quite correct.

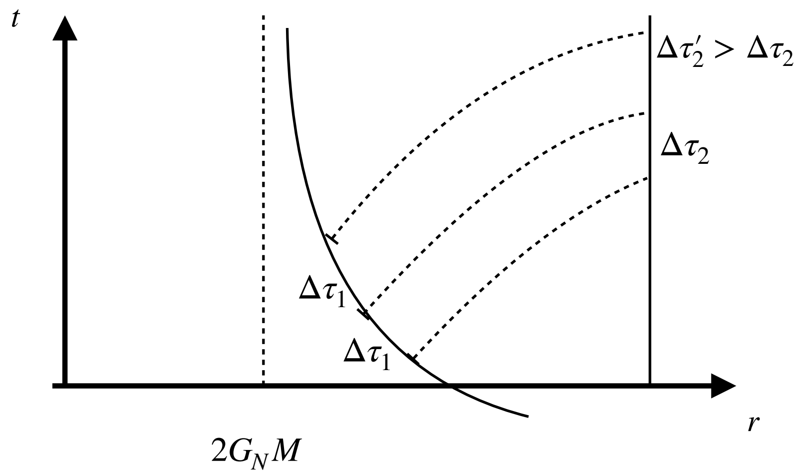


Figure 19: A beacon freely falling into a black hole emits signals at intervals of proper time $\Delta\tau_1$. An observer at fixed r receives these signals at a successively longer time intervals $\Delta\tau_2$.

The fact that we never see them reach $r = 2G_NM$ is a meaningful statement but the fact that their trajectory in the t-r plane never reaches there is not: it is highly dependent on our coordinate system. We want to change coordinates to some that are better behaved at $r = r_S$. Note that we can solve (6.90) by introducing the *tortoise coordinate* r_*

$$r_* = r + 2G_N M \log\left(\frac{r - 2G_N M}{2G_N M}\right), \tag{6.91}$$

then

$$t = \pm r_* + \text{constant}\,, \tag{6.92}$$

and we see that this is well adapted to null radial geodesics. The plus sign corresponds to out-going geodesics and the negative to in-going geodesics[28]. Next we introduce a pair of null coordinates further adapted to the null geodesics:

$$v = t + r_*\,, \qquad u = t - r_*\,. \tag{6.93}$$

In these coordinates the null radial geodesics are simply $u = \text{const}$ or $v = \text{const}$. We can write the metric in these new coordinates. First consider the metric in (v, r) coordinates and then we will study (u, r) coordinates before biting the bullet and using (v, u) coordinates.

Ingoing Eddington–Finkelstein coordinates Eliminating t via $t = v - r_*(r)$ we find

$$\mathrm{d}s^2 = -\left(1 - \frac{2G_N M}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}s^2(S^2)\,. \tag{6.94}$$

This is the Schwarzschild solution in *ingoing Eddington–Finkelstein coordinates.* Even though the metric coefficient g_{vv} vanishes at $r = 2G_N M$ there is no real degeneracy. The determinant of the metric is

$$\det g = \det\begin{pmatrix} -\left(1 - \frac{2G_N M}{r}\right) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} = -r^4\sin^2\theta\,. \tag{6.95}$$

The cross terms stops the metric from being degenerate at the horizon. The metric is still degenerate at $r = 0$ and $\theta = 0, \pi$ however the latter are just the usual pole problems of the S^2 and nothing to worry about. This is the benefit of the Eddington–Finkelstein coordinates, the radial coordinate can be extended beyond the horizon.

To build further intuition we can look at the behaviour of light rays. We saw that the null radial geodesics were given by (6.92). The outgoing geodesics are

$$u = t - r_* = \text{const}\,. \tag{6.96}$$

[28]The quick way to see this is to note that as $r \to \infty$ we have $r_* \to \infty$ and therefore we need the plus sign for out-going geodesics so that the radial direction increases with time.

Eliminating t in favour of v we have that the outgoing geodesics satisfy $v = 2r_* + \text{const}$. The solutions of this equation have a different behaviour depending on whether they are inside the horizon or outside. For $r > 2G_N M$ we can use the original definition of r_* in (6.91) to get

$$v = 2r + 4G_N M \log\left(\frac{r - 2G_N M}{2G_N M}\right) + \text{const}\,. \tag{6.97}$$

The Log term goes bad when $r < 2G_N M$, however we can simply modify the coordinate to take the norm of the argument of the log, so that

$$r_* = r + 2G_N M \log\left|\frac{r - 2G_N M}{2G_N M}\right|\,. \tag{6.98}$$

This means that r_* is multi-valued. Outside the horizon it takes values $r_* \in (-\infty, \infty)$ while inside the horizon it takes values $r_* \in (-\infty, 0)$. The singularity sits at $r_* = 0$. Outgoing geodesics inside the horizon obey

$$v = 2r + 4G_N M \log\left(\frac{2G_N M - r}{2G_N M}\right) + \text{const}\,. \tag{6.99}$$

Finally note that $r = 2G_N M$ is itself a null geodesic. This information can be captured in a *Finkelstein diagram*. It is designed so that ingoing null rays travel at 45°. This is simple to do if we label the coordinates of the diagram by t and r_*, however since r_* is not single valued we use r instead. We define a new temporal coordinate t_* by the requirement

$$v = t + r_* = t_* + r\,. \tag{6.100}$$

Thus ingoing null rays travel at 45° in the (t_*, r)-plane. See figure 20

The outgoing null geodesics that sit outside the horizon tend to infinity, whereas those inside the horizon don't actually go out, but rather go towards the singularity at $r = 0$. Each hits the singularity at some finite t_*. We can draw lightcones on the Finekstein diagram. These are regions which are bounded by the in-going and out-going future pointing null geodesics. Any massive particle must follow a timelike path and this must then sit within these lightcones. We see that the lights cones get tipped as we get closer to the horizon, and then once inside the horizon there is no way of getting back out. The causal structure of spacetime prevents this. The term black hole really refers to this area inside the horizon $r < 2G_N M$, any observer outside the horizon can never known what is happening inside the black hole.

We can also see what happens if we watch someone fall into a black hole. The person falls through the horizon without realising anything is wrong. However as they fall the light

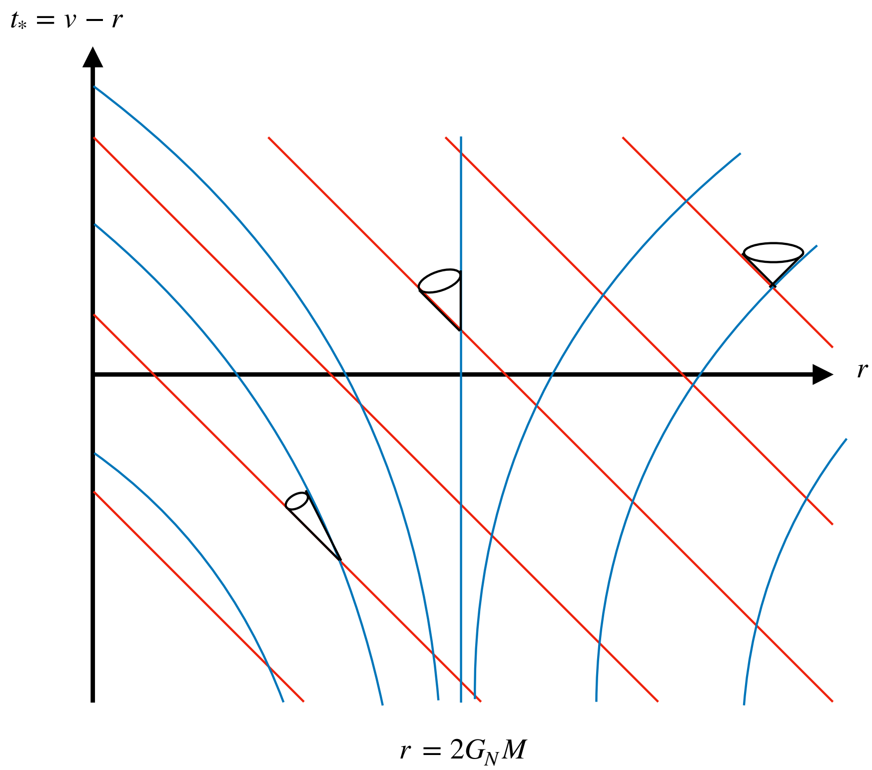


Figure 20: The Finkelstein diagram in in-going coordinates. The ingoing null geodesics are in red while the outgoing are in blue. Inside the horizon the outgoing geodesics never go past the horizon.

signals that come back to us take longer and longer to reach us. The actions of the in-falling person become increasingly slowed as they approach the horizon. In this way we continue to see the person forever, but we know nothing about their fate past the horizon. Since the light returns to us from a deeper and deeper gravitational well it appears increasingly red-shifted to us.

Out-going Eddington–Finkelstein coordinates We can also extend the exterior of the Schwarzschild black hole by replacing the time coordinate with the null coordinate

$$u = t - r_* \,. \tag{6.101}$$

Surfaces of constant u correspond to outgoing radial null geodesics. After the change of coordinates we have

$$\mathrm{d}s^2 = -\left(1 - \frac{2G_N M}{r}\right)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2\mathrm{d}s^2(S^2)\,. \tag{6.102}$$

This is the Schwarzschild solution in *out-going Eddington–Finkelstein coordinates*. The only difference is in the sign of the cross term. This seemingly trivial modification changes the interpretation drastically.

As before the metric is smooth at the horizon and we can continue the metric down to the singularity at $r = 0$. However the region $r < 2G_N M$ now describes a different part of spacetime from the analogous region in ingoing Eddington–Finkelstein coordinates.

We again look at the ingoing and outgoing null radial geodesics. This time we pick coordinates so that the outgoing geodesics travel at 45°. This means that we take r and $t_* = u + r$ to be the axes.

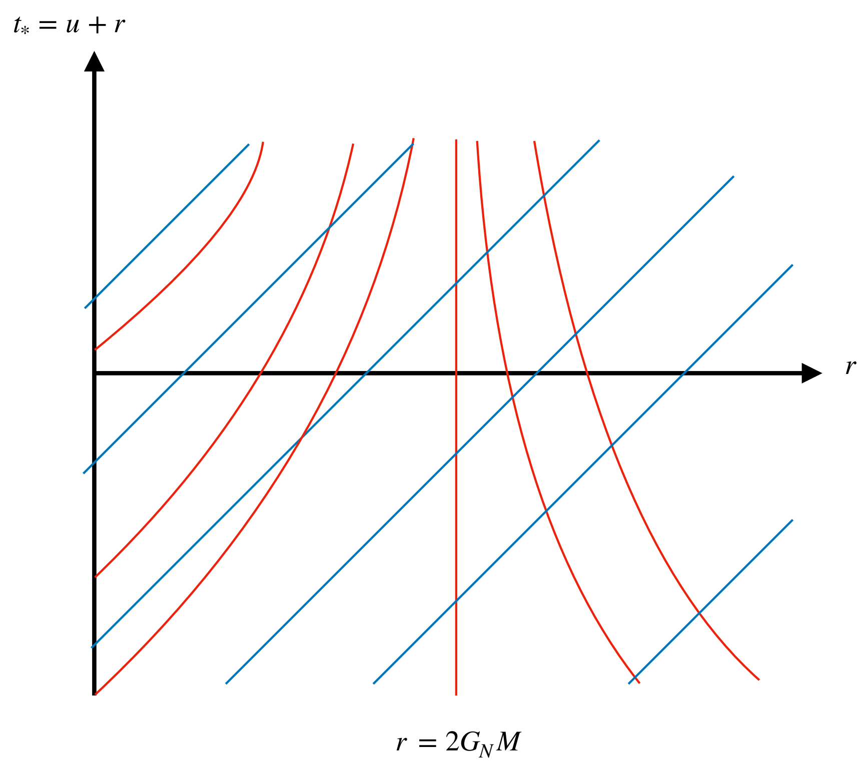


Figure 21: The Finkelstein diagram in out-going coordinates. The ingoing null geodesics are in red while the outgoing are in blue. Inside the horizon the ingoing geodesics never go past the horizon.

This time the ingoing null geodesics have the interesting property. Those which start outside are unable to reach the singularity, instead they pile up at the horizon. Those that start behind the horizon move towards the horizon, once again piling up there. What happens to massive particles that sit inside the horizon? Their trajectories must lie inside the future pointing light-cones. They cannot stay inside the horizon and the causal structure of spacetime requires them to be ejected outside of the horizon. This is a *white hole*, an object which expels matter. This is the time reversal of a black hole, indeed the difference is purely a minus sign. Moreover if we flip white-hole upside down we get the black hole.

White holes are perfectly acceptable solutions of general relativity. Indeed they are implied by the time reversal invariance of Einstein's equations. However white holes are not physically relevant since in contrast to a black hole they cannot be formed by collapsing matter.

6.3.1 Kruskal spacetime

We have seen that we can extend the $r \in (2G_N M, \infty)$ coordinate in two ways so that we gain the region $r \in (0, 2G_N M]$ which corresponds to two different parts of spacetime. We can write the Schwarzschild metric using both null (u, v)-coordinates, the metric is

$$\mathrm{d}s^2 = -\left(1 - \frac{2G_N M}{r}\right)\mathrm{d}u\mathrm{d}v + r^2\mathrm{d}s^2(S^2)\,, \tag{6.103}$$

where r is a function of $u-v$. In these coordinates the metric is again degenerate at $r = 2G_N M$ so we need to perform another change of coordinates. We can introduce the *Kruskal-Szekeres coordinates*,

$$U = -\exp\left(-\frac{u}{4G_N M}\right), \qquad V = \exp\left(\frac{v}{4G_N M}\right), \tag{6.104}$$

which are both null coordinates. The Schwarzschild black hole is parametrised by $U < 0$ and $V > 0$. Outside the horizon they satisfy

$$UV = -\exp\left(\frac{r_*}{2G_N M}\right) = \frac{2G_N M - r}{2G_N M}\exp\left(\frac{r}{2G_N M}\right), \tag{6.105}$$

and similarly

$$\frac{U}{V} = -\exp\left(-\frac{t}{2G_N M}\right). \tag{6.106}$$

The metric is then

$$\mathrm{d}s^2 = -\frac{32(G_N M)^3}{r}\mathrm{e}^{-\frac{r}{2G_N M}}\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}s^2(S^2)\,, \tag{6.107}$$

with $r(U, V)$ defined by inverting (6.105). The original Schwarzschild metric covers just $U < 0$ and $V > 0$ however there is no obstruction to extending $U, V \in \mathbb{R}$. Nothing bad happens at $r = 2G_N M$, the metric is smooth and non-degenerate. The Kruskal spacetime is the maximal extension of the Schwarzschild solution.

The Kruskal Diagram To find the location of the horizon in the new coordinates we can use equation (6.105). We see that this is at

$$r = 2G_N M \quad \Rightarrow \quad U = 0 \text{ or } V = 0\,. \tag{6.108}$$

The horizon is not just one null surface but 2 which intersect at $U = V = 0$. On the other hand the singularity is at

$$r = 0 \quad \Rightarrow \quad UV = 1\,. \tag{6.109}$$

This hyperbola has two disconnected, one with $U, V > 0$ and the other with $U, V < 0$. The former corresponds to the singularity of the black hole and the latter the singularity of the white hole, see figure 22 We can define $T = \frac{1}{2}(U + V)$ and $X = \frac{1}{2}(V - U)$ as the vertical and

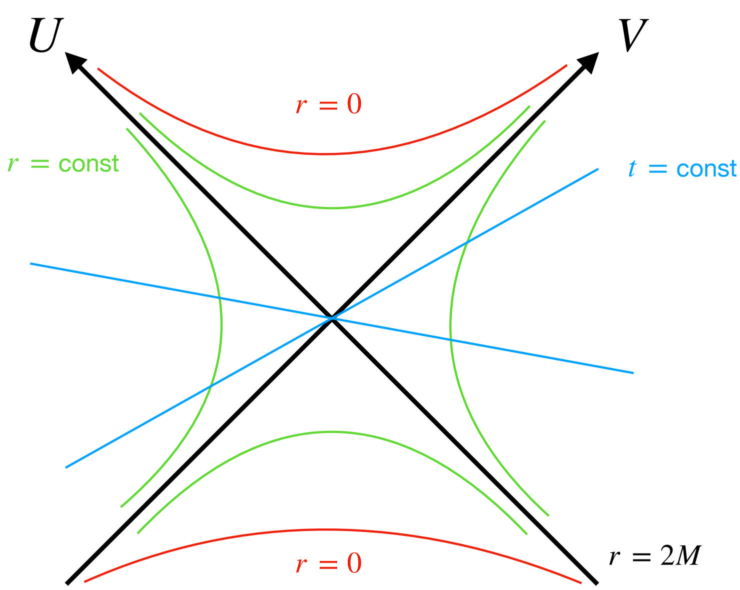


Figure 22: The Kruskal diagram. The U and V axes have been rotated 45°. They are the locations of the horizons at $r = 2G_N M$ and the red lines are the singularities at $r = 0$. Lines of constant r are in green and lines of constant t are in blue.

horizontal lines respectively. Lines of constant r are given by UV =constant while lines of constant t are U/V =constant.

We see that the singularity is spacelike. Once you pass through the horizon the singularity lies in your future. You cannot avoid the singularity once you cross the horizon. Similarly the singularity of the white hole lies in the past, one could think of this as the singularity of the Big Bang.

We can understand three quadrants of the four. The right quadrant is the exterior of the black hole, the top quadrant is the black hole interior and the bottom quadrant is the interior of the white hole. The left hand quadrant is in fact another copy of the black hole exterior, it is just covered by $U > 0$ and $V < 0$. To see this write

$$U = +\exp\left(-\frac{u}{4G_N M}\right), \quad V = -\exp\left(\frac{v}{4G_N M}\right). \tag{6.110}$$

Undoing all the coordinate transformations we see that this is precisely the metric of the Schwarzschild solution again.

Our spacetime contains two asymptotically flat regions joined by a black hole. Note that it is not possible for an observer to cross from one to the other, nor to send a signal from one region to the other. The causal structure of spacetime forbids this.

One could ask what the spatial geometry that connects the two regions is. Fix the $t = 0$ slice of Kruskal spacetime ($U = V = 0$). In our original Schwarzschild solution the spatial geometry is

$$\mathrm{d}s^2 = \left(1 - \frac{2G_N M}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}s^2(S^2)\,, \tag{6.111}$$

which is valid for $r > 2G_N M$. There is another copy of this that describes the geometry of the left-hand side and we can glue these two together at $r = 2G_N M$, giving a worm-hole like geometry. This is known as the *Einstein–Rosen bridge*. Before getting excited about travelling through the black hole you cannot travel through the worm-hole as the paths are space-like not time-like.

7 Cosmology

We have only considered one solution of Einstein's equations so far in these lectures, we will consider another which describes the evolution of the universe. The basic idea behind this model is that the universe is pretty much the same everywhere. Since we inhabit an orbit close, in cosmological terms, to a star we do not see the similarity between our situation and the desolate cold of deep space and this assumption may seem somewhat crazy. This assumption is applied to the very largest scales, where local variations in density are averaged over. There are a number of observations which support this assumption. The most clear way of seeing this is by looking at the Cosmic Background Radiation (CMB), see figure 23. The microwave background radiation is not perfectly smooth but the deviations from regularity are of the order 10^{-15} or less. The radiation is consistent with that of a blackbody spectrum radiated in all directions. The spectrum has been redshifted due to the expansion of the universe and today the average temperature is $2.725K$.

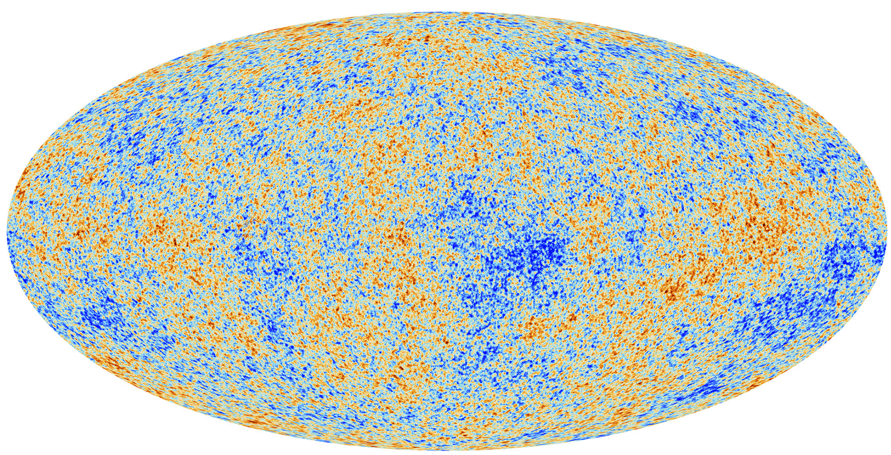


Figure 23: The anisotropies of the CMB as observed by Planck. It is a snapshot of the oldest light in the universe, coming from when the universe was just 380000 years old. It shows tiny temperature fluctuations that correspond to regions of slightly different densities and it is these regions which were the seeds for the stars and galaxies we see today. (Credit ESA for the picture).

7.1 FRW metric

We want to formalise this notion of the same everywhere in a more mathematical way. A manifold may have the properties of being *isotropic* and/or *homogeneous*: these are the necessary mathematical concepts which formalise our "same in every direction" comment.

Homogeneous A spacetime is spatially homogeneous if there exist a one-parameter family of space-like hypersurfaces Σ_t foliating spacetime, such that for each t and for any points $p, q \in \Sigma_t$ there exists an isometry of the spacetime metric $g_{\mu\nu}$ which takes p into q.

Isotropic A spacetime is isotropic at the point p if, for each pair of unit tangent vectors $X, Y \in T_p(M)$ there is an isometry which maps X to Y.

A spacetime can be isotropic around a point without being homogeneous. Conversely a spacetime can be homogenous without being isotropic ($\mathbb{R} \times S^2$ for example). If, however, a spacetime is isotropic around every point then it is homogeneous. Likewise if it is isotropic around any point, and homogeneous then it is isotropic everywhere.

Since there is ample observational data for isotropy (recall this is data about a point) and we are not so self-centred to think we are the centre of the universe we should assume that it is also homogeneous. The utility of these assumptions relies on the fact that a space which is both isotropic and homogeneous is maximally symmetric. (Think of isotropy as generalised rotations and homogeneity as generalised translations). This implies that the space has the maximal number of Killing vectors. Now spacetime itself should not be maximally symmetric, we want it to evolve, instead we want spatial slices to be maximally symmetric.

For a maximally symmetric space with metric $g_{\mu\nu}$ the Riemann tensor takes the form

$$R_{\mu\nu\rho\sigma} = \kappa(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \,, \tag{7.1}$$

where κ is a normalised measure of the Ricci scalar

$$\kappa = \frac{R}{n(n-1)} \,, \tag{7.2}$$

which must be constant. These spaces are classified and for us the difference will arise in the sign of κ, either positive, negative or 0. We will consider our spacetime to be of the form $\mathbb{R} \times \Sigma$ with metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}s^2(\Sigma) \,, \tag{7.3}$$

with t a time-like coordinate and $a(t)$ a function known as the *scale factor.* The metric used here which is free of cross terms with $\mathrm{d}t$ is known as *co-moving* coordinates. An observer who stays at fixed coordinate in Σ is said to be a *comoving observer.* Only a comoving observer sees the universe as isotropic. On Earth we are not quite comoving due to our motion around the sun.

We want a maximally symmetric 3d space, we can write the metric in the form

$$\mathrm{d}s^2(\Sigma) = \frac{\mathrm{d}r^2}{1 - kr^2} + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\big) \,, \tag{7.4}$$

with $k = \{-1, 0, +1\}$.[29] The case $k = -1$ gives a constant negative curvature metric and is sometimes called *open*. The $k = 0$ case corresponds to no curvature on Σ and is sometimes called *flat*, while the case $k = +1$ corresponds to positive curvature and is sometimes called *closed*. Note that the $k = 1$ case is the only one which is compact (unless one makes certain identifications of the coordinates). We then have the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\Big[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\big)\Big]\,. \tag{7.5}$$

This is the *Friedmann–Robertson–Walker metric* (FRW).

To understand why $a(t)$ is called a scale factor consider the distance between two observers, one at $r = 0$ and another at $r = r_0$. Then the spatial distance between them is

$$d_{\text{prop}} = \int_0^{r_0} \sqrt{g_{rr}}\mathrm{d}r = a(t)\int_0^{r_0} \frac{\mathrm{d}r}{\sqrt{1 - kr^2}} \equiv a(t) f(r_0)\,. \tag{7.6}$$

We see that the distance depends on the scale factor. We can look at the relative speed at which distance is changing with respect to time, we have

$$\dot{d}_{\text{prop}}(t) = \dot{a}(t) f(r_0) = \frac{\dot{a}}{a} d_{\text{prop}}(t) \equiv H(t) d_{\text{prop}}(t)\,, \tag{7.7}$$

with

$$H(t) = \frac{\dot{a}(t)}{a(t)}\,, \tag{7.8}$$

the *Hubble parameter*. The value of the Hubble parameter at present is the Hubble constant H_0. Current measurements give $H_0 = 70 \pm 10$ km/sec/Mpc. (Mpc is a megaparsec, $\sim 3.09 \times 10^{22} m$). Cosmology took off as a subject when the relative motions of the galaxies was first measured. We cannot actually determine the relative velocities of the galaxies now, i.e. at the same cosmological time, since we only have information about them at the time that the light left them. We are therefore not deducing $a(t)$ as it is now but rather as it was in the past. By looking at galaxies further away we can deduce the past history of $a(t)$.

7.1.1 Cosmological red-shift

Cosmological red-shift has a different origin to the gravitational red-shift we saw previously, however we can work it out in a similar manner. Assume that the light reaching us is on purely radial geodesics. Then we have

$$0 = -\mathrm{d}t^2 + \frac{a(t)^2}{1 - kt^2}\mathrm{d}r^2\,, \tag{7.9}$$

[29] Different values can be reduced to one of these three cases by redefining the radial coordinate r.

and therefore

$$\frac{\mathrm{d}t}{a(t)} = -\frac{\mathrm{d}r}{\sqrt{1-kr^2}}\,, \tag{7.10}$$

where we picked the $-$ sign for the incoming radial geodesic (paths of decreasing r). The time emission t_1 and reception t_0 of the photon are given by

$$\int_{t_0}^{t_1} \frac{\mathrm{d}t}{a(t)} = -\int_{r_0}^{0} \frac{\mathrm{d}r}{\sqrt{1-kr^2}} \equiv f(r_0)\,. \tag{7.11}$$

Suppose that the next wave crest is emitted at time $t_1 + \delta t_1$ and received at $t_0 + \delta t_0$. Then since t is the proper time of stationary observers $\delta t_1 = \omega_1^{-1}$ and $\delta t_0 = \omega_0^{-1}$, with ω_i the frequency. Since the second photon leaves from r_0 and arrives at $r = 0$ it must also satisfy

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{\mathrm{d}t}{a(t)} = f(r_0)\,. \tag{7.12}$$

If δt_i are small then

$$\begin{aligned} f(r_0) &= \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{\mathrm{d}t}{a(t)} = \left(\int_{t_1}^{t_0} + \int_{t_0}^{t_0+\delta t_0} - \int_{t_1}^{t_1+\delta t_1}\right)\frac{\mathrm{d}t}{a(t)} \sim \int_{t_1}^{t_0} \frac{\mathrm{d}t}{a(t)} + \frac{\delta t_0}{a(t_0)} - \frac{\delta t_1}{a(t_1)} \\ &= f(r_0) + \frac{\delta t_0}{a(t_0)} - \frac{\delta t_1}{a(t_1)}\,. \end{aligned} \tag{7.13}$$

Therefore we have

$$\frac{\delta t_0}{a(t_0)} \sim \frac{\delta t_1}{a(t_1)} \quad \Rightarrow \quad \omega_0 \sim \frac{a(t_1)}{a(t_0)}\omega_1\,. \tag{7.14}$$

The change in frequency is directly given by the ratio of the scale factors from when the light was emitted and when the light was received. The standard cosmologists definition of red-shift is through

$$z = \frac{\omega_1}{\omega_0} - 1 = \frac{a(t_0)}{a(t_1)} - 1\,. \tag{7.15}$$

Red shift is a direct measure of the change in separation of galaxies during the time the photon has taken to reach us. If a galaxy is at redshift 5 for example then it is 6 times further away than when the photon was emitted. Red shift does not give any direct information about the distance of the source, nor does it need to be faithful indicator of distance. Sources at different distances can have the same or similar red-shifts. If there was a period of time where the scale factor was essentially constant then any photons emitted during this period would appear to have the same red-shift. Similarly if there was a period of the scale factor decreasing then increasing again then sources at very different distances could give the same red-shift factor.

7.2 The Friedmann equations

Note that the Christoffel symbol ${\Gamma^i}_{tt} = 0$ and therefore the paths $\vec{x}$ =const are geodesics. The role of $a(t)$ is to change distances over time. There is a redundancy in the metric. If we rescale the coordinates as $a \to \lambda a$, $r \to \lambda$ and $k \to \lambda^{-2}k$ we leave the metric invariant. Of course now we are no longer fixed to take $k \in \{-1, 0, 1\}$. The non-zero components of the Ricci tensor are

$$\begin{aligned} R_{tt} &= -3\frac{\ddot{a}}{a}\,, \\ R_{rr} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2\kappa}{1 - kr^2}\,, \\ R_{\theta\theta} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2\kappa)\,, \\ R_{\phi\phi} &= r^2\sin^2\theta(a\ddot{a} + 2\dot{a}^2 + 2\kappa)\,. \end{aligned} \tag{7.16}$$

It follows that the Ricci scalar is then

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2}\right]. \tag{7.17}$$

The FRW metric is determined by the behaviour of $a(t)$. We want to plug this into Einstein's equations to derive the so called *Friedmann equations* which relates the scale factor to the energy-momentum of the universe. We choose to model the matter as a perfect fluid. If a fluid is isotropic in one frame and leads to an isotropic metric then it must be that the fluid is a rest in co-moving coordinates. The four-velocity is then

$$U^\mu = (1, 0, 0, 0)\,, \tag{7.18}$$

and the energy momentum tensor is

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}\,. \tag{7.19}$$

With one index raised this becomes

$${T^\mu}_\nu = \mathrm{diag}(-\rho, p, p, p)\,, \tag{7.20}$$

and the trace is

$$T = {T^\mu}_\mu = -\rho + 3p\,. \tag{7.21}$$

Before plugging into Einstein's equations it is useful to consider the conservation of the energy momentum tensor, in particular for the first component. We have

$$\begin{aligned} 0 &= \nabla_\mu {T^\mu}_0 \\ &= -\dot{\rho} - 3\frac{\dot{a}}{a}(\rho + p)\,. \end{aligned} \tag{7.22}$$

7.2.1 Equation of state

To make progress we choose an equation of state, that is a relationship between p and ρ. The perfect fluids relevant to cosmology satisfy

$$p = w\rho\,, \tag{7.23}$$

with w a constant independent of time. The conservation of energy becomes

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}\,. \tag{7.24}$$

When w is constant this can be integrated to give

$$\rho \propto a^{-3(1+w)}\,. \tag{7.25}$$

For the vacuum to be stable[30] we need to pick $|w| \leq 1$. The two most popular cosmological fluids are known as *matter* and *radiation*.

Matter is any set of collision-less non-relativistic particles which have zero pressure $p_M = 0$, i.e. $w = 0$. Examples include stars and galaxies for which the pressure is negligible. Matter also goes by the name of *dust* and universe whose energy density is mostly due to matter are known as *matter-dominated* universes. The energy density of matter falls off as

$$\rho_M \propto a^{-3}\,, \tag{7.26}$$

which is just interpreted as the decrease in number density of particles as the universe expands. For matter the energy density is dominated by the rest-energy which is proportional to the number density.

Radiation may be used to describe actual electromagnetic radiation or massive particles moving at relativistic velocities, close to the speed of light. The trace of the energy-momentum tensor of the electromagnetic field vanishes and therefore this fixes

$$p_R = \frac{1}{3}\rho_R \qquad \Rightarrow \quad w = \frac{1}{3}. \tag{7.27}$$

In a *radiation dominated* universe the energy density falls off as

$$\rho_R \propto a^{-4}\,. \tag{7.28}$$

Thus the energy density of photons falls off slightly faster than that of matter. To understand why observe that the number density of photons decreases in the same way as for the slow

[30] This is beyond the scope of the course but one can read about this in Carroll chapter 4.

moving massive particles, but in addition they lose energy due to cosmological red-shift of the previous section. When a is small radiation will dominate, while as a increases dust will dominate.

Vacuum energy also takes the form of a perfect fluid, that is a cosmological constant. In this case $p_\Lambda = -\rho_\Lambda$ and the energy density is constant,

$$\rho_\Lambda \propto a^0 \,. \tag{7.29}$$

Since the energy density of both matter and radiation decreases as the universe expands if there is a non-zero vacuum energy it tends to dominate over the long term so long as the universe does not start contracting. If the vacuum energy begins to dominate then we say that the universe becomes *vacuum-dominated*. Examples of this are the maximal symmetric spaces de Sitter and anti-de Sitter.

7.2.2 Deriving the Friedmann equations

We can now substitute this into the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \,. \tag{7.30}$$

The $\mu\nu = 00$ components give

$$\frac{3\dot{a}^2}{a^2} + \frac{3k}{a^2} - \Lambda = 8\pi G_N \rho, , \tag{7.31}$$

while the $\mu\nu = ij$ components give

$$\frac{2\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \Lambda = -8\pi G_N p \,. \tag{7.32}$$

There is only one distinct condition from the spatial part because of our isotropic assumption. From a linear combination of the two equations we find

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{4\pi G_N}{3}(\rho + 3p) \,. \tag{7.33}$$

Note that the conservation of the energy momentum tensor,

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \,, \tag{7.34}$$

can be obtained from these two equations.

Equations (7.31) and (7.33) are known as the Friedmann equations and metrics of the form (7.5) satisfying these equations are FRW universes. If we know the dependence of ρ on a then the first can be solved.

7.3 Cosmological solutions

Let us consider some solutions. Before trying to solve anything let us analyse the behaviour of the function. With our equation of state the Friedmann equation becomes

$$\begin{aligned}\dot{a}^2 &= \frac{\Lambda a^2}{3} - k + \frac{8\pi G_N}{3}\rho a^2 \\ &= \frac{\Lambda a^2}{3} - k + \frac{C}{a^{1+3w}}\,,\end{aligned} \tag{7.35}$$

where C is a constant such that $8\pi G_N \rho = Ca^{-3(1+w)}$.

We now want to analyse the form of $a(t)$. Note that qualitatively there is very little difference between dust and radiation, radiation is a little more dominant for small a but otherwise the overall structure is the same.

- For small a $\dot{a}^2$ is dominated by the term $Ca^{-3(1+w)}$ and therefore $|\dot{a}| \to \infty$ as $a \to 0$. This is then a period of rapid expansion or contraction. We have

  $$\dot{a}^2 \sim a^{-3(1+w)}\,, \quad \Rightarrow \quad \dot{a} \sim \pm\sqrt{C}a^{-\frac{1+3w}{2}}\,, \tag{7.36}$$

  which can be solved to give

  $$a(t) \sim \text{constant}|t|^{\frac{2}{3(1+w)}}\,. \tag{7.37}$$

  In both cases $a(t)$ will expand from zero to finite size, or collapse from finite size to 0 in finite time.

- For large a the behaviour depends on the sign of Λ and if this vanishes then on k.

We can now consider in more detail various cases.

7.3.1 Solutions with $k = 0$

Let us set $k = 0$. This is the most likely value for the current universe.

We can now distinguish the different behaviours depending on the sign of Λ, see figure 24.

$\Lambda > 0$ For $\Lambda > 0$ $\dot{a}^2$ is never negative and therefore $\dot{a}$ must always be positive or negative. For $\dot{a} > 0$ a starts off small with a rapid expansion which slows down to a minimum rate of expansion and then the rate of expansion increases again. See figure 25a.

For $\dot{a} < 0$ then the evolution is the opposite. a starts off large, collapsing quickly before the rate of collapse slows to a minimum before speeding up once again until the universe collapses again. See figure 25b.

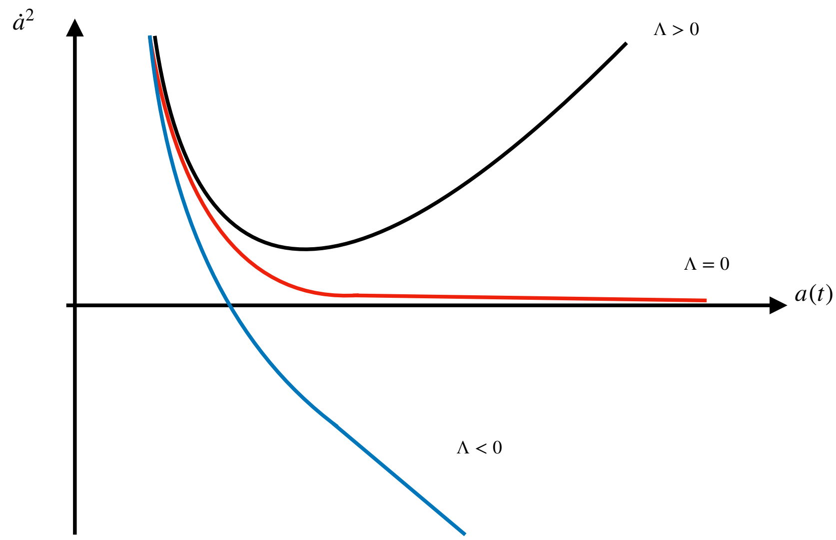


Figure 24: A plot of $\dot{a}^2$ for $k = 0$. Note the unphysical region for $\Lambda < 0$.

$\Lambda = 0$ As before $\dot{a}^2$ is always positive so $\dot{a}$ cannot change sign. For $\dot{a} > 0$ the universe starts off at zero size expands rapidly before the rate of expansion decreases, tending to zero but never reaching it. The opposite sign for $\dot{a}$ is the time reversal of this. See figure 25c.

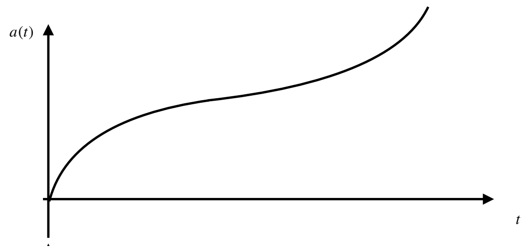
$\Lambda < 0$ In this case there is a critical value $a = a_c$ wt which $\dot{a} = 0$. One can show that at this point $\ddot{a} < 0$ and therefore if a is initially increasing it slows until it reaches a_c and then starts to decrease. The universe begins expanding before reaching a critical size before contracting again, all in finite time. See figure 25d.

We can in fact explicitly solve for $a(t)$. For dust, $w = 0$ we find

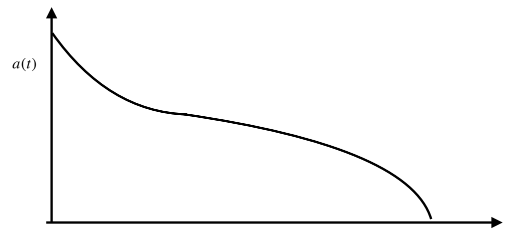
$$a(t) = \begin{cases} \left(\frac{3C}{\Lambda}\right)^{1/3} \sinh^{2/3}\left(\frac{\sqrt{3\Lambda}}{2}t\right) & \Lambda > 0\,, \\ \left(\frac{3\sqrt{C}}{2}\right)^{2/3} t^{2/3} & \Lambda = 0\,, \\ \left(-\frac{3C}{\Lambda}\right)^{1/3} \sin^{2/3}\left(\frac{\sqrt{-3\Lambda}}{2}t\right) & \Lambda < 0\,. \end{cases} \tag{7.38}$$

For radiation, $w = \frac{1}{3}$ we have

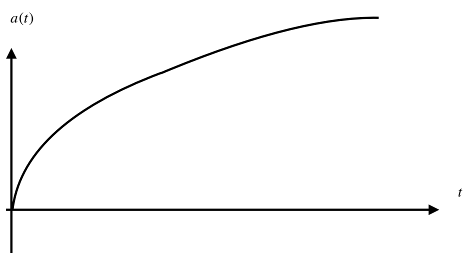
$$a(t) = \begin{cases} \left(\frac{3C}{\Lambda}\right)^{1/4} \sinh^{1/2}\left(\frac{\sqrt{3\Lambda}}{2}t\right) & \Lambda > 0\,, \\ \sqrt{2\sqrt{C}}t^{1/2} & \Lambda = 0\,, \\ \left(-\frac{3C}{\Lambda}\right)^{1/4} \sin^{1/2}\left(\frac{\sqrt{-3\Lambda}}{2}t\right) & \Lambda < 0\,. \end{cases} \tag{7.39}$$



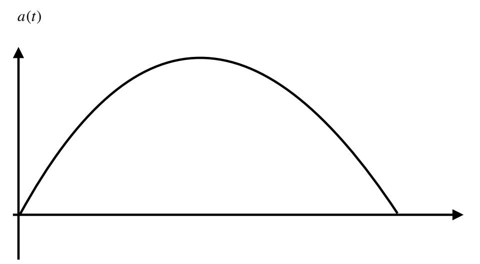
(a) For $\Lambda > 0$ with an initial increasing scale factor.



(b) For $\Lambda > 0$ with an initial decreasing scale factor.



(c) For $\Lambda = 0$ with an initial increasing scale factor.



(d) For $\Lambda < 0$, notice that the universe ends in a big crunch.

Figure 25: various plots of the scale factor for $k = 0$ and different choices of the cosmological constant.

7.3.2 Solutions with $\Lambda = 0$

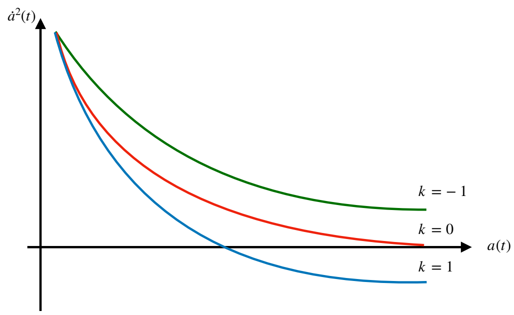
We can now consider keeping k free, (well we can arrange for $k \in \{-1, 0, 1\}$ without loss of generality) and set the cosmological constant to vanish. We can again plot the qualitative features of $a(t)$.

- We have that for $k = 1$ there is a maximum value of a for which $\dot{a}^2$ is positive or zero and so we end up with an initial phase of expansion before reaching the critical value and then a subsequent contraction.

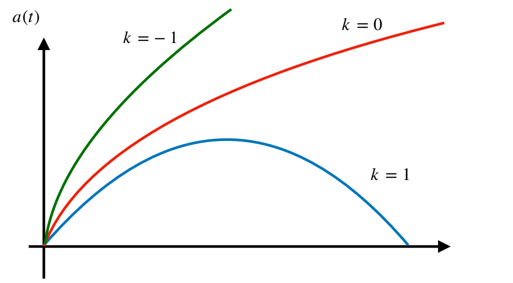
- If $k = 0$ or $k = -1$ then the universe continues to expand, but at different rates. For $a \to \infty$ we have that when $k = -1$ we have $\dot{a}^2 \to 1$ while for $k = 0$ we have $\dot{a} \to 0$.

We have plotted $\dot{a}^2$ in figure 26a while a is plotted in figure 26b.

One can again find full solutions to these equations however they are somewhat tedious to work out and best expressed in terms as parametric functions, for this reason we omit this.



(a) Plot of $\dot{a}^2$ as a function of $a(t)$.



(b) Plot of a as a function of t for the various choices of k.

7.3.3 The Big Bang

All of the solutions we have constructed have a region where $a = 0$. One can show that this is a generic feature of the Friedmann equations. From (7.33) we see that if the matter obeys the *strong energy condition*

$$\rho + 3p \geq 0\,, \tag{7.40}$$

then there is a singularity at a finite time t_{BB} where $a(t_{BB}) = 0$. This follows since the acceleration is necessarily negative. The universe is therefore decelerating, meaning it must have been accelerating faster at some point. If $\ddot{a} = 0$ then $a(t) = H_0 t + \text{const}$.

Suppose that $\ddot{a} = 0$, then $a(t) = H_0 t + \text{const}$. This is the dotted line shown in figure 27. If this is the case then the Big bang occurs at $t_0 - t_B B = H_0^{-1}$. The strong energy condition ensures that $\ddot{a} \leq 0$ and so the dashed line provides an upper bound on the scale factor. In such a universe the Big Bang must occur at $t_0 - t_B B \leq H_0^{-1}$.

The Big Bang refers to the creation of the universe from a singular state, not an explosion of matter into a pre-existing spacetime. One may wonder whether this singularity is an artefact of our choice of initial assumptions however it has been shown (by Hawking in his PhD thesis) that a singularity is a necessity even in the absence of such assumptions, given the strong energy condition.

The strong energy condition is obeyed by all conventional matter, including dust and radiation. However there are substances which violate it, leading to an accelerating universe. The single component pieces above still have a big bang however the above argument cannot rule out the possibility of more complicated solutions which avoid the Big Bang. In fact the leading theory at the moment is that in the very first moments after the Big Bang there was a period of exponential expansion.

All of the cosmological models we use predict a time in the past where the scale factor

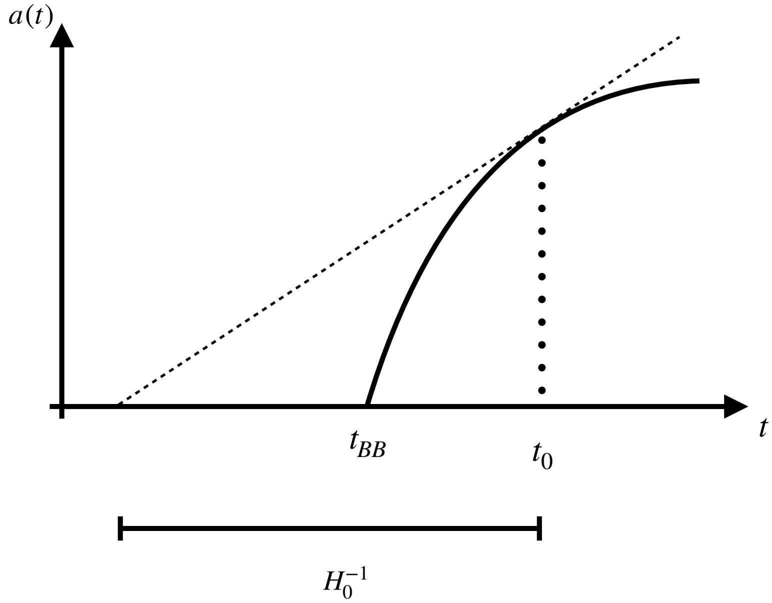


Figure 27: A plot of the scale factor showing the inevitability of the Big Bang.

vanishes. The Big bang is a point in time not in space, it happens everywhere in space. We can get an estimate for the age of the universe by Taylor expanding $a(t)$ and truncating to linear order. Recall that we fixed $a(t_0) = 1$ then

$$a(t) \sim 1 + H_0(t - t_0)\,. \tag{7.41}$$

This gives the estimate

$$t_0 - t_B B = H_0^{-1} \sim 4.4 \times 10^{17} s \sim 1.4 \times 10^{10} \text{ years}\,. \tag{7.42}$$

This is close to the 13.8 billion years which is widely accepted to be the age of the universe. Strictly speaking we should not trust the solution at $a(t_{BB}) = 0$ since the metric is singular there. Any matter in the universe will be squeezed into an infinite density object. In such a regime our classical equations are no longer any good and we need a quantum theory of gravity. Despite much effort such a theory of quantum gravity is lacking and so we are unable to answer many questions. Did time begin at t_{BB}? Was there a previous phase of a contracting universe and we are another bounce?

7.3.4 Cosmological horizon

The existence of a special time t_{BB} means that there is a limit as to how far back we can look into the past. Let us set $t_{BB} = 0$ in the following.

The speed of light sets an upper bound on the local propagation velocity of any signal so at a given time t an observer at $r = 0$ can receive signals emitted at time t_1 only from radial coordinates $r < r_1$ where r_1 is the radial coordinate from which light signals emitted at time t_1 would just reach $r = 0$ at time t. We can determine r_1 as

$$\int_0^{r_1} \frac{\mathrm{d}r}{\sqrt{1-kr^2}} = \int_{t_1}^{t} \frac{\mathrm{d}t'}{a(t')} \,. \tag{7.43}$$

If the t' integral diverges as $t_1 \to 0$ then it is in principle possible to receive signals emitted at sufficiently early times from any comoving particle in the universe. On the other hand if the t'-integral converges at $t_1 \to 0$ then our vision is limited by a so-called *particle horizon*: it is possible to receive signals from a comoving particles that lie within the radial coordinate $r_H(t)$ defined by

$$\int_0^{r_H(t)} \frac{\mathrm{d}r}{\sqrt{1-kr^2}} = \int_{t_1}^{t} \frac{\mathrm{d}t'}{a(t')} \,. \tag{7.44}$$

The proper distance is

$$d_H(t) = a(t) \int_0^{r_H(t)} \frac{\mathrm{d}r}{\sqrt{1-kr^2}} = a(t) \int_{t_1}^{t} \frac{\mathrm{d}t'}{a(t')} \,. \tag{7.45}$$

From (7.31) if ρ grows faster than $a^{-2-\epsilon}$ as $a \to 0$ then there will be a particle horizon.

We can play a similar game and ask if there are regions we will never see even if we wait long enough. If the t' integral diverges as $t \to \infty$ then in principle it is possible to receive signals from any event in the universe if we wait long enough. On the other hand if this is finite then it is only possible to receive signals for which

$$\int_0^{r_1} \frac{\mathrm{d}r}{\sqrt{1-kr^2}} \leq \int_{t_1}^{t_{\max}} \frac{\mathrm{d}t'}{a(t')} \,. \tag{7.46}$$

Here $t_{\max}$ can either be ∞ or the value of the next contraction to $a(t_{\max}) = 0$. This is known as an event horizon. It behaves in a similar way to falling inside the event horizon of a black hole, we will never be able to communicate with someone beyond the even-horizon.

This leads to some problems. We have assumed an isotropic universe, this is despite widely separated points being completely outside the event horizon of other points. Distinct patches of the CMB sky were causally disconnected. How then did they know ahead of time to coordinate their evolution (so that the CMB background looks isotropic) in the right way even though they were never in causal contact? One way of fixing this is by considering a period of inflation: an era of acceleration $\ddot{a} > 0$ in the very early universe, which is driven by some component other than matter or radiation.

Acknowledgments

It is a pleasure to thank all the students who have attended these lectures. You have made this an enjoyable course to teach and all your questions are much appreciated. I hope that you have enjoyed the course and learnt some of the amazing things that GR has to offer.

A Euler–Lagrange equations in field theory

General relativity is an example of a classical field theory. We want to understand how to compute the equations of motion for such a field theory. Let us begin with the computation of the equations of motion in classical mechanics before building up to the field theory version.

A.1 Classical mechanics and Euler–Lagrange

Consider a single particle in one dimension with coordinate $q(t)$. The equations of motion are computed by using the *principle of least action*. We need to define a functional of an *action* S

$$S = \int \mathrm{d}t L(q, \dot{q}) \,, \tag{A.1}$$

and find the critical points of S. Here $L(q, \dot{q})$ is known as the *Lagrangian*, and typically for point-particle mechanics takes the form:

$$L = \text{Kinetic Energy} - \text{Potential Energy} \,. \tag{A.2}$$

From your classical mechanics course you will have derived the *Euler–Lagrange equations*:

$$\frac{\partial L}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) \,, \tag{A.3}$$

see below for a recap of its derivation.

Consider a trajectory between the points q_1 and q_2 with $q(t_1) = q_1$ and $q(t_2) = q_2$. Let us deform the trajectory by

$$q(t) \to q(t) + \epsilon \delta q(t) \,, \tag{A.4}$$

whilst keeping the end-points fixed, that is $\delta q(t_1) = \delta q(t_2) = 0$. We have introduced the parameter ϵ which we take to be small. Consider now the action for this shifted path:

$$\begin{aligned} S[q(t) + \epsilon \delta q(t)] &= \int_{t_1}^{t_2} L\big(q + \epsilon \delta q, \dot{q} + \epsilon \tfrac{\mathrm{d}}{\mathrm{d}t} \delta q\big) \mathrm{d}t \\ &= \int_{t_1}^{t_2} \Big[L(q, \dot{q}) + \epsilon \Big\{ \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{\mathrm{d}}{\mathrm{d}t} \delta q \Big\} + \mathcal{O}(\epsilon^2)\Big] \mathrm{d}t \\ &= S[q(t)] + \epsilon \left[\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\right) \delta q \mathrm{d}t + \left[\frac{\partial L}{\partial \dot{q}} \delta q\right]_{t_1}^{t_2} \right] + \mathcal{O}(\epsilon^2) \,. \end{aligned} \tag{A.5}$$

Note that due to our boundary conditions the last term vanishes. The first order variation is then

$$\begin{aligned}\delta S &= S(q+\delta q) - S(q) \\ &= \epsilon \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\right)\delta q \mathrm{d}t\end{aligned} \tag{A.6}$$

and the requirement that this be an extremum, $\delta S = 0$ then implies

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0\,, \tag{A.7}$$

which is the Euler–Lagrange equation.

A.2 Classical field theory and Euler–Lagrange

Having reviewed the Euler–Lagrange equations for classical mechanics consider the field theory version. We replace the single coordinate $q(t)$ by a set of spacetime dependent fields $\Phi^i(x^\mu)$, and take the action S to be a functional of these fields. The Lagrangian is expressed in terms of a *Lagrange density*, $\mathcal{L}$, which we take to be a function of the fields Φ^i and their spacetime derivatives, $\partial_\mu \Phi^i$:

$$L = \int \mathrm{d}^3 x \mathcal{L}(\Phi^i, \partial_\mu \Phi^i)\,. \tag{A.8}$$

In principle the fields could be tensorial but we will suppress these details here, as we will see tensorial objects are necessary when considering the geodesics of GR. Similar to the classical mechanics case reviewed above we consider a small variation of the fields:

$$\Phi^i \to \Phi^i + \epsilon \delta \Phi^i\,, \qquad \partial_\mu \Phi^i \to \partial_\mu \Phi^i + \epsilon \partial_\mu(\delta\Phi^i)\,, \tag{A.9}$$

with ϵ our small parameter once again. The boundary conditions are similar to before with $\delta\Phi^i$ vanishing on the boundary of spacetime. The the first order variation of the action is

$$\begin{aligned}\delta S &= \epsilon \int \mathrm{d}^4 x \left[\frac{\partial \mathcal{L}}{\partial \Phi^i}\delta\Phi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\partial_\mu(\delta\Phi^i)\right] \\ &= \epsilon \int \mathrm{d}^4 x \left[\frac{\partial \mathcal{L}}{\partial \Phi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\right)\right]\delta\Phi^i + \epsilon \int \mathrm{d}^4 x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\delta\Phi^i\right)\,.\end{aligned} \tag{A.10}$$

Due to our boundary conditions the last term vanishes and for a stationary point, $\delta S = 0$ we obtain the Euler–Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \Phi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\right) = 0\,. \tag{A.11}$$

Exercise A.1: Euler–Lagrange equation examples

- Compute the Euler–Lagrange equations for a scalar field with Lagrangian density:

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\,. \tag{A.12}$$

- Compute the Euler–Lagrange equations for electromagnetism with Lagrangian density:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu\,, \tag{A.13}$$

  with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

  Hint: You should vary with respect to A and not F

B Solutions to exercises

In this appendix we provide solutions to the various exercises which are not covered in the tutorial problems.

B.1 Exercises to chapter 2

B.1.1 Exercise 2.1

1. Under two successive boosts along the x direction we have

$$\begin{aligned}
\Lambda_{\text{final}} &= \Lambda_1^{\text{Boost}} \Lambda_1^{\text{Boost}} \\
&= \begin{pmatrix} \cosh\phi_1 & -\sinh\phi_1 & 0 & 0 \\ -\sinh\phi_1 & \cosh\phi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cosh\phi_2 & -\sinh\phi_2 & 0 & 0 \\ -\sinh\phi_2 & \cosh\phi_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cosh\phi_1\cosh\phi_2 + \sinh\phi_1\sinh\phi_2 & -\cosh\phi_1\sinh\phi_2 - \sinh\phi_1\cosh\phi_2 & 0 & 0 \\ -\cosh\phi_1\sinh\phi_2 - \sinh\phi_1\cosh\phi_2 & \cosh\phi_1\cosh\phi_2 + \sinh\phi_1\sinh\phi_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cosh(\phi_1+\phi_2) & -\sinh(\phi_1+\phi_2) & 0 & 0 \\ -\sinh(\phi_1+\phi_2) & \cosh(\phi_1+\phi_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cosh\phi_{\text{final}} & -\sinh\phi_{\text{final}} & 0 & 0 \\ -\sinh\phi_{\text{final}} & \cosh\phi_{\text{final}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{B.1}$$

   We see that we end up with another boost in the x-direction and that the rapidity of two successive boosts is additive: $\phi_{\text{final}} = \phi_1 + \phi_2$.

2. We have

$$\begin{aligned}
e^{i\theta T^{\text{Rotation}}} &= 1_{4\times 4} + \sum_{k=2}^{\infty} \frac{(i\theta)^k}{k!} (T^{\text{Rotation}})^k \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m \theta^{2m}}{(2m)!} (T^{\text{Rotation}})^{2m} + i \sum_{m=0}^{\infty} \frac{(-1)^m \theta^{2m+1}}{(2m+1)!} (T^{\text{Rotation}})^{2m+1} .
\end{aligned} \tag{B.2}$$

Now

$$(T^{\text{Rotation}})^{2k} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad (T^{\text{Rotation}})^{2k+1} = T^{\text{Rotation}} \tag{B.3}$$

and therefore we have

$$\begin{aligned} e^{i\theta T^{\text{Rotation}}} &= 1_{4\times 4} + \sum_{m=2}^{\infty} \frac{(-1)^m \theta^{2m}}{(2m)!} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + iT^{\text{Rotation}} \sum_{m=0}^{\infty} \frac{(-1)^m \theta^{2m+1}}{(2m+1)!} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} , \tag{B.4}$$

where in the last step we use the series expansion of the trigonometric functions. A similar computation gives the boost.

3. Now consider the commutator of the generator of a boost along the x-direction with the generator of a boost along the y-direction. We have

$$\begin{aligned} [T_x^{\text{Boost}}, T_y^{\text{Boost}}] &= T_x^{\text{Boost}} T_y^{\text{Boost}} - T_y^{\text{Boost}} T_x^{\text{Boost}} \\ &= \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= -iT_{xy}^{\text{Rotation}} \end{aligned} \tag{B.5}$$

4. Compute the commutator of the generators (2.10) for a boost along x and rotation in the x-y-plane.

   We have the result:

$$[T_x^{\text{Boost}}, T_{xy}^{\text{Rotation}}] = -iT_y^{\text{Boost}} \tag{B.6}$$

5. Compute the commutator of the generators ((2.10))of a boost along x and rotation in the y-z-plane.

   In this case we find that the commutator vanishes.

Let $K_i \equiv T_i^{\text{Boost}}$ and $L_i = \frac{1}{2}\epsilon_{ijk} T_{jk}^{\text{Rotation}}$ then we find that the commutators are:

$$[K_i, K_j] = -i\epsilon_{ijk} L_k \, , \quad [K_i, L_j] = i\epsilon_{ijk} K_k \, , \quad [L_i, L_j] = i\epsilon_{ijk} L_k \, . \tag{B.7}$$

B.1.2 Exercise 2.2

We have just computed the addition of the rapidity. The velocity, in terms of the rapidity is (2.12)

$$v = \frac{x}{t} = \tanh\phi\,. \tag{B.8}$$

We saw that the rapidity was additive. We have that after two boosts with rapidity/velocity (ϕ_i/v_i) we have

$$\begin{aligned} v_3 &= \tanh\phi_3 \\ &= \tanh(\phi_1 + \phi_2) \\ &= \frac{v_1 + v_2}{1 + v_1 v_2}\,. \end{aligned} \tag{B.9}$$

This uses well known trig identities.

B.1.3 Exercise 2.3

We want to take the proper time defined in (2.16)

$$\Delta\tau = \int_{\lambda_1}^{\lambda_2} \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda\,, \tag{B.10}$$

and show that under $\lambda \to \lambda(\sigma)$ the proper time is left invariant.

$$\begin{aligned} \Delta\tau &= \int_{\lambda_1}^{\lambda_2} \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda \\ &= \int_{\lambda_1(\sigma_1)}^{\lambda_2(\sigma_2)} \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\frac{\mathrm{d}\eta}{\mathrm{d}\lambda}}\frac{\mathrm{d}\lambda}{\mathrm{d}\sigma}\mathrm{d}\sigma \\ &= \int_{\eta_1}^{\eta_2} \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}}\mathrm{d}\sigma\,, \end{aligned} \tag{B.11}$$

and therefore it is reparametrisation invariant. Note that the square root is important. Any other power would not be reparametrisation invariant!

B.1.4 Exercise 2.4

Show that it is always possible to find a parametrisation of a timelike curve so that it satisfies (2.17), and moreover that it is unique up to constant shifts.

Let us take a time-like curve. It must therefore satisfy

$$-f^2 = \eta_{\mu\nu}\frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu(\lambda)}{\mathrm{d}\lambda}\,. \tag{B.12}$$

We can make a change of reparametrisation which leads to

$$\begin{aligned} -f(x)^2 &= \eta_{\mu\nu}\frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu(\lambda)}{\mathrm{d}\lambda} \\ &= \eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda} \end{aligned} \tag{B.13}$$

Bringing the factor of $\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}$ to the other side we have

$$-f(x)^2\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\sigma}\right]^2 = \eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\,. \tag{B.14}$$

We see that if we solve

$$f(x(\lambda(\sigma)))\frac{\mathrm{d}\lambda}{\mathrm{d}\sigma} = 1\,, \tag{B.15}$$

then we have the correct normalised curve.

This is just a first order ODE which has a unique solution given a boundary condition. The boundary condition only fixes σ up to a constant which is the constant shift.

To see that σ is just the proper time we need to substitute this into (2.16) and everything falls into place.

B.1.5 Exercise 2.5

We take

$$\Lambda_\mu{}^\nu \equiv \eta_{\mu\rho}\eta^{\nu\sigma}\Lambda^\rho{}_\sigma\,, \tag{B.16}$$

then

$$\begin{aligned} \Lambda_\mu{}^\nu\Lambda^\tau{}_\nu &= \eta_{\mu\rho}\eta^{\nu\sigma}\Lambda^\rho{}_\sigma\Lambda^\tau{}_\nu \\ &= \eta_{\mu\rho}\eta^{\rho\tau} \\ &= \delta^\tau_\mu \end{aligned} \tag{B.17}$$

B.1.6 Exercise 2.6

- If V^ν transforms as a vector then

  $$V'^\mu = \Lambda^\mu{}_\nu V^\nu\,. \tag{B.18}$$

  Transforming $\eta_{\mu\nu}V^\nu$ we would have

  $$\begin{aligned} (\eta_{\mu\nu}V^\nu)' &= \eta'_{\mu\nu}V'^\nu \\ &= \eta_{\tau\kappa}\Lambda_\mu{}^\tau\Lambda_\nu{}^\kappa\Lambda^\nu{}_\sigma V^\sigma \\ &= \Lambda_\mu{}^\tau\eta_{\tau\kappa}\delta^\kappa_\sigma V^\sigma \\ &= \Lambda_\mu{}^\tau(\eta_{\tau\sigma}V^\sigma) \end{aligned} \tag{B.19}$$

  We have used the result from exercise (2.5) in going to the third line.

- This works in the same way as the one above.

B.1.7 Exercise 2.7

Integrating equation (2.56) over the volume in a sphere of radius r, centred about the center of symmetry, and whose surface contains all of the mass one finds

$$\int_{B^3(r)} \mathrm{d}^3x \nabla \cdot F = 4\pi G \int_{B^3(r)} \mathrm{d}^3x \rho(r) = -4\pi G M\,, \tag{B.20}$$

where M is the total mass. The divergence theorem allows us to express the left-hand side as a surface integral over the sphere of radius r giving

$$\int_r \mathrm{d}\vec{A} \cdot F = -4\pi G M\,. \tag{B.21}$$

Due to the spherical symmety F can only depend on r and point only in a radial direction. This implies that the surface integral is simply

$$\int_r \mathrm{d}\vec{A} \cdot F = 4\pi r^2 |F|\,, \tag{B.22}$$

where $|F|$ is the magnitude of F. Therefore if e_r is a unit vector in the radial direction then

$$F = -\frac{GM}{r^2} e_r\,, \tag{B.23}$$

and depends only on M. We can now solve for Φ, this is of course only defined up to a constant and if we choose the constant so that Φ vanishes at ∞ then we end up with the simple expression

$$\Phi(r) = -\frac{GM}{r}\,. \tag{B.24}$$

We learn that it does not matter how the mass is distributed in a spherically symmetric configuration. It could be localised at the centre or in spherically symmetric shells. The potential only depends on the total mass.

B.2 Exercises in section 3

B.2.1 Exercise 3.1

The discrete topology is the obtained by taking a collection of all subsets of the topological space.

The usual topology is obtained by taking all open sets (a, b) and their unions. Imagine we were to change the 3rd condition in the definition of a topological space to include an infinite subcollection of I. Then we could take the subsets

$$U_n = (a, b + \tfrac{1}{n})\,. \tag{B.25}$$

Taking the infinite intersection we have

$$\cap_{n=1}^{\infty}(a, b + \tfrac{1}{n}) = (a, b]\,. \tag{B.26}$$

This implies that $\{b\}$ is in the topology and this holds for all b. Therefore the topology is reduced to the discrete one.

B.2.2 Exercise 3.2

To show that the Tangent space at p is a vector space we need to show a number of properties. It boils down to show that if $X_1, X_2 \in T_p(M)$ and $\alpha \in \mathbb{R}$ then $X_1 + \alpha X_2 \in T_p(M)$. By definition both X_i are linear transformations on functions. We then need to show $X_1 + \alpha X_2$ satisfied the Leibniz property and is therefore a tangent vector too. For functions f, g we have

$$\begin{aligned}
(X_1 + \alpha X_2)(fg) &= X_1(fg) + \alpha X_2(fg) \\
&= fX_1(g) + gX_1(f) + \alpha(gX_2(f) + fX_2(g)) \\
&= f(X_1 + \alpha X_2)(g) + g(X_1 + \alpha X_2)(f)\,,
\end{aligned} \tag{B.27}$$

hence $f(X_1 + \alpha X_2)(g) \in T_p(M)$

B.2.3 Exercise 3.3

We want to show that the flow defines a commutative group. This turns out to be simple since

$$\begin{aligned}
\sigma_t(\sigma_s(x)) &= \sigma_{t+s}(x) \\
&= \sigma_{s+t}(x) \\
&= \sigma_s(\sigma_t(x))\,,
\end{aligned} \tag{B.28}$$

and therefore it commutes. Note that we have used that $\mathbb{R}$ is commutative. Since it is 1-dimensional there are two options, $\mathbb{R}$ or S^1.

B.2.4 Exercise 3.4

These are given in problem sheet 2.

B.2.5 Exercise 3.5

Let $X, Y, Z \in \mathcal{X}(M)$ be vector fields. We have

$$\mathcal{L}_X \mathcal{L}_Y Z = \mathcal{L}_X([Y, Z]) = [X, [Y, Z]]\,. \tag{B.29}$$

The Lie bracket satisfies the Jacobi identity, which after some rearranging can be written in the form

$$[X,[Y,Z]] - [Y,[X,Y]] = -[[X,Y],Z]\,. \tag{B.30}$$

Using our expression above for the Lie derivative we have

$$\mathcal{L}_X\mathcal{L}_Y Z - \mathcal{L}_Y\mathcal{L}_X Z = \mathcal{L}_{[X,Y]}Z\,, \tag{B.31}$$

as required.

B.2.6 Exercise 3.6

We want to show:

1. That the Lie derivative satisfies:

$$\mathcal{L}_X(t_1 + t_2) = \mathcal{L}_X t_1 + \mathcal{L}_X t_2\,, \tag{B.32}$$

   where t_1 and t_2 are tensor fields of the same type.

2. That

$$\mathcal{L}_X(t_1 \otimes t_2) = \big(\mathcal{L}_X t_1\big) \otimes t_2 + t_1 \otimes \big(\mathcal{L}_X t_2\big)\,, \tag{B.33}$$

   with t_1 and t_2 tensors of arbitrary type.

For the first by the way we constructed the Lie derivative for vector fields, one forms and scalars this is obvious.

For the second we have to do a little work. We will show it in a single example, but it will be obvious how it generalises. Take a vector field Y and one-form ω and construct the tensor product $Y \otimes \omega$. Then $(Y \otimes \omega)|_{\sigma_\epsilon(x)}$ is mapped onto a tensor at x by the action of $(\sigma_{-\epsilon})^* \otimes (\sigma_\epsilon)^*$. Therefore

$$[(\sigma_{-\epsilon})^* \otimes (\sigma_\epsilon)^*](Y \otimes \omega)|_{\sigma_\epsilon(x)} = [(\sigma_{-\epsilon})_* Y \otimes (\sigma_\epsilon)^*\omega]|_x\,. \tag{B.34}$$

We therefore have

$$\begin{aligned}\mathcal{L}_X &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[[(\sigma_{-\epsilon})_* Y \otimes (\sigma_\epsilon)^*\omega]|_x - (Y \otimes \omega)|_x\right] \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[(\sigma_{-\epsilon})_* Y \otimes \{(\sigma_\epsilon)^*\omega - \omega\} + \{(\sigma_{-\epsilon})_* Y - Y\} \otimes \omega\right] \\ &= Y \otimes \mathcal{L}_X\omega + \mathcal{L}_X Y \otimes \omega\,.\end{aligned} \tag{B.35}$$

For other tensors the computation works similarly.

B.2.7 Exercise 3.7

From the properties of the wedge product show that for $\xi \in \Omega^q_p(M)$, $\eta \in \Omega^r_p(M)$ and $\omega \in \Omega^s_p(M)$ that

$$\begin{aligned} \xi \wedge \eta &= (-1)^{qr} \eta \wedge \xi \,, \\ \xi \wedge \xi &= 0 \qquad \text{if } q \text{ odd} \,, \\ (\xi \wedge \eta) \wedge \omega &= \xi \wedge (\eta \wedge \omega) \,. \end{aligned} \tag{B.36}$$

Recall that the action is given by:

$$(\omega \wedge \xi)(V_1, ..., V_{q+r}) = \frac{1}{q!r!} \sum_{\sigma \in S_{q+r}} \mathrm{sign}(\sigma) \omega\big(V_{\sigma(1)}, ..., V_{\sigma(q)}\big) \xi\big(V_{\sigma(q+1)}, ..., V_{\sigma(q+r)}\big) \,. \tag{B.37}$$

Then if we have two forms ξ, η of degree q, r respectively then

$$\begin{aligned} (\xi \wedge \eta)(V_1, ..., V_{q+r}) &= \frac{1}{q!r!} \sum_{\sigma \in S_{q+r}} \mathrm{sign}(\sigma) \xi(V_{\sigma(1)}, ..., V_{\sigma(q)}) \eta(V_{\sigma(q+1)}, ..., V_{\sigma(q+r)}) \\ &= \frac{1}{q!r!} \sum_{\sigma \in S_{q+r}} \mathrm{sign}(\sigma) \eta(V_{\sigma(q+1)}, ..., V_{\sigma(q+r)}) \xi(V_{\sigma(1)}, ..., V_{\sigma(q)}) \end{aligned} \tag{B.38}$$

We now want to move the indices on the V's to be in the canonical order. Assume $q \geq r$, the alternative is easily checked too. If we exchange an index we pick up a sign from the permutation. We can flip the indices in η with the first r indices in the ξ. This will introduce r signs.

$$\begin{aligned} (\xi \wedge \eta)(V_1, ..., V_{q+r}) &= (-1)^r \frac{1}{q!r!} \sum_{\sigma \in S_{q+r}} \mathrm{sign}(\sigma) \eta(V_{\sigma(1)}, ..., V_{\sigma(r)}) \xi(V_{\sigma(q+1)}, ..., V_{\sigma(r+q)}, ..., V_{\sigma(r+1)}, ... V_{\sigma(q)}) \\ &= (-1)^r (-1)^{r(q-r)} \frac{1}{q!r!} \sum_{\sigma \in S_{q+r}} \mathrm{sign}(\sigma) \eta(V_{\sigma(1)}, ..., V_{\sigma(r)}) \xi(V_{\sigma(r+1)}, ... V_{\sigma(r+q)}) \\ &= (-1)^{r(q-r-1)} (\eta \wedge \xi)(V_1, ..., V_{q+r}) \\ &= (-1)^{rq} (-1)^{-r(r+1)} (\eta \wedge \xi)(V_1, ..., V_{q+r}) \\ &= (-1)^{rq} (\eta \wedge \xi)(V_1, ..., V_{q+r} \,, \end{aligned} \tag{B.39}$$

where we used that $(-1)^{-r(r+1)} = 1$.

This proves the first, the second follows simply from the first since

$$\xi \wedge \xi = (-1)^{r^2} \xi \wedge \xi = (-1)^r \xi \wedge \xi = \begin{cases} \xi \wedge \xi & r \in 2\mathbb{Z} \\ -\xi \wedge \xi & r \notin 2\mathbb{Z} \end{cases} \tag{B.40}$$

It then follows for odd dimensional form it vanishes.

The associativity property follows from the associativity of functions and some messing with permutations. Let us use the shorthand $[V] = (V_1, ..., V_{q+r+s})$ then

$$((\xi \wedge \eta) \wedge \omega)[V] = \frac{1}{s!}\frac{1}{(q+r)!}\sum_{\sigma \in S_{q+r+s}} \text{sign}(\sigma)(\xi \wedge \eta)(V_{\sigma(1)}, ..., V_{\sigma(q+r)})\omega(V_{\sigma(q+r+1)}, ..., V_{\sigma(q+r+s)}) \tag{B.41}$$

We now want to simplify the permutations. We can decompose the permutation group S_{q+r+s} into residual classes of the subgroup $S_{q+r} \subset S_{q+r+s}$. These are permutations which act as the identity on the final s indices. Let us all the set of these residual classes $\mathcal{C}$ and let $R \in \mathcal{C}$ be one of these residual classes. Each $R \in \mathcal{C}$ is isomorphic to S_{q+r}. Let us pick one class R and within the class a particular permutation σ_R. Then each element $\sigma \in R$ can be decomposed as $\sigma = \sigma_R \otimes \pi$ where $\pi \in S_{q+r}$. We therefore have that the sum over S_{q+r+s} may be written as:

$$\begin{aligned} ((\xi \wedge \eta) \wedge \omega)[V] = \frac{1}{s!}\frac{1}{(q+r)!}&\sum_{R \in \mathcal{C}} \text{sign}(\sigma_R)\left[\sum_{\sigma \in R} \text{sign}(\pi)(\xi \wedge \eta)(V_{\sigma(1)}, ..., V_{\sigma(q+r)})\right] \\ &\times \omega(V_{\sigma_R(q+r+1)}, ..., V_{\sigma_R(q+r+s)}) \end{aligned} \tag{B.42}$$

The terms in the brackets are all equal because they are permutations of π from a fixed ordering given by σ_R. Since each residual class R is isomorphic to S_{q+r} there are $(q+r)!$ terms. Thus,

$$\begin{aligned} ((\xi \wedge \eta) \wedge \omega)[V] &= \frac{1}{s!}\sum_{R \in \mathcal{C}} \text{sign}(\sigma_R)(\xi \wedge \eta)(V_{\sigma_R(1)}, ..., V_{\sigma_R(q+r)})\omega(V_{\sigma_R(q+r+1)}, ..., V_{\sigma_R(q+r+s)}) \\ &= \frac{1}{q!r!s!}\sum_{R \in \mathcal{C}} \text{sign}(\sigma_R)\sum_{\tau \in S_{q+r}} \text{sign}(\tau)\xi(V_{\tau(\sigma_R(1))}, ..., V_{\tau(\sigma_R(q))}) \\ &\times \eta(V_{\tau(\sigma_R(q+1))}, ..., V_{\tau(\sigma_R(q+r))})\omega(V_{\sigma_R(q+r+1)}, ..., V_{\sigma_R(q+r+s)}) \end{aligned} \tag{B.43}$$

We need to use a similar trick to before. All permutations $\sigma \in S_{q+r+s}$ can be decomposed as $\sigma = \tau \circ \sigma_R$ and since τ acts on the last s indices as the identity we have $\sigma = \sigma_R$ for these. Thus we obtain:

$$\begin{aligned} ((\xi \wedge \eta) \wedge \omega)[V] = \frac{1}{q!r!s!}&\sum_{\sigma \in S_{q+r+s}} \text{sign}(\sigma)\xi(V_{\sigma(1)}, ..., V_{\sigma(q)}) \\ &\times \eta(V_{\sigma(q+1)}, ..., V_{\sigma(q+r)})\omega(V_{\sigma(q+r+1)}, ..., V_{\sigma(q+r+s)}) \end{aligned} \tag{B.44}$$

We can play the same game starting from $(\xi \wedge (\eta \wedge \omega))[V]$. It is clear though that the difference is that we decompose in terms of the residual class of $S_{r+s} \subset S_{q+r+s}$ instead and we end up with the exact same result. We have therefore shown associativity.

B.2.8 Exercise 3.8

Show that for $\xi \in \Omega_p^q(M)$, $\eta \in \Omega_p^r(M)$ we have

$$\mathrm{d}(\xi \wedge \eta) = \mathrm{d}\xi \wedge \eta + (-1)^q \xi \wedge \mathrm{d}\eta\,. \tag{B.45}$$

We can do this in coordinate free notation or using a coordinate basis, we choose the latter since it is simpler though the former is equally doable.

Let

$$\xi = \frac{1}{q!}\xi_{\mu_1 ...,\mu_q}\mathrm{d}x^{\mu_1} \wedge .. \wedge \mathrm{d}x^{\mu_q}\,, \qquad \eta = \frac{1}{r!}\eta_{\mu_1 ...\mu_r}\mathrm{d}x^{\mu_1} \wedge ... \wedge \mathrm{d}x^{\mu_r}\,, \tag{B.46}$$

then

$$\begin{aligned}
\mathrm{d}(\xi \wedge \eta) &= \mathrm{d}(\xi_{\mu_1 ...\mu_q}\eta_{\nu_1 ...\nu_r}\mathrm{d}x^{\mu_1} \wedge ..\mathrm{d}x^{\mu_q} \wedge \mathrm{d}x^{\nu_1} \wedge .... \wedge \mathrm{d}x^{\nu_r}) \\
&= \frac{1}{q!r!}\frac{\partial}{\partial x^\sigma}(\xi_{\mu_1 ...\mu_q}\eta_{\nu_1 ...\nu_r})\mathrm{d}x^\sigma \wedge \mathrm{d}x^{\mu_1} \wedge ..\mathrm{d}x^{\mu_q} \wedge \mathrm{d}x^{\nu_1} \wedge .... \wedge \mathrm{d}x^{\nu_r}) \\
&= \frac{1}{q!r!}\left(\frac{\partial}{\partial x^\sigma}\xi_{\mu_1 ...\mu_q}\right)\eta_{\nu_1 ...\nu_r}\mathrm{d}x^\sigma \wedge \mathrm{d}x^{\mu_1} \wedge ..\mathrm{d}x^{\mu_q} \wedge \mathrm{d}x^{\nu_1} \wedge .... \wedge \mathrm{d}x^{\nu_r}) \\
&+ (-1)^q \frac{1}{q!r!}\xi_{\mu_1 ...\mu_q}\left(\frac{\partial}{\partial x^\sigma}\eta_{\nu_1 ...\nu_r}\right)\mathrm{d}x^{\mu_1} \wedge ..\mathrm{d}x^{\mu_q} \wedge \mathrm{d}x^\sigma \wedge \mathrm{d}x^{\nu_1} \wedge .... \wedge \mathrm{d}x^{\nu_r}) \\
&= \mathrm{d}\xi \wedge \eta + (-1)^q \xi \wedge \mathrm{d}\eta\,.
\end{aligned} \tag{B.47}$$

B.2.9 Exercise 3.9

$$\begin{aligned}
\mathrm{i}_X^2 &= 0\,, \\
\mathrm{i}_X(\omega \wedge \eta) &= \mathrm{i}_X\omega \wedge \eta + (-1)^r \omega \wedge \mathrm{i}_X\eta\,, \\
\mathrm{i}_{[X,Y]}\omega &= X(\mathrm{i}_Y\omega) - Y(\mathrm{i}_X\omega)\,, \\
\mathcal{L}_X \mathrm{i}_X\omega &= \mathrm{i}_X\mathcal{L}_X\omega\,.
\end{aligned} \tag{B.48}$$

1. For the first if we contract into a q-form ω we have

$$\mathrm{i}_X^2\omega = \frac{1}{(q-2)!}X^{\mu_1}X^{\mu_2}\omega_{\mu_1\mu_2\nu_1 ...\nu_{q-2}}\mathrm{d}x^{\nu_1} \wedge ... \wedge \mathrm{d}x^{\nu_{q-2}} \tag{B.49}$$

   Since the two X's are symmetric and the two indices of the p-form are antisymmetric this vanishes.

2. In components we have:

$$\begin{aligned}
\mathrm{i}_{V_1}(\omega \wedge \eta)(V_2, ..., V_{q+r}) &= (\omega \wedge \eta)(V_1, ..., V_{q+r}) \\
&= \frac{1}{q!r!}\sum_{\sigma \in S_{q+r}} \mathrm{sign}(\sigma)\omega(V_{\sigma(1)}, ..., V_{\sigma(q)})\eta(V_{\sigma(q+1)}, ..., V_{\sigma(q+r)})
\end{aligned} \tag{B.50}$$

We now want to identify the ones with the V_1 and the ones without.

$$\begin{aligned}
\mathrm{i}_{V_1}(\omega\wedge\eta)(V_2,....,V_{q+r}) = \frac{1}{q!r!}\sum_{\sigma\in S_{q+r-1}} \Big[& q\omega(V_1V_{\sigma(2)},...,V_{\sigma(q)})\eta(V_{\sigma(q+1)},...,V_{\sigma(q+r)}) \\
&+ (-1)^q r\omega(V_{\sigma(2)},...,V_{\sigma(q+1)})\eta(V_1,V_{\sigma(q+2)},...,V_{\sigma(q+r)})\Big] \\
&= (\mathrm{i}_{V_1}\omega)\wedge\eta + (-1)^q\omega\wedge(\mathrm{i}_{V_1}\eta)\,.
\end{aligned} \tag{B.51}$$

3. Using that on a form
$$\mathcal{L}_X\omega = (\mathrm{i}_X\mathrm{d} + \mathrm{d}\mathrm{i}_X)\omega\,, \tag{B.52}$$
we have
$$\begin{aligned}
\mathcal{L}_X\mathrm{i}_X\omega &= (\mathrm{i}_X\mathrm{d} + \mathrm{d}\mathrm{i}_X)\mathrm{i}_X\omega \\
&= \mathrm{i}_X(\mathrm{d}\mathrm{i}_X\omega) \\
&= \mathrm{i}_X\mathcal{L}_X\omega
\end{aligned} \tag{B.53}$$

B.2.10 Exercise 3.10

Stoke's theorem states that in 3d, given a vector field $\vec{F}$ and a smooth oriented surface in $\mathbb{R}^3$ with boundary $\partial\Sigma = B$ then

$$\int_\Sigma(\nabla\times\vec{F})\cdot\mathrm{d}\Sigma = \oint_{\partial\Sigma}\vec{F}\cdot\mathrm{d}\Gamma \tag{B.54}$$

We need to identify the vector field F as a 1-form and the curl as a two form given by the exterior derivative of F. We have that

$$\vec{F}\cdot\mathrm{d}\Gamma = F_x\mathrm{d}x + F_y\mathrm{d}y + F_z\mathrm{d}z \equiv F\,, \tag{B.55}$$

similarly

$$\mathrm{d}F = (\partial_xF_y - \partial_yF_x)\mathrm{d}x\wedge\mathrm{d}y + (\partial_yF_z - \partial_zF_y)\mathrm{d}y\wedge\mathrm{d}z + (\partial_zF_x - \partial_xF_z)\mathrm{d}z\wedge\mathrm{d}x \tag{B.56}$$

this is nothing other than
$$\mathrm{d}F = (\nabla\times\vec{F})\cdot\mathrm{d}S\,, \tag{B.57}$$

and therefore putting everything together we have the claimed result.

C Exercise to section 4

C.0.1 Exercise 4.1

We take the embedding of the sphere into $\mathbb{R}^3$ given via equation (4.11). Plugging this parametrisation into the metric on $\mathbb{R}^3$

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\,, \tag{C.1}$$

we find

$$\begin{aligned} \mathrm{d}s^2(S^2) &= [\mathrm{d}(\sin\theta\cos\phi))]^2 + [\mathrm{d}(\sin\theta\sin\phi)]^2 + [\mathrm{d}\cos\theta]^2 \\ &= \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\,. \end{aligned} \tag{C.2}$$

C.0.2 Exercise 4.2

1. Let us take the torsion tensor defined via (4.58). Then

$$\begin{aligned} T(\omega; fX, Y) &= \omega(\nabla_{fX}Y - \nabla_Y(fX) - [fX, Y]) \\ &= \omega(f\nabla_X Y - Y[f]X - f[X,Y] + Y[f]X) \\ &= \omega(f(\nabla_X Y - \nabla_Y X - [X,Y])) \\ &= f\omega(\nabla_X Y - \nabla_Y X - [X,Y]) \\ &= fT(\omega; X, Y)\,. \end{aligned} \tag{C.3}$$

For f on the other arguments it follows identically to above on Y and trivially on ω. We also need to show that if we take sums of two then it all becomes linear as well.

$$\begin{aligned} T(\omega; X_1 + X_2, Y) &= \omega(\nabla_{X_1+X_2}Y - \nabla_Y(X_1 + X_2) - [X_1 + X_2, Y]) \\ &= \omega(\nabla_{X_1}Y + \nabla_{X_2}Y - \nabla_Y X_1 - \nabla_Y X_2 - [X_1, Y] - [X_2, Y]) \\ &= \omega(\nabla_{X_1}Y - \nabla_Y X_1 - [X_1, Y]) + \omega(\nabla_{X_2}Y - \nabla_Y X_2 - [X_2, Y]) \\ &= T(\omega; X_1, Y) + T(\omega; X_2, Y)\,, \end{aligned} \tag{C.4}$$

with a similar result for other arguments. This shows that T is a multi-linear map in all indices. Note that we have used the established results that ω is a linear map, ∇_X a linear map and properties of the connection to show this.

2. We now want to show that this is a tensor by checking that it transforms correctly in coordinate components. In components we have that

$$T^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu}\,, \tag{C.5}$$

we have already seen the transformation of the connection coefficients under a coordinate transformation in (4.51),

$$\tilde{\Gamma}^{\mu}{}_{\nu\rho} = (\Lambda^{-1})^{\mu}{}_{\kappa}\Lambda^{\sigma}{}_{\rho}\Lambda^{\tau}{}_{\nu}\Gamma^{\kappa}{}_{\sigma\tau} + (\Lambda^{-1})^{\mu}{}_{\kappa}\Lambda^{\sigma}{}_{\rho}\partial_{\sigma}\Lambda^{\kappa}{}_{\nu} \tag{C.6}$$

We therefore have

$$\begin{aligned}
\tilde{T}^{\rho}{}_{\mu\nu} &= \tilde{\Gamma}^{\rho}{}_{\mu\nu} - \tilde{\Gamma}^{\rho}{}_{\nu\mu} \\
&= (\Lambda^{-1})^{\rho}{}_{\kappa}\Lambda^{\sigma}{}_{\nu}\Lambda^{\tau}{}_{\mu}\Gamma^{\kappa}{}_{\sigma\tau} + (\Lambda^{-1})^{\rho}{}_{\kappa}\Lambda^{\sigma}{}_{\nu}\partial_{\sigma}\Lambda^{\kappa}{}_{\mu} \\
&- \left[(\Lambda^{-1})^{\rho}{}_{\kappa}\Lambda^{\sigma}{}_{\nu}\Lambda^{\tau}{}_{\mu}\Gamma^{\kappa}{}_{\tau\sigma} + (\Lambda^{-1})^{\rho}{}_{\kappa}\Lambda^{\sigma}{}_{\mu}\partial_{\sigma}\Lambda^{\kappa}{}_{\nu}\right] \\
&= (\Lambda^{-1})^{\rho}{}_{\kappa}\Lambda^{\sigma}{}_{\nu}\Lambda^{\tau}{}_{\mu}(\Gamma^{\kappa}{}_{\tau\sigma} - \Gamma^{\kappa}{}_{\sigma\tau}) - \frac{\partial x^{\sigma}}{\partial y^{[\nu}}\frac{\partial x^{\kappa}}{\partial y^{\mu]}}\frac{\partial^2 y^{\rho}}{\partial x^{\sigma}\partial x^{\kappa}} \\
&(\Lambda^{-1})^{\rho}{}_{\kappa}\Lambda^{\sigma}{}_{\nu}\Lambda^{\tau}{}_{\mu}T^{\kappa}{}_{\tau\sigma}
\end{aligned} \tag{C.7}$$

which is as claimed. We have used the coordinate expression for the Λ's to simplify the derivative term and then that the contraction of anti-symmetric indices with symmetric indices vanishes.

C.0.3 Exercise 4.3

Covered in problem sheet 3.

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