PS1 Q1

(a) $\{a_n\}_{n\in\mathbb{N}_0}$ is an asymptotic sequence as $\varepsilon \to 0$ iff

$$\frac{a_{n+1}(\varepsilon)}{a_n(\varepsilon)} \to 0 \text{ or/ } a_{n+1}(\varepsilon) = o(a_n(\varepsilon)) \quad \forall n\in\mathbb{N}_0.$$

(b) $\sum_{n=0}^{\infty} a_n(\varepsilon)$ is an asymptotic expansion of a function $f(\varepsilon)$ as $\varepsilon\to 0$

iff $$\frac{f(\varepsilon) - \sum_{n=0}^{N} a_n(\varepsilon)}{a_N(\varepsilon)} \to 0 \text{ or/ } f(\varepsilon) - \sum_{n=0}^{N} a_n(\varepsilon) = o(a_N(\varepsilon)) \quad \forall n\in\mathbb{N}_0$$

(c) $$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} \text{ for } |x|<1$$

$$\Rightarrow \log(1-\log\varepsilon) = \log\left(1+\log\left(\tfrac{1}{\varepsilon}\right)\right)$$
$$= \log\left(\log\left(\tfrac{1}{\varepsilon}\right)\right) + \log\left(1+\frac{1}{\log(1/\varepsilon)}\right)$$
$$\sim \log\left(\log\left(\tfrac{1}{\varepsilon}\right)\right) + \sum_{n=1}^{\infty}\frac{(-1)^n}{n\log(1/\varepsilon)^n} \quad \text{as } \varepsilon\to 0^+$$

$\therefore$ $a_0 = \log(\log(\frac{1}{\varepsilon}))$ and $a_n = \dfrac{(-1)^n}{n\log(1/\varepsilon)^n}$ for $n\in\mathbb{N}$.

(d) $$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n \text{ for } |x|<1$$

$$\Rightarrow \exp\left(\frac{-1}{\varepsilon^2+\varepsilon^3}\right) = \exp\left(-\frac{1}{\varepsilon^2}\sum_{n=0}^{\infty}(-1)^n\varepsilon^n\right)$$
$$= \exp\left(-\frac{1}{\varepsilon^2}\left(1-\varepsilon+\varepsilon^2-\varepsilon^3+\dots\right)\right)$$
$$= \exp\left(-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}-1+\sum_{n=1}^{\infty}(-1)^{n+1}\varepsilon^n\right)$$
$$= \exp\left(-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}-1\right)\exp\left(\sum_{n=1}^{\infty}(-1)^{n+1}\varepsilon^n\right)$$
$$= \exp\left(-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}-1\right)\sum_{m=0}^{\infty}\frac{1}{m!}\left(\sum_{n=1}^{\infty}(-1)^{n+1}\varepsilon^n\right)^m \quad \text{for } |\varepsilon|<1$$

$\therefore$ $a_n = b_n\varepsilon^n\left(-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}-1\right)$ for $n\in\mathbb{N}_0$ where $b_n = O(1)$ as $\varepsilon\to 0$.

(a) $x^3 + x - \varepsilon = 0$ as $\varepsilon \to 0$

Iterative method:

$\varepsilon = 0 \Rightarrow x^3 + x = 0 \Rightarrow x = 0, \pm i$

For the root near $x=0$: rewrite as $x = \varepsilon - x^3$ ie $g(x;\varepsilon) = \varepsilon - x^3$

so that $x_{n+1} = g(x_n;\varepsilon) = \varepsilon - x_n^3$, with $x_0 = 0$.

Then
$$x_1 = \varepsilon$$
$$x_2 = \varepsilon - \varepsilon^3$$
$$x_3 = \varepsilon - (\varepsilon - \varepsilon^3)^3 \sim \varepsilon - \varepsilon^3 + 3\varepsilon^5 + \ldots$$
$$x_4 = \varepsilon - (\varepsilon - \varepsilon^3 + 3\varepsilon^5 + \ldots)^3 \sim \varepsilon - \varepsilon^3 + 3\varepsilon^5 + \ldots$$

no change

$\therefore$ $x \sim \varepsilon - \varepsilon^3 + 3\varepsilon^5 + \ldots$ as $\varepsilon \to 0$

For the roots close to $x = \pm i$: rewrite as $x^2 = \frac{\varepsilon}{x} - 1 \Rightarrow x = \pm i\sqrt{1 - \frac{\varepsilon}{x}}$

ie $g(x;\varepsilon) = \pm i\left(1 - \frac{\varepsilon}{x}\right)^{\frac{1}{2}}$ so that $x_{n+1} = \pm i\left(1 - \frac{\varepsilon}{x_n}\right)^{\frac{1}{2}}$ with $x_0 = \pm i$.

Then $x_1 = \pm i\left(1 \mp \frac{\varepsilon}{i}\right)^{\frac{1}{2}} \sim \pm i\left(1 \mp \frac{\varepsilon}{2i} \pm \ldots\right) \sim \pm i - \frac{\varepsilon}{2} + \ldots$

$$x_2 = \pm i\left(1 - \frac{\varepsilon}{\pm i \mp \frac{\varepsilon}{2} + \ldots}\right)^{\frac{1}{2}} \sim \pm i - \frac{\varepsilon}{2} \pm \frac{3i\varepsilon^2}{8} + \ldots$$

$$x_3 = \pm i\left(1 - \frac{\varepsilon}{\pm i - \frac{\varepsilon}{2} \pm \frac{3i\varepsilon^2}{8} + \ldots}\right)^{\frac{1}{2}} \sim \pm i - \frac{\varepsilon}{2} + \frac{3i\varepsilon^2}{8} + \ldots$$

$\therefore$ $x \sim \pm i - \frac{\varepsilon}{2} \pm \frac{3i\varepsilon^2}{8} + \ldots$ as $\varepsilon \to 0$

Expansion method $x \sim x_0 + \varepsilon x_1 + \cdots$ as $\varepsilon \to 0$

Substitute into $x^3 + x - \varepsilon = 0$:

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots)^3 + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots) - \varepsilon = 0$$

$O(\varepsilon^0)$: $\quad x_0^3 + x_0 = 0 \quad \Rightarrow \quad x_0 = 0, +i, -i$

$O(\varepsilon^1)$: $\quad 3x_0^2 x_1 + x_1 - 1 = 0 \quad \Rightarrow \quad x_1 = \dfrac{1}{3x_0^2+1} = 1, -\frac{1}{2}, -\frac{1}{2}$

$O(\varepsilon^2)$: $\quad 3x_0 x_1^2 + 3x_0^2 x_2 + x_2 = 0 \quad \Rightarrow \quad x_2 = \dfrac{-3x_0 x_1^2}{3x_0^2+1} = 0, \frac{3i}{8}, -\frac{3i}{8}$

Hence for the roots close to $x = \pm i$, we have

$$x \sim \pm i - \frac{\varepsilon}{2} \pm \frac{3i\varepsilon^2}{8} + \ldots \quad \text{as } \varepsilon \to 0.$$

For the root close to $x = 0$ we need to go to higher order:

$O(\varepsilon^3)$: $\quad x_1^3 + x_3 = 0 \quad \Rightarrow \quad x_3 = -x_1^3 = -1$

$O(\varepsilon^4)$: $\quad x_4 = 0 \quad \Rightarrow \quad x_4 = 0$

$O(\varepsilon^5)$: $\quad 3x_1^2 x_3 + x_5 = 0 \Rightarrow x_5 = -3x_1^2 x_3 = 3$

Hence, $\quad x \sim \varepsilon - \varepsilon^3 + 3\varepsilon^5 + \ldots \quad \text{as } \varepsilon \to 0.$

(b) $\varepsilon^3 x^2 + \varepsilon x + 1 = 0$

Analytic solution $x = \dfrac{-\varepsilon \pm \sqrt{\varepsilon^2 - 4\varepsilon^3}}{2\varepsilon^3} = \dfrac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon^2}$

Expand for $|\varepsilon| < \frac{1}{4}$ to give $x = \frac{-1}{2\varepsilon^3}\left[-1 \,(1 - 2\varepsilon - 2\varepsilon^2 - 4\varepsilon^3 + \dots)\right]$

$$\therefore \quad x \sim \begin{cases} -\frac{1}{\varepsilon} - 1 - 2\varepsilon + \dots \\ -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} + 1 + \dots \end{cases} \quad \text{as } \varepsilon \to 0.$$

(expansions converge for $|\varepsilon| < \frac{1}{4}$)

Check via rescaling and expanding: we take $x = \frac{X}{\varepsilon^3}$ so that
$X^2 + X + \varepsilon = 0$.

Expand using $X = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \dots$ as $\varepsilon \to 0$

$$(X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \dots)^2 + (X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \dots) + \varepsilon = 0$$

$O(\varepsilon^0)$: $X_0^2 + X_0 = 0 \Rightarrow X_0 = -1, 0$

$O(\varepsilon^1)$: $2X_0X_1 + X_1 + 1 = 0 \Rightarrow X_1 = 1, -1$

$O(\varepsilon^2)$: $2X_0X_2 + X_1^2 + X_2 = 0 \Rightarrow X_2 = 1, -1$

$O(\varepsilon^3)$: $2X_0X_3 + 2X_1X_2 + X_3 = 0 \Rightarrow X_3 = 2, -2$

gives the same expansions as above.

NB No other scaling can be used to regularise this problem.

(c) $\varepsilon^2 x^3 + x^2 + 2x + \varepsilon = 0$

Find roots by taking different scalings which give different terms in the dominant balance...)

Balance terms ① and ② : $x = \frac{1}{\varepsilon^2} X \Rightarrow X^3 + X^2 + 2\varepsilon^2 X + \varepsilon^5 = 0$

Let $X = X_0 + \varepsilon X_1 + \cdots$ and substitute:

$O(\varepsilon^0)$: $X_0^3 + X_0^2 = 0 \Rightarrow X_0 = 0, 0, -1$

$O(\varepsilon^1)$: $3X_0^2 X_1 + 2X_0 X_1 = 0 \Rightarrow X_1 = ?, ?, 0$

$O(\varepsilon^2)$: $3X_0^2 X_2 + 3X_0 X_1^2 + 2X_0 X_2 + 2X_0 = 0 \Rightarrow X_2 = ?, ?, 2$

Hence ∃ a root of the form $x \sim -\frac{1}{\varepsilon^2} + 2 + \cdots$ as $\varepsilon \to 0$.

To find the other roots we need a different rescaling/dominant balance.

Balance terms ② and ③ : let $x = x_0 + \varepsilon x_1 + \cdots$ and substitute

$O(\varepsilon^0)$: $x_0^2 + 2x_0 = 0 \Rightarrow x_0 = 0, -2$

$O(\varepsilon^1)$: $2x_0 x_1 + 2x_1 = -1 \Rightarrow x_1 = -\frac{1}{2}, +\frac{1}{2}$

$O(\varepsilon^2)$: $x_0^3 + 2x_0 x_2 + x_1^2 + 2x_2 = 0 \Rightarrow x_2 = -\frac{1}{8}, -\frac{31}{8}$

Hence two further roots are
$$x \sim -2 + \tfrac{1}{2}\varepsilon + \cdots$$
$$x \sim -\tfrac{1}{2}\varepsilon - \tfrac{1}{8}\varepsilon^2 + \cdots$$
as $\varepsilon \to 0$

Finally, balance terms ③ and ④ : let $x = \varepsilon X \Rightarrow \varepsilon^4 X^4 + \varepsilon X^2 + 2X + 1 = 0$

Let $X = X_0 + \varepsilon X_1 + \cdots$ and substitute

$O(\varepsilon^0)$: $2X_0 + 1 = 0 \Rightarrow X_0 = -\frac{1}{2}$

$O(\varepsilon^1)$: $X_0^2 + 2X_1 = 0 \Rightarrow X_1 = \frac{1}{8}$

Hence the final root is of the form $x \sim -\frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \cdots$ as $\varepsilon \to 0$.

(a) (i) $x^3 + \varepsilon(ax+b) = 0$ as $\varepsilon \to 0$ with $a, b = O(1)$.

For $\varepsilon = 0$ we have $x^3 = 0 \Rightarrow$ roots are small $\Rightarrow \varepsilon a x = o(\varepsilon)$ as $\varepsilon \to 0$.

$\therefore \quad x^3 \sim -b\varepsilon$ as $\varepsilon \to 0$

$\Rightarrow x \sim (\varepsilon b)^{1/3} e^{2m\pi i/3} + \ldots$ as $\varepsilon \to 0$ for $m = 0, 1, 2$.

(ii) $\varepsilon x^3 + ax + b = 0$ as $\varepsilon \to 0$ with $a, b = O(1)$

Balance terms ② and ③ $\Rightarrow x \sim \frac{-b}{a} + \ldots$ as $\varepsilon \to 0$.

Balance terms ① and ②: scale $x = \frac{1}{\sqrt{\varepsilon}} X \Rightarrow X^3 + aX + b\sqrt{\varepsilon} = 0$

$\Rightarrow \exists$ two other roots $X \sim \pm\sqrt{-a} + \ldots$ as $\varepsilon \to 0$

$\Rightarrow x \sim \pm\left(\frac{-a}{\varepsilon}\right)^{\frac{1}{2}} + \ldots$ as $\varepsilon \to 0$

1b) $\underbrace{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}-1-x+\frac{x^2}{2} = -\frac{\varepsilon}{6}$, as $\varepsilon \to 0$

$$= \sin x + \cos x = \left(x - \frac{x^3}{3!} + \cdots\right) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$

$$\therefore \left(x - \frac{x^3}{3!} + \cdots\right) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) - 1 - x + \frac{1}{2}x^2 = -\frac{\varepsilon}{6}$$

$$\left(\cancel{x} - \frac{x^3}{3!} + \cdots\right) + \left(\cancel{1} - \cancel{\frac{x^2}{2!}} + \frac{x^4}{4!} + \cdots\right) - \cancel{1} - \cancel{x} + \cancel{\frac{1}{2}x^2} = -\frac{\varepsilon}{6}$$

$$\Rightarrow -\frac{x^3}{6} + \frac{x^4}{24} + O(x^6) = -\frac{\varepsilon}{6}$$

Scale $x = \varepsilon^{\frac{1}{3}} X \Rightarrow X^3 - \frac{\varepsilon^{\frac{1}{3}}}{4} X^4 + O(\varepsilon) = 1$ as $\varepsilon \to 0$

Expand: $X = X_0 + \varepsilon^{\frac{1}{3}} X_1 + \ldots$ and substitute to give

$$\left(X_0 + \varepsilon^{\frac{1}{3}} X_1 + \varepsilon^2 X_2 + \cdots\right)^3 - \frac{\varepsilon^{\frac{1}{3}}}{4}\left(X_0 + \varepsilon^{\frac{1}{3}} X_1 + \varepsilon^2 X_2 + \cdots\right)^4 + O(\varepsilon) = 1 \quad \text{as } \varepsilon \to 0$$

$O(\varepsilon^0)$: $X_0^3 = X \Rightarrow X_0 = 1$ (want real solution nearest $x=0$ only)

$O(\varepsilon^{\frac{1}{3}})$: $3X_0^2 X_1 - \frac{1}{4} X_0^4 = 0 \Rightarrow X_1 = \frac{1}{12}$

Hence $x \sim \varepsilon^{\frac{1}{3}} + \frac{1}{12}\varepsilon^{\frac{2}{3}} + \ldots$ as $\varepsilon \to 0$.

(c) Consider $\{a_n(\varepsilon)\}_{n\geq 0}$ where $a_0(\varepsilon) = \log(\frac{1}{\varepsilon})$ and $a_{n+1} = \log(a_n)$ $n \geq 0$

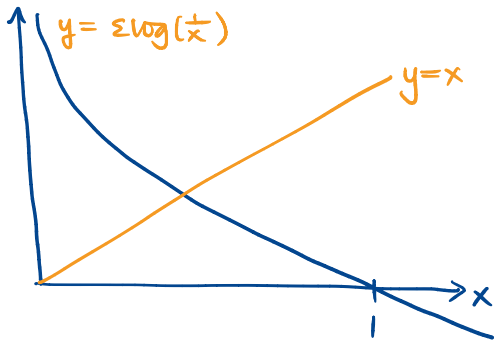
Then $\dfrac{a_{n+1}(\varepsilon)}{a_n(\varepsilon)} = \dfrac{\log(a_n(\varepsilon))}{a_n(\varepsilon)} \to 0$ as $a_n(\varepsilon) \to \infty$ (ie as $\varepsilon \to 0^+$)

$\therefore$ $\{a_n(\varepsilon)\}_{n\geq 0}$ forms an asymptotic sequence as $\varepsilon \to 0$.

Consider $x = \varepsilon \log(\frac{1}{x})$ as $\varepsilon \to 0$. Then $\log(\frac{1}{x})$ varies more slowly than x

$\Rightarrow$ take $g(x;\varepsilon) = \varepsilon\log(\frac{1}{x})$ so that $x_{n+1} = g(x_n;\varepsilon) = \varepsilon \log(\frac{1}{x_n})$.

and take $x_0 = \varepsilon$.



Then $x_1 = \varepsilon \log(\frac{1}{\varepsilon})$

$$x_2 = \varepsilon\log\left(\frac{1}{\varepsilon\log(\frac{1}{\varepsilon})}\right) = \varepsilon\log\left(\frac{1}{\varepsilon}\right) - \varepsilon\log\left(\log\left(\frac{1}{\varepsilon}\right)\right)$$

$$x_3 = \varepsilon\log\left(\frac{1}{\varepsilon\log(\frac{1}{\varepsilon})\left[1 - \frac{\log(\log(\frac{1}{\varepsilon}))}{\log(\frac{1}{\varepsilon})}\right]}\right)$$

$$= \varepsilon\log\left(\frac{1}{\varepsilon}\right) - \varepsilon\log\left(\log\left(\frac{1}{\varepsilon}\right)\right) + \frac{\varepsilon\log(\log(\frac{1}{\varepsilon}))}{\log(\frac{1}{\varepsilon})} + O\left(\frac{\varepsilon(\log(\frac{1}{\varepsilon}))^2}{\log(\frac{1}{\varepsilon})}\right)$$

[NB we need to go to x_4 to make sure the first three terms are fixed...]

$$x \sim \varepsilon\log\left(\frac{1}{\varepsilon}\right) - \varepsilon\log\left(\log\left(\frac{1}{\varepsilon}\right)\right) + \frac{\varepsilon\log(\log(\frac{1}{\varepsilon}))}{\log(\frac{1}{\varepsilon})} + \dots \quad \text{as } \varepsilon \to 0^+.$$

PS1 Q4

$$I(\varepsilon) = \int_0^\infty \frac{e^{-t}}{1+\varepsilon t}\,dt = \frac{e^{\frac{1}{\varepsilon}}}{\varepsilon}\int_{\frac{1}{\varepsilon}}^\infty \frac{e^{-t}}{t}\,dt \quad (\text{for } \varepsilon>0)$$

(a) Let $u = \frac{1}{t} \Rightarrow \frac{du}{dt} = -\frac{1}{t^2}$ and $\frac{dv}{dt} = e^{-t} \Rightarrow v = -e^{-t}$

$$I(\varepsilon) = \frac{e^{\frac{1}{\varepsilon}}}{\varepsilon}\left\{\left[\frac{-e^{-t}}{t}\right]_{\frac{1}{\varepsilon}}^\infty - \int_{\frac{1}{\varepsilon}}^\infty \frac{1}{t^2}e^{-t}\,dt\right\}$$

$$= \frac{e^{\frac{1}{\varepsilon}}}{\varepsilon}\left\{\varepsilon e^{-\frac{1}{\varepsilon}} - \int_{\frac{1}{\varepsilon}}^\infty \frac{1}{t^2}e^{-t}\,dt\right\} \quad \Rightarrow \text{true for } N=1.$$

Assume true for $N=1,\dots,K$, and consider

$$(-1)^K K!\int_{\frac{1}{\varepsilon}}^\infty \frac{e^{-t}}{t^{K+1}}\,dt = (-1)^K K!\left\{\left[\frac{1}{t^{K+1}}e^{-t}\right]_{\frac{1}{\varepsilon}}^\infty - \int_{\frac{1}{\varepsilon}}^\infty \frac{(K+1)}{t^{K+2}}e^{-t}\,dt\right\}$$

$-dv/dt \Rightarrow v = e^{-t}$

$\uparrow u \Rightarrow \frac{du}{dt} = \frac{-(K+1)}{t^{K+2}}$

$$= (-1)^K K!\,\varepsilon^{K+1}e^{-\frac{1}{\varepsilon}} + (-1)^{K+1}(K+1)!\int_{\frac{1}{\varepsilon}}^\infty \frac{e^{-t}}{t^{K+2}}\,dt$$

Hence true for $N=K+1$, and so true $\forall N\geq 1$ by induction. //

(b) Write

$$I(\varepsilon) = \underbrace{\sum_{n=1}^{N}(-1)^{n-1}(n-1)!\,\varepsilon^{n-1}}_{=\sum_{n=0}^{N-1}(-1)^n n!\,\varepsilon^n} + \frac{(-1)^N N!\,e^{\frac{1}{\varepsilon}}}{\varepsilon}\int_{\frac{1}{\varepsilon}}^\infty \frac{e^{-t}}{t^{N+1}}\,dt$$

Consider the 'remainder' term:

$$\left|\int_{\frac{1}{\varepsilon}}^\infty \frac{e^{-t}}{t^{N+1}}\,dt\right| < \varepsilon^{N+1}\left|\int_{\frac{1}{\varepsilon}}^\infty e^{-t}\,dt\right| = e^{-\frac{1}{\varepsilon}}\varepsilon^{N+1}$$

Then,

$$\left|I(\varepsilon) - \sum_{n=0}^{N-1}(-1)^n n!\,\varepsilon^n\right| = \left|\frac{(-1)^N N!\,e^{\frac{1}{\varepsilon}}}{\varepsilon}\int_{\frac{1}{\varepsilon}}^\infty \frac{e^{-t}}{t^{N+1}}\,dt\right| < N!\,\varepsilon^N$$

$$\therefore \frac{\left|I(\varepsilon) - \sum_{n=0}^{N-1}(-1)^n n!\,\varepsilon^n\right|}{\left|(-1)^{N-1}(N-1)!\,\varepsilon^{N-1}\right|} < N\varepsilon \to 0 \text{ as } \varepsilon\to 0 \text{ for } \forall N\in\mathbb{N}.$$

Hence $$I(\varepsilon) \sim \sum_{n=0}^{\infty}(-1)^n n!\,\varepsilon^n \quad \text{as } \varepsilon\to 0.$$

(c) Consider $S_N(\varepsilon) = \sum_{n=0}^{N-1} (-1)^n n! \, \varepsilon^n$ as $N \to \infty$, $\forall \varepsilon > 0$.

Plots show that $|I(0.2) - S_N(0.2)|$ minimal for $N=5$

$|I(0.1) - S_N(0.1)|$ minimal for $N=10$

corresponds to the optimal truncation

NB $a_n(\varepsilon) = (-1)^n n! \, \varepsilon^n \Rightarrow \dfrac{|a_{n+1}(\varepsilon)|}{|a_n(\varepsilon)|} = (n+1)\varepsilon$ which grows when $(n+1) > \frac{1}{\varepsilon}$

As a result, given $0 < \varepsilon \ll 1$, the optimal truncation is at $N(\varepsilon)$ where

$$N(\varepsilon).\varepsilon \leqslant 1 < (N(\varepsilon)+1)\varepsilon.$$

In this case, the remainder is

$$R_{N(\varepsilon)}(\varepsilon) = \left| (-1)^{N(\varepsilon)} N(\varepsilon)! \, \frac{e^{\frac{1}{\varepsilon}}}{\varepsilon} \int_{\frac{1}{\varepsilon}}^{\infty} \frac{e^{-t}}{t^{N(\varepsilon)+1}} \, dt \right| \sim \sqrt{\frac{\pi}{2\varepsilon}} \, e^{-\frac{1}{\varepsilon}} \text{ as } \varepsilon \to 0^+$$

[via Laplace's method – sheet 2.]

$\therefore$ the error is in fact exponentially small with the optimal truncation!

PS1Q5

$$I(x) = \int_x^\infty t^\alpha e^{-t^\beta}\,dt = \int_x^\infty \underbrace{\left(-\beta t^{\beta-1} e^{-t^\beta}\right)}_{\frac{dv}{dt} \Rightarrow v = e^{-t^\beta}} . \underbrace{\left(-\tfrac{1}{\beta} t^{\alpha-\beta+1}\right)}_{u \Rightarrow \frac{du}{dt} = \frac{\alpha-\beta+1}{-\beta} t^{\alpha-\beta}} dt$$

$$\therefore\ I(x) = \left[-\tfrac{1}{\beta} t^{\alpha-\beta+1} . e^{-t^\beta}\right]_x^\infty + \underbrace{\frac{\alpha-\beta+1}{\beta}}_{\neq 0} \int_x^\infty t^{\alpha-\beta} e^{-t^\beta}\,dt \qquad (x>0)$$

$$= \tfrac{1}{\beta} x^{\alpha-\beta+1} e^{-x^\beta} + \frac{\alpha-\beta+1}{\beta} \underbrace{\int_x^\infty t^{\alpha-\beta} e^{-t^\beta}\,dt}_{I_1(x)}$$

Then $\quad 0 < I_1(x) \leq \dfrac{1}{x^\beta} \displaystyle\int_x^\infty t^\alpha e^{-t^\beta}\,dt = \dfrac{I(x)}{x^\beta} \qquad (x>0).$

ie $\quad |I_1(x)| \ll |I(x)|$ as $x \to \infty$

$\Rightarrow\ I(x) \sim \tfrac{1}{\beta} e^{\alpha-\beta+1} e^{-x^\beta}$ as $x \to \infty$

(b) $$J = \int_{x^\gamma}^{\infty} e^{-xt^3}\,dt = \int_{x^{\gamma+\frac13}}^{\infty} e^{-s^3} . x^{-\frac13} ds \qquad (x>0)$$

$$t = x^{-\frac13} s$$

$$\Rightarrow \quad dt = x^{-\frac13} ds$$

(i) Suppose $\gamma > -\frac13$: then $\underbrace{x^{\gamma+\frac13}}_{\text{lower limit}} \to \infty$ as $x \to \infty$, and so

we can apply the result from part (a) with $x \mapsto x^{\gamma+\frac13}$

and $\alpha = 0$, $\beta = 3$ to give

$$J \sim \frac13 x^{(\gamma+\frac13)(0-3+1)} e^{-x^{(\gamma+\frac13).3}} . \, x^{-\frac13} \quad \text{as } x \to \infty$$

ie $$J \sim \frac13 x^{-(2\gamma+1)} e^{-x^{(3\gamma+1)}} \quad \text{as } x \to \infty.$$

(ii) Suppose $\gamma < \frac13$: then $x^{\gamma+\frac13} \to 0$ as $x \to \infty$. We can then use the hint to get

$$J = x^{-\frac13}\left[\Gamma\left(\tfrac43\right) - \underbrace{\int_0^{x^{\gamma+\frac13}} e^{-s^2} ds}_{= O(x^{\gamma+\frac13}) \text{ as } x\to\infty}\right] \qquad (x>0)$$

ie $$J \sim x^{-\frac13}\,\Gamma\left(\tfrac43\right) \quad \text{as } x \to \infty.$$

(a) $I(x) = \int_0^x e^{t^3}\,dt$ as $x \to \infty$

Try IBPs as is: let $I(x) = \int_0^x \underbrace{3t^2 e^{t^3}}_{\frac{dv}{dt}} \cdot \underbrace{\frac{1}{3t^2}}_{u}\,dt$

$\frac{dv}{dt} \Rightarrow v = e^{t^3}$ $\quad u \Rightarrow \frac{du}{dt} = -\frac{2}{3}t^{-3}$

Then, $I(x) = \left[\frac{1}{3t^2}e^{t^3}\right]_0^x + \int_0^x \frac{2}{3}t^{-3}e^{t^3}\,dt = \infty$ ie this method fails.

So, re-write as $I(x) = \int_0^a e^{t^3}\,dt + \int_a^x e^{t^3}\,dt$ for some $a>0$

change variables: let $s = t^3 \Rightarrow I(x) = \frac{1}{3}\int_0^{x^3} s^{-\frac{2}{3}} e^s\,ds$

$\frac{ds}{dt} = 3t^2 = 3s^{\frac{2}{3}}$

Let $J_n(x) := \int_1^{x^3} \underbrace{s^{-n}}_{u} \underbrace{e^s}_{\frac{dv}{ds}}\,ds = \left[s^{-n}e^s\right]_1^{x^3} + \int_1^{x^3} n s^{-(n+1)} e^s\,ds = \frac{e^{x^3}}{x^{3n}} - e + nJ_{n+1}(x)$

$$\therefore\ J_{\frac{2}{3}}(x) = \frac{e^{x^3}}{x^2} - e + \frac{2}{3}J_{\frac{5}{3}}(x)$$

$$= \frac{e^{x^3}}{x^2} - e + \frac{2}{3}\left[\frac{e^{x^5}}{x^5} - e + \frac{5}{3}J_{\frac{8}{3}}(x)\right]$$

$$= \frac{e^{x^3}}{x^2} + \frac{2e^{x^3}}{3x^5} - \frac{5}{3}e + \frac{10}{9}\left[J_{\frac{8}{3}}(x) - e + \frac{8}{3}J_{\frac{11}{3}}(x)\right]$$

$$= \frac{e^{x^3}}{x^2} + \frac{2e^{x^3}}{3x^5} + \frac{10e^{x^3}}{9x^8} - \frac{25}{9}e + \frac{80}{27}J_{\frac{11}{3}}(x)$$

$$\left|J_{\frac{11}{3}}(x)\right| = \left|\int_1^{x^3}\frac{e^s}{s^{11/3}}\,ds\right| < \left.\frac{e^s}{s^{11/3}}\right|_{s=x^3}\int_1^{x^3} ds = \frac{e^{x^3}}{x^8} \text{ as } x\to\infty$$

Hence, $I(x) = \frac{1}{3}\int_0^1 s^{-\frac{2}{3}} e^s\,ds + \frac{1}{3}J_{\frac{2}{3}}(x)$

$$\Rightarrow I(x) = \frac{e^{x^3}}{3x^2} + \frac{2e^{x^3}}{9x^5} + O\left(\frac{e^{x^3}}{x^8}\right) \text{ as } x\to\infty$$

(b) $I(x) = \int_0^\infty \underbrace{te^{-t^2}}_{u} \cos(xt)\,dt$ as $x \to \infty$.

$\Rightarrow \frac{du}{dt} = (1-2t^2)e^{-t^2}$ $\qquad \frac{dv}{dt} = \quad v = \frac{1}{x}\sin(xt)$

$$= \left[te^{-t^2} \cdot \frac{1}{x}\sin(xt) \right]_0^\infty - \int_0^\infty (1-2t^2)e^{-t^2} \cdot \frac{1}{x}\sin(xt)\,dt$$

$$= -\frac{1}{x}\int_0^\infty \underbrace{(1-2t^2)e^{-t^2}}_{u} \underbrace{\sin(xt)}_{\frac{dv}{dt}}\,dt \Rightarrow v = -\frac{1}{x}\cos(xt)$$

$\frac{du}{dt} = -4t - 2t(1-2t^2))e^{-t^2} = (-6t + 4t^3)e^{-t^2}$

$$= -\frac{1}{x}\left\{ \left[(1-2t^2)e^{-t^2} \cdot -\frac{1}{x}\cos(xt) \right]_0^\infty + \int_0^\infty (6t-4t^3)e^{-t^2} \cdot \frac{1}{x}\cos(xt)\,dt \right\}$$

$$= -\frac{1}{x^2} + \frac{1}{x^2}\int_0^\infty \underbrace{(6t-4t^3)e^{-t^2}}_{u} \cos(xt)\,dt$$

$u \Rightarrow \frac{du}{dt} = (6 - 12t^2 - 2t(6t - 4t^3))e^{-t^2}$

$\frac{dv}{dt} = \cos(xt) \Rightarrow v = \frac{1}{x}\sin(xt)$

$$= -\frac{1}{x^2} + \frac{1}{x^2}\left\{ \left[(6t-4t^3)e^{-t^2} \cdot \frac{1}{x}\sin(xt) \right]_0^\infty - \underbrace{\int_0^\infty \frac{1}{x}\sin(xt) \cdot \frac{du}{dt}\,dt}_{x^2 R(x)} \right\}$$

$$= -\frac{1}{x^2} + R(x)$$

$$|R(x)| = \frac{1}{x^3}\left| \int_0^\infty (6 - 24t^2 + 8t^4)e^{-t^2} \sin(xt)\,dt \right| \leq \frac{c}{x^3}$$

with $c = \int_0^\infty |(6 - 24t^2 + 8t^4)e^{-t^2}|\,dt$

$\therefore\ I(x) = -\frac{1}{x^2} + O\left(\frac{1}{x^3}\right)$ as $x \to \infty$.