C5.11 Solutions

1. (a) $\mu \frac{\partial I}{\partial z} = -\rho\kappa(I - B)$ $\qquad \mu = \cos\theta \quad d\mu = -\sin\theta\, d\theta.$

$$I_+ = \int_0^{2\pi}\int_0^{\pi/2} I \sin\theta\, d\theta\, d\phi = 2\pi \int_0^1 I\, d\mu$$

$$I_- = \int_0^{2\pi}\int_{\pi/2}^{\pi} I \sin\theta\, d\theta\, d\phi = 2\pi \int_{-1}^0 I\, d\mu$$ (2)

Approximate $\int_0^1 \mu I\, d\mu \approx \frac{1}{2} I_+$ and $\int_{-1}^0 \mu I\, d\mu \approx -\frac{1}{2} I_-$ B.

Integrate radiative transfer equation ($\int_0^1 d\mu$ & $\int_{-1}^0 d\mu$)

$$\Rightarrow \quad \frac{1}{2}\frac{\partial I_+}{\partial z} = -\rho\kappa(I_+ - B) \qquad \text{with } B = \frac{\sigma T^4}{\pi}.$$ (3)

$$-\frac{1}{2}\frac{\partial I_-}{\partial z} = -\rho\kappa(I_- - B)$$

$\pi I_+ = \sigma T_S^4$ at $z = 0$ is Stefan Boltzmann law for radiative emission from ground.
(T_s = surface temp) (3)

$I_- \to 0$ as $z \to \infty$ means no incoming longwave radiation from space

(b) $\frac{d\tau}{dz} = -\rho\kappa \qquad \tau \to 0$ as $z \to \infty \qquad \Rightarrow \quad \tau = \int_z^\infty \rho\kappa\, dz$

$\frac{1}{2}\frac{dI_+}{d\tau} = I_+ - \frac{\sigma T^4}{\pi}$ with $I_+ = \frac{\sigma T_s^4}{\pi}$ at $\tau = \tau_s = \int_0^\infty \rho\kappa\, dz.$

$$\left(\frac{1}{2}\frac{dI_-}{d\tau} = I_- - \frac{\sigma T^4}{\pi} \right)$$ (3)

integrating factor $e^{-2\tau}$ $\Rightarrow \quad \frac{d}{d\tau}(I_+ e^{-2\tau}) = -\frac{2\sigma T^4}{\pi} e^{-2\tau}$

so $\quad I_+ e^{-2\tau} - \frac{\sigma T_s^4}{\pi} e^{-2\tau_s} = -\int_{\tau_s}^{\tau} \frac{2\sigma T^4}{\pi} e^{-2\tau'}\, d\tau'$ N. (straightforward)

$$\Rightarrow \quad \boxed{I_+ = \frac{\sigma T_s^4}{\pi} e^{-2(\tau_s - \tau)} + \int_\tau^{\tau_s} \frac{2\sigma T^4}{\pi} e^{-2(\tau' - \tau)}\, d\tau'}$$ (3)

$$\Rightarrow \boxed{\gamma = \frac{\pi I_+(0)}{\sigma T_s^4} = e^{-2\tau_s} + \int_0^{\tau_s} 2\left(\frac{T}{T_s}\right)^4 e^{-2\tau'}\, d\tau'}$$ (1)

)
$$\rho c_p \frac{dT}{dz} - \frac{dp}{dz} = 0. \qquad p = \rho \frac{RT}{M}$$

$$\Rightarrow \quad \frac{dT}{dp} = \frac{1}{\rho c_p} = \frac{RT}{c_p p M}$$

B.

$$\Rightarrow \quad \ln T = \frac{R}{c_p M} \ln p + \text{constant}$$

$$\frac{T}{T_s} = \left(\frac{p}{p_s}\right)^{R/c_p M}$$ (4)

d) If pressure hydrostatic then $\frac{dp}{dz} = -\rho g$ & $\frac{d\tau}{dz} = -\rho \kappa$. $\Rightarrow \frac{dp}{d\tau} = \frac{g}{\kappa}$ so p & τ linearly related.

$p = \frac{g}{\kappa}\tau$ (since p and τ both 0 as $z \to \infty$).

$$\Rightarrow \frac{T}{T_s} = \left(\frac{\tau}{\tau_s}\right)^{R/c_p M}$$ (2)

[If $\tau_s \ll 1$ then integral term in γ is small and $\gamma \simeq e^{-2\tau_s} \simeq 1$. ~~(1)~~]

N

If $\tau_s \gg 1$, then first term $e^{-2\tau_s}$ is exponentially small, and integral is approximately

$$\gamma \simeq \int_0^\infty 2\left(\frac{\tau}{\tau_s}\right)^{4R/c_p M} e^{-2\tau}\, d\tau \qquad \tau \to \tfrac{1}{2}\hat{\tau}$$

$$= \frac{1}{(2\tau_s)^{4R/c_p M}} \underbrace{\int_0^\infty \hat{\tau}^{4R/c_p M} e^{-\hat{\tau}}\, d\hat{\tau}}_{\Gamma\left(1 + \frac{4R}{c_p M}\right)}$$ ~~(3)~~. (4)

2. (a) $\dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(hu) = r$ conservation of mass

$\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = gS - \dfrac{fu^2}{h} - g\dfrac{\partial h}{\partial x}$ conservation of momentum

gravitational force (S = slope) turbulent drag. pressure force (hydrostatic pressure)

(3)

B

Negative values of r might represent seepage into the underlying bed or evaporation. (2). (S)

Scale $x \sim L$, $hu \sim R_0 L = Q$, $gS \sim \dfrac{fu^2}{h}$, $t \sim \dfrac{x}{u}$

$$\Rightarrow \left(\frac{gSR_0L}{f}\right)^{1/3} = [u]. \qquad [h] = \frac{R_0 L}{[u]}. \qquad [t] = \frac{L}{[u]}.$$

Then $\dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(hu) = r$ $\qquad \delta = \dfrac{[h]}{SL}$

$\delta F^2\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x}\right) = 1 - \dfrac{u^2}{h} - \delta\dfrac{\partial h}{\partial x}$ $\qquad F^2 = \dfrac{[u]^2}{g[h]}.$ (4)

(b) If $F \ll 1$, $\delta \ll 1$, second eqn $\Rightarrow$ $u = h^{1/2}$

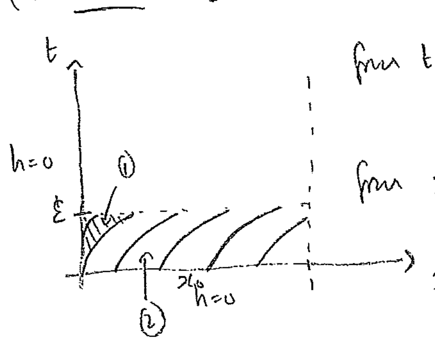
$$\Rightarrow \frac{\partial h}{\partial t} + \frac{3}{2}h^{1/2}\frac{\partial h}{\partial x} = r$$

B.

$h = 0$ at $t = 0$. $(x > 0)$ Initial condition (4).

$h = 0$ at $x = 0$ $(t > 0)$. Boundary condition

(c) $\underline{0 < t < \varepsilon}$ Characteristics have $\dot{t} = 1$ $\quad \dot{x} = \frac{3}{2}h^{1/2}$ $\quad \dot{h} = \frac{1}{\varepsilon}$



from $t=0$: $t = 0$ $\quad x = x_0$ ← parameter $\quad h = 0$ are initial data

$\Rightarrow$ (1) $\boxed{h = \frac{1}{\varepsilon}t.}$ $\quad x = x_0 + \dfrac{1}{\varepsilon^{1/2}}t^{3/2}$ $\quad$ for $x > \varepsilon^{-1/2}t^{3/2}$ (2

from $x=0$: $t = t_0$ ← parameter. $\quad x = 0$ $\quad h = 0$

$\Rightarrow$ $h = \frac{1}{\varepsilon}(t - t_0)$ $\quad x = \dfrac{1}{\varepsilon^{1/2}}(t-t_0)^{3/2}$ $\quad$ (2) $\Rightarrow \boxed{h = \varepsilon^{-2/3}x^{2/3}}$ (2

for $x < \varepsilon^{-1/2}t^{3/2}$

At $t = \varepsilon$, $h \begin{cases} 1 & x > \varepsilon \\ \left(\dfrac{x}{\varepsilon}\right)^{2/3} & x < \varepsilon. \end{cases}$

S

$t > \varepsilon$ Characteristics now $\dot t = 1 \quad \dot x = \frac{3}{2}h^{1/2} \quad \dot h = -1.$

parameter.

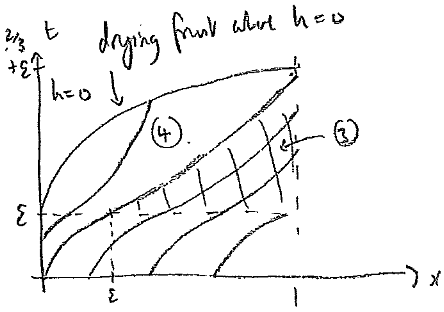
from $x > \varepsilon$, have $t = \varepsilon,\ x = x_1 \quad h = 1$

$$\Rightarrow (3)\ \boxed{h = 1 - t + \varepsilon} \quad x = x_1 + \left(1 - (1 - t + \varepsilon)^{3/2}\right).$$

h reaches zero when $t = 1 + \varepsilon$. (2) ($\Rightarrow x > 1$, so irrelevant).

from $x < \varepsilon$, $t = \varepsilon$, $x = x_1$, $h = \left(\frac{x_1}{\varepsilon}\right)^{2/3}$

$$\Rightarrow (4)\ \boxed{h = \left(\frac{x_1}{\varepsilon}\right)^{2/3} - t + \varepsilon \qquad x = x_1 + \left(\frac{x_1}{\varepsilon} - \left(\left(\frac{x_1}{\varepsilon}\right)^{2/3} - t + \varepsilon\right)^{3/2}\right)}$$



$\hookrightarrow$ h reaches zero when $t = \varepsilon + \left(\frac{x_1}{\varepsilon}\right)^{2/3}$. (for $\varepsilon < t < \varepsilon + 1$)

so (eliminate x_1) $x_1 = \varepsilon(t-\varepsilon)^{3/2}$

$$\Rightarrow x = \varepsilon(t-\varepsilon)^{3/2}\left(1 + \frac{1}{\varepsilon}\right)$$
$$= (1+\varepsilon)(t-\varepsilon)^{3/2}$$

(reaches $x=1$ at $t = \varepsilon + (1+\varepsilon)^{-2}$) (2)

is all from solution (3).

At $x = 1$, $h = \begin{cases} \frac{t}{\varepsilon} & 0 < t < \varepsilon \\ 1 - t + \varepsilon & \varepsilon < t < 1 + \varepsilon - \varepsilon^{2/3} \\ ? & 1 + \varepsilon - \varepsilon^{2/3} < t < \varepsilon + (1+\varepsilon)^{-2/3} \\ 0 & t > (1+\varepsilon)^{-2/3} + \varepsilon \end{cases}$ so $h = \begin{cases} 1 - t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$ in limit $\varepsilon \to 0$.

$$\Rightarrow \boxed{Q = h^{3/2} = \begin{cases} (1-t)^{3/2} & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}}$$ (2).

3. (a) $0 = -p_x + \tau_z$ $\quad$ $p = \tau = 0$ at $z = s$.

$0 = -p_z - \rho g$.

$u_z = 2A\tau^n$ $\quad$ $u = C\tau^m$ at $z = b$.

$\boxed{\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = a}$ $\quad$ $q = \int_b^s u\,dz$ $\quad$ (2)

vertical momentum $\Rightarrow$ $p = \rho g(s-z)$.

horizontal momentum $\Rightarrow$ $\tau = -\rho g s_x (s-z)$ $\rightarrow$ $\tau_b = -\rho g s_x h$ $\quad$ $h = s-b$. $\quad$ (4)

flow law $\Rightarrow$ $u = u_b + \int_b^z 2A(-\rho g s_x)^n (s-z)^n\,dz$

$\boxed{m=1}$

$$= C(-\rho g s_x h)^m + \frac{2A(-\rho g s_x)^n}{n+1}\left[h^{n+1} - (s-z)^{n+1}\right] \qquad B.$$

$$q = \int_b^s u\,dz = C(\rho g)^m h^{m+1}(-s_x)^m + \frac{2A(\rho g)^n(-s_x)^n}{n+1}\left(h^{n+2} - \frac{1}{n+2}h^{n+2}\right)$$

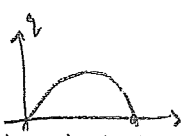
$$= C(\rho g)^m h^{m+1}(-s_x)^m + \frac{2A(\rho g)^n}{n+2}(-s_x)^n h^{n+2} \qquad (4)$$

$$\Rightarrow \boxed{\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left[C(\rho g)^m h^{m+1}(-s_x)^m + \frac{2A(\rho g)^n}{n+2} h^{n+2}(-s_x)^n\right] = a.}$$

non-dimensionalise with $x \sim [x]$, $a \sim [a]$, choose $[h]$ so that $\frac{C(\rho g)^m [h]^{2m+1}}{[x]^{m+1}} = [a]$.

and $[t]$ so that $\frac{[h]}{[t]} = [a]$, $\quad$ $\boxed{\alpha = \frac{2A(\rho g)^{n-m}}{(n+2)C}\,\frac{[h]^{2(n-m)}}{[x]^{(n-m)}}}$ $\quad$ (2)

(b) $b = 0$, $a = 1-x$. $\quad$ $q = \int_0^x a = x - \frac{1}{2}x^2$ $\quad$ so



(2)

$q = 0$ at $x = 2$ $\Rightarrow$ ice sheet extent is 2.

If $\alpha \ll 1$, $m = 1$ $\quad$ $q = -h^2 h_x = -\frac{1}{3}(h^3)_x$ $\quad$ so $\quad$ $h^3 = \int_x^2 3\left(x - \frac{1}{2}x^2\right)dx$ $\quad$ ($h = 0$ at $x = 2$)

At $x = 0$; $\quad$ $h^3 = \left[\frac{3}{2}x^2 - \frac{1}{2}x^3\right]_0^2 = 6 - 4 = 2$ $\Rightarrow$ $h = 2^{1/3}$ $\quad$!

$\uparrow$

max thickness $\quad$ (4).

(c) Archimedes $\Rightarrow$ mantle exerts upwards force equal to the weight of the amount displaced.

If $b=0$ in absence of ice, this force is $-\rho_m g b$. (per unit area).

This must balance weight of the ice. $\rho g h$, so we need $\boxed{-\rho_m b = \rho h.}$ (2).

Flux becomes $q = -h^2 s_x = -h^2(b+h)_x = -\left(1-\frac{\rho}{\rho_m}\right)h^2 h_x$.

So now $\underbrace{\left(1-\frac{\rho}{\rho_m}\right)}_{<1} h^3 = \int_x^2 3\left(x-\frac{1}{2}x^2\right)dx$. i.e. h is increased. (3)

At $x=0$ $\left(1-\frac{\rho}{\rho_m}\right)h^3 = 2$. $\Rightarrow$ $h = 2^{1/3}\left(1-\frac{\rho}{\rho_m}\right)^{-1/3}$

If $\rho_m \approx 3\rho$ then $1-\frac{\rho}{\rho_m} \approx \frac{2}{3}$ so $h \approx 2^{1/3}\left(\frac{2}{3}\right)^{-1/3} = 3^{1/3}$ (2).