2 Asymptotic approximations

2.1 Convergence

Series $\sum_{n=0}^{\infty} f_n(z)$ is said to converge at fixed z iff, given arbitrary $\varepsilon > 0$, we can find $N_0(z, \varepsilon)$ s.t. $\left| \sum_{n=M}^{N} f_n(z) \right| < \varepsilon \;\; \forall\, M, N > N_0$

Series $\sum_{n=0}^{\infty} f_n(z)$ is said to converge to a function $f(z)$ at a fixed value of z, iff given arbitrary $\varepsilon > 0$, we can find $N_0(z, \varepsilon)$ s.t.

$$\left| \sum_{n=0}^{N} f_n(z) - f(z) \right| < \varepsilon \;\; \forall\, N > N_0.$$

$\therefore$ Series converges iff terms decay sufficiently rapidly as $n \to \infty$.

NB Not always all that useful, in the sense that we can have series that don't converge but nonetheless provide very good approximations!

Example $\quad \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt \quad (z \in \mathbb{C})$

analytic in entire complex plane
$\Rightarrow$ has Taylor series expansion with infinite radius of convergence (converges $\forall t$) $\Rightarrow e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!}$

So, integrate term by term (can swap $\Sigma, \int$) to give

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)\, n!} = \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \frac{z^9}{216} - \frac{z^{11}}{1320} + \dots \right)$$

For an accuracy of 10^{-5}: $\quad z = 2$ — 16 terms
$z = 3$ — 31 terms
$z = 5$ — 75 terms !!

Also, intermediate terms get very large (lots of cancellation of +ve/−ve terms)
$\Rightarrow$ issues with round off from a computational perspective.

Problem: truncated sums are very different from the converged limit (ie approximation doesn't get better with more terms, unless you have a lot!)

Alternative approach: write $\operatorname{erf}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt$

↳ It's going to give a divergent series, but the approximation will be much better!

Integrate by parts: $\int_z^\infty e^{-t^2} dt = \int_z^\infty \underbrace{\frac{1}{2t}}_{u} \cdot \underbrace{2te^{-t^2}}_{dv/dt} dt = \frac{e^{-z^2}}{2z} - \int_z^\infty \frac{e^{-t^2}}{2t^2} dt$

$\Rightarrow du/dt = \frac{-1}{2t^2}$ $\quad v = -e^{-t^2}$

Continuing the integration by parts:

$$\operatorname{erf}(z) = 1 - \frac{e^{-z^2}}{z\sqrt{\pi}} \left(1 - \frac{1}{2z^2} + \frac{1.3}{(2z^2)^2} - \frac{1.3.5}{(2z^2)^3} + \dots \right)$$

diverges $\forall z$: has radius of convergence equal to zero.

Truncated series very useful: for $z = 2.5$ then three terms $\Rightarrow$ accuracy of 10^{-5}. At $z = 3$ we only need two terms (recall 31 for the other series !!)

Why does it work? ① The leading term is almost correct

② Adding subsequent terms gets us closer (because the terms are of decreasing size, at least initially...)

and the early ones are usually enough.

This is an asymptotic series (that is not convergent).

2.2 Asymptoticness

For a sequence: $\{f_n(\varepsilon)\}_{n \in \mathbb{N}_0}$ is said to be asymptotic if, $\forall n \geq 1$,

$$\frac{f_n(\varepsilon)}{f_{n-1}(\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0.$$

(ie ratio of successive terms $\to 0$ as $\varepsilon \to 0$).

$\sum_{n=0}^\infty f_n(\varepsilon)$ is said to be an asymptotic approximation to (or asymptotic expansion of) a function $f(\varepsilon)$ as $\varepsilon \to 0$ if, $\forall N \geq 0$,

$$\frac{f(\varepsilon) - \sum_{n=0}^N f_n(\varepsilon)}{f_N(\varepsilon)} \to 0 \text{ as } \varepsilon \to 0$$

ie remainder smaller than the last term included once ε sufficiently small.

We usually write $f \sim \sum_{n=0}^{\infty} f_n(\varepsilon)$ as $\varepsilon \to 0$. (and don't generally worry about getting more than the first few terms...)

Often $\underbrace{f \sim \sum_{n=0}^{\infty} a_n \varepsilon^n}_{\text{asymptotic power series}}$ ie the $f_n(\varepsilon)$ are powers of ε x coefficient.

2.3 Order notation

'Big O' $f = O(g)$ iff $\exists\, K>0, \varepsilon_0>0$ s.t. $|f| < K|g| \;\forall\, \varepsilon < \varepsilon_0$.

'little o' $f = o(g)$ as $\varepsilon \to 0 \Rightarrow \frac{f}{g} \to 0$ as $\varepsilon \to 0$. (stronger statement: $\forall\, \delta>0, \exists\, \varepsilon_0$ s.t. $|f| \le \delta|g| \;\forall\, \varepsilon<\varepsilon_0$.)

Hence ① $f_n(\varepsilon)$ is an asymptotic sequence iff $f_n = o(f_{n-1})$

② $f \sim \sum_{n=0}^{\infty} f_n$ iff $f - \sum_{n=0}^{N} f_n = o(f_N) \;\forall\, N \geq 0$.

'ord' $f(\varepsilon) = \mathrm{ord}(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ iff $\exists\, K \in \mathbb{R}\setminus\{0\}$ s.t. $\frac{f(\varepsilon)}{g(\varepsilon)} \to K$ as $\varepsilon \to \varepsilon_0$

NB $f(\varepsilon) = O(g(\varepsilon)) \not\Rightarrow f(\varepsilon) = \mathrm{ord}(g(\varepsilon))$

(but common to write O instead of ord when it's clear what the meaning is from the context. Eg. "$X = \delta(\varepsilon) X$ with $\delta(\varepsilon) \to 0$, $X = \mathrm{ord}(1)$ as $\varepsilon \to 0$".

Examples

$\sin x = O(x)$ as $x \to 0$, $\sin x = O(1)$ as $x \to \infty$, $\sin x = o(1)$ as $x \to 0$

$\log x = O(x)$ as $x \to \infty$, $\log x = o(x)$ as $x \to \infty$, $\log x = o(x^{-\delta})$ as $x \to 0$ for any $\delta > 0$.

2.4 Uniqueness and manipulation of asymptotic series

If a function has an asymptotic approximation in terms of an asymptotic sequence, then that approximation is unique for that particular sequence.

If we have $f \sim \sum_{n=0}^{\infty} a_n \delta_n(\varepsilon)$ for given $\{\delta_n(\varepsilon)\}_{n\in\mathbb{N}_0}$, then

$$a_K = \lim_{\varepsilon \to 0} \frac{f(\varepsilon) - \sum_{n=0}^{K-1} a_n \delta_n(\varepsilon)}{\delta_K(\varepsilon)}$$

(evaluate inductively)

NB uniqueness – for a given sequence. BUT, a sequence may have many asymptotic approximations, each in terms of a different sequence.

Eg. $\tan(\varepsilon) \sim \varepsilon + \frac{\varepsilon^3}{3} + \frac{2\varepsilon^5}{15} + \ldots$

$$\sim \sin\varepsilon + \frac{1}{2}(\sin\varepsilon)^3 + \frac{3}{8}(\sin\varepsilon)^5 + \ldots$$

$$\sim \varepsilon\cosh\left(\sqrt{\tfrac{2}{3}}\,\varepsilon\right) + \frac{31}{270}\left(\varepsilon\cosh\left(\sqrt{\tfrac{2}{3}}\,\varepsilon\right)\right)^2 + \ldots$$

NB uniqueness – also for a given function: two functions can share the same asymptotic approximation because they differ by an amount smaller than the last term included.

Eg.
$$e^{\varepsilon} \sim \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \quad \text{as } \varepsilon \to 0$$

$$e^{\varepsilon} + e^{-\frac{1}{\varepsilon^2}} \sim \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \quad \text{as } \varepsilon \to 0^+$$

Two functions that share the same asymptotic power series can only differ by a function which is not analytic.

because two analytic functions with the same power series are identical.

NB Asymptotic approximations can be naively added, subtracted, multiplied or divided.

NB we can substitute one asymptotic series into another.

↳ we need to take care when doing this with exponentials though!

eg. $f(z) = e^{z^2}$ and $z(\varepsilon) = \frac{1}{\varepsilon} + \varepsilon$

$$\Rightarrow f(z(\varepsilon)) = e^{(\frac{1}{\varepsilon}+\varepsilon)^2} = e^{\frac{1}{\varepsilon^2}} e^2 e^{\varepsilon^2} = e^{\frac{1}{\varepsilon^2}} e^2 \underbrace{\left(1 + \varepsilon^2 + \frac{\varepsilon^4}{4} + \ldots\right)}_{1+O(\varepsilon^2)}$$

But, if we only take the leading term in z ie let $z \approx \frac{1}{\varepsilon}$ then we miss the factor e^2

To avoid this issue: need to calculate exponents to $O(1)$, not just leading order. (sin, cos are exponentials here too..)

NB we can integrate asymptotic expansions term by term wrt. $\varepsilon \Rightarrow$ correct asymptotic expansion of the integral.

BUT we can't (in general) differentiate with safety.

often higher order terms that we have neglected become important.

eg. $f(\varepsilon) = \varepsilon \cos\left(\frac{1}{\varepsilon}\right) = O(\varepsilon)$ as $\varepsilon \to 0$

$f'(\varepsilon) = \cos\left(\frac{1}{\varepsilon}\right) + \frac{1}{\varepsilon}\sin\left(\frac{1}{\varepsilon}\right) = O\left(\frac{1}{\varepsilon}\right)$ as $\varepsilon \to 0$.

But when we differentiate the asymptotic expansion:

$$\frac{d}{d\varepsilon}\left[\varepsilon\left(1 + \frac{\varepsilon^2}{2} + \dots\right)\right] = 1 \text{ instead of } O\left(\frac{1}{\varepsilon}\right)!$$

2.5 Numerical use of divergent series

- usually the first few terms in a sequence are enough (for a desired accuracy).
  - and if we need better accuracy we just add more terms.
    → this is problematic if the series is divergent! Clearly, we should stop when the terms start getting larger — known as the optimal truncation.

2.6 Parametric expansions

More generally, we will want to consider eg $f(x;\varepsilon)$ ie functions that depend also on x.

- eg have a differential equation in x, which depends on small parameter ε (hence parametric expansion..).

We usually write the asymptotic expansion as

$$f(x;\varepsilon) \sim \sum_{n=0}^{\infty} \underbrace{a_n(x)}_{\text{coefficients depend on } x} \delta_n(\varepsilon) \quad \text{as } \varepsilon \to 0$$

$$\Leftrightarrow \frac{1}{\delta_N(\varepsilon)}\left[f(x;\varepsilon) - \sum_{n=0}^{N} a_n(x)\delta_n(\varepsilon)\right] \to 0 \quad \text{as } \varepsilon \to 0.$$

3 Asymptotic approximation of integrals

– Range of different approaches to approximate integrals with either very large or very small parameters.

3.1 Integration by parts

(have already seen this for erf(z))

Example 1 If $f(\varepsilon)$ is differentiable near $\varepsilon=0$ then we can study local behaviour of $f(\varepsilon)$ near $\varepsilon=0$ using IBPs.

$$f(\varepsilon) = f(0) + \int_0^\varepsilon f'(x)\,dx$$

↑ assuming f' differentiable near $x=0$.

IBP: let $\frac{dv}{dx}=1$ and $v=(x-\varepsilon)$, $u=f'(x) \Rightarrow \frac{du}{dx}=f''(x)$

Then

$$f(\varepsilon) = f(0) + \underbrace{\left[(x-\varepsilon)f'(\varepsilon)\right]_0^\varepsilon}_{\varepsilon f'(0)} + \int_0^\varepsilon (\varepsilon-x)f''(x)\,dx$$

repeat $N-1$ times...

$$= \sum_{n=0}^{N} \frac{\varepsilon^n f^{(n)}(0)}{n!} + \underbrace{\frac{1}{N!}\int_0^\varepsilon (\varepsilon-x)^N f^{(N+1)}(x)\,dx}_{\text{remainder, } R_N}$$

If R_N exists $\forall N$ and sufficiently small $\varepsilon>0$ then

$$f(\varepsilon) \sim \sum_{n=0}^{\infty} \frac{\varepsilon^n f^{(n)}(0)}{n!} \quad \text{as } \varepsilon\to 0$$

If the series converges then it's just the Taylor expansion of f about $\varepsilon=0$:)

Example 2 $I(x) = \int_x^\infty e^{-t^4}\,dt$ ← want an expansion as $x\to\infty$. (There's no Taylor series to help!)

Write $$I(x) = \int_x^\infty e^{-t^4}\,dt = \int_x^\infty \underbrace{\frac{-1}{4t^3}}_{u}\cdot\underbrace{(-4t^3)e^{-t^4}}_{\frac{dv}{dt}}\,dt$$

$\frac{du}{dt} = \frac{3}{4}t^{-4}$ $\qquad \frac{dv}{dt} \Rightarrow v = e^{-t^4}$

Then $I(x) = \left[\frac{-e^{-t^4}}{4t^3} \right]_x^\infty - \frac{3}{4} \int_x^\infty \frac{1}{t^4} e^{-t^4} dt$

$$= \frac{e^{-x^4}}{4x^3} - \frac{3}{4} \int_x^\infty \underbrace{\frac{1}{t^4} e^{-t^4}} dt$$

This term is much smaller than the original integrand (e^{-t^4})

more formally:

The first term is the leading order asymptotic approximation because

$$J = \int_x^\infty \frac{1}{t^4} e^{-t^4} dt < \frac{1}{x^4} \int e^{-t^4} dt = \frac{1}{x^4} I(x) \ll I(x) \text{ as } x \to \infty$$

In fact,

$$J < \frac{1}{x^4} e^{-x^4} \int_x^\infty e^{-(t^4 - x^4)} dt$$

$\leftarrow (t-x)(t+x)(x^2+t^2)$ — let $u = t-x$ in the integral

$$= \frac{1}{x^4} e^{-x^4} \int_0^\infty e^{-u(u+2x)((u+x)^2+x^2)} du$$

$$< \frac{1}{x^4} e^{-x^4} \int_0^\infty e^{-u^4} du$$

$$< \frac{1}{x^4} e^{-x^4} \underbrace{\int_0^\infty e^{-u^2} du}_{O(1)}$$

So,

$$J = O\left(\frac{e^{-x^4}}{x^4} \right)$$

and $I(x) \sim \dfrac{e^{-x^4}}{4x^3}$ as $x \to \infty$.

(can get more terms by repeated integration...)

Example 3 (Sometimes IBPs fails!)

$$I(x) = \int_0^x t^{-\frac{1}{2}} e^{-t} dt$$

$u = t^{-\frac{1}{2}} \Rightarrow \frac{du}{dt} = -\frac{1}{2} t^{-\frac{3}{2}}$, $\frac{dv}{dt} = e^{-t} \Rightarrow v = -e^{-t}$

Taking a naive approach gives

$$I(x) = \left[-t^{-\frac{1}{2}} e^{-t} \right]_0^x - \frac{1}{2} \int_0^x \underbrace{t^{-\frac{3}{2}}}_{\text{not integrable}} e^{-t} dt$$

$\Rightarrow$ end up with '$\infty - \infty$'

But, it's pretty simple to fix! Let

$$I(x) = \underbrace{\int_0^\infty t^{-\frac{1}{2}} e^{-t} dt}_{\text{can evaluate by letting } u = t^{\frac{1}{2}}\text{, answer is } \Gamma(\frac{1}{2}) = \sqrt{\pi}} - \underbrace{\int_x^\infty t^{-\frac{1}{2}} e^{-t} dt}_{\text{can integrate by parts}}$$

$\leftarrow$ because contributions from end points vanishes.

$$I(x) = \sqrt{\pi} - \underbrace{\int_x^\infty t^{-\frac{1}{2}} e^{-t}\,dt}$$

rewrite as $+\int_x^\infty t^{-\frac{1}{2}} \frac{d}{dt}(e^{-t})\,dt$ $\quad \frac{dv}{dt} = e^{-t}$ $\quad u = t^{-\frac{1}{2}}$

$v = -e^{-t}$ $\quad \frac{du}{dt} = -\frac{1}{2}t^{-\frac{3}{2}}$

$$= \sqrt{\pi} - \frac{e^{-x}}{\sqrt{x}} + \frac{1}{2}\underbrace{\int_x^\infty t^{-\frac{3}{2}} e^{-t}\,dt}$$

power of the integrand is reduced, so we anticipate that $e^{-x}/\sqrt{x}$ is the leading term and this term gives the correction

$$< \frac{1}{x^{\frac{3}{2}}}\int_x^\infty e^{-t}\,dt = \frac{e^{-x}}{x^{3/2}} \ll \frac{e^{-x}}{x^{\frac{1}{2}}}$$

which is indeed the case :)

General rule Integration by parts will <u>not</u> work if the contribution from one of the limits of integration is much larger than the size of the integral.

(For the example above, $I(x)$ is finite $\forall\, x > 0$, but in the integral, the endpoint $t=0$ has a singularity — and it's made worse by differentiating!).

Example 4 (Another example of a failure).

$$I(x) = \int_0^\infty e^{-xt^2}\,dt = \frac{1}{2}\sqrt{\frac{\pi}{x}} \qquad (x>0).$$

(compute the integral directly using $u = \sqrt{x}\,t$)

IBPs:
$$I(x) = \int_0^\infty \left(\frac{-1}{2xt}\right)\left(-2xte^{-xt^2}\right)dt$$

$u = \frac{-1}{2xt} \Rightarrow \frac{du}{dt} = \frac{1}{2xt^2}$

$\frac{dv}{dt} = -2xte^{-xt^2} \Rightarrow v = e^{-xt^2}$

$$= \left[\frac{-e^{-xt^2}}{2xt}\right]_0^\infty - \int_0^\infty \frac{1}{2xt^2} e^{-xt^2}\,dt$$

both of these are problematic, and there's no simple fix!

NB IBPs will also not work when the dominant contribution to the integral comes from an interior point rather than an end point.

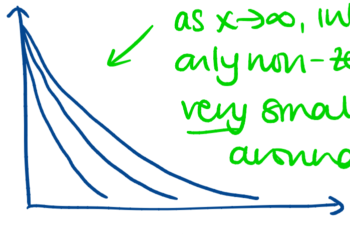
Summary – IBPs is simple, and gives an explicit error term that can be bounded. BUT, limited applicability.

3.2 Laplace's method

General technique for integrals of the form $I(x) = \int_a^b f(t) e^{x\varphi(t)} dt$ as $x \to \infty$.

($f(t), \varphi(t)$ — real, continuous functions, and $[a,b]$ generally an interval on the real line.)

Example $I(x) = \int_0^{10} \frac{e^{-xt}}{(1+t)} dt$ as $x \to \infty$



as $x \to \infty$, integrand only non-zero on a very small region around $x=0$.

Split the range of integration:

$$I(x) = \underbrace{\int_0^{\varepsilon} \frac{e^{-xt}}{(1+t)} dt}_{I_1} + \underbrace{\int_{\varepsilon}^{10} \frac{e^{-xt}}{(1+t)} dt}_{I_2}$$

We want to choose ε s.t. the dominant contribution comes from $I_1 \Rightarrow$ choose ε small but $\varepsilon \gg \frac{1}{x}$ ie $\frac{1}{x} \ll \varepsilon \ll 1$.

$$I_2 < \int_{\varepsilon}^{10} e^{-xt} dt$$

$$= \underbrace{e^{-x\varepsilon}}_{} - e^{-10x}$$

$x\varepsilon \gg 1$ $\Rightarrow$ negligible

Hence I_2 is asymptotically small compared to I_1

Then

$$I_1(x) = \int_0^{\varepsilon} \frac{e^{-xt}}{(1+t)} dt$$

let $s = xt \Rightarrow dt = \frac{1}{x} ds$

$$= \int_0^{x\varepsilon} \frac{e^{-s}}{1 + s/x} \cdot \frac{1}{x} ds$$

$$= \frac{1}{x} \int_0^{x\varepsilon} e^{-s} \left(\sum_{n=0}^{\infty} \left(\frac{-s}{x}\right)^n \right) ds$$

swap $\int, \Sigma$ since inside radius of convergence of Σ.

$$= \frac{1}{x} \sum_{n=0}^{\infty} \underbrace{\left[\int_0^{x\varepsilon} s^n e^{-s} ds \right]}_{} \frac{(-1)^n}{x^n}$$

$$= \int_0^{\infty} s^n e^{-s} ds - \underbrace{\int_{\varepsilon x}^{\infty} s^n e^{-s} ds}_{=k_n,\ \text{anticipate small}}$$

$$= n! - k_n$$

↑ by iteration (or seeing that it's a Γ function!)

Looking at k_n: $k_n = \int_{x\varepsilon}^{\infty} s^n e^{-s} ds = \underbrace{(x\varepsilon)^n e^{-x\varepsilon}}_{\sim \text{exponentially small}} + n \int_0^{x\varepsilon} s^{n-1} e^{-s} ds$ (IBPs)

$$= n! \int_{x\varepsilon}^{\infty} e^{-s} ds + \text{exponentially small terms}$$

also exp. small! $= n!\, e^{-x\varepsilon} + \text{exp. small terms.}$

$\therefore$ k_n always exponentially small compared to the rest of the terms.

Putting it all back together

$$I(x) \sim I_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^{n+1}} \quad \text{as } x \to \infty$$

3.3 Watson's Lemma

(Generally very useful, can be used to justify Laplace's method)

Consider $\quad I(x) = \int_0^b f(t) e^{-xt} \, dt \quad (b>0)$.

Suppose that ① $f(t)$ is continuous on $[0, b]$

② has the asymptotic expansion $f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n t^{\beta n}$ as $t \to 0^+$

where $\alpha > -1$ and $\beta > 0$, so the integral converges at $t=0$.

(NB If $b = \infty$ then we also need $f(t) \ll e^{ct}$ as $t \to \infty$ for some $c>0$ so that the integral converges at $t = \infty$.)

Watson's lemma states that

$$I(x) \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha + \beta n + 1}} \quad \text{as } x \to \infty.$$

$\Gamma(m) = \int_0^{\infty} t^{m-1} e^{-t} dt$
$= (m-1)!$ for $m \in \mathbb{N}$.

Derivation - essentially the same as in the example, so long as the asymptotic series is uniformly convergent in a neighbourhood of $t=0$ (often the case). If not - can't exchange $\int, \Sigma$ so we work with a finite number of terms, N, and show that, for any N,

$$I(x) = \sum_{n=0}^{N} \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha + \beta n + 1}} + O\left(\frac{1}{x^{\alpha + \beta N + 1}}\right) \quad \text{as } x \to \infty$$

3.4 Asymptotic expansion of general Laplace integrals

$$I(x) = \int_a^b f(t) e^{x\phi(t)} dt$$ ← dominant contribution from region where $\phi(t)$ is the largest.

Three cases: ① max. @ $t=a$
② max. @ $t=b$
③ max. @ $t=c$ with $a<c<b$ (ie interior point).

In each case we argue: (i) dominant contribution from around max. of ϕ - so reduce domain of integration to this region.
(ii) expand f, ϕ in Taylor series about max here.
(iii) rescaling the integration variable means we can replace integration limits by ∞ and introduce only exp. small errors.

make integral as simple as possible.

Case ① – max. @ $t=a$

$$I(x) = \underbrace{\int_a^{a+\varepsilon} f(t)e^{x\varphi(t)}dt}_{I_1} + \underbrace{\int_{a+\varepsilon}^{b} f(t)e^{x\varphi(t)}dt}_{I_2}$$

max. of $\varphi(t)$ @ $t=a$
$\Rightarrow$ assume $\varphi'(a)<0$,
and also that
$f(a)\neq 0$, $\varphi''(a)\neq 0$.

new ε as a small parameter – yet suff. large that dominant contribution to the integral from I_1.

Need to work out how big ε must be in order for $|I_1| \gg |I_2|$.

consider $\underbrace{e^{x\varphi(a+\varepsilon)}}_{\text{size of } e^{x\varphi(t)} \text{ at the 'start' of } I_2} \ll \underbrace{e^{x\varphi(a)}}_{\text{largest value of } e^{x\varphi(t)}}$ ← we want this to hold!

Taylor expand to write $\varphi(a+\varepsilon) = \varphi(a) + \varepsilon\varphi'(a) + \dots$

$$\Rightarrow e^{x\varepsilon\underbrace{\varphi'(a)}_{\varphi'(a)<0}} \ll 1 \quad \Rightarrow \quad x\varepsilon \ll 1 \text{ i.e. } \boxed{0 < \tfrac{1}{x} \ll \varepsilon}$$

consider the first integral:

$$I_1(x) = \int_a^{a+\varepsilon} f(t)e^{x\varphi(t)}dt$$ } expand f, φ as asymptotic series about $x=a$

$$= \int_a^{a+\varepsilon} [f(a)+(t-a)f'(a)+\dots]\exp\{x[\varphi(a)+(t-a)\varphi'(a)+\underbrace{\tfrac{1}{2}(t-a)^2\varphi''(a)}_{\text{we want to Taylor expand this term}}+\dots]\}dt$$

$$e^{\frac{1}{2}x(t-a)^2\varphi''(a)} = 1 + \frac{x(t-a)^2}{2}\varphi''(a) + \dots$$

Taylor expansion requires $x(t-a)^2 \ll 1$ – but t is at most $a+\varepsilon$ $\Rightarrow$ require $x(t-a)^2 < x\varepsilon^2 \ll 1$

Hence second constraint on ε: $\boxed{\varepsilon \ll \frac{1}{\sqrt{x}}}$

All together ↓

$$\therefore \boxed{\frac{1}{x} \ll \varepsilon \ll \frac{1}{\sqrt{x}}}$$

Therefore

$$I_1(x) = e^{x\varphi(a)} \int_a^{a+\varepsilon} [f(a)+(t-a)f'(a)+\dots]\, e^{x(t-a)\varphi'(a)} \Big[1+ \underbrace{\tfrac{x^2(t-a)^2}{2}\varphi''(a)+\dots}\Big]\, dt$$

we want to be able to conclude that this term is asymptotically small...

Let $x(t-a) = s \quad \Rightarrow \quad dt = \frac{1}{x} ds$

then

$$I_1(x) = \frac{e^{x\varphi(a)}}{x} \int_0^{\varepsilon x} [f(a) + O(\tfrac{s}{x})]\, e^{s\varphi'(a)} \left[1 + O\left(\tfrac{s^2}{x^2}\right)\right] ds$$

$$= \frac{e^{x\varphi(a)} f(a)}{x} \Big[\underbrace{\int_0^{\varepsilon x} e^{s\varphi'(a)}\, ds} \Big] \Big[1 + O\left(\tfrac{1}{x}\right) \Big]$$

← from higher order terms

Here we can now replace εx with ∞ because any exponentially small terms are being neglected.

$$\int_0^{\infty} e^{s\varphi'(a)} ds = -\frac{1}{\varphi'(a)}$$

$$\therefore \boxed{I(x) \sim -\frac{f(a)\, e^{x\varphi(a)}}{x\varphi'(a)}}$$

Case ②: max @ $t=b$ $\quad [\varphi'(b)>0,\ f(b)\neq 0,\ \varphi''(b)\neq 0]$

↳ very similar argument shows $\boxed{I(x) \sim \dfrac{f(b)\, e^{x\varphi(b)}}{x\varphi'(b)}}$

Case ③: max @ $t=c$, $a<c<b$ $\quad \varphi'(c)=0,\ \underbrace{\varphi''(c)<0}_{\text{ie max @ } c},\ \varphi'''(c)\neq 0,\ f(c)\neq 0$

Split the integral up:

$$I(x) = \underbrace{\int_a^{c-\varepsilon} f(t)\, e^{x\varphi(t)}\, dt}_{I_1} + \underbrace{\int_{c-\varepsilon}^{c+\varepsilon} f(t)\, e^{x\varphi(t)}\, dt}_{I_2} + \underbrace{\int_{c+\varepsilon}^{b} f(t)\, e^{x\varphi(t)}\, dt}_{I_3}$$

↑ want to find conditions on ε such that I_2 dominates.

For I_2 to be dominant, we want $e^{x\varphi(c+\varepsilon)} \ll e^{x\varphi(c)}$ so that I_3 is small.

Taylor expand: $\varphi(c+\varepsilon) = \varphi(c) + \underbrace{\varepsilon\varphi'(c)}_{=0} + \underbrace{\tfrac{1}{2}\varepsilon^2\varphi''(c)} + \dots$

hence for $e^{x\varphi(c+\varepsilon)} \ll e^{x\varphi(c)}$

we need $e^{x\varepsilon^2\varphi''(c)/2} \ll 1$

$\Leftrightarrow \; x\varepsilon^2 \ll 1$ (NB $\varphi''(c) < 0$ since c is a max.)

A similar argument then shows that I_1 also small for $x\varepsilon^2 \ll 1$.

ie $\varepsilon \gg \frac{1}{\sqrt{x}}$

Then, consider I_2. Taylor expand:

$$\varphi(t) \sim \varphi(c) + (t-c)\underbrace{\varphi'(c)}_{=0} + \tfrac{1}{2}(t-c)^2\varphi''(c) + \tfrac{1}{6}(t-c)^3\varphi'''(c) + \dots$$

$$f(t) \sim f(c) + (t-c)f'(c) + \dots$$

and substitute into I_2:

$$I_2 = \int_{c-\varepsilon}^{c+\varepsilon} [f(c) + O(t-c)]\, e^{x\varphi(c)} e^{x(t-c)^2\varphi''(c)/2} \underbrace{\left[1 + O\left(\frac{x(t-c)^3}{6}\varphi'''(c)\right)\right]} dt$$

here we have Taylor expanded $e^{x(t-c)^3\varphi'''(c)/6}$. To ensure higher order terms are small we need $x(t-c)^3 \ll 1$

ie $x\varepsilon^3 \ll 1$

which gives a second constraint

$$\boxed{\text{In summary, we need } \frac{1}{x^{\frac{1}{2}}} \ll \varepsilon \ll \frac{1}{x^{\frac{1}{3}}}}$$

we actually need x quite large to get a clear separation of scales.

eg. $x = 8 \Rightarrow \frac{1}{2\sqrt{2}} \ll \varepsilon < \frac{1}{2}$

ie x needs to be much larger to get an accurate expansion for integrals of this term.

Now we rescale the integration variable to make the integral as simple as possible: Let $\sqrt{x}(t-c) = s \Rightarrow dt = \frac{1}{\sqrt{x}} ds$

$$I_2(x) = \frac{f(c)e^{x\varphi(c)}}{\sqrt{x}} \int_{-\sqrt{x}\varepsilon}^{+\sqrt{x}\varepsilon} e^{s^2\varphi''(c)/2} \left(1 + O\left(\frac{s}{\sqrt{x}}\right)\right)\left(1 + O\left(\frac{s^3}{\sqrt{x}}\right)\right) ds$$

since $\frac{1}{x^{\frac{1}{2}}} \ll \varepsilon \ll \frac{1}{x^{\frac{1}{3}}}$ we have $\varepsilon x^{\frac{1}{2}} \gg 1$ so we can approximate as $\int_{-\infty}^{\infty}$ with only exponentially small corrections

will generate a correction that is $O\left(\frac{1}{\sqrt{x}}\right)$ since $\int_{-\infty}^{\infty} s e^{-s^2} ds \sim O(1)$

Putting everything together:

$u^2 = -\frac{s^2\varphi''(c)}{2}$

$$I_2(x) \sim \frac{f(c)e^{x\varphi(c)}}{\sqrt{x}} \sqrt{\frac{2}{-\varphi''(c)}} \int_{-\infty}^{\infty} e^{-u^2}du \left(1+o\left(\frac{1}{\sqrt{x}}\right)\right)$$

$$\Rightarrow \quad I(x) \sim \frac{2\pi f(c)e^{x\varphi(c)}}{\sqrt{-x\varphi''(c)}} \quad //$$

3.5 Method of stationary phase

– used for cases where $\varphi(t) = i\psi(t)$ where $\psi(t)$ real

ie. $$I(x) = \int_a^b f(t)\, e^{\underbrace{ix\psi(t)}}\, dt$$

exponent is purely imaginary
⇒ behaves very differently
from the previous integrals

3.5.1 Riemann-Lebesgue lemma

If $\int_a^b |f(t)|\,dt < \infty$ and $\psi(t)$ ctsly differentiable for $a \le t \le b$ and not constant on any subinterval in $a \le t \le b$ then $\int_a^b f(t)e^{ix\psi(t)}dt \to 0$ as $x \to \infty$.

– NB useful when integrating by parts

Example

$$I(x) = \int_0^1 \frac{e^{ixt}}{1+t}\,dt = \left[\frac{-i}{x(1+t)}e^{ixt}\right]_0^1 - \frac{i}{x}\int_0^1 \frac{e^{ixt}}{(1+t)^2}\,dt$$

$u = \frac{1}{1+t} \Rightarrow \frac{du}{dt} = \frac{-1}{(1+t)^2}$, $\frac{dv}{dt} = e^{ixt} \Rightarrow v = \frac{-i}{x}e^{ixt}$

$$= \frac{-ie^{ix}}{2x} + \frac{i}{x} - \underbrace{\frac{i}{x}\int_0^1 \frac{e^{ixt}}{(1+t)^2}\,dt}$$

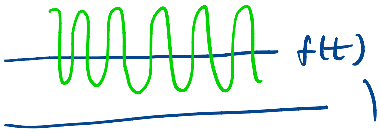
→ 0 as $x \to \infty$ by RLL
ie this term is $o\left(\frac{1}{x}\right)$
(subdominant and so
first term is the start
of an asymptotic exp.)

Why is the RLL true?

First - think about eg $\psi(t) = t$ so that $I(x) = \int_a^b f(t)\, \underbrace{e^{ixt}}\, dt$

oscillates more and more rapidly as $x \to \infty$

everything just 'cancels out' $\to$



looks locally flat

What about more general functions: Taylor expand near $t = t_0$:

$$\psi(t) = \psi(t_0) + (t - t_0)\psi'(t_0) + \tfrac{1}{2}(t - t_0)^2 \psi''(t_0) + \dots$$

Then $$e^{ix\psi(t)} = \underbrace{e^{ix\psi(t_0)}}_{\text{constant}} \underbrace{e^{ix\psi'(t_0)(t - t_0) + \dots}}_{\text{oscillating component}}$$

$\therefore$ Period of oscillation close to $t = t_0$ is $\approx \dfrac{2\pi}{x|\psi'(t_0)|}$, provided $\psi'(t_0) \neq 0$.

$\to 0$ as $x \to \infty$ ——||——

ie increasingly fast oscillations which cancel each other out as $x \to \infty$. (regardless of the fn $f(t)$).

NB only exception is if $\psi'(t_0)$ is very small - then, unless x is very large, the period of oscillation will be large and cancellation won't work.

(If $\psi'(t_0) = 0$ then cancellations won't occur $\Rightarrow$ dominant contributions to the integral when $|\psi'(t_0)| = 0$)

↳ this is how we will generate asymptotic approximations to integrals of this form!

PTO.

Suppose that $\psi'(c)=0$ with $a<c<b$ and $\psi'(t)\neq 0$ for $a\le t<c$ and $c<t\le b$. Also, assume $f(c)\neq 0$ and $\psi''(t)\sim \text{ord}(1)$ in a neighbourhood of c.

As before, to make progress, we split the range of integration up:

$$I(x)=\underbrace{\int_a^{c-\varepsilon} f(t)e^{ix\psi(t)}dt}_{I_1}+\underbrace{\int_{c-\varepsilon}^{c+\varepsilon} f(t)e^{ix\psi(t)}dt}_{I_2}+\underbrace{\int_{c+\varepsilon}^{b} f(t)e^{ix\psi(t)}dt}_{I_3}$$

Expect I_2 to dominate as $x\to\infty$ for sufficiently well chosen $\varepsilon\ll 1$

First consider I_2 (we will see the constraints on ε needed for I_2 to dominate...)

Expand ψ and f about $t=c$:

$$f(t)\sim f(c)+(t-c)f'(c)+\dots$$

$$\psi(t)\sim\psi(c)+(t-c)\cancel{\underbrace{\psi'(c)}_{=0}}+\tfrac{1}{2}(t-c)^2\psi''(c)+\tfrac{1}{6}(t-c)^3\psi'''(c)+\dots$$

Then

$$I_2(x)=\int_{c-\varepsilon}^{c+\varepsilon}\underbrace{[f(c)+(t-c)f'(c)+\dots]}_{\text{ie } f(c)+O(t-c)}e^{ix[\psi(c)+\frac{1}{2}(t-c)^2\psi''(c)+\underbrace{\frac{1}{6}(t-c)^3\psi'''(c)+\dots}_{O((t-c)^3)}]}dt$$

$$=e^{ix\psi(c)}\int_{c-\varepsilon}^{c+\varepsilon}[f(c)+O(t-c)]e^{ix(t-c)^2\psi''(c)/2}\underbrace{[1+O((t-c)^3x)]}_{\substack{\text{Taylor series expansion,}\\ \text{valid if } \varepsilon^3x\ll 1\\ \text{ie } \varepsilon\ll \frac{1}{x^{1/3}}}}dt$$

let $s=\sqrt{x}(t-c)$

$$=\frac{e^{ix\psi(c)}}{\sqrt{x}}\int_{-\varepsilon\sqrt{x}}^{+\varepsilon\sqrt{x}}[f(c)+\underbrace{O(\tfrac{s}{\sqrt{x}})}_{\text{subleading}}]e^{is^2\psi''(c)/2}[1+\underbrace{O(\tfrac{s^3}{\sqrt{x}})}_{\text{subleading}}]ds$$

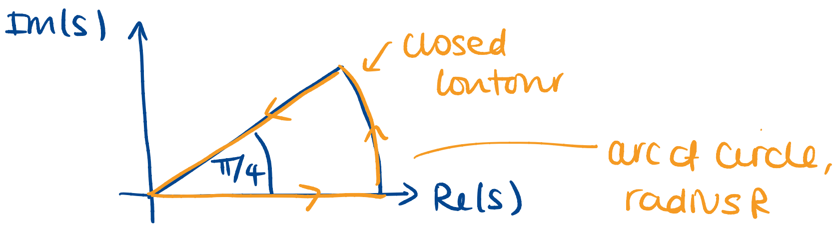
(we will consider the corrections later – but both subleading...)

Want also to replace the limits by $\pm\infty$ – requires $\varepsilon\sqrt{x}\gg 1$ ie $\varepsilon\ll\frac{1}{\sqrt{x}}$

Together we have $\frac{1}{x^{\frac{1}{2}}}\ll\varepsilon\ \frac{1}{x^{\frac{1}{3}}}$ (again, need x quite large to get clear sep. of scales)

Then $I_2(x) \sim \dfrac{e^{ix\psi(c)} f(c)}{\sqrt{x}} \underbrace{\displaystyle\int_{-\infty}^{\infty} e^{is^2\psi''(c)/2}\,ds.}_{\text{use contour integration}}$

Suppose $\psi''(c) > 0$ (the case $\psi''(c) < 0$ is very similar...)



By Cauchy,

$$0 = \int_{\triangleleft} e^{is^2\psi''(c)/2}\,ds = \left(\int_{\rightarrow} + \int_{\swarrow} + \underbrace{\int_{\uparrow}}_{\substack{\to 0 \text{ as } R\to\infty \\ \text{by Jordan's lemma}}} \right) e^{is^2\psi''(c)/2}\,ds$$

(we want to consider ↑ in the limit $R \to \infty$...)

i.e. $\displaystyle\int_{\rightarrow} = -\int_{\swarrow} = \int_{\nearrow}$

Then,

$$\int_0^\infty e^{is^2\psi''(c)/2}\,ds = \underbrace{\int_0^\infty e^{-p^2\psi''(c)/2} e^{i\pi/4}\,dp}_{\text{on } \nearrow \text{ let } s = e^{i\pi/4}p}$$

$$= e^{i\pi/4} \cdot \sqrt{\frac{2\pi}{\psi''(c)}}$$ (since left with a Gaussian integral to evaluate)

If we have $\psi''(c) < 0$, use an angle of $-\pi/4$ which gives the general result:

$$\int_0^\infty e^{is^2\psi''(c)/2}\,ds = e^{i\pi/4 \cdot \operatorname{sign}(\psi''(c))} \cdot \sqrt{\frac{2\pi}{|\psi''(c)|}}$$

(NB since we assumed $\psi \sim \mathrm{Ord}(1)$ near c, we don't need to consider $\psi''(c) = 0$.)

$$\therefore \quad I_2(x) \sim \underbrace{\frac{e^{ix\psi(c)} f(c)}{\sqrt{x}}}_{\text{factored out in calculations}} \cdot \underbrace{\sqrt{\frac{2\pi}{|\psi''(c)|}}\, e^{i\pi/4\, \operatorname{sign}(\psi''(c))}}_{\text{from evaluating the integral}} \quad (*)$$

so the magnitude of the corrections need to be considered in relation to this term!

We now need to consider

(1) The size of I_1, I_3

(2) The size of the neglected terms in the integral I_2

First consider the contribution from I_1:

$$I_1(x) = \int_a^{c-\varepsilon} f(t) e^{ix\psi(t)}\,dt = \int_a^{c-\varepsilon} \underbrace{\frac{f(t)}{ix\psi'(t)}}_{u} \underbrace{\frac{d}{dt}\left(e^{ix\psi(t)}\right)}_{dv/dt} dt$$

$$\text{(IBPs)} \quad = \left[\frac{f(t)}{ix\psi'(t)} \cdot e^{ix\psi(t)}\right]_a^{c-\varepsilon} - \frac{1}{x}\underbrace{\int_a^{c-\varepsilon} e^{ix\psi(t)} \frac{d}{dt}\left(\frac{f(t)}{i\psi'(t)}\right) dt}$$

$\to 0$ as $x \to \infty$ by RLL

hence the term is $o\left(\frac{1}{x}\right)$.

$$\sim O\left(\frac{1}{\underbrace{x\psi'(c-\varepsilon)}}\right)$$

dominant term since $\psi'(c) = 0 \Rightarrow \psi'(c-\varepsilon)$ relatively small.

(more concretely, $\psi'(c-\varepsilon) = \underbrace{\psi'(c)}_{=0} - \underbrace{\varepsilon\psi''(c)}_{\text{ord}(1)} + \cdots$)

$$\sim O\left(\frac{1}{x\varepsilon}\right)$$

$$\sim O\left(\frac{1}{\varepsilon\sqrt{x}} \cdot \frac{1}{\sqrt{x}}\right)$$

then $\varepsilon\sqrt{x} \gg 1 \Rightarrow$ this term gives a genuinely small correction

The method for I_3 proceeds in exactly the same way...

<u>Next</u> consider the correction from the change of limits

– we have added terms of the form

$$\int_{\varepsilon\sqrt{x}}^{\infty} e^{is^2\psi''(c)/2}\,ds = \int_{\varepsilon\sqrt{x}}^{\infty} \underbrace{\frac{1}{is\psi''(c)}}_{u} \cdot \underbrace{is\psi''(c)\, e^{is^2\psi''(c)/2}}_{dv/ds}\,ds$$

$$= \left[\frac{1}{is\psi''(c)} e^{is^2\psi''(c)/2}\right]_{\varepsilon\sqrt{x}}^{\infty} - \underbrace{\int_{\varepsilon\sqrt{x}}^{\infty} \frac{-1}{is^2\psi''(c)} e^{is^2\psi''(c)/2}\,ds}$$

$$\sim O\left(\frac{1}{\varepsilon\sqrt{x}}\right)$$

↑ this is then small compared to the original integral ⊕

this is a smaller correction due to the $\frac{1}{s^2}$ compared to $\frac{1}{s}$ in the original integral, and $\varepsilon\sqrt{x} \gg 1$.

Finally, consider the correction from making the Taylor series expansions

From the Taylor series expansion of f, we have (for $n \geq 1$) terms of the form

$$\frac{1}{\sqrt{x}} \int_{-\varepsilon\sqrt{x}}^{+\varepsilon\sqrt{x}} \frac{s^n}{x^{n/2}} e^{is^2\psi''(c)/2} ds = \frac{1}{\sqrt{x}} \cdot \frac{1}{x^{n/2}} (\sqrt{x}\varepsilon)^{n-1} \sim \frac{\varepsilon^{n-1}}{x}$$ ← small relative to (*1)

using $\int_{-\varepsilon\sqrt{x}}^{+\varepsilon\sqrt{x}} s^n e^{is^2\psi''(c)/2} ds = O\left((\sqrt{x}\varepsilon)^{n-1}\right)$ by IBPs n−1 times (or induction).

and also, from the Taylor series expansion of the exponential

$$\frac{1}{\sqrt{x}} \int_{-\varepsilon\sqrt{x}}^{+\varepsilon\sqrt{x}} \frac{(s^3)^n}{x^{n/2}} e^{is^2\psi''(c)/2} ds = \frac{1}{\sqrt{x}} \frac{1}{x^{n/2}} \left(\sqrt{x}\varepsilon\right)^{3n-1} \sim \frac{(\varepsilon^3 x)^n}{x\varepsilon} \sim \frac{\varepsilon(\varepsilon^3 x)^n}{x\varepsilon^2}$$

since $\varepsilon^2 x \gg 1$ this term is also small, relative to the $O(1)$ term it is compared against.

Hence, in summary,

$$I(x) \sim \frac{\sqrt{2\pi}\, f(c)\, e^{ix\psi(c)}\, e^{i\pi/4 \cdot \mathrm{sign}(\psi''(c))}}{\sqrt{x}\, |\psi''(c)|^{1/2}} + O\left(\frac{1}{x\varepsilon}\right) \quad \text{as } x \to \infty$$

NB – the errors are only algebraically small, <u>not</u> exponentially small, as in Laplace's method.

– higher order corrections very difficult to get since they come from the whole range of integration (again – in contrast to Laplace's method where the full asymptotic expansion depends only on the local region since the errors are algebraically small ...).

3.6 Method of steepest descents

$$I(x) = \int_c f(t) e^{x\varphi(t)}\, dt$$

t - complex variable

$x \in \mathbb{R}$ - interested in the expansion as $x \to \infty$

$f(t), \varphi(t)$ - complex and holomorphic $\Rightarrow$ analytic.

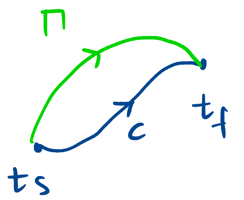
C - contour in the complex plane.

NB Laplace's method and the method of stationary phase are just special cases of the MOSD.

Caution we might expect (based on what we have seen for Laplace's method) that the important contribution as $x \to \infty$ comes from places where $\mathrm{Re}(\varphi)$ is max. - this would give $I \sim f(t_0) e^{x\varphi(t_0)} \sqrt{\frac{2\pi}{-\lambda\varphi''(t_0)}}$.

This is a huge overestimate because it ignores all the cancellation from oscillations due to $\mathrm{Im}(\varphi)$. We can, in fact, see that the estimate is wrong by deforming C - won't change the value of I, but it will (in general) change max $\mathrm{Re}(\varphi)$.

KEY IDEA $I(x)$ will be unchanged by deforming C to a new contour Γ with the same start and end points - this is because f, φ are analytic and $x \in \mathbb{R}$, so there are no poles or residues in $C \cup -\Gamma$.



We are free to choose Γ to make the integral as simple to compute as possible. We will find a contour Γ on which $\mathrm{Im}(\varphi(t))$ is piecewise constant ie find Γ_j, V_j s.t. $\Gamma = \bigcup_j \Gamma_j$ and $\mathrm{Im}(\varphi(t)) = V_j$ on Γ_j. Then

$$I(x) = \sum_j e^{ixV_j} \underbrace{\int_{\Gamma_j} f(t) e^{x\mathrm{Re}(\varphi(t))}\, dt}_{\text{can now evaluate using Laplace's method}}$$

So, we need to understand how to deform $C \to \Gamma$.

Let $\varphi(t) = u(\xi,\eta) + iv(\xi,\eta)$ with $t = \xi + i\eta$.

Since $\varphi(t)$ is holomorphic, the Cauchy Riemann equations hold.

$\Rightarrow \quad u_\xi = v_\eta$ and $u_\eta = -v_\xi$

As such: (we know that)
① $\nabla u \cdot \nabla v = u_\xi v_\xi + u_\eta v_\eta = 0 \Rightarrow \nabla u \perp \nabla v$ (perpendicular)

② $\nabla v \perp$ to contours of constant v
$\Rightarrow$ contours of constant $v \parallel \nabla u$

③ ∇u points in the direction where u increases at the fastest rate.

④ $-\nabla u$ points in the direction where u decreases at the fastest rate.

Hence, contours with $v =$ constant give a path of steepest ascent / descent of u.

Let's consider the landscape of $u(\xi,\eta)$:

by the CREs we have $u_{\xi\xi} + u_{\eta\eta} = (v_\eta)_\xi + (-v_\xi)_\eta = 0$ ie u harmonic.

Hence, by the maximum principle, u cannot have a maximum or minimum in the interior of the domain.

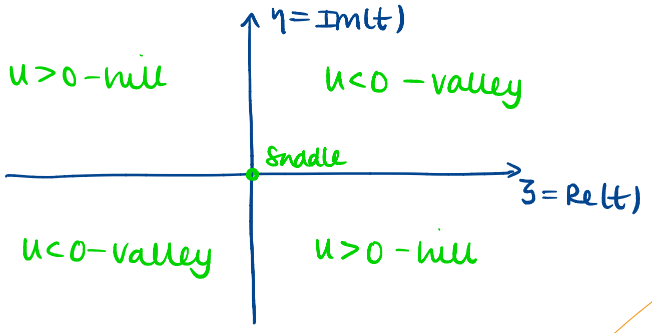
[NB exception is if we consider a point where u is singular, or a branch point, where φ is not holomorphic.] might be some such cases in the examples.

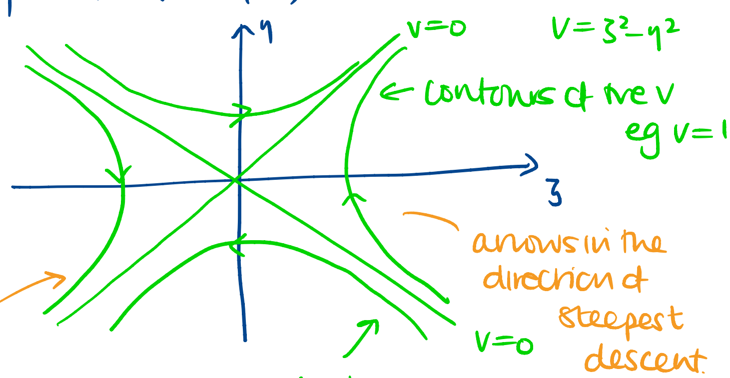
$\therefore$ At a stationary point: $u_\xi = 0$, $u_\eta = 0$, and we have a saddle.

ie the general structure of the landscape of u is hills and valleys at infinity with saddle points in the interior of the complex plane.

<u>Example</u> $\varphi(t) = it^2 = i(\zeta + i\eta)^2 = \underbrace{-2\zeta\eta}_{u} + i\underbrace{(\zeta^2 - \eta^2)}_{v}$ $\quad (t = \zeta + i\eta)$

Then, $\nabla u = -2(\eta, \zeta)$ => saddle point at $(0,0)$

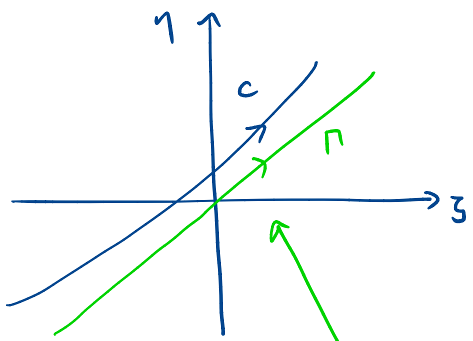




contours of -ve v eg $v = -1$

These are paths of steepest descent for u, with v = constant.

We want to consider a contour C which joins two valleys of u:



then we want to deform C to a new contour Γ that has the same start and end points (in each valley).

↳ In the valleys with u negative then the contribution to the overall integral will be exponentially small since

$$e^{x\,\mathrm{Re}(\varphi)} = e^{xu}.$$

We want to find paths on which $v = \mathrm{Im}(\varphi)$ is constant eg. $\underline{v = 0}$

Then, $$I(x) = \int_C f(t)e^{x\varphi(t)}\,dt = \int_\Gamma f(t)\,e^{x\,\mathrm{Re}(\varphi(t))}\,dt$$

↑ since on Γ we have $v =$ constant

↗ for the sketched example, $v = 0$ here.

> $\mathrm{Re}(\varphi(t)) = u$ has valleys at either end of Γ => contributions from the end points will be exponentially small.

∴ Main contribution to the integral – from $u = \mathrm{Re}(\varphi(t))$ at (around) the saddle point. Can evaluate using Laplace's method.

Method of steepest descent is

1. Deform the contour to union of steepest descent (v=constant) contours through the end points and any relevant saddle points.
2. Evaluate local contribution from saddle point, and the local contribution from the end points, using Laplace's method.

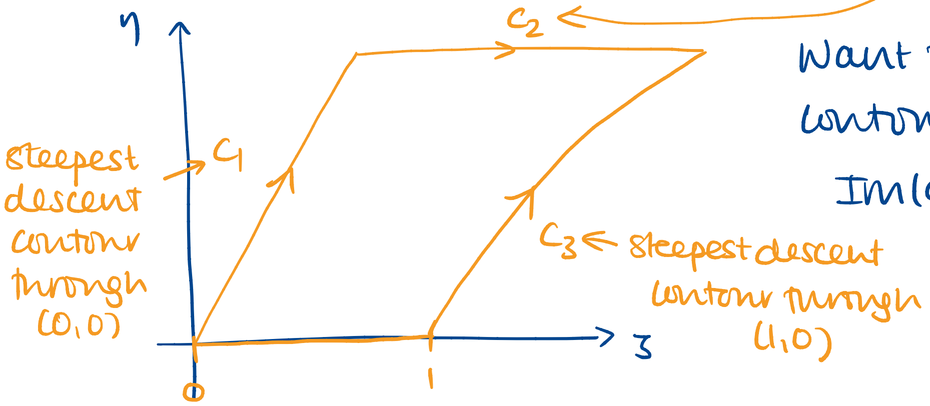
NB – could have deformed to Γ s.t. $\mathrm{Re}(\varphi)$ = constant and applied the method of stationary phase - but we have seen that Laplace's method is far superior (can generate all terms in the asymptotic series, and neglected 'terms' are exp. small.
(Actually, for a stationary phase integral, the best approach is to transform to the steepest descent contour...)

<u>Example</u> $\quad I(x) = \int_0^1 e^{x\varphi(t)}\,dt$ as $x\to\infty$ with $\varphi(t) = it^2$ and $t = \xi + i\eta$.

$\therefore$ As in the previous example,

$$\varphi = \underbrace{-2\xi\eta}_{u=\mathrm{Re}(\varphi)} + i\underbrace{(\xi^2-\eta^2)}_{v=\mathrm{Im}(\varphi)}$$

join C_1 and C_3 using C_2 with $\eta = R$ so that as η gets larger, C_2 heads to a valley and generates a sub-leading term.



Want to deform our original contour to one where

$$\mathrm{Im}(\varphi) = \xi^2 - \eta^2 = \text{constant}$$

$$C_1(R) = \{\xi(1+i) : \xi\in[0,R]\}$$

$$C_2(R) = \{\xi + iR : \xi\in[R, \sqrt{R^2+1}\,]\}$$

$$C_3(R) = \{\sqrt{1+\eta^2} + i\eta : \eta\in[0,R]\}$$

Then we write the original integral as

$$I(x) = \left[\int_{C_1(R)} + \int_{C_2(R)} - \int_{C_3(R)} \right] e^{ixt^2}\, dt$$

Along C_2 : $|e^{ixt^2}| = |e^{x(-2\xi\eta + i(\xi^2-\eta^2))}|$ s.t. $\eta = R$ (by defn on C_2)

$$= |e^{-2x\xi R}| \quad \text{with } \xi \geq R$$

$$\sim O(e^{-2xR^2})$$

$\to 0$ as $R \to \infty$ $\Rightarrow$ only an exponentially small contribution.

Along C_1 : as $R \to \infty$ with $t = \xi(1+i) \Rightarrow dt = d\xi(1+i)$

$$\int_{C_1(\infty)} e^{ixt^2}\, dt = \int_0^\infty e^{\underbrace{ix\xi^2(1+i)^2}} (1+i)\, d\xi$$

$ix\xi^2(1+i)^2 = ix\xi^2(1+2i-1)$
$= -2x\xi^2$

$$= (1+i)\int_0^\infty e^{-2x\xi^2}\, d\xi$$

Gaussian integral, let $u = \sqrt{2x}\,\xi$

$$= \frac{e^{i\pi/4}}{2}\sqrt{\frac{\pi}{x}}$$

Along C_3 : as $R \to \infty$ with $t = \sqrt{1+\eta^2} + i\eta \Rightarrow t^2 = 1 + 2\eta\sqrt{1+\eta^2}\, i$

$$\int_{C_3(\infty)} e^{ixt^2}\, dt = \int_0^\infty e^{ix[1+2\eta\sqrt{1+\eta^2}\, i]} \frac{dt}{d\eta}\, d\eta$$

$$= \underbrace{e^{ix}\int_0^\infty e^{x\varphi(\eta)} f(\eta)\, d\eta}$$

for $\varphi(\eta) = -2\eta\sqrt{1+\eta^2}$
$f(\eta) = \dfrac{dt}{d\eta} = i + \dfrac{\eta}{\sqrt{1+\eta^2}}$

use Laplace's method to evaluate the asymptotic approximation (consider the real and imaginary parts of $f(\eta)$ separately).

But, to get to a quicker answer, note that on $C_3(\infty)$ $t = \xi + i\eta$ with $\xi^2 - \eta^2 = 1$ $\Rightarrow$ $t^2 = \xi^2 - \eta^2 + 2i\xi\eta = 1 + 2i\eta(1+\eta^2)^{\frac{1}{2}}$ $\eta \in [0,\infty)$.

$\Rightarrow$ suggests to re-parameterise C_3 as $t^2 = 1 + is$ with $s \in [0,\infty)$

(since $\mathrm{Im}(t^2)$ is monotonic increasing from $0 \to \infty$)

$$\int_{C_3(\infty)} e^{ixt^2}\,dt = \int_0^\infty e^{ix - xs}\frac{dt}{ds}\,ds \qquad t = (1+is)^{\frac{1}{2}} \Rightarrow \frac{dt}{ds} = \frac{i}{2}(1+is)^{-\frac{1}{2}}$$

$$= \frac{1}{2}ie^{ix}\int_0^\infty e^{-xs}\frac{1}{(1+is)^{\frac{1}{2}}}\,ds$$

apply Watson's lemma to get this, with $a_n = \dfrac{(-1)^n\Gamma(n+\frac{1}{2})}{\Gamma(n+1)\sqrt{\pi}}$

$$= \frac{1}{2}ie^{ix}\sum_{n=0}^{\infty}\frac{a_n\Gamma(n+1)}{x^{n+1}}$$

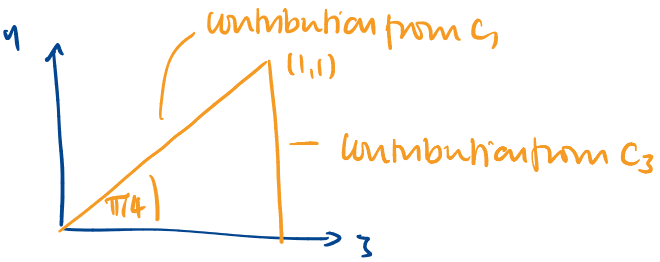
combining C_1, C_2, C_3 gives

$$I(x) \sim \frac{1}{2}e^{i\pi/4}\sqrt{\frac{\pi}{x}} - \frac{ie^{ix}}{2\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+\frac{1}{2})}{x^{n+1}}$$

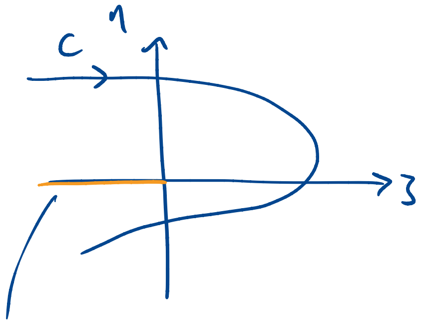
(neglecting exponentially small terms from C_2)

Note – local contributions dominate – so to generate the asymptotic approximations we simply need the <u>tangents</u> to the steepest descent paths

e.g.



Example $I(x) = \int_c e^s s^{-x}\, ds$ as $x \to \infty$



branch cut for $\ln s$ given by $\{\mathrm{Re}(s)<0,\ \mathrm{Im}(s)=0\}$

First need to find the saddle points of the integrand: write $e^s s^{-x} = e^{s - x\ln s}$

saddle point: $\frac{d}{ds}\left(e^{s-x\ln s}\right) = 0$

$\iff 1 - \frac{x}{s} = 0$

ie $x = s$

Note that the location of the saddle point depends on the asymptotic parameter x — not good!

$\Rightarrow$ make a change of variables to fix the saddle point

let $s = tx$ so that when $s = x$, $t = 1$ and the saddle is fixed.

Then, $I(x) = x \int_{c^*} e^{\overbrace{tx - x\log(tx)}}\, dt$ $= tx - x\log t - x\log x$

$$= x^{1-x} \int_{c^*} e^{x\varphi(t)}\, dt \quad \text{with } \varphi(t) = t - \log t$$

$t = \xi + i\eta \Rightarrow \varphi = \xi + i\eta - \underbrace{\log r - i\theta}_{(r,\theta) = \text{polars for } (\xi,\eta)}$

stationary point for φ: $\varphi'(t) = 1 - \frac{1}{t} \Rightarrow \varphi'(t) = 0$ for $t = 1$.

↳ hence, from the CREs we have a saddle for $u = \mathrm{Re}(\varphi)$ at $t = 1$.

So, deform c^* to go through this point (the saddle @ $t = 1$)

$u = \mathrm{Re}(\varphi) = \overbrace{r\cos\theta}^{\xi} - \log r$

$v = \mathrm{Im}(\varphi) = \underbrace{r\sin\theta}_{\eta} - \theta$

At the saddle point $(t=1)$: $\theta = 0$, $r = 1$

hence $v = 0$ on the path of steepest descent through the saddle point @ $t = 1$

On this path, $r = \frac{\theta}{\sin\theta}$ for $\theta \in (-\pi, \pi)$

$v = 0$

$u = r\cos\theta - \log r = \theta\cot\theta - \log\left(\frac{\theta}{\sin\theta}\right)$

$= \theta\cot\theta - \log\theta + \log(\sin\theta) := u(\theta)$

Take the contour integral, parametrised by θ: will be along a deformed C^* such that we go through the saddle point on the path of steepest descent.

$$I(x) = x^{1-x}\int_{-\pi}^{\pi} e^{xu(\theta)}\frac{dt}{d\theta}\,d\theta$$

$t = \xi + i\eta = re^{i\theta}$

with $r(\theta) = \dfrac{\theta}{\sin\theta}$

$\Rightarrow \dfrac{dt}{d\theta} = [r'(\theta)+ir(\theta)]e^{i\theta}$

$$= x^{1-x}\int_{-\pi}^{\pi} e^{x\underbrace{\{\theta\cot\theta-\log\theta+\log|\sin\theta|\}}_{\varphi(\theta)}}\underbrace{[r'(\theta)+ir(\theta)]e^{i\theta}}_{f(\theta)}\,d\theta.$$

ie $I(x)$ in the form of a Laplace integral, with $\varphi(\theta)$ taking its maximum at $\theta=0$ (which is the saddle point location).

$\Rightarrow$ use Laplace's method to generate the approximation to the integral for large x.

$$\therefore\ I(x) \sim \frac{x^{1-x}\sqrt{2\pi}\,f(0)\,e^{x\varphi(0)}}{\sqrt{-\varphi''(0)}} \quad \text{as } x\to\infty.$$

$$\underbrace{r(\theta) = \frac{\theta}{\sin\theta}}_{} = \frac{\theta}{\theta-\frac{1}{3!}\theta^3+\dots} = 1+\frac{1}{6}\theta^2+o(\theta^3) \Rightarrow f(0)=i$$

only need local behaviour around $\theta=0 \Rightarrow$ Taylor expand.

$$\varphi(\theta) = \theta\cot\theta - \log\left(\frac{\theta}{\sin\theta}\right)$$

$$= \frac{\theta\left(1-\frac{1}{2!}\theta^2+\dots\right)}{\theta-\frac{1}{3!}\theta^3+\dots} - \log\left(1+\frac{1}{6}\theta^2+\dots\right)$$

$$= 1-\frac{1}{2}\theta^2+o(\theta^3) \Rightarrow \begin{array}{l}\varphi(0)=1\\ \varphi''(0)=-1\end{array}$$

$$\therefore\ I(x)\sim i\,x^{\frac{1}{2}-x}e^{x}\sqrt{2\pi} \quad \text{as } x\to\infty.$$

NB can use this example to deduce Stirling's approximation!

$$\Gamma(x)\sim\sqrt{2\pi}\,x^{x-\frac{1}{2}}e^{-x}$$

(since we computed $\frac{1}{\Gamma(x)}$ ...).

3.7 Splitting the range of integration

↳ idea: we can split the range of integration and use different approximations in each part.

Example $I = \int_0^{\pi/4} \frac{1}{\varepsilon^2 + \sin^2\theta}\, d\theta$ as $\varepsilon \to 0^+$

Regions ① $\theta = O(\varepsilon) \Rightarrow$ integrand is $O(\varepsilon^{-2})$ and contribution to the integral is $O(\frac{1}{\varepsilon})$.

② $\theta = O(1) \Rightarrow$ integrand is $O(1)$ and contribution to the integral is $O(1)$.

(so expect the local contribution to dominate...). the $\theta \sim O(1)$ term will contribute at higher order...

As before, we split the region of integration (at δ, with $\underline{\varepsilon \ll \delta \ll 1}$)

$$I = \underbrace{\int_0^{\delta} \frac{1}{\varepsilon^2 + \sin^2\theta}\, d\theta}_{I_1} + \underbrace{\int_{\delta}^{\pi/4} \frac{1}{\varepsilon^2 + \sin^2\theta}\, d\theta}_{I_2}$$

$\downarrow$ $\theta = \varepsilon u$

$$I_1 = \int_0^{\delta/\varepsilon} \frac{\varepsilon}{\underbrace{\varepsilon^2 + \sin^2(\varepsilon u)}} du \sim \int_0^{\delta/\varepsilon} \frac{\varepsilon}{\varepsilon^2 + \varepsilon^2 u^2 - \frac{1}{3}\varepsilon^4 u^4 + \dots} du$$

$\Rightarrow$ safe to Taylor expand given $u\varepsilon \in (0, \delta)$ and $\delta \ll 1$.

$\downarrow$ expand

$$\sim \int_0^{\delta/\varepsilon} \left(\underbrace{\frac{1}{\varepsilon(1+u^2)}}_{\text{leading order}} + \frac{\varepsilon u^4}{\underbrace{3(1+u^2)^2}} + \dots \right) du$$

$\frac{u^4}{(1+u^2)^2} \sim O(1)$ over range $u \in (0, \delta/\varepsilon)$

$\Rightarrow O\left(\varepsilon \cdot \frac{\delta}{\varepsilon}\right) = O(\delta)$.

$$= \frac{1}{\varepsilon}\tan^{-1}\left(\frac{\delta}{\varepsilon}\right) + O(\delta)$$

$$= \frac{1}{\varepsilon}\left[\frac{\pi}{2} - \frac{\varepsilon}{\delta} + O\left(\frac{\varepsilon^3}{\delta^3}\right)\right] + O(\delta)$$

$$= \frac{\pi}{2\varepsilon} - \frac{1}{\delta} + O\left(\frac{\varepsilon^3}{\delta^3}\right) + O(\delta)$$

For I_2:

$$I_2 = \int_\delta^{\pi/4} \frac{1}{\varepsilon^2+\sin^2\theta}\,d\theta \sim \int_\delta^{\pi/4}\left(\frac{1}{\sin^2\theta} - \frac{\varepsilon^2}{\sin^4\theta} + \cdots\right)d\theta$$

here $\sin^2\theta$ dominates ε^2 since $\sin\theta \sim \delta$ and so $\sin^2\theta \sim \delta^2 \gg \varepsilon^2$

dominant contribution to the integral will be from around $\theta = \delta$, where $\sin^4\theta \sim \delta^4$. Then $\frac{\varepsilon^2}{\sin^4\theta} \sim \frac{\varepsilon^2}{\delta^4}$ which will integrate to $O(\varepsilon^2/\delta^3)$

more formally,

$$\frac{1}{\varepsilon^2+\sin^2\theta} = \frac{1}{\sin^2\theta}\left(\frac{1}{1+\frac{\varepsilon^2}{\sin^2\theta}}\right) = \frac{1}{\sin^2\theta}\left(1 - \frac{\varepsilon^2}{\sin^2\theta} + \cdots\right)$$

so then

$$I_2(x) = \left[-\cot\theta\right]_\delta^{\pi/4} + O\left(\frac{\varepsilon^2}{\delta^3}\right)$$

$$= -1 + \frac{1}{\delta} + O(\delta) + O\left(\frac{\varepsilon^2}{\delta^3}\right)$$

with the requirement then that $\frac{\varepsilon^2}{\delta^3} \ll 1$ ie

$$\underline{\varepsilon^{2/3} \ll \delta \ll 1}$$

$\therefore$ Putting it all back together:

$$I(x) \sim \frac{\pi}{2\varepsilon} - 1 + O(1)$$

comes from I_1, where we predicted a contribution of $O(\frac{1}{\varepsilon})$

comes from I_2, where we predicted a contribution of $O(1)$.

Note that the $\pm\frac{1}{\delta}$ terms cancel – this needs to happen in order that the result does not depend on the specific choice of δ used to partition the integral!