PS2 Q1

As $x \to \infty$

$\int_0^{\pi/2} e^{ix\cos t}\,dt$ – method of stationary phase for first term (then method of steepest descents for more..)

$\int_0^1 \ln t\, e^{ixt}\,dt$ – method of steepest descents (note that using IBPs and the method of stationary phase does not work...)

$\int_0^x t^{-\frac{1}{2}} e^{-t}\,dt$ – write as $\int_0^\infty t^{-\frac{1}{2}}e^{-t}\,dt - \int_x^\infty t^{-\frac{1}{2}}e^{-t}\,dt$ and then use IBPs for the second integral

$\int_0^{\pi/2} e^{-x\sin^2 t}\,dt$ – Laplace's method

$\int_0^1 e^{ixe^{-1/t}}\,dt$ – method of steepest descents with $s = e^{-1/t}$

As $x \to 0^+$

$\int_0^{10} \frac{e^{-xt}}{1+t}\,dt$ – Taylor expand the integrand and integrate term-by-term

$\int_0^{\pi/2} \frac{1}{\sqrt{\cos^2 t + x\sin^2 t}}\,dt$ – write as $\int_0^{\pi/2-\delta} + \int_{\pi/2-\delta}^{\pi/2}$ where $x \ll \delta \ll 1$

$\int_0^1 \frac{\sin(xt)}{t}\,dt$ – Taylor expand and integrate term-by-term.

$\int_x^\infty t^{a-1}e^{-t}\,dt$ – write as $\int_0^\infty - \int_0^x$ and then Taylor expand and integrate term-by-term for the second integral when $\mathrm{Re}(a) > 0$. (NB v. tricky o/w!)

$\int_0^1 \frac{\ln t}{x+t}\,dt$ – write as $\int_0^\delta + \int_\delta^1$ where $x \ll \delta \ll 1$.

PS2 Q2 (as $x \to \infty$)

$$I_1(x) = \int_{-1}^{1} e^{-x\cosh t}\,dt = \underbrace{\int_{-1}^{-\varepsilon} e^{-x\cosh t}\,dt}_{I_{11}} + \underbrace{\int_{-\varepsilon}^{\varepsilon} e^{-x\cosh t}\,dt}_{I_{12}} + \underbrace{\int_{\varepsilon}^{1} e^{-x\cosh t}\,dt}_{I_{13} = I_1 \text{ (by symmetry)}}$$

$\varepsilon \ll 1$

- Split integral this way because $\cosh t$ has a maximum at $t=0$.

$$I_{11}(x) = \int_{\varepsilon}^{1} e^{-x\cosh t}\,dt = O\left(e^{-x\cosh \varepsilon}\right) = O\left(e^{-x}e^{-x\varepsilon^2/2}\right)$$

$$I_{12}(x) = \int_{-\varepsilon}^{+\varepsilon} e^{-x\cosh t}\,dt = \int_{-\varepsilon}^{+\varepsilon} e^{-x(1+\frac{1}{2}t^2+O(t^4))}\,dt$$

ok since $\varepsilon \ll 1$

$$= e^{-x}\int_{-\varepsilon}^{+\varepsilon} e^{-\frac{1}{2}xt^2} e^{O(xt^4)}\,dt$$

ok provided $xt^4 \ll 1$

$$= e^{-x}\int_{-\varepsilon}^{+\varepsilon} e^{-\frac{1}{2}xt^2}\left[1+O(xt^4)\right]dt$$

$s = \sqrt{\frac{x}{2}}\,t$

$$= e^{-x}\int_{-\varepsilon\sqrt{\frac{x}{2}}}^{+\varepsilon\sqrt{\frac{x}{2}}} e^{-s^2}\left[1+O(s^4/x)\right]\cdot\sqrt{\frac{2}{x}}\,ds$$

$$= e^{-x}\sqrt{\frac{2}{x}}\left\{\int_{-\varepsilon\sqrt{x/2}}^{+\varepsilon\sqrt{x/2}} e^{-s^2}\,ds + O\left(\frac{1}{x}\right)\right\}$$

ok provided $\varepsilon\sqrt{\frac{x}{2}} \gg 1$

$$= e^{-x}\sqrt{\frac{2}{x}}\left\{\int_{-\infty}^{\infty} e^{-s^2}\,ds + O\left(\frac{1}{x}\right)\right\}$$

$$= e^{-x}\sqrt{\frac{2}{x}}\left\{\sqrt{\pi} + O\left(\frac{1}{x}\right)\right\} \quad \text{as } x \to \infty$$

Then for $\varepsilon\sqrt{x} \gg 1$ we have $I_{11}(x) \ll I_{12}(x)$ as $x \to \infty$.

$$\therefore\ I_1(x) \sim \sqrt{\frac{2\pi}{x}}\,e^{-x} \quad \text{as } x \to \infty$$

NB we need to select ε s.t. $x^{-\frac{1}{2}} \ll \varepsilon \ll x^{-\frac{1}{4}}$

$$I_2(x) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-x(t^2 - \sin^2 t)}\, dt$$

Let $\varphi(t) = \sin^2 t - t^2$. Then $\varphi(0) = 0$ and $\varphi(t) < 0$ for $t \neq 0$. Hence $\varphi(t)$ has an interior maximum @ $t = 0$. Note that

$\varphi'(t) = 2\sin t \cos t - 2t$ and so $\varphi'(0) = 0 \Rightarrow$ degenerate case.

$$\varphi(t) = \left(t - \tfrac{1}{3}t^3 + o(t^5)\right)^2 - t^2 = -\tfrac{1}{3}t^4 + o(t^6) \text{ as } t \to 0.$$

Split the integral as

$$I_2(x) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sim dt = \underbrace{\int_{-\frac{\pi}{2}}^{-\varepsilon} \sim dt}_{I_{21}} + \underbrace{\int_{-\varepsilon}^{+\varepsilon} \sim dt}_{I_{22}} + \underbrace{\int_{+\varepsilon}^{\frac{\pi}{2}} \sim dt}_{I_{23} = I_{21} \text{ by symmetry.}} \quad (\text{where } \varepsilon \ll 1)$$

$$I_{21}(x) = \int_{+\varepsilon}^{\frac{\pi}{2}} e^{x\varphi(t)}\, dt = o\left(e^{x\varphi(\varepsilon)}\right) = o\left(e^{-x\varepsilon^4}\right)$$

$$I_{22}(x) = \int_{-\varepsilon}^{+\varepsilon} e^{-x\left(\frac{1}{3}t^4 + o(t^6)\right)}\, dt \quad \text{ok since } \varepsilon \ll 1$$

$$\Big\downarrow \text{ ok provided } xt^6 \ll 1$$

$$= \int_{-\varepsilon}^{+\varepsilon} e^{-\frac{1}{3}xt^4}\left(1 + o(xt^6)\right) dt$$

$$\Big\downarrow\ s = \left(\tfrac{x}{3}\right)^{\frac{1}{4}} t$$

$$= \int_{-\left(\frac{x}{3}\right)^{1/4}\varepsilon}^{+\left(\frac{x}{3}\right)^{\frac{1}{4}}\varepsilon} e^{-s^4}\left(1 + o\left(x^{-\frac{1}{2}}s^6\right)\right) \cdot \left(\tfrac{3}{x}\right)^{\frac{1}{4}} ds$$

$$\Big\downarrow \text{ provided } \left(\tfrac{x}{3}\right)^{\frac{1}{4}}\varepsilon \gg 1$$

$$= \left(\frac{3}{x}\right)^{\frac{1}{4}} \left\{ \int_{-\infty}^{+\infty} e^{-s^4}\, ds + o\left(x^{-\frac{1}{2}}\right) \right\}$$

$$\Big\downarrow \text{ by hint on problem sheet}$$

$$= \left(\frac{3}{x}\right)^{\frac{1}{4}} \left\{ \frac{2\Gamma(1/4)}{4} + o\left(x^{-\frac{1}{2}}\right) \right\}$$

For $x^{\frac{1}{4}}\varepsilon \gg 1$ then we have $I_{21}(x) \ll I_{22}(x)$ as $x \to \infty$

$$\therefore\ I_2(x) \sim \frac{\Gamma(1/4)}{2} \cdot \left(\frac{3}{x}\right)^{\frac{1}{4}} \text{ as } x \to \infty$$

NB we need to select ε s.t. $x^{-\frac{1}{4}} \ll \varepsilon \ll x^{-\frac{1}{6}}$.

$$I_3(x) = \int_0^\infty e^{-2t - x/t^2}\,dt$$

$$\frac{d}{dt}\left(2t + \frac{x}{t^2}\right) = 2 - \frac{2x}{t^3}$$ $\Rightarrow$ the maximum moves as x is varied.

$\Rightarrow$ Make a change of variables: let $y = x^{\frac{1}{3}}$ and $t = x^{\frac{1}{3}}u$ so that

$$I_3(x) = \int_0^\infty e^{-y\underbrace{(2u+\frac{1}{u^2})}_{\varphi(u)}}\, y\,du$$

Then $\varphi(u) = -2u + \frac{1}{u^2}$ $\Rightarrow$ $\varphi'(u) = -2 - \frac{2}{u^3}$ and $\varphi'(u) = 0$ for $u = 1$.

Expand $\varphi(u)$ about $u=1$: $\varphi(u) = \varphi(1) + (u-1)\underbrace{\varphi'(1)}_{=0} + \frac{1}{2}(u-1)^2\varphi''(1) + O((u-1)^3)$

$$= -3 - 3(u-1)^2 + O((u-1)^3)$$

Split the range of integration:

$$I_3(y) = \underbrace{\int_0^{1-\varepsilon} \sim du}_{I_{31}} + \underbrace{\int_{1-\varepsilon}^{1+\varepsilon} \sim du}_{I_{32}} + \underbrace{\int_{1+\varepsilon}^{\infty} \sim du}_{I_{33}} \qquad (\varepsilon \ll 1)$$

$$I_{31}(y) = \int_0^{1-\varepsilon} y e^{-y(2u+1/u^2)}\,du = O\left(ye^{-y\varphi(1-\varepsilon)}\right) = O\left(ye^{-3y}e^{-3y\varepsilon^2}\right)$$

$$I_{33}(y) = \int_{1+\varepsilon}^{\infty} y e^{-y(2u+1/u^2)}\,du = O\left(ye^{-y\varphi(1+\varepsilon)}\right) = O\left(ye^{-3y}e^{-3y\varepsilon^2}\right)$$

$$I_{32}(y) = \int_{1-\varepsilon}^{1+\varepsilon} y e^{-y(3+3(u-1)^2+O((u-1)^3))}\,du$$ (ok since $\varepsilon \ll 1$)

$$= ye^{-3y}\int_{1-\varepsilon}^{1+\varepsilon} e^{-3y(u-1)^2}\left(1 + O\left(y(u-1)^3\right)\right)du$$ provided $y\varepsilon^3 \ll 1$

$$= ye^{-3y}\int_{-\varepsilon\sqrt{3y}}^{+\varepsilon\sqrt{3y}} e^{-s^2}\left(1 + O\left(y^{-\frac{1}{2}}s^3\right)\right)\frac{1}{\sqrt{3y}}\,ds$$ $\sqrt{3y}(u-1) = s$

$$= \sqrt{\frac{y}{3}}\,e^{-3y}\left\{\int_{-\infty}^{\infty} e^{-s^2}\,ds + O\left(y^{-\frac{1}{2}}\right)\right\}$$ provided $\varepsilon\sqrt{3y} \gg 1$

Then, for $\varepsilon\sqrt{y} \gg 1$ we have $I_{31}(y), I_{33}(y) \ll I_{32}(y)$ as $y \to \infty$.

Hence $I_3(y) \sim \sqrt{\frac{y}{3}}\, e^{-3y} \cdot \sqrt{\pi}$

$$= \sqrt{\frac{\pi}{3}}\, x^{1/6} e^{-3x^{1/3}} \quad \text{as } x \to \infty.$$

NB we need to select ε s.t. $x^{-\frac{1}{6}} \ll \varepsilon \ll x^{-\frac{1}{9}}$.

PS2 Q3

$$J_1(x) = \int_0^1 e^{ixt^2}\cosh(t^2)\,dt \qquad \text{as } x\to\infty.$$

Then $\psi(t) = t^2$ and $\psi'(t) = 2t \Rightarrow \psi'(t) = 0$ for $t = 0$.

Hence split the region of integration as

$$J_1(x) = \underbrace{\int_0^{\varepsilon} \sim dt}_{J_{11}} + \underbrace{\int_{\varepsilon}^1 \sim dt}_{J_{12}} \qquad \text{for } \varepsilon \ll 1.$$

$$J_{11}(x) = \int_0^{\varepsilon} (1 + O(t^4))\, e^{ixt^2}\,dt \qquad \text{(OK since } \varepsilon \ll 1)$$

$$= \int_0^{\varepsilon\sqrt{x}} (1 + O(s^4/x^2))\, e^{is^2}\cdot \frac{1}{\sqrt{x}}\,ds \qquad s = \sqrt{x}\,t$$

$$= \frac{1}{\sqrt{x}}\int_0^{\infty} e^{is^2}\,ds + O\left(\frac{1}{\sqrt{x}\cdot \varepsilon\sqrt{x}}\right) \qquad \text{provided } \varepsilon\sqrt{x} \gg 1$$

(✱)

NB we can do $\int_0^{\varepsilon\sqrt{x}} \to \int_0^{\infty}$ because

$$\frac{1}{\sqrt{x}}\int_{\varepsilon\sqrt{x}}^{\infty} e^{is^2}\,ds = \frac{1}{\sqrt{x}}\int_{\varepsilon\sqrt{x}}^{\infty} \underbrace{\frac{1}{2is}}_{u}\cdot \underbrace{2is\,e^{is^2}}_{dv/ds}\,ds$$

$$= \frac{1}{\sqrt{x}}\left\{\left[\frac{1}{2is}e^{is^2}\right]_{\varepsilon\sqrt{x}}^{\infty} - \int_{\varepsilon\sqrt{x}}^{\infty} \frac{-1}{2is^2}e^{is^2}\,ds\right\}$$

$$= \underbrace{\frac{1}{2i\varepsilon x}e^{i\varepsilon^2 x}}_{O\left(\frac{1}{\varepsilon x}\right)} - \underbrace{\frac{1}{2i\sqrt{x}}\int_{\varepsilon\sqrt{x}}^{\infty} \frac{1}{s^2}e^{is^2}\,ds}_{O\left(\frac{1}{\varepsilon x}\right)}$$

Further, (✱) comes from

$$\frac{1}{x^{5/2}}\int_0^{\varepsilon\sqrt{x}} s^4 e^{is^2}\,ds = O\left(\frac{(\varepsilon\sqrt{x})^3}{x^{5/2}}\right) = O\left(\frac{\varepsilon^3}{x}\right) \ll 1 \text{ provided } \varepsilon^3 \ll x.$$

from IBPs and looking at the first term...

$$J_{12}(x) = \int_\varepsilon^1 \underbrace{\frac{\cosh(t^2)}{2ixt}}_{u} \cdot \underbrace{2ixt\, e^{ixt^2}}_{dv/dt}\, dt$$

$$= \left[\frac{\cosh(t^2)}{2ixt} e^{ixt^2} \right]_\varepsilon^1 - \underbrace{\int_\varepsilon^1 \frac{\partial}{\partial t}\left(\frac{\cosh(t^2)}{2ixt} \right) e^{ixt^2}\, dt}_{= o\left(\frac{1}{x}\right) \text{ as } x \to \infty \text{ by RLL}}$$

$$= O\left(\frac{1}{\varepsilon x} \right)$$

Hence $J_1(x) \sim \dfrac{e^{i\pi/4}}{2} \sqrt{\dfrac{\pi}{x}}$ as $x \to \infty$

NB we need to select $x^{-\frac{1}{2}} \ll \varepsilon \ll x^{-\frac{1}{3}}$ $\left(\Rightarrow x^{-\frac{1}{2}} \gg \frac{1}{\varepsilon x} \right)$.

$$J_2(x) = \mathrm{Re}[J_4(x)] = \mathrm{Re}\left[\int_0^1 \tan t\, e^{ixt^4}\,dt\right] \text{ as } x\to\infty$$

Then $\psi(t) = t^4$ has $\psi'(0)=0$, so we split the region of integration as:

$$J_4(x) = \underbrace{\int_0^\varepsilon \sim dt}_{J_{41}} + \underbrace{\int_\varepsilon^1 \sim dt}_{J_{42}} \qquad (\varepsilon \ll 1)$$

$$J_{41}(x) = \int_0^\varepsilon (t + o(t^3))\, e^{ixt^4}\,dt \qquad \text{(ok since } \varepsilon \ll 1)$$

$$\downarrow \quad s = x^{1/4} t$$

$$= \int_0^{x^{1/4}\varepsilon} \left(\frac{s}{x^{1/4}} + o\left(\frac{s^3}{x^{3/4}}\right)\right) e^{is^4}\cdot x^{-\frac{1}{4}}\,ds$$

$$\downarrow \quad \text{provided } x^{4}\varepsilon \gg 1$$

$$= x^{-\frac{1}{2}} \int_0^\infty s e^{is^4}\,ds + o\left(\frac{1}{\varepsilon^2 x}\right)$$

NB the above holds since

$$x^{-\frac{1}{2}} \int_{x^{1/4}\varepsilon}^\infty s e^{is^4}\,ds = x^{-\frac{1}{2}} \int_{x^{1/4}\varepsilon}^\infty \underbrace{\frac{1}{4is^2}}_{u}\cdot \underbrace{4is^3 e^{is^4}}_{dv/ds}\,ds$$

$$= x^{-\frac{1}{2}} \left\{ \underbrace{\left[\frac{1}{4is^2} e^{is^4}\right]_{x^{1/4}\varepsilon}^\infty}_{=o\left(\frac{1}{x^{\frac{1}{2}}\varepsilon^2}\right)} - \underbrace{\int_{x^{1/4}\varepsilon}^\infty \frac{-2}{4is^3} e^{is^4}\,ds}_{=o\left(\frac{1}{x^{1/2}\varepsilon^2}\right)} \right\}$$

$$= o\left(\frac{1}{x\varepsilon^2}\right)$$

and also

$$\frac{1}{x}\int_0^{x^{1/4}\varepsilon} s^3 e^{is^4}\,ds = \frac{1}{4ix}\left[e^{is^4}\right]_0^{x^{1/4}\varepsilon} = o\left(\frac{1}{x}\right)$$

$$J_{42}(x) = \int_\varepsilon^1 \underbrace{\frac{\tan t}{4ixt^3}}_{u}\cdot \underbrace{4ixt^3 e^{ixt^4}}_{dv/dt}\,dt$$

$$J_{42}(x) = \left[\frac{\tan t}{4ixt^3} e^{ixt^4} \right]_\varepsilon^1 - \underbrace{\int_\varepsilon^1 \frac{\partial}{\partial t}\left(\frac{\tan t}{4ixt^3}\right) e^{ixt^4}\, dt}_{= o\left(\frac{1}{x}\right) \text{ as } x\to\infty \text{ by RLL}}$$

$$= O\left(\frac{1}{x\varepsilon^2}\right)$$

Hence

$$J_4 \sim x^{-\frac{1}{2}} \int_0^\infty s e^{is^4} ds = \frac{x^{-\frac{1}{2}} e^{i\pi/4} \Gamma\left(\frac{1}{2}\right)}{4}$$

$$\therefore \quad J_2(x) \sim \frac{\cos\left(\frac{\pi}{4}\right)\Gamma\left(\frac{1}{2}\right)}{4} x^{-\frac{1}{2}} \quad \text{as } x \to \infty.$$

$$J_3(x) = \int_0^1 e^{ix(t-\sin t)}dt$$

Let $\psi(t) = t - \sin t \Rightarrow \psi'(t) = 1 - \cos t$ and hence $t=0$ is the only stationary point. We have $\psi(t) = t - [t - \frac{1}{3!}t^3 + O(t^5)]$

$$= \frac{1}{3!}t^3 + O(t^5)$$

Hence we split the region of integration:

$$J_3(x) = \underbrace{\int_0^\varepsilon \sim dt}_{J_{31}} + \underbrace{\int_\varepsilon^1 \sim dt}_{J_{32}}$$

$$J_{31}(x) = \int_0^\varepsilon e^{ix[\frac{1}{6}t^3 + O(t^5)]} dt \qquad \text{(ok since } \varepsilon \ll 1)$$

$$= \int_0^{\varepsilon(\frac{x}{6})^{1/3}} e^{is^3}\left[1 + O\left(x\left(\frac{s}{x^{1/3}}\right)^5\right)\right]\left(\frac{6}{x}\right)^{\frac{1}{3}} ds \qquad s = \left(\frac{x}{6}\right)^{\frac{1}{3}}t$$

(provided $x\varepsilon^5 \ll 1$)

$$= \left(\frac{6}{x}\right)^{\frac{1}{3}} \int_0^\infty e^{is^3}ds + O\left(\frac{1}{\varepsilon^2 x}\right) \qquad \text{(provided } x^{\frac{1}{3}}\varepsilon \gg 1)$$

NB the above holds since

$$x^{-\frac{1}{3}} \int_{\varepsilon(\frac{x}{6})^{1/3}}^\infty e^{is^3}ds = x^{-\frac{1}{3}} \int_{\varepsilon(\frac{x}{6})^{1/3}}^\infty \underbrace{\frac{1}{3is^2}}_{u} \cdot \underbrace{3is^2 e^{is^3}}_{dv/ds} ds$$

$$= x^{-\frac{1}{3}}\left\{\left[\frac{1}{3is^2}e^{is^3}\right]_{\varepsilon(\frac{x}{6})^{1/3}}^\infty - \int_{\varepsilon(\frac{x}{6})^{1/3}}^\infty \frac{\partial}{\partial s}\left(\frac{1}{3is^2}\right)e^{is^3}ds\right.$$

$$= O\left(\frac{1}{\varepsilon^2 x}\right)$$

and also

$$\frac{1}{x}\int_0^{\varepsilon(\frac{x}{6})^{1/3}} s^5 e^{is^3}ds = \frac{1}{x}\int_0^{\varepsilon(\frac{x}{6})^{1/3}} \underbrace{\frac{s^3}{3i}}_{u} \cdot \underbrace{3is^2 e^{is^3}}_{dv/ds} ds$$

$$= \frac{1}{x}\left\{\left[\frac{s^3}{3i}e^{is^3}\right]_0^{\varepsilon(\frac{x}{6})^{1/3}} - \int_0^{\varepsilon(\frac{x}{6})^{1/3}} \frac{s^2}{i}e^{is^3}ds\right.$$

$$= O(\varepsilon^3) + O\left(\frac{1}{x}\right)$$

$$\therefore \quad J_3(x) \sim \left(\frac{6}{x}\right)^{1/3} \int_0^\infty e^{is^3}\,ds$$

$$= \left(\frac{6}{x}\right)^{1/3} \frac{e^{i\pi/6}\,\Gamma\left(\frac{1}{3}\right)}{3}$$

$$= \left(\frac{2}{9}\right)^{1/3} \Gamma\left(\frac{1}{3}\right) e^{i\pi/6}\, x^{-\frac{1}{3}} \text{ as } x \to \infty$$

NB we need to select $x^{-\frac{1}{3}} \ll \varepsilon \ll x^{-\frac{1}{5}}$.

PS2 Q4

$$I(x) = \int_{-1}^{1} (1-t^2)^N e^{ixt}\, dt$$

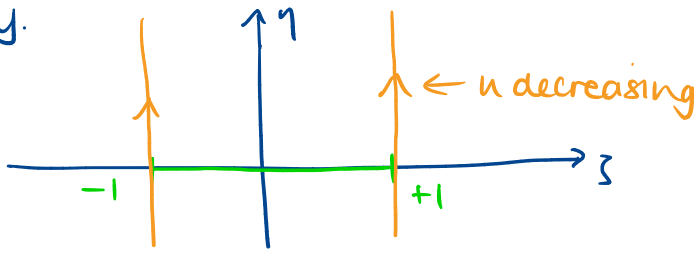
N – integer, and contour of integration is a line segment from $-1 \to 1$.

(a) $\varphi(t) = it = i(\xi + i\eta) \Rightarrow u(\xi,\eta) = -\eta,\ v(\xi,\eta) = \xi$

On a steepest descent contour: $v(\xi,\eta) = \xi =$ constant

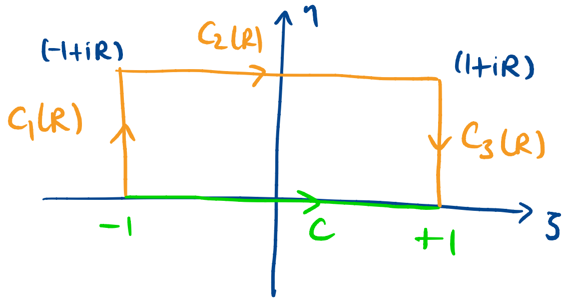
$\nabla u = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ is tangent to the steepest descent contour, and

$-\nabla u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a tangent in the direction in which u decreases most rapidly.



(b) Since the integrand is holomorphic in $t \in \mathbb{C}$, we can deform the contour to give

$$I(x) = \left\{ \int_{C_1} + \int_{C_2} + \int_{C_3} \right\} f(t) e^{x\varphi(t)} dt$$



On $C_2(R)$: $\left|f(t)e^{x\varphi(t)}\right| = |f(t)|\, e^{-xR} \to 0$ as $R \to \infty$.

$$\therefore\ I(x) = \int_{-1}^{-1+i\infty} (1-t^2)^N e^{ixt}\, dt + \int_{1}^{1+i\infty} (1-t^2)^N e^{ixt}\, dt$$

$$= I_-(x) + I_+(x)$$

(c) $$I_\pm(x) = \int_{\pm1}^{\pm1+i\infty} f(\pm1+is)\, e^{ix(\pm1+is)}\, i\, ds \qquad s>0,\ t = \pm1+is$$

$$= ie^{\pm ix} \int_{\pm1}^{\pm1+i\infty} f(\pm1+is)\, e^{-xs}\, ds$$

where $f(\pm1+is) = (1-(\pm1+is)^2)^N = \begin{cases} (-is)^N (2+is)^N & \oplus \\ (2+is)^N (is)^N & \ominus \end{cases}$

Using Laplace's method: we write

$$I_\pm(x) = i e^{\pm ix}\left\{\int_0^\varepsilon + \int_\varepsilon^\infty\right\} f(\pm 1 + is)\, e^{-xs}\, ds \qquad (\varepsilon \ll 1)$$

Consider I_+ first:

$$\int_0^\varepsilon f(1+is) e^{-xs} ds = \int_0^\varepsilon (-is)^N (2+is)^N e^{-xs} ds$$

$$= (-2i)^N \int_0^\varepsilon \left(s^N + O(s^{N+1})\right) e^{-xs} ds \quad \Big\downarrow \text{ ok since } \varepsilon \ll 1.$$

$$= \frac{(-2i)^N}{x^{N+1}}\left\{\int_0^{\varepsilon x} u^N e^{-u} du + O\left(\frac{1}{x}\right)\right\} \quad \Big\downarrow \text{ letting } s = \frac{u}{x}$$

$$= \frac{(-2i)^N}{x^{N+1}}\left\{\underbrace{\int_0^\infty u^N e^{-u} du}_{N!} + O\left(\frac{1}{x}\right)\right\} \quad \Big\downarrow \text{ provided } \varepsilon x \gg 1$$

$$= \frac{(-2i)^N N!}{x^{N+1}} + O\left(\frac{1}{x^{N+2}}\right)$$

$$\int_\varepsilon^\infty f(1+is) e^{-xs} ds = O\left(e^{-\varepsilon x}\right)$$

$$\therefore \quad I_+(x) \sim \frac{i e^{ix} (-2i)^N N!}{x^{N+1}} \text{ as } x \to \infty$$

NB Need to select $x^{-1} \ll \varepsilon \ll 1$.

Similarly, $I_-(x) \sim \dfrac{i e^{-ix} (2i)^N N!}{x^{N+1}}$ as $x \to \infty$

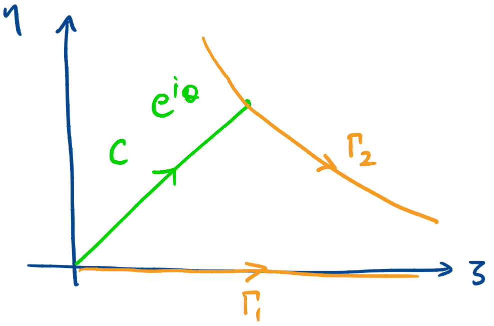
Hence

$$I(x) \sim \frac{2^N i^{N+1} N!}{x^{N+1}}\left(e^{-ix} - (-1)^N e^{ix}\right) \quad \text{as } x \to \infty.$$

(NB - this is real - as required by symmetry!).

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-z^2}\,dz = \frac{2r}{\sqrt{\pi}}\int_0^{e^{i\theta}} e^{-r^2t^2}\,dt \quad \text{with } z = re^{i\theta} \text{ and } s = rt$$

$$\varphi(t) = -t^2 = -(\zeta+i\eta)^2 = \underbrace{\eta^2-\zeta^2}_{u(\zeta,\eta)} - \underbrace{2\zeta\eta i}_{v(\zeta,\eta)}$$



Contour of steepest descent through $(0,0)$ is $\eta = 0$.

Contour of steepest descent through $t = e^{i\theta}$ is $2\zeta\eta = 2\cos\theta\sin\theta = \sin 2\theta$

$(\theta \in (0, \pi/2) \Rightarrow \zeta, \eta > 0)$

Then, by the deformation theorem,

$$\mathrm{erf}(z) = \Big(\int_{\Gamma_1} - \int_{\Gamma_2}\Big)\frac{2r}{\sqrt{\pi}}e^{r^2\varphi(t)}\,dt$$

$\Gamma_1 = \Gamma_1(r)$ $\quad$ $\Gamma_2 = \Gamma_2(r,\theta)$

$$I_1(r) = \frac{2r}{\sqrt{\pi}}\int_0^\infty e^{-r^2\zeta^2}\,d\zeta = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}\,du = 1$$

with $u = r\zeta$

For I_2, we have

$$\Gamma_2 = \Big\{\overbrace{\zeta + i\frac{\sin 2\theta}{2\zeta}}^{t},\ \zeta > \cos\theta\Big\}$$

$2\zeta\eta = \sin 2\theta$

$\Rightarrow \eta = \dfrac{\sin 2\theta}{2\zeta}$

and

$$\varphi(t) = \underbrace{\eta^2-\zeta^2}_{u} + \underbrace{(-2\zeta\eta i)}_{iv}$$
$$= \eta^2-\zeta^2 - i\sin 2\theta$$

which gives

$$\eta'(\zeta) = -\frac{\sin 2\theta}{2\zeta^2}$$

$$I_2(r,\theta) = \frac{2r}{\sqrt{\pi}}\int_{\cos\theta}^\infty e^{r^2(\eta^2-\zeta^2-i\sin 2\theta)}\cdot(1+\eta'(\zeta)i)\,d\zeta$$

$$= \frac{2r}{\sqrt{\pi}}e^{-r^2 i\sin 2\theta}\int_{\cos\theta}^\infty F(\zeta)e^{r^2\Phi(\zeta)}\,d\zeta \qquad F(\zeta) = 1 - i\frac{\sin 2\theta}{2\zeta^2}$$

$$\Phi(\zeta) = \frac{\sin^2 2\theta}{4\zeta^2}$$

Since Γ_2 is a contour of steepest descent then $\Phi(\zeta)$ is a decreasing function of ζ on Γ_2

$\Rightarrow$ Apply Laplace's method to give

$$I_2(r,\theta) = \frac{-2r}{\sqrt{\pi}} e^{-r^2 i \sin 2\theta} \frac{F(\cos\theta)\, e^{r^2 \Phi(\cos\theta)}}{r^2\, \Phi'(\cos\theta)} \quad \text{as } r\to\infty.$$

$$F(\cos\theta) = \frac{e^{-i\theta}}{\cos\theta}, \quad \Phi(\cos\theta) = -\cos 2\theta, \quad \Phi'(\cos\theta) = \frac{-2}{\cos\theta}$$

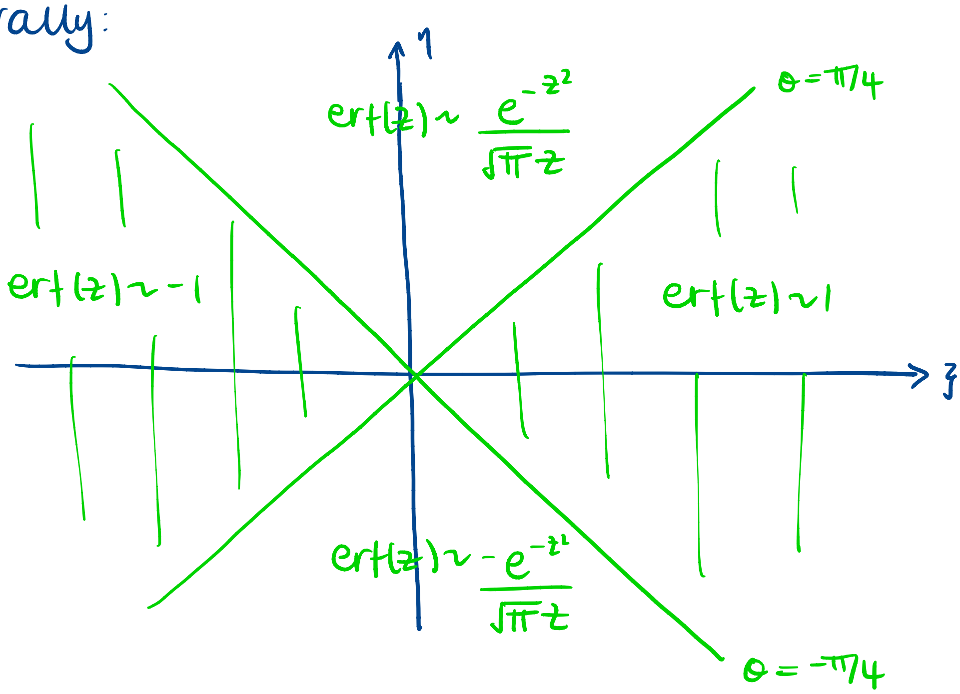
$$\Rightarrow \quad I_2(r,\theta) \sim \frac{1}{\sqrt{\pi}\, r e^{i\theta}}\, e^{-r^2 e^{2i\theta}} \quad \text{as } r\to\infty.$$

Hence $I_1(r) \sim 1$ and $I_2(r,\theta) \sim \frac{1}{\sqrt{\pi} z} e^{-z^2}$ as $r = |z| \to \infty$
for $0 < \theta = \arg(z) < \frac{\pi}{2}$

$$|I_2(r)| \sim \frac{1}{r} e^{-r^2 \cos 2\theta} = \begin{cases} \ll 1 & \text{for } 0 < \theta \le \frac{\pi}{4} \\ \gg 1 & \text{for } \frac{\pi}{4} < \theta < \frac{\pi}{2} \end{cases} \quad \text{as } z \to \infty$$

$$\therefore \quad \operatorname{erf}(z) = \begin{cases} 1 & \text{for } 0 < \theta \le \frac{\pi}{4} \\ -\frac{1}{\sqrt{\pi} z} e^{-z^2} & \text{for } \frac{\pi}{4} < \theta < \frac{\pi}{2} \end{cases} \quad \text{as } |z| \to \infty.$$

More generally:



- Different asymptotic expansions in different regions → Stokes Phenomena
- While e^{-z^2} is active, it has an essential singularity at ∞
- $\theta = \pm\frac{\pi}{2}$ – Stokes' lines (across which topology of SD contour changes).
- $|\theta| = \frac{\pi}{4}, \frac{5\pi}{4}$ – anti Stokes' lines (across which dominance of end point and saddle point changes).

PS2 Q6

$$I(\varepsilon) = \int_0^1 \frac{f(x)}{x+\varepsilon}\,dx \quad \text{as } \varepsilon \to 0^+ \text{ with } f \text{ smooth.}$$

$$= \underbrace{\int_0^\delta \frac{f(x)}{x+\varepsilon}\,dx}_{I_1} + \underbrace{\int_\delta^1 \frac{f(x)}{x+\varepsilon}\,dx}_{I_2} \quad \text{where } 0<\varepsilon \ll \delta \ll 1$$

$$I_1(\varepsilon) = \int_0^{\delta/\varepsilon} \frac{f(\varepsilon y)}{y+1}\,dy \quad (\text{Letting } x = \varepsilon y)$$

$$= \int_0^{\delta/\varepsilon} \frac{1}{y+1}\left[f(0) + \varepsilon y f'(0) + O(\varepsilon^2)\right] dy$$

ok since $\varepsilon y \ll \delta \ll 1$.

$$= \left[f(0)\ln(y+1)\right]_0^{\delta/\varepsilon} + O(\delta)$$

$O(\varepsilon \cdot \frac{\delta}{\varepsilon})$

$$= f(0)\ln\left(1+\frac{\delta}{\varepsilon}\right) + O(\delta)$$

$$= f(0)\ln\left(\frac{\delta}{\varepsilon}\right) + f(0)\ln\left(1+\frac{\varepsilon}{\delta}\right) + O(\delta)$$

$$= -f(0)\ln\varepsilon + f(0)\ln\delta + O\left(\delta, \frac{\varepsilon}{\delta}\right)$$

$$I_2(\varepsilon) = \int_\delta^1 \frac{f(x)}{x+\varepsilon}\,dx$$

$$= \int_\delta^1 \frac{f(x)}{x(1+\varepsilon/x)}\,dx$$

$$= \int_\delta^1 \frac{f(x)}{x}\left(1 - \frac{\varepsilon}{x} + O(\varepsilon^2)\right)dx$$

ok since $\frac{\varepsilon}{x} < \frac{\varepsilon}{\delta} \ll 1$

$$= \int_\delta^1 \frac{f(x)-f(0)}{x}\,dx + \int_\delta^1 \frac{f(0)}{x}\,dx + \dots$$

$$= \int_\delta^1 \frac{f(x)-f(0)}{x}\,dx - f(0)\ln\delta + \dots$$

$$\therefore\ I(\varepsilon) \sim -f(0)\ln\varepsilon + f(0)\ln\delta + \int_\delta^1 \frac{f(x)-f(0)}{x}\,dx - f(0)\ln\delta + \dots$$

$$\sim -f(0)\ln\varepsilon + \int_0^1 \frac{f(x)-f(0)}{x}\,dx + \dots \quad \text{as } \varepsilon \to 0^+.$$