

(a) $\ddot{x} + \varepsilon \dot{x} + x = 0$ as $\varepsilon \to 0^+$

let $x = x(t, T)$ with $T = \varepsilon t$ $\Rightarrow$ $\frac{d}{dt} \mapsto \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}$

substituting: $x_{tt} + 2\varepsilon x_{tT} + \varepsilon^2 x_{TT} + \varepsilon(x_t + \varepsilon x_T) + x = 0$

Expand: $x \sim x_0(t, T) + \varepsilon x_1(t, T) + \cdots$ and collect terms

$O(\varepsilon^0)$: $x_{0tt} + x_0 = 0 \Rightarrow x_0 = \frac{1}{2}(A(T)e^{it} + \bar{A}(T)e^{-it})$

$\therefore x \sim \frac{1}{2}(A(T)e^{it} + \bar{A}(t)e^{-it})$ as $\varepsilon \to 0^+$ with $T = \varepsilon t = O(1)$.

$O(\varepsilon^1)$: $x_{1tt} + x_1 = -2x_{0tT} - x_{0t}$

$= -(iA_T e^{it} - i\bar{A}_T e^{-it}) - \frac{1}{2}(iAe^{it} - i\bar{A}e^{-it})$

$= -i(A_T + \frac{1}{2}A)e^{it} + \text{c.c.}$

Suppress secular terms $(e^{\pm it}) \iff A_T + \frac{1}{2}A = 0$

let $A = Re^{i\theta}$ so that $R_T e^{i\theta} + Ri\theta_T e^{i\theta} + \frac{1}{2}Re^{i\theta} = 0$

$\therefore \theta_T = 0 \Rightarrow \theta = \text{constant} = \theta_0$ $\qquad (\theta_0, R_0 \in \mathbb{R})$

$R_T = -\frac{1}{2}R \Rightarrow R = R_0 e^{-\frac{1}{2}T}$

$\therefore x_0 = R_0 e^{-\frac{1}{2}T}\cos(t + \theta_0)$

Exact solution: $x = r_0 e^{-\frac{1}{2}\varepsilon t}\cos\left((1 - \frac{\varepsilon^2}{4})^{1/2} t + \theta_0\right)$ $\qquad (r_0, \theta_0 \in \mathbb{R})$

$\sim r_0 e^{-\frac{1}{2}T}\cos(t + \theta_0 - \frac{1}{8}\varepsilon^2)$

$\therefore x - x_0 \sim O(\varepsilon)$ for $t = O(\frac{1}{\varepsilon})$.

(b) $\ddot{x} + x = \varepsilon x^3$ as $\varepsilon \to 0^+$.

Let $x = x(t, T)$ with $T = \varepsilon t$, and write $x \sim x_0(t,T) + \varepsilon x_1(t,T) + \ldots$

$$x_{tt} + 2\varepsilon x_{tT} + \varepsilon^2 x_{TT} + x = \varepsilon x^3$$

Substitute and collect terms.

$O(\varepsilon^0)$: $x_{0_{tt}} + x_0 = 0 \Rightarrow x_0(t,T) = \frac{1}{2}\left(A(T)e^{it} + \bar{A}(t)e^{-it}\right)$

$O(\varepsilon^1)$: $x_{1_{tt}} + x_{1_t} = -2x_{0_{tT}} - x_0^3$

$$= -\left(iA_T e^{it} - i\bar{A}_T e^{-it}\right) - \frac{1}{8}\left(Ae^{it} + \bar{A}e^{-it}\right)^3$$

$$= \left[-iA_T + \frac{3}{8}A^2\bar{A}\right]e^{it} + \text{c.c.} + \text{non-secular terms.}$$

Hence to suppress secular terms we need $iA_T = \frac{3}{8}A^2\bar{A}$

Let $A = Re^{i\theta} \Rightarrow i(R_T + iR\theta_T) = \frac{3}{8}R^3$

$\therefore \quad R_T = 0 \Rightarrow R(T) = R_0$

$R\theta_T = -\frac{3}{8}R^3 \Rightarrow \theta_T = -\frac{3}{8}R_0^2 \Rightarrow \theta = -\frac{3}{8}R_0^2 T + \theta_0$

$$\Rightarrow \quad A(T) = R_0 e^{i(\theta_0 - \frac{3}{8}R_0^2 T)}$$

$$= A_0 e^{-\frac{3}{8}|A_0|^2 T}$$

(c) $\ddot{x} + \varepsilon(x^2 - \mu)\dot{x} + x = 0$ as $\varepsilon \to 0^+$

let $x = x(t, T)$ with $T = \varepsilon t$ and write $x = x_0(t,T) + \varepsilon x_1(t,T) + \ldots$

$$x_{tt} + 2\varepsilon x_{tT} + \varepsilon^2 x_{TT} + \varepsilon(x^2 - \mu)(x_t + \varepsilon x_T) + x = 0.$$

$O(\varepsilon^0)$: $\quad x_{0_{tt}} + x_0 = 0 \Rightarrow x_0(t,T) = \frac{1}{2}\left(A(T)e^{it} + \bar{A}(T)e^{-it}\right)$

$$O(\varepsilon^1): \quad x_{1_{tt}} + x_1 = -2x_{0_{tT}} - (x_0^2 - \mu)x_{0_t}$$

$$= -(iA_T e^{it} - i\bar{A}_T e^{-it})$$

$$- \left[\tfrac{1}{4}(Ae^{it} + \bar{A}e^{-it})^2 - \mu\right](iA_T e^{it} - i\bar{A}_T e^{-it})$$

$$= \left[-iA_T - \tfrac{1}{4}A^2\left(-\tfrac{i}{2}\bar{A}\right) - \left(\tfrac{2}{4}A\bar{A} - \mu\right)\tfrac{iA}{2}\right]e^{it} + c.c.$$

+ non-secular terms

Suppress non-secular terms by taking

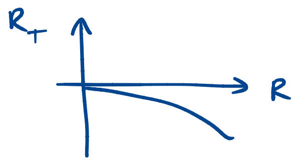
$$-2iA_T + \tfrac{i}{4}A^2\bar{A} - i\left(\tfrac{1}{2}A\bar{A} - \mu\right)A = 0$$

ie $\quad 2A_T = \left(\mu - \frac{|A|^2}{4}\right)A$

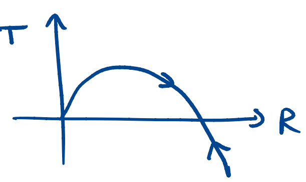
let $A = Re^{i\theta} \Rightarrow \quad 2(R_T + i\theta_T R) = (\mu - \tfrac{1}{4}R^2)R$

$\therefore \quad \theta_T = 0 \Rightarrow \theta = \theta_0$

$$2R_T = (\mu - \tfrac{1}{4}R^2)R$$



$\mu < 0$
$\Rightarrow R_T < 0$
$\Rightarrow R(T) \to 0$



$\mu > 0$
$\Rightarrow R(T) \to 2\sqrt{\mu}$ as $T \to \infty$

$\therefore$ For $\mu < 0$ – system tends to a steady state with $R = 0$, whilst for $\mu > 0$, solution tends to a periodic orbit with period 2π and amplitude $2\sqrt{\mu}$ (at leading order).

$\Leftarrow$ called a Hopf bifurcation.

$\ddot{x} + (1+\varepsilon)x = \cos t$ as $\varepsilon \to 0^+$.

let $x = x(t,T)$ with $T = \varepsilon t$ and expand as $x \sim x_0(t,T) + \varepsilon(t,T) + \dots$

$\Rightarrow \quad x_{tt} + 2\varepsilon x_{tT} + \varepsilon^2 x_{TT} + (1+\varepsilon)x = \cos t$

$O(\varepsilon^0)$: $x_{0tt} + x_0 = \frac{1}{2}(e^{it} + e^{-it})$ ← cannot suppress secular terms!

suggests we have the scaling wrong!

↳ Try instead $x(t,T) \sim \frac{1}{\varepsilon} x_0(t,T) + x_1(t,T) + \dots$

$O(\varepsilon^{-1})$: $x_{0tt} + x_0 = 0 \Rightarrow x_0 = \frac{1}{2}(A(T)e^{it} + \bar{A}(T)e^{-it})$

$$
\begin{aligned}
O(\varepsilon^0): \quad x_{1tt} + x_1 &= -2x_{0tT} - x_0 + \tfrac{1}{2}(e^{it} + e^{-it}) \\
&= -(iA_T e^{it} - i\bar{A}_T e^{-it}) - \tfrac{1}{2}(Ae^{it} + \bar{A}e^{-it}) \\
&\quad + \tfrac{1}{2}(e^{it} + e^{-it}) \\
&= \left[-iA_T - \tfrac{1}{2}A + \tfrac{1}{2}\right]e^{it} + c.c.
\end{aligned}
$$

suppress secular terms : $-iA_T = \frac{1}{2}(A-1)$

let $A = 1 + Re^{i\theta}$ $\Rightarrow$ $i(R_T + iR\theta_T) = \frac{1}{2}R$

$\therefore$ $R_T = 0 \Rightarrow R = R_0$ $\qquad R\theta_T = \frac{1}{2}R \Rightarrow \theta = \frac{1}{2}T + \theta_0$

Hence $A(T) = 1 + R_0 e^{\frac{1}{2}T + \theta_0}$

$\Rightarrow \quad x(t,T) \sim |A(T)| \cos(t + \arg(A(T)))$

$x_0(t,T)$ is periodic with period $2\pi \iff A(0) = 1$

(5)

(b) $\ddot{x} + (1+\varepsilon)x + k\varepsilon x^3 = \cos t \quad$ as $\varepsilon \to 0^+$

Proceed as in (a): let $x = x(t,T)$ with $T = \varepsilon t$ and expand as

$$x(t,T) \sim \frac{1}{\varepsilon} x_0(t,T) + x_1(t,T) + \dots$$

$O(\varepsilon^{-1})$: $\quad x_{0_{tt}} + x_0 = 0 \Rightarrow x_0 = \frac{1}{2}(A(T)e^{it} + \bar{A}(T)e^{-it})$

$O(\varepsilon^0)$: $\quad x_{1_{tt}} + x_1 = -2x_{0_{tT}} - x_0 + \frac{1}{2}(e^{it} + e^{-it}) - kx_0^3$

$$= \left[-iA_T - \frac{1}{2}A + \frac{1}{2} - \frac{3}{8}kA^2\bar{A}\right]e^{it} + c.c.$$

$$+ \text{ non-secular terms}$$

Suppress secular terms: $\quad -iA_T - \frac{1}{2}A + \frac{1}{2} - \frac{3}{8}kA^2\bar{A} = 0$

Let $A(T) = \alpha(T) + i\beta(T)$ so that $x_0(t,T) = \alpha(T)\cos t - \beta(T)\sin t$

$$\Rightarrow -i(\alpha_T + i\beta_T) - \frac{1}{2}(\alpha + i\beta) + \frac{1}{2} - \frac{3}{8}k(\alpha+i\beta)^2(\alpha - i\beta) = 0$$

$$\therefore \quad \beta_T + \frac{1}{2}(1-\alpha) - \frac{3}{8}k(\alpha^2+\beta^2)\alpha = 0 \qquad (1)$$

$$-\alpha_T - \frac{1}{2}\beta - \frac{3}{8}k(\alpha^2+\beta^2)\beta = 0 \qquad (2)$$

Then the solution is periodic with period $t \Leftrightarrow \alpha, \beta$ constant (leading order multiple scales)

From (2), with $k>0$, then $\alpha_T = 0 \Rightarrow -\frac{1}{2}\beta - \frac{3}{8}k(\alpha^2+\beta^2)\beta = 0$

So $\beta = 0$ or $k(\alpha^2+\beta^2) = -\frac{4}{3}$

Cannot happen if $k>0$ ※

From (1) and $\beta = 0$ we have

$$\frac{1}{2}(1-\alpha) - \frac{3}{8}k\alpha^2 = 0$$

Hence we need $A(0) \in \mathbb{R}$ with $A(0) + \frac{3}{4}kA(0)^3 = 1.$

PS4 Q3

$$\frac{d}{dx}\left(D\left(x,\tfrac{x}{\varepsilon}\right)\frac{du}{dx}\right) = f\left(x,\tfrac{x}{\varepsilon}\right)$$ as $\varepsilon \to 0^+$

with $D(x,X) > 0$ and $f(x,X)$ smooth and periodic in $X = \frac{x}{\varepsilon}$ with period one.

(a) let $u = u(x,X)$, $x = \varepsilon X$ $\Rightarrow$ $\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial X}$

$$\Rightarrow \left(\frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial X}\right)\left(D(x,X)\left(\frac{\partial u}{\partial x} + \frac{1}{\varepsilon}\frac{\partial u}{\partial X}\right)\right) = f(x,X)$$

Expand: $u(x,X) = u_0(x,X) + \varepsilon u_1(x,X) + \cdots$ and collect terms

$O(\varepsilon^0)$: $(D u_{0X})_X = 0$

$O(\varepsilon^1)$: $(D(u_{1X} + u_{0x}))_X + (D u_{0X})_x = 0$

$O(\varepsilon^2)$: $(D(u_{2X} + u_{1x}))_X + (D(u_{1X} + u_{0x}))_x = f$

(b) $(D u_{0X})_X = 0 \Rightarrow D u_{0X} = a_1(x)$

$$u_0 = a_0(x) + a_1(x)\int_0^X \frac{ds}{D(x,s)}$$

Given $u_0(x,X)$ is periodic in X with period one, then we have

$$a_0(x) = u_0(x,0) = u_0(x,1) = a_0(x) + a_1(x)\int_0^1 \frac{ds}{D(x,s)}.$$

Then, since $D(x,s) > 0$, $\int_0^1 \frac{ds}{D(x,s)} > 0$ and so $a_1(x) \equiv 0$.

$\therefore$ $u_0(x,X) = a_0(x)$ ie a fn of x only.

From the $O(\varepsilon^1)$ eqn, $D(u_{1X} + u_{0x}) = b_1(x)$

$$u_1 = b_0(x) - u_{0x}X + b_1(x)\int_0^X \frac{ds}{D(x,s)}$$

Given $u_1(x,X)$ is periodic in X with period one, then we have

$$b_0(x) = u_1(x,0) = u_1(x,1) = b_0(x) - a_0'(x) + b_1(x)\int_0^1 \frac{ds}{D(x,s)}$$

$$\therefore \quad b_1(x) = a_0'(x)\left[\int_0^1 \frac{ds}{D(x,s)}\right]^{-1} = \hat{D}(x)\,a_0'(x) \qquad (*)$$

At $O(\varepsilon^2)$: $\quad (D(u_{2X} + u_{1x}))_X = f - b_{1x}$

$$\Rightarrow \underbrace{D(u_{2X} + u_{1x})}_{} = c_1(x) + \int_0^X f(x,s)\,ds - b_{1x} X$$

Both u_1 and u_2 periodic in X with period one

$$\left.\begin{aligned} \text{Then } u_{2X} &= \lim_{h\to 0} \frac{u_2(x, X+h) - u_2(x,X)}{h} \\ u_{1x} &= \lim_{h\to 0} \frac{u_1(x+h, X) - u_1(x,X)}{h} \end{aligned}\right\} \text{periodic in } X \text{ with period one}$$

$$\left.\begin{aligned} \therefore \quad c_1(x) &= D(u_{2X} + u_{1x})\big|_{X=0} \\ &= D(u_{2X} + u_{1x})\big|_{X=1} \\ &= c_1(x) + \int_0^1 f(x,s)\,ds - b_{1x} \end{aligned}\right] \Rightarrow b_{1x} = \int_0^1 f(x,s)\,ds$$

Hence, the homogenised equation for $u_0(x)$ is (using ⊛)

$$\frac{d}{dx}\left(\hat{D}(x)\frac{du_0}{dx}\right) = \hat{f}(x) \quad \text{with} \quad \hat{D}(x) = \left(\int_0^1 \frac{ds}{D(x,s)}\right)^{-1}$$

harmonic average of D over one period

$$\hat{f}(x) = \int_0^1 f(x,s)\,ds$$

↗ note the different averages for the diffusivity and the net production term.

PS4 Q4

WKB expansion: $y(x) = A(x)e^{i\mu(x)/\varepsilon}$

$$y'(x) = e^{i\mu(x)/\varepsilon}\left[A' + \frac{iA\mu'}{\varepsilon}\right]$$

$$y''(x) = e^{i\mu(x)/\varepsilon}\left[A'' + \frac{iA\mu''}{\varepsilon} + \frac{2iA'\mu'}{\varepsilon} - \frac{A(\mu')^2}{\varepsilon^2}\right]$$

(a) $\varepsilon^2 y'' + xy = 0$ as $\varepsilon \to 0^+$ with $x > 0$.

$$\varepsilon^2 A'' + iA\mu''\varepsilon + 2iA'\mu'\varepsilon - A(\mu')^2 + xA = 0$$

Expand: $A(x) \sim A_0(x) + \varepsilon A_1(x) + \dots$ and collect terms:

$O(\varepsilon^0)$: $-A_0(\mu')^2 + xA_0 = 0 \Rightarrow \mu' = \pm x^{1/2} \Rightarrow \mu = \pm \frac{2}{3}x^{3/2}$

(for $A_0 \neq 0$)

$O(\varepsilon^1)$: $-A_1(\mu')^2 + 2iA_0'\mu' + iA_0\mu'' + xA_1 = 0$

Imaginary part: $2A_0' x^{\frac{1}{2}} + A_0 \cdot \frac{1}{2}x^{-\frac{1}{2}} = 0$

$$\Rightarrow \frac{A_0'}{A_0} = -\frac{1}{4x}$$

$$\ln|A_0| = c_1 - \frac{1}{4}\ln x \qquad (c_1 \in \mathbb{R})$$

$$\therefore A_0 = \frac{c_2}{x^{1/4}}$$

Hence, $y_+(x) = \frac{c_2^+}{x^{1/4}} e^{2ix^{3/2}/3\varepsilon}$, $y_-(x) = \frac{c_2^-}{x^{1/4}} e^{-2ix^{3/2}/3\varepsilon}$ as $\varepsilon \to 0^+$

(b) $\varepsilon^2 y'' - xy = 0$ for $x > 0$

$$\Rightarrow u = \pm\frac{2ix^{3/2}}{3} \text{ and } A_0 = \frac{c_2^\pm}{x^{1/4}}$$

Hence $y_+(x) = \frac{c_2^+}{x^{1/4}} e^{-2x^{3/2}/3\varepsilon}$, $y_-(x) = \frac{c_2^-}{x^{1/4}} e^{+2x^{3/2}/3\varepsilon}$ as $\varepsilon \to 0^+$

Valid for $u = O(1)$ as $\varepsilon \to 0^+ \Rightarrow$ loses validity when $x = O(\varepsilon^{2/3})$.

PS4 Q5

$\varepsilon y'' + y' + xy = 0$ as $\varepsilon \to 0^+$ with $0 < x < 1$ and $y(0) = 0$, $y(1) = 1$.

(a) WKB expansion: $y(x) = e^{S(x)/\varepsilon}$ with $S(x) = S_0 + \varepsilon S_1 + \dots$

$$y'(x) = \frac{S'(x)}{\varepsilon} e^{S(x)/\varepsilon} \quad \text{and} \quad y''(x) = \left[\frac{(S')^2}{\varepsilon^2} + \frac{S''}{\varepsilon} \right] e^{S(x)/\varepsilon}$$

$$\Rightarrow \quad (S')^2 + S' + \varepsilon(S'' + x) = 0$$

Expanding and collecting terms:

$O(\varepsilon^0)$: $\quad (S_0')^2 + S_0' = 0 \Rightarrow S_0' = 0$ or $S_0' = -1$

$\therefore \quad S_0(x) = A_1, \quad S_0(x) = B_1 - x \quad (A_1, B_1 \in \mathbb{R})$

$O(\varepsilon^1)$: $\quad 2S_0' S_1' + S_1' + S_0'' + x = 0$

$S_0(x) = A_1 \Rightarrow S_1' = -x \Rightarrow S_1(x) = A_2 - \frac{1}{2}x^2 \quad (A_2 \in \mathbb{R})$

$S_0(x) = B_1 - x \Rightarrow S_1' = x \Rightarrow S_1(x) = B_2 + \frac{1}{2}x^2 \quad (B_2 \in \mathbb{R})$

$\therefore$ General solution is $y \sim A_3 e^{-\frac{1}{2}x^2} + B_3 e^{-\frac{x}{\varepsilon} + \frac{1}{2}x^2} \quad (A_3, B_3 \in \mathbb{R})$

Boundary conditions: $y(0) = 0 \Rightarrow A_3 \sim B_3$

$y(1) = 1 \Rightarrow A_3 e^{-\frac{1}{2}} + B_3 e^{-\frac{1}{\varepsilon} + \frac{1}{2}} \sim 1$

$$\therefore \quad A_3 \sim -B_3 \sim \frac{1}{e^{-1/2} - e^{-1/\varepsilon + 1/2}} = \frac{e^{\frac{1}{2}}}{1 - e^{1 - 1/\varepsilon}}$$

Hence
$$y \sim \frac{e^{(1-x^2)/2} - e^{-x/\varepsilon + (1+x^2)/2}}{1 - e^{1-1/\varepsilon}} \quad \text{as } \varepsilon \to 0^+$$

(b) $\varepsilon y'' + y' + xy = 0$ for $0<x<1$ with $y(0)=0$ and $y(1)=1$

- Seek a BL at $x=1$: $x = 1+\varepsilon X$, $y(x) = Y(X)$ with $X<0$, $X = \text{ord}(1)$

$$\Rightarrow \quad Y'' + Y' + \varepsilon(1+\varepsilon X)Y = 0 \quad \Rightarrow \quad Y_0(X) = c_1 + c_2 e^{-X} \quad (c_1, c_2 \in \mathbb{R})$$

then, matching with the outer solution with require $Y_0(-\infty)$ finite and hence $c_2 = 0$ so that $Y_0(X) = c_1 = 1$ and there is no BL at $x=1$.

- Seek a BL at $x=0$: $x = \varepsilon X$, $y(x) = Y(X)$ with $X>0$, $X = \text{ord}(1)$.

$$\Rightarrow Y'' + Y' + \varepsilon^2 XY = 0 \quad \text{with} \quad Y(0) = 0$$

Expand as $Y \sim Y_0 + \varepsilon Y_1 + \dots$ and collect terms:

$O(\varepsilon^0)$ $\quad Y_0'' + Y_0' = 0$ with $Y_0(0) = 0 \Rightarrow Y_0(X) = E_1(1-e^{-X})$

$(E_1 \in \mathbb{R})$

- Solution in outer region:

Expand as $y \sim y_0 + \varepsilon y_1 + \dots$ and collect terms:

$O(\varepsilon^0)$: $\quad y_0' + xy_0 = 0$ with $y_0(1) = 1$

$$\Rightarrow \frac{y_0'}{y_0} = -x \Rightarrow \ln|y_0| = D_1 - \tfrac{1}{2}x^2$$

$$\therefore \quad y_0(x) = e^{(1-x^2)/2}$$

Matching: $(1 \text{ to } 0) = e^{(1-x^2)/2}$ $\quad$ | $\quad (1 \text{ t } i) = E_1(1-e^{-X})$

$= e^{(1-\varepsilon^2 X^2)/2}$ $\quad$ | $\quad = E_1(1-e^{-x/\varepsilon})$

$\sim e^{\frac{1}{2}}$ $\quad$ | $\quad \sim E_1$

$$\Rightarrow E_1 = e^{\frac{1}{2}}$$

Composite expansion: $y \sim y_0(x) + Y_0(x/\varepsilon) - (1 \text{ t } i)(1 \text{ t } o)$

$$= e^{(1-x^2)/2} - e^{\frac{1}{2} - \frac{x}{\varepsilon}} \quad \text{as } \varepsilon \to 0^+$$

PS4 Q6

(a) $\varepsilon^2 y'' + (1-x)y = 0$ as $\varepsilon \to 0^+$ for $x > 0$ with $y(0) = 1$, $y(\infty) = 0$

Let $x = 1 + \varepsilon^{2/3} X$ and $y(x) = Y(X)$ $\Rightarrow$ $Y'' - XY = 0$

for $X > -\varepsilon^{-2/3}$

$\therefore$ $Y(X) = a\,Ai(X) + b\,Bi(X)$ $(a, b \in \mathbb{R})$

with $Y(-\varepsilon^{-2/3}) = 1$, $Y(\infty) = 0$

$Y(\infty) = 0 \Rightarrow b = 0$, $Y(-\varepsilon^{-2/3}) = 1 \Rightarrow a\,Ai(-\varepsilon^{-2/3}) = 1$

$$\therefore\ y(x) = Y(X) = \frac{Ai(X)}{Ai(-\varepsilon^{-2/3})} = \frac{Ai(\varepsilon^{-\frac{2}{3}}(x-1))}{Ai(-\varepsilon^{-\frac{2}{3}})}$$

(b) WKB expansion: $y(x) = A(x)e^{i\varphi(x)/\varepsilon}$

$\Rightarrow$ $y'(x) = \left(\frac{iA\varphi'}{\varepsilon} + A'\right)e^{i\varphi/\varepsilon}$, $y''(x) = \left(-\frac{A(\varphi')^2}{\varepsilon^2} + \frac{2iA'\varphi'}{\varepsilon} + \frac{iA\varphi''}{\varepsilon} + A''\right)e^{i\varphi/\varepsilon}$

$\therefore$ $-A(\varphi')^2 + \varepsilon(2iA'\varphi' + iA\varphi'') + \varepsilon^2 A'' + (1-x)A = 0$

Expand as $A \sim A_0 + \varepsilon A_1 + \dots$ and collect terms:

$O(\varepsilon^0)$: $-A_0(\varphi')^2 + (1-x)A_0 = 0 \Rightarrow \varphi' = \pm(1-x)^{\frac{1}{2}}$ $(A_0 \neq 0)$

$\varphi = \pm\frac{2}{3}(1-x)^{\frac{3}{2}} + c_1$

$O(\varepsilon^1)$: $-A_1(\varphi')^2 + 2iA_0'\varphi' + iA_0\varphi'' + (1-x)A_1 = 0$

$\Rightarrow$ $(A_0^2\varphi')' = 0$ (collecting imaginary parts)

$\Rightarrow$ $A_0^2 = \frac{\tilde{c}_2}{\varphi'}$

$\Rightarrow$ $A_0 = \frac{c_2}{(1-x)^{1/4}}$

character of solution depends on whether $x>1$ or $x<1$.

RH outer solution $(x>1)$

$y(\infty)=0 \Rightarrow$ need to eliminate the growing solution and so

$$y(x) \sim \frac{c_1}{(x-1)^{1/4}} e^{-\frac{2}{3\varepsilon}(x-1)^{3/2}} \quad \text{as } \varepsilon \to 0^+ \text{ with } x>1,\ x-1=\text{ord}(1)$$

$(c_1 \in \mathbb{R})$

LH outer solution $(0<x<1)$

- Both roots $\left(\varphi = \pm\frac{2}{3}(1-x)^{3/2}\right)$ admissible. Write solution as

$$y \sim \frac{c_2}{(1-x)^{1/4}} \sin\left(\frac{2}{3\varepsilon}(1-x)^{3/2}+\alpha_2\right) \quad \text{as } \varepsilon \to 0^+ \text{ with } 0<x<1$$

and $x=\text{ord}(1)$.

$(c_2, \alpha_2 \in \mathbb{R})$

BC: $y(0)=1 \Rightarrow c_2 \sin\left(\frac{2}{3\varepsilon}+\alpha_2\right)=1$

$$\therefore \quad y(x) \sim \frac{\operatorname{cosec}\left(\frac{2}{3\varepsilon}+\alpha_2\right)}{(1-x)^{1/4}} \sin\left(\frac{2}{3\varepsilon}(1-x)^{3/2}+\alpha_2\right)$$

Inner region (near $x=1$)

Note that, due to factors $(1-x)^{-1/4}$, both outer solutions are unbounded as $x \to 1^{\pm}$. So seek an inner solution of the form

$$y=\delta(\varepsilon)^{-1/4}\, Y(X) \quad \text{with} \quad x=1+\delta(\varepsilon)X$$

$$\Rightarrow \quad Y''-XY=0 \quad \text{provided } \delta(\varepsilon)=\varepsilon^{2/3}$$

$$\therefore \quad Y(X) = c_3 \operatorname{Ai}(X) + c_4 \operatorname{Bi}(X) \qquad (c_3, c_4 \in \mathbb{R})$$

NB $\quad \operatorname{Ai}(X) \sim \dfrac{1}{2\sqrt{\pi}X^{1/4}} e^{-2/3 X^{3/2}}$ and $\operatorname{Bi}(X) \sim \dfrac{1}{\sqrt{\pi}X^{1/4}} e^{2/3 X^{3/2}}$

as $X \to \infty$

Matching inner solution $(X \to \infty)$ with RH outer solution $(x \to 1^+)$

Use an intermediate variable: $x-1=\varepsilon^{\alpha}\hat{x}=\varepsilon^{2/3}X \quad (0<\alpha<\frac{2}{3})$

$(\hat{x}>0)$

$$X = \frac{\hat{x}}{\varepsilon^{2/3-\alpha}} \to \infty \text{ as } \varepsilon \to 0^+ \text{ with } \hat{x} > 0,\ \hat{x} = \operatorname{ord}(1)$$

$$\delta^{-\frac{1}{4}} Y\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right) = \frac{c_3}{\varepsilon^{1/6}} \mathrm{Ai}\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right) + \frac{c_4}{\varepsilon^{1/6}} \mathrm{Bi}\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right)$$

$$\sim \frac{c_3}{\varepsilon^{1/6}} \frac{1}{2\sqrt{\pi}\left(\hat{x}/\varepsilon^{2/3-\alpha}\right)^{1/4}} e^{-2/3\left(\hat{x}/\varepsilon^{2/3-\alpha}\right)^{3/2}}$$

$$+ \frac{c_4}{\varepsilon^{1/6}} \frac{1}{\sqrt{\pi}\left(\hat{x}/\varepsilon^{2/3-\alpha}\right)^{1/4}} e^{2/3\left(\hat{x}/\varepsilon^{2/3-\alpha}\right)^{3/2}}$$

and

$$y(x) \sim \frac{c_1}{(x-1)^{1/4}} e^{-2/3\varepsilon (x-1)^{3/2}} \qquad x = 1 + \varepsilon^\alpha \hat{x} \to 1^+ \text{ as } \varepsilon \to 0^+$$

with $\hat{x} > 0,\ \hat{x} = \operatorname{ord}(1)$

$$\therefore\ y(1+\varepsilon^\alpha \hat{x}) \sim \frac{c_1}{(\varepsilon^\alpha \hat{x})^{1/4}} e^{-2/3\varepsilon (\varepsilon^\alpha \hat{x})^{3/2}}$$

Matching $\Rightarrow$ $c_4 = 0$ and $c_1 = \dfrac{c_3}{2\sqrt{\pi}}$

Matching inner solution $(X \to -\infty)$ with LH outer solution $(x \to 1^-)$

$$X = \frac{\hat{x}}{\varepsilon^{2/3-\alpha}} \to -\infty \text{ as } \varepsilon \to 0^+ \text{ with } \hat{x} < 0,\ \hat{x} = \operatorname{ord}(1).$$

$$\delta^{-\frac{1}{4}} Y\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right) \sim \frac{c_3}{\varepsilon^{1/6}} \mathrm{Ai}\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right)$$

$$\sim \frac{c_3}{\varepsilon^{1/6}} \frac{1}{\sqrt{\pi}\left(-\hat{x}/\varepsilon^{2/3-\alpha}\right)} \sin\left(\frac{2}{3}\left(\frac{-\hat{x}}{\varepsilon^{2/3-\alpha}}\right)^{3/2} + \frac{\pi}{4}\right)$$

$$x = 1 + \varepsilon^\alpha \hat{x} \to 1^+ \text{ as } \varepsilon \to 0^+ \text{ with } \hat{x} < 0,\ \hat{x} = \operatorname{ord}(1)$$

$$y(1+\varepsilon^\alpha \hat{x}) \sim \frac{\operatorname{cosec}\left(\frac{2}{3\varepsilon} + \alpha_2\right)}{(-\varepsilon^\alpha \hat{x})^{1/4}} \sin\left(\frac{2}{3\varepsilon}(-\varepsilon^\alpha \hat{x})^{3/2} + \alpha_2\right)$$

Matching $\Rightarrow$ $\alpha_2 = \dfrac{\pi}{4}$ (WLOG) and $\dfrac{c_3}{\sqrt{\pi}} = \operatorname{cosec}\left(\dfrac{2}{3\varepsilon} + \dfrac{\pi}{4}\right)$