C5.5 Perturbation Methods Lecture Notes

Introduction

- Important to be able to make precise approximations to solutions of problems across applied mathematics
- Two methods – numerical methods
  – analytical (asymptotic) methods

  } not in competition, but <u>complementary</u>.

- { Perturbation methods – good for situations where some parameter(s) is large or small.
  Numerical methods – work best when all parameters are $O(1)$.

- Agreement between the two is reassuring, but often analytical methods provide more insight.

- Aim of this course: to provide an introduction to a range of methods that can be used to better understand the nature of solutions to applied maths problems. Note that it's more of an art than a science in many ways – we will learn the guidelines but experience is everything!

Chapter 1 Algebraic equations

Example Suppose we want to solve $x^2 + \varepsilon x - 1 = 0$ where ε is a small parameter.

In this case, we can solve to give

$$x = \tfrac{1}{2}\left[-\varepsilon \pm \sqrt{\varepsilon^2+4}\right] = -\frac{\varepsilon}{2} \pm \underbrace{\sqrt{1+\left(\tfrac{\varepsilon}{2}\right)^2}}_{1+\frac{1}{2}\left(\frac{\varepsilon}{2}\right)^2+\cdots}.$$

The binomial theorem gives

$$x = \begin{cases} +1 - \dfrac{\varepsilon}{2} + \dfrac{\varepsilon^2}{8} - \dfrac{\varepsilon^4}{128} + \cdots \\ -1 - \dfrac{\varepsilon}{2} - \dfrac{\varepsilon^2}{8} + \dfrac{\varepsilon^4}{128} + \cdots \end{cases}$$

with convergence for $\left|\frac{\varepsilon}{2}\right| < 1$ ie $|\varepsilon| < 2$.

- Most important is the 'quality' of the expansion, in the sense of how good the truncated expansions are at approximating the roots when ε is small.

For $\varepsilon = 0.1$

$x \sim 1.0$	1 term	
0.95	2 terms	
0.95125	3 terms	
0.951249	4 terms	
$= 0.95124922\ldots$	exact.	

We aim to get better and better approximations as we take more terms.

Here, we first found the solution and then approximated. But usually we won't know the solution, so we will need to make the approximation first!

1.1 The iterative method

We want to iteratively find solutions by letting $x_{n+1} = g(x_n; \varepsilon)$

Then if x^* is a root we have $x^* = g(x^*; \varepsilon)$, and if $|x_n - x^*| \ll 1$ we have

$$
\begin{aligned}
x_{n+1} - x^* &= g(x_n; \varepsilon) - x^* \\
&= g(x^* + (x_n - x^*); \varepsilon) - x^* \\
&= \underbrace{(g(x^*; \varepsilon) - x^*)}_{=0} + (x_n - x^*) g'(x^*; \varepsilon) + \dots
\end{aligned}
$$

Hence, whether the iteration converges, and how quickly it converges, depends on $|g'(x^*; \varepsilon)|$.

For our example problem, $x^2 + \varepsilon x - 1 = 0$, we take (for the +ve root)

$$g(x; \varepsilon) = \sqrt{1 - \varepsilon x} \quad \text{so that} \quad x_{n+1} = \sqrt{1 - \varepsilon x_n}.$$

We have
$$g'(x^*; \varepsilon) = \frac{-\varepsilon/2}{\sqrt{1 - \varepsilon x^*}} \approx -\frac{\varepsilon}{2}$$

we know that $x^* \approx 1$ ie $\varepsilon x^* \ll 1$

Hence the iteration converges - at each round we get approximately a factor of $\varepsilon/2$ closer.

Now we need to think about where to start (this will potentially affect the ability of the iteration to converge.

A sensible choice for the starting point, x_0, is the solution for $\varepsilon = 0$. Here, we have $x_0 = 1$.

(binomial expansion)

$$\Rightarrow \quad x_1 = \sqrt{1 - \varepsilon} = \underbrace{1 - \frac{\varepsilon}{2}}_{\text{correct to } O(\varepsilon)} \underbrace{- \frac{\varepsilon^2}{8} - \frac{\varepsilon^3}{16}}_{\text{higher order terms incorrect.}} + \dots$$

Hence going forward we only need to keep the first two terms: $x_1 = 1 - \frac{\varepsilon}{2}$.

$$x_2 = \sqrt{1-\varepsilon\left(1-\frac{\varepsilon}{2}\right)} = 1 - \frac{\varepsilon}{2}\left(1-\frac{\varepsilon}{2}\right) - \frac{\varepsilon^2}{8}\left(1-\frac{\varepsilon}{2}\right)^2 - \frac{\varepsilon^3}{16}\left(1-\frac{\varepsilon}{2}\right)^3 + \cdots$$

$$= \underbrace{1 - \frac{\varepsilon}{2} + \frac{\varepsilon^2}{8}}_{\text{correct to } O(\varepsilon^2)} + \underbrace{\frac{\varepsilon^3}{16} + \cdots}_{\text{higher order terms incorrect.}}$$

Notes

- At each iteration, more and more terms are correct, but more and more work is required!
- The only way to check a term is correct is to proceed to the next iteration and see if it changes.
- For fast convergence, we want $|g'(x^*;\varepsilon)|$ small. More generally, we try to choose $g(x;\varepsilon)$ s.t. $g'(x^*;\varepsilon)$ exists and $|g'(x^*;\varepsilon)| \to 0$ as $\varepsilon \to 0$.
- The usual procedure is to place the dominant term on the LH side. (As we will see later, the dominant term can be adjusted by scaling.)

1.2 Expansion method (much more common)

Here, we set $\varepsilon = 0$ and find the unperturbed roots ($x = \pm 1$). Then, we pose an expansion about one of the roots of the form

$$x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \dots \qquad \text{(+ve root)}$$

Need to find the x_i, which are independent of ε.

We substitute the expansion into the original equation ($x^2 + \varepsilon x - 1 = 0$):

$$(1 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \dots)^2 + \varepsilon(1 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \dots) - 1 = 0$$

Expand to give

$$1 + 2x_1\varepsilon + (x_1^2 + 2x_2)\varepsilon^2 + (2x_1x_2 + 2x_3)\varepsilon^3 + \dots + \varepsilon + \varepsilon^2 x_1 + \varepsilon^3 x_2 + \dots - 1 = 0$$

Collect terms of the same order in ε together:

$$(1-1) + (2x_1+1)\varepsilon + (x_1^2 + 2x_2 + x_1)\varepsilon^2 + (2x_1x_2 + 2x_3 + x_2)\varepsilon^3 + \dots = 0$$

Equate coefficients in powers of ε: (we can do this because the approximation is valid for any suff. small ε)

ε^0: $1 - 1 = 0$ ✓✓ automatically satisfied since we started with the correct value.

ε^1: $2x_1 + 1 = 0 \Rightarrow x_1 = -\frac{1}{2}$

ε^2: $x_1^2 + 2x_2 + x_1 = 0 \Rightarrow x_2 = \frac{1}{8}$

ε^3: $2x_1x_2 + 2x_3 + x_2 = 0 \Rightarrow x_3 = 0$

Note: The expansion method is easier than the iterative method when working to high orders. However, we might not know the form of the expansion a priori (and we do need to assume one!) – if we use the wrong expansion, the method will break down (we will see such examples later on).

1.3 Singular perturbations

(Another example where finding perturbation solutions becomes difficult!)

What is a singular perturbation? Consider the problem

$$\varepsilon x^2 + x - 1 = 0$$

← for $\varepsilon = 0$ there is one root ($x = 1$), but for $\varepsilon \neq 0$ there are two roots.

This is an example of a singular perturbation problem in which the limit solution ($\varepsilon = 0$) differs in an important way from the limit $\varepsilon \to 0$. (Problems which are not singular are regular.)

To see what is happening, let's look at the exact solutions:

$$\varepsilon x^2 + x - 1 = 0 \quad \Rightarrow \quad x = \frac{1}{2\varepsilon}\left[-1 \pm \sqrt{1+4\varepsilon}\right]$$

For small ε, we can expand the $\sqrt{\ }$ term to give

$$x = \begin{cases} 1 - \varepsilon + 2\varepsilon^2 - 5\varepsilon^4 + \dots \\ -\frac{1}{\varepsilon} - 1 + \varepsilon - 2\varepsilon^2 + 5\varepsilon^4 + \dots \end{cases}$$

valid for $|4\varepsilon| < 1$ ie $|\varepsilon| < \frac{1}{4}$.

↑ This term means that the second root tends to $x = -\infty$ as $\varepsilon \to 0$ – this is a key feature of these types of problems.

Let's see what happens when we try to use the two methods we have looked at so far.

Iterative method

There are two options

① $g(x;\varepsilon) = 1 - \varepsilon x^2$ (1st root)

② $g(x;\varepsilon) = \dfrac{1-x}{\varepsilon x}$ (2nd root)

WHY??

Recall that we need $g'(x;\varepsilon)$ small close to the root for this to work.

① $\frac{d}{dx}(1-\varepsilon x^2) = 2\varepsilon x$ ← This will be small near $x=0$, but not near $x=-\frac{1}{\varepsilon}$

② $\frac{d}{dx}\left(\frac{1-x}{\varepsilon x}\right) = \frac{-\varepsilon}{\varepsilon^2 x^2}$ ← This is small near $x=-\frac{1}{\varepsilon}$ but not near $x=0$.

- To do this, we needed an idea of the scale of the root. We will see what to do when this isn't the case later on.

Expansion method

To capture the second root, we take $x = \frac{x_{-1}}{\varepsilon} + x_0 + \varepsilon x_1 + \dots$ (other root is as before)

Substitute into the equation ($\varepsilon x^2 + x - 1 = 0$):

$$\varepsilon\left(\frac{x_{-1}}{\varepsilon} + x_0 + \varepsilon x_1 + \dots\right)^2 + \left(\frac{x_{-1}}{\varepsilon} + x_0 + \varepsilon x_1 + \dots\right) - 1 = 0$$

Expand: $\frac{1}{\varepsilon}x_{-1}^2 + 2x_{-1}x_0 + \varepsilon(2x_{-1}x_0 + x_0^2) + \dots + \frac{1}{\varepsilon}x_{-1} + x_0 + \varepsilon x_1 + \dots - 1 = 0$

Collect terms in powers of ε:

singular root

$\frac{1}{\varepsilon}$: $x_{-1}^2 + x_{-1} = 0 \Rightarrow x_{-1} = -1$ or 0 ← regular root

ε^0: $2x_{-1}x_0 + x_0 - 1 = 0 \Rightarrow x_0 = -1$ | $x_0 = 1$

ε^1: $2x_{-1}x_0 + x_0^2 + x_1 = 0 \Rightarrow \underbrace{x_1 = 1}_{\text{singular}}$ | $\underbrace{x_1 = -1}_{\text{regular}}$

Regular root: $x = 1 - \varepsilon + \dots$

Singular root: $x = -\frac{1}{\varepsilon} - 1 + \varepsilon + \dots$

1.4 Rescaling the equation

For singular problems - a useful idea is to rescale the original equation.

For the previous problem ($\varepsilon x^2 + x - 1 = 0$) we let $x = \frac{X}{\varepsilon}$ so that

$$X^2 + X - \varepsilon = 0$$ ← this is now a regular problem

∴ The problem of finding the correct starting point can be viewed as the problem of finding the right re-scaling to regularise the singular problem.

There are some different approaches...

1.4.1 systematic approach: general rescaling

(need this for both the expansion and iterative approaches)

Pose a general scale factor and let

$$x = \underbrace{\delta(\varepsilon)}_{\text{scale factor}} X$$ ← strictly order 1 as $\varepsilon \to 0$.

This gives $\varepsilon\delta^2 X^2 + \delta X - 1 = 0$.

We consider the dominant balance in the equation as $\delta(\varepsilon)$ varies (small → large).

① $\delta(\varepsilon) \ll 1$

$$\underbrace{\underbrace{\varepsilon\delta^2 X^2}_{\text{small}} + \underbrace{\delta X}_{\text{small}} - 1}_{\text{Cannot balance the zero on the RHS}} = 0$$

② $\delta(\varepsilon) = 1$

$$\underbrace{\varepsilon\delta^2 X^2}_{\text{small}} + \underbrace{\delta X}_{\sim X} - 1 = 0 \Rightarrow X = 1 + \text{small}$$

This is the regular root (which we got w/o scaling).

③ $1 \ll \delta(\varepsilon) \ll \frac{1}{\varepsilon}$

$$\frac{\varepsilon\delta^2 X^2 + \delta X - 1}{\delta} = \underbrace{\text{small} + X + \text{small}}$$

LHS/δ

$$\frac{\varepsilon\delta^2 X^2}{\delta} = \underbrace{\varepsilon\delta}_{\ll 1} X^2 \Rightarrow \text{small}$$

Can only balance the zero on the RHS if $X = 0$, but this violates assumption $X \sim O(1)$.

As we keep increasing δ, we see that the dominance of the δX term will be broken when $\delta = \frac{1}{\varepsilon}$ (since then $\varepsilon\delta^2X^2$ also relevant).

④ $\delta(\varepsilon) = \frac{1}{\varepsilon}$

$$\frac{\varepsilon\delta^2X^2 + \delta X - 1}{\varepsilon\delta^2} = X^2 + X + \text{small}$$

LHS$/\varepsilon\delta^2$

$\frac{\delta}{\varepsilon\delta^2} \sim O(1)$ and $\frac{1}{\varepsilon\delta^2} = o(\varepsilon)$

$\therefore$ Balance is $X^2 + X = X(X+1) = 0 \Rightarrow X = -1$ (as per the singular root)

So, if we rescale $x = \frac{X}{\varepsilon}$ then we can find a regular expansion in X, or, we don't rescale, but include the $\frac{X-1}{\varepsilon}$ term.

⑤ $\delta \gg \frac{1}{\varepsilon}$

$$\frac{\varepsilon\delta^2X^2 + \delta X - 1}{\varepsilon\delta^2} = \underbrace{X^2 + \text{small} + \text{small}}$$

$\frac{\delta}{\varepsilon\delta^2} \ll 1$, $\frac{1}{\varepsilon\delta^2} \ll 1$

Cannot balance the zero on the RHS with $X \sim O(1)$ ✗

SUMMARY: we proceed by varying δ from small to large in order to identify dominant balances.

Scalings that yield dominant balances are known as distinguished limits.

1.4.2 Alternative approach: pairwise comparison

- Pairwise comparison of terms - quicker when you only have a small number of terms!

- To get a sensible answer, we need at least two terms to be in balance.

↑ Here 1st + 2nd, 1st and 3rd, 2nd and 3rd.

$$\underbrace{\varepsilon\delta^2X^2}_{1} + \underbrace{\delta X}_{2} - \underbrace{1}_{3} = 0$$

1 and 2 $\quad \varepsilon\delta^2 \sim \delta \Rightarrow \delta \sim \frac{1}{\varepsilon}$ ie $x = \frac{X}{\varepsilon}$ (and 1, 2 dominate 3)

(gives singular root)

1 and 3 $\quad \varepsilon\delta^2 \sim 1 \Rightarrow \delta = \frac{1}{\sqrt{\varepsilon}}$ BUT this doesn't give a dominant balance because 2 then dominates.

2 and 3 $\quad \delta \sim 1$ ie no rescaling needed. (2, 3 dominate 1)

(gives regular root).

1.5 Non-integral powers

(powers might not always be integers!)

Example $(1-\varepsilon)x^2 - 2x + 1 = 0$ with $\varepsilon \ll 1$.

We know that $x = \dfrac{1 \pm \sqrt{\varepsilon}}{1-\varepsilon} = (1 \pm \varepsilon^{\frac{1}{2}})\underbrace{(1 - \varepsilon + \varepsilon^2 + \dots)}_{\text{binomial expansion}}$

$$= 1 \pm \varepsilon^{\frac{1}{2}} - \varepsilon \pm \varepsilon^{\frac{3}{2}} + \dots$$

Setting $\varepsilon = 0$ gives $x = 1$ as the double root (sign of danger to come!).
We will proceed as usual (knowing something will go wrong) to see what happens.

Pose the expansion $x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$

Substitute into the equation ($(1-\varepsilon)x^2 - 2x + 1 = 0$)

$$(1-\varepsilon)(1 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)^2 - 2(1 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots) + 1 = 0$$

Expanding:

$$1 + 2x_1\varepsilon + (2x_2 + x_1^2)\varepsilon^2 + \dots - \varepsilon - 2x_1\varepsilon^2 + \dots - 2 - 2x_1\varepsilon - 2x_2\varepsilon^2 + \dots + 1 = 0$$

Coefficients of powers of ε:

$O(\varepsilon^0)$: $1 - 2 + 1 = 0$ ✓ (since we started with the correct value, $x = 1$, at $\varepsilon = 0$).

$O(\varepsilon^1)$: $2x_1 - 1 - 2x_1 = 0$ — cannot be satisfied by any value of x_1 (except $x_1 = \infty$ in some sense...)

The cause of the difficulty: look at the exact solution

$x = \dfrac{1}{1 \pm \sqrt{\varepsilon}}$ — for the largest root $x = 1 + \varepsilon^{\frac{1}{2}} + \varepsilon + \varepsilon^{\frac{3}{2}} + \dots$

We should have expanded in powers of $\varepsilon^{\frac{1}{2}}$!

(This is what the $x_1 = \infty$ constraint is hinting at: the scaling on x_1 is too small...)

And, in retrospect, we could have noticed/guessed that a change in x of order $\sqrt{\varepsilon}$ would be required for an order ε change in the LHS at its minimum...

Instead, pose the expansion $x = 1 + \varepsilon^{\frac{1}{2}} x_{\frac{1}{2}} + \varepsilon x_1 + \dots$

Substitute into the equation $((1-\varepsilon)x^2 - 2x + 1 = 0)$:

$$(1-\varepsilon)\left(1+\varepsilon^{\frac{1}{2}}x_{\frac{1}{2}} + \varepsilon x_1 + \dots\right)^2 - 2\left(1+\varepsilon^{\frac{1}{2}}x_{\frac{1}{2}} + \varepsilon x_1 + \dots\right) + 1 = 0$$

Expand:
$$\left\{\begin{array}{l} 1 + 2x_{\frac{1}{2}}\varepsilon^{\frac{1}{2}} + (2x_1 + x_{\frac{1}{2}}^2)\varepsilon + (2x_{\frac{3}{2}} + 2x_{\frac{1}{2}}x_1)\varepsilon^{\frac{3}{2}} + \dots \\ -\varepsilon - 2x_{\frac{1}{2}}\varepsilon^{\frac{3}{2}} + \dots - 2 - 2x_{\frac{1}{2}}\varepsilon^{\frac{1}{2}} - 2x_1\varepsilon - 2x_{\frac{3}{2}}\varepsilon^{\frac{3}{2}} + \dots + 1 \end{array}\right\} = 0$$

Comparing coefficients of ε:

$O(\varepsilon^0)$: $1 - 2 + 1 = 0$ ✓✓ (we had the correct guess for the x_0 term)

$O(\varepsilon^{\frac{1}{2}})$: $2x_{\frac{1}{2}} - 2x_{\frac{1}{2}} = 0$ ✓

$O(\varepsilon^1)$: $2x_1 + x_{\frac{1}{2}}^2 - 1 - 2x_1 = 0 \Rightarrow x_{\frac{1}{2}}^2 = 1 \Rightarrow x_{\frac{1}{2}} = \pm 1$

$O(\varepsilon^{\frac{3}{2}})$: $2x_{\frac{3}{2}} + 2x_{\frac{1}{2}}x_1 - 2x_{\frac{1}{2}} - 2x_{\frac{3}{2}} = 0 \Rightarrow x_1 = 1$ for both roots

NB – each term is determined at a higher level than we might have anticipated...

1.6 Finding the right expansion sequence

How can we determine the expansion sequence if we don't have the exact solution?

Pose $x = 1 + \delta_1(\varepsilon) x_1$ with $\delta_1(\varepsilon) \ll 1$

Substitute into $(1-\varepsilon)x^2 - 2x + 1 = 0$

$$(1-\varepsilon)(1+\delta_1 x_1)^2 - 2(1+\delta_1 x_1) + 1 = 0$$

$$\Rightarrow \quad 1 + 2\delta_1 x_1 + \delta_1^2 x_1^2 - \varepsilon - 2\delta_1 x_1 \varepsilon - \delta_1^2 x_1^2 \varepsilon - 2 - 2\delta_1 x_1 + 1 = 0$$

Simplify:

$$\cancel{1} + \cancel{2\delta_1 x_1} + \delta_1^2 x_1^2 - \varepsilon - 2\delta_1 x_1 \varepsilon - \delta_1^2 x_1^2 \varepsilon - \cancel{2} - \cancel{2\delta_1 x_1} + \cancel{1} = 0$$

$$\Rightarrow \quad \underbrace{\delta_1^2 x_1^2}_{1} - \underbrace{\varepsilon}_{2} - \underbrace{2\delta_1 x_1 \varepsilon}_{3} - \underbrace{\delta_1^2 x_1^2 \varepsilon}_{4} = 0$$

Play dominant balance game again:

$\varepsilon\delta_1 \ll \varepsilon$ so 3 $\ll$ 2 and 4 $\ll$ 2 $\Rightarrow$ leading terms are $\underbrace{\delta_1^2 x_1^2}_{1}$, $\underbrace{\varepsilon}_{2}$

To get a sensible balance: need $\delta_1 = \varepsilon^{\frac{1}{2}}$.

In this case, $\cancel{\varepsilon} x_1^2 - \cancel{\varepsilon} - 2\varepsilon^{\frac{3}{2}} x_1 - \varepsilon^2 x_1^2 = 0$

$$x_1^2 - 1 = 0 \Rightarrow x_1 = \pm 1.$$

To proceed to higher order – repeat: let $x = 1 + \varepsilon^{\frac{1}{2}} + \delta_2 x_2$ with $\delta_2 \ll \varepsilon^{\frac{1}{2}} = \delta_1(\varepsilon)$.

Substitute into $(1-\varepsilon)x^2 - 2x + 1 = 0$:

$$(1-\varepsilon)(\underbrace{1 + \varepsilon^{\frac{1}{2}} + \delta_2 x_2)^2}_{1 + 2\varepsilon^{\frac{1}{2}} + 2\delta_2 x_2 + 2\varepsilon^{\frac{1}{2}}\delta_2 x_2 + \varepsilon + \delta_2^2 x_2^2} - 2(1 + \varepsilon^{\frac{1}{2}} + \delta_2 x_2) + 1 = 0$$

$$\Rightarrow \left\{ \begin{aligned} &[1 + 2\varepsilon^{\frac{1}{2}} + 2\delta_2 x_2 + 2\varepsilon^{\frac{1}{2}}\delta_2 x_2 + \varepsilon + \delta_2^2 x_2^2] \\ &- [\varepsilon + 2\varepsilon^{\frac{3}{2}} + \varepsilon^2 + 2\varepsilon\delta_2 x_2 + 2\varepsilon^{3/2}\delta_2 x_2 + \varepsilon\delta_2^2 x_2^2] \\ &- [2 + 2\varepsilon^{\frac{1}{2}} + 2\delta_2 x_2] + 1 \end{aligned} \right\} = 0$$

Simplifying

$$\left\{\begin{array}{l}[\cancel{1}+\cancel{2\varepsilon^{\frac{1}{2}}}+\cancel{2\delta_2 x_2}+2\varepsilon^{\frac{1}{2}}\delta_2 x_2+\cancel{\varepsilon}+\delta_2^2 x_2^2] \\ -[\cancel{\varepsilon}+2\varepsilon^{\frac{3}{2}}+\varepsilon^2+2\varepsilon\delta_2 x_2+2\varepsilon^{3/2}\delta_2 x_2+\varepsilon\delta_2^2 x_2^2] \\ -[\cancel{\varepsilon}+\cancel{2\varepsilon^{\frac{1}{2}}}+\cancel{2\delta_2 x_2}]+\cancel{1}\end{array}\right\} = 0$$

$$\Rightarrow \underbrace{\underset{1}{2\varepsilon^{\frac{1}{2}}\delta_2 x_2}+\underset{2}{\delta_2^2 x_2^2}-\underset{3}{2\varepsilon^{\frac{3}{2}}}}_{}-\underset{4}{\varepsilon^2}-\underset{5}{2\varepsilon\delta_2 x_2}-\underset{6}{2\varepsilon^{\frac{3}{2}}\delta_2 x_2}-\underset{7}{\varepsilon\delta_2^2 x_2^2}=0$$

4 << 3   5 << 1   6 << 1   7 << 2

only three terms to consider ☺

Then, since $\delta_2 \ll \varepsilon^{\frac{1}{2}}$ 2 << 1 and so we must have the dominant terms 1 ~ 3 ie $\varepsilon^{\frac{1}{2}}\delta_2 = \varepsilon^{\frac{3}{2}}$

$$\Rightarrow \delta_2 = \varepsilon.$$

Then, $2\varepsilon^{\frac{1}{2}}\delta_2 x_2 - 2\varepsilon^{\frac{3}{2}} = 0 \Rightarrow x_2 = 1$

and $x = 1 + \varepsilon^{\frac{1}{2}} + \varepsilon \ldots$

1.7 Iterative method

- Can be very useful in cases where the expansion sequence isn't known!

Recall: $(1-\varepsilon)x^2 - 2x + 1 = 0 \Longrightarrow x^2 - 2x + 1 = \varepsilon x^2$

$$(x-1)^2 = \varepsilon x^2$$

$\Rightarrow$ let $g(x;\varepsilon) = 1 \pm \sqrt{\varepsilon}\,x$ so that $x_{n+1} = 1 \pm \sqrt{\varepsilon}\,x_n$.

Starting with $x_0 = 1$ (+ve root):

$$\left\{\begin{array}{l} g(x;\varepsilon) = 1+\sqrt{\varepsilon}\,x \\ g'(x;\varepsilon) = \sqrt{\varepsilon} \\ \quad \to 0 \text{ as } \varepsilon \to 0 \end{array}\right.$$

✓✓

$x_1 = 1+\sqrt{\varepsilon}$

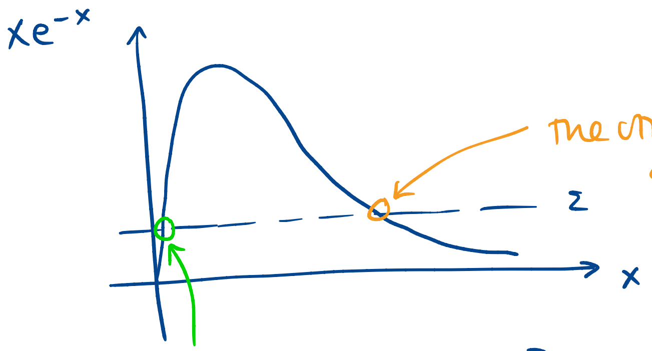
$x_2 = 1+\sqrt{\varepsilon}(1+\sqrt{\varepsilon})$

$= 1+\sqrt{\varepsilon}+\varepsilon$

generates terms <u>very</u> quickly compared to the expansion method!

1.7.1 Logarithms (sometimes the expansion is not even in powers of ε!)

Consider the transcendental eqn $\quad xe^{-x} = \varepsilon \quad (0<\varepsilon\ll 1)$



The other root gets large as $\varepsilon \to 0$ and is more difficult to find.

one root close to $x=0$ - easy to approximate

$\varepsilon=0$ has $x=0$ as a solution
- suggests trying $x = 0+\varepsilon x_1+\varepsilon^2 x_2+\dots$
and expanding e^{-x}.

Consider an iterative procedure for the large root:

$$xe^{-x} = \varepsilon \Rightarrow \frac{1}{x}e^{x} = \frac{1}{\varepsilon}$$

$$-\log x + x - \log\left(\tfrac{1}{\varepsilon}\right) = 0$$

$$\text{ie}\quad g(x;\varepsilon) = \log\left(\tfrac{1}{\varepsilon}\right) + \log x$$

$$x_{n+1} = \log\left(\tfrac{1}{\varepsilon}\right) + \log x$$

Why this? Because when $x=\log\left(\frac{1}{\varepsilon}\right)$ we have $xe^{-x} = \varepsilon\log\left(\frac{1}{\varepsilon}\right) \gg \varepsilon$ but when $x = 2\log\left(\frac{1}{\varepsilon}\right)$ $xe^{-x} = 2\varepsilon^2\log\left(\frac{1}{\varepsilon}\right) \ll \varepsilon$ ie over this range x is slowly varying whilst e^{-x} is rapidly varying.

Note that for large x, $x \gg \log x$

$\Rightarrow$ root is roughly around $x = \log\left(\frac{1}{\varepsilon}\right)$.

We have $g'(x;\varepsilon) = \frac{1}{x}$ which is small for $x \sim \log\left(\frac{1}{\varepsilon}\right)$ //

(However $g'(x;\varepsilon)$ is not that small so unless ε is very small we might expect to see relatively slow convergence...)

Taking $x_0 = \log\left(\frac{1}{\varepsilon}\right) \Rightarrow x_1 = \log\left(\frac{1}{\varepsilon}\right) + \log\left(\log\left(\frac{1}{\varepsilon}\right)\right)$

$$x_2 = \log\left(\tfrac{1}{\varepsilon}\right) + \log\left[\log\left(\tfrac{1}{\varepsilon}\right) + \log\left(\log\left(\tfrac{1}{\varepsilon}\right)\right)\right]$$

Simplify by writing $\log(1+\delta) \simeq \delta$ for $|\delta| \ll 1 \dots$

Note that we would actually need to calculate x_3 to check the first two terms are correct...

↳ see the printed lecture notes!

NB Difficult sequence to guess! Also, having terms such as $\log(\log(\frac{1}{\varepsilon}))$ means the asymptotic approximation is only a good approximation for very small values of ε.

(Normally, we'd hope to get away with $\varepsilon = 0.5$ or 0.1 but here $\varepsilon = 10^{-9}$ gives $\log(\log(\frac{1}{\varepsilon})) \approx 3$!!)