SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.2

ELASTICITY AND PLASTICITY

TRINITY TERM 2016

WEDNESDAY, 1 JUNE 2016, 2.30pm to 4.00pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

Do not turn this page until you are told that you may do so

1. In two-dimensional polar coordinates, the two components of Cauchy's equation for plane strain read

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0$$

and

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + 2\frac{\tau_{r\theta}}{r} = 0$$

while the compatibility of strains equation may be written

$$\nabla^2[\mathrm{Tr}(\mathcal{T})] = 0$$

where $\mathcal{T} = (\tau_{ij})$ is the stress tensor.

(a) [5 marks] Verify that an Airy stress function, $\mathcal{A}(r,\theta)$, defined such that

$$\tau_{rr} = \frac{1}{r}\frac{\partial \mathcal{A}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \mathcal{A}}{\partial \theta^2}, \quad \tau_{\theta\theta} = \frac{\partial^2 \mathcal{A}}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \mathcal{A}}{\partial \theta}\right)$$

identically satisfies the components of Cauchy's equation, and find a non-trivial equation solved by $\mathcal{A}$.

(b) [9 marks] An infinite elastic medium occupies the region $r \geqslant a$ and is subject to an isotropic tension $T_{\mathrm{out}} > 0$ as $r \to \infty$ and a normal traction $T_{\mathrm{in}} > T_{\mathrm{out}}$ at $r = a$.

(i) Find the non-zero components of the stress tensor, τ_{rr} and $\tau_{\theta\theta}$, in $r > a$.

(ii) Find a condition on the ratio $T_{\mathrm{in}}/T_{\mathrm{out}}$ for which at least one of the principal stress components is compressive within the elastic medium.

(c) [11 marks] A crude model for the locomotion of a biological cell assumes that the cell is circular, with a radius a, and sits on an elastic substrate subject to an isotropic tension, T_{out}, far from the cell. The cell 'crawls' in a particular direction by exerting a normal traction $T_{\mathrm{in}} + \Delta T \cos 2\theta$ on its outer boundary ($r = a$), with $\Delta T > 0$. However, the cell does not exert any tangential traction at $r = a$. In the remainder of this question, you will repeat the analysis of part (b) to find the regions of compression in this non-axisymmetric problem.

(i) By seeking Airy stress functions of the form $\mathcal{A}(r,\theta) = r^k \cos n\theta$ with n a non-negative integer, find the general Airy stress function $\mathcal{A}(r,\theta) \propto \cos n\theta$. (You should be careful to explicitly give results for any values of n that require special consideration.)

(ii) What boundary conditions on $\mathcal{A}$ are appropriate at $r = a$ and as $r \to \infty$?

(iii) Find the stress field τ_{rr}, $\tau_{r\theta}$ and $\tau_{\theta\theta}$ resulting from the applied traction $T_{\mathrm{in}} + \Delta T \cos 2\theta$. Is the largest compression in the azimuthal direction found where the applied traction is largest or smallest?

[*Throughout this question, you may use without proof that*

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2},$$

in two-dimensional polar coordinates.]

2. This question concerns the small displacement, dynamic motion of a one-dimensional beam of mass ρ per unit length and constant bending stiffness B. Throughout the question, you may neglect any body forces.

   (a) [6 marks] Starting from first principles, but assuming the constitutive relationship $M = -B\partial^2 w/\partial x^2$ between the bending moment and the curvature of the beam, derive the dynamic beam equation

   $$\rho \frac{\partial^2 w}{\partial t^2} = -B\frac{\partial^4 w}{\partial x^4} + T\frac{\partial^2 w}{\partial x^2}$$

   for small vertical displacements of the beam, $w(x, t)$. Here T is the tension within the beam. In the course of your answer, you should specify more precisely what is meant by 'small displacements' here.

   (b) [10 marks] A beam has simply supported ends at $x = 0$ and $x = L$; this beam is dynamically compressed by the action of a compressive load $T = -P$.

      (i) Seek a solution of the dynamic beam equation of the form $w(x, t) = \mathrm{e}^{\sigma t} f(x)$, and find an expression for the growth rate σ.
      [*You may find it helpful to chose a particular form for $f(x)$ motivated by the boundary conditions on the beam.*]

      (ii) Find an expression describing which deformation modes are unstable, and show that the fastest growing unstable mode has wavelength

      $$\lambda = \frac{2L}{n}$$

      where

      $$n = \left\lfloor \frac{L}{\sqrt{2}\pi} \left(\frac{P}{B}\right)^{1/2} \right\rfloor \quad \text{or} \quad n = \left\lceil \frac{L}{\sqrt{2}\pi} \left(\frac{P}{B}\right)^{1/2} \right\rceil.$$

      (Here $\lfloor x \rfloor$ and $\lceil x \rceil$, denote the floor and ceiling of x, respectively, i.e. $\lfloor x \rfloor$ is the largest integer $m \leqslant x$ and $\lceil x \rceil$ is the smallest integer $m \geqslant x$.)

   (c) [9 marks] To play a 'musical ruler' one generally holds one end of a ruler to a table, or other flat surface, while causing the other end of the ruler to vibrate. The remainder of this question seeks to determine the frequency of the resulting oscillations. You should model the ruler as a beam of bending stiffness B, linear density ρ and length L.

      (i) What are the appropriate boundary conditions in this scenario?

      (ii) Seek a solution of the dynamic beam equation $w(x, t) = \mathrm{e}^{\mathrm{i}\omega t} g(x)$, solving for the unknown function $g(x)$ in terms of the vibration frequency ω.

      (iii) Find an equation satisfied by ω and show that the lowest frequency of the ruler satisfies

      $$\frac{\pi^2}{4} < \omega \left(\frac{\rho L^4}{B}\right)^{1/2} < \pi^2.$$

3. (a) [6 marks] Assuming plane strain in the x-y plane, calculate the shear stress on a surface with unit normal $\mathbf{n} = (\cos\theta, \sin\theta, 0)^T$ and show that the maximum shear stress S is given by

$$S = \sqrt{\tau_{xy}^2 + \frac{(\tau_{yy} - \tau_{xx})^2}{4}}$$

as θ varies.

What is the Tresca condition for a material with yield stress τ_y?

(b) [8 marks] An elastic beam occupies the region $-\infty < x < \infty$, $-h/2 < y < h/2$. A displacement $\mathbf{u}(x, y) = [u(x, y), v(x, y), 0]^T$ is imposed with

$$u(x, y) = -\kappa x y$$
$$v(x, y) = \tfrac{1}{2}\kappa x^2 + \frac{\lambda}{2(\lambda + 2\mu)}\kappa y^2$$

for a constant κ (the curvature of the beam), where λ and μ are the Lamé constants.

(i) Calculate the stress components τ_{xx}, τ_{yy} and τ_{xy} and show that the above displacement field is admissible (i.e. that it satisfies Cauchy's equation with no body force and satisfies the appropriate boundary conditions).

(ii) What is the bending moment, M, required to impose this displacement?

(c) [11 marks] For the beam described in part (b):

(i) Find the critical curvature, κ_y, at which yield first occurs, as well as the y coordinates of the point(s) where yield first occurs.

(ii) Find the size of the yielded region(s) for $\kappa > \kappa_y$ assuming perfect plasticity.

(iii) Calculate the bending moment, M, required to impose a displacement with $\kappa > \kappa_y$ and show that $M \to \tau_y h^2/2$ in the limit $\kappa \to \infty$.

Draft Marks

1.1 C5.2 - Elasticity + Plasticity.

Comments

a) We have the components of Cauchy's equation:

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\tau_{rr}-\tau_{\theta\theta}}{r} = 0$$

and:

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{2\tau_{r\theta}}{r} = 0.$$

Bookwork

Letting $\tau_{rr} = \frac{1}{r}\frac{\partial A}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2}$, $\tau_{\theta\theta} = \frac{\partial^2 A}{\partial r^2}$, $\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial A}{\partial \theta}\right)$

we have:

$$0 \overset{?}{=} \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial A}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left[-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial A}{\partial \theta}\right)\right] + \frac{1}{r^2}A_r + \frac{A_{\theta\theta}}{r^3} - \frac{A_{rr}}{r}$$

$$= -\frac{A_r}{r^2} + \cancel{\frac{A_{rr}}{r}} - \cancel{\frac{2}{r^3}A_{\theta\theta}} + \cancel{\frac{1}{r^2}A_{r\theta\theta}} - \frac{1}{r}\left(-\cancel{\frac{1}{r^2}A_{\theta\theta}} + \cancel{\frac{1}{r}A_{r\theta\theta}}\right)$$

$$+ \cancel{\frac{A_r}{r^2}} + \cancel{\frac{A_{\theta\theta}}{r^3}} - \cancel{\frac{A_{rr}}{r}} = 0 \quad \text{(as required).}$$

Covered in PS2, Q2.

and $-\frac{\partial^2}{\partial r^2}\left[\frac{1}{r}A_\theta\right] + \frac{1}{r}A_{rr\theta} - \frac{2}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}A_\theta\right)$

$$= -\cancel{\frac{2}{r^3}A_\theta} + \cancel{\frac{2}{r^2}A_{r\theta}} - \cancel{\frac{1}{r}A_{rr\theta}} + \cancel{\frac{1}{r}A_{rr\theta}} + \cancel{\frac{2}{r^3}A_\theta} - \cancel{\frac{2}{r^2}A_{r\theta}}$$

$= 0$ (as required).

In cylindrical coordinates, compatibility of strains gives:

$$\nabla^2(\tau_{ii}) = 0 = \nabla^2(\nabla^2 A)$$

5 $$\Rightarrow \nabla^4 A = 0.$$

B

b) In axisymmetry, we have, $\nabla^2 A = \frac{1}{r}\frac{d}{dr}\left(r\frac{dA}{dr}\right)$

~~so that~~:

Letting $A = r^k$ we have: $\nabla^2 A = k^2 r^{k-2}$

$$\Rightarrow k^2(k-2)^2 r^{k-4} = 0.$$

Covered in lectures

Hence $A(r) = \alpha_0 + \alpha_1 \log r + \alpha_2 r^2 + \alpha_3 r^2 \log r$.

3 $\alpha_0 = 0$ wlog (Airy stress fn only determined up to linear functions of x and y).

1.2.

In axisymmetric problem we have boundary conditions:

$$\tau_{rr} \to \tau_{out} \quad \text{as} \quad r \to \infty$$

$$\tau_{rr} = \tau_{in} \quad \text{at } r = a.$$

But
$$\tau_{rr} = \frac{1}{r}\frac{\partial A}{\partial r}$$
$$= \frac{1}{r}\left[\frac{\alpha_1}{r} + 2\alpha_2 r + \alpha_3\left(2r\log r + r\right)\right].$$

As $r \to \infty$ $\quad \tau_{rr} \to 2\alpha_2 + \alpha_3\left(1 + 2\log r\right)$

$$\Rightarrow \alpha_3 = 0 \text{ and } \alpha_2 = \tau_{out}/2.$$

At $r = a$:
$$\tau_{rr} = \frac{\alpha_1}{a^2} + \tau_{out}$$
$$\tau_{rr} = \tau_{in} \Rightarrow \alpha_1 = \left(\tau_{in} - \tau_{out}\right)a^2$$

$$\therefore \quad \tau_{rr} = \tau_{out} + \left(\tau_{in} - \tau_{out}\right)a^2/r^2.$$

Also:
$$\tau_{\theta\theta} = \frac{\partial^2 A}{\partial r^2} = -\frac{\alpha_1}{r^2} + 2\alpha_2$$

$$\Rightarrow \tau_{\theta\theta} = \tau_{out} - \left(\tau_{in} - \tau_{out}\right)\frac{a^2}{r^2}.$$

5

Clearly $\tau_{\theta\theta}$ is smallest (closest to going into compression) at $r = a$:

$$\min_r \tau_{\theta\theta} = 2\tau_{out} - \tau_{in}.$$

Hence only have ~~wrinkles~~ compression if $2\tau_{out} - \tau_{in} < 0$

or $\tau_{in}/\tau_{out} > 2$.

1

Comments

S

Lamé problem solved directly from stress-strain in lectures

N

Draft Marks

1.3

c) i) Seek solutions of $\nabla^4 A = 0$ of the form: $A = r^k \cos n\theta$ Comm

$\Rightarrow$ ~~$\nabla^2 A = k^2 r^{k-2} \cos n\theta$~~. $\nabla^2 A = (k^2 - n^2) r^{k-2} \cos n\theta$. S

and ~~$\nabla^4 A = k^2 (\ldots$~~ $\nabla^4 A = (k^2 - n^2)\left[(k-2)^2 - n^2\right] \cos n\theta$.

so: $k = \pm n$

and $k = 2 \pm n$.

Only degeneracy is when $n = 0$ (covered in part b).

and when $n = 1 \Rightarrow A = \alpha_0 r + \frac{\alpha_1}{r} + \alpha_2 r \log r + \alpha_3 r^3$.

3 Standard, covered in lectures

ii) No tangential stress $\Rightarrow \underline{t}\, \tau\, \underline{n} = 0 = (0 \ 1)\begin{pmatrix} \tau_{rr} & \tau_{r\theta} \\ \tau_{r\theta} & \tau_{\theta\theta} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ N

$= \tau_{r\theta}$. at $r = a$.

(Note not $\tau_{\theta\theta} = 0$!). $= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial A}{\partial \theta}\right)$

Similar to before, normal traction $\underline{n}\, \tau\, \underline{n} = \tau_{in} + \Delta\tau \cos 2\theta$ B

$\| \ \tau_{rr}$.

$\| \ \frac{1}{r}\frac{\partial A}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2}$

2

iii) We clearly need: $A = A_0(r) + A_2(r) \cos 2\theta$ to satisfy these boundary conditions. N

Also clear that $A_0 = (\tau_{in} - \tau_{out}) a^2 \log r + \frac{\tau_{out} r^2}{2}$

From c)i) we have $A_2(r) = \alpha_0 r^{-2} + \alpha_1 + \alpha_2 r^2 + \alpha_4 r^4$.

so{ as $r \to \infty$ $\tau_{rr} \to \tau_{out} + \left[\frac{-2\alpha_0}{r^4} + 2\alpha_2 + 4\alpha_4 r^2 - \frac{4}{r^2}\left(\alpha_0 r^{-2} + \alpha_1 + \alpha_2 r^2 + \alpha_4 r^4\right)\right] \cos 2\theta$.

$\Rightarrow \alpha_4 = \alpha_2 = 0$.

Comments

1.4

At $r=a$, $\tau_{rr} = \tau_{in} + \left[-\frac{6\alpha_0}{a^4} - \frac{4\alpha_1}{a^2}\right]\cos 2\theta$.

$$= \tau_{in} + \Delta\tau \cos 2\theta.$$

$\therefore$

$$\frac{6\alpha_0}{a^4} + \frac{4\alpha_1}{a^2} = -\Delta\tau.$$

Also, $\tau_{r\theta} = +2\frac{\partial}{\partial r}\left[\frac{1}{r}\left(\alpha_0 r^{-2} + \alpha_1\right)\right]\sin 2\theta$.

so :

$$-\frac{3\alpha_0}{a^4} - \frac{\alpha_1}{a^2} = 0. \Rightarrow \alpha_1 = -\frac{3\alpha_0}{a^2}.$$

$$\Rightarrow \alpha_0 = \frac{a^4}{6}\left[-\Delta\tau + 12\frac{\alpha_0}{a^4}\right]$$

$$\Rightarrow \alpha_0 = \Delta\tau a^4/6$$

$$\alpha_1 = -\frac{3\Delta\tau a^2}{6} = -\frac{\Delta\tau a^2}{2}.$$

Then:

$$\tau_{rr} = \tau_{out} + (\tau_{in} - \tau_{out})\frac{a^2}{r^2} + \left[-\frac{\Delta\tau a^4}{r^4} + 2\frac{\Delta\tau a^2}{r^2}\right]\cos 2\theta.$$

$$\tau_{\theta\theta} = \tau_{out} - (\tau_{in} - \tau_{out})\frac{a^2}{r^2} + \Delta\tau\frac{a^4}{r^4}\cos 2\theta.$$

$$\tau_{r\theta} = \cancel{2}\sin 2\theta\left[-\frac{\Delta\tau a^4}{\cancel{2}r^4} + \frac{\Delta\tau a^2}{\cancel{2}r^2}\right]$$

$$= \Delta\tau \sin 2\theta\left(\left(\frac{a}{r}\right)^2 - \left(\frac{a}{r}\right)^4\right).$$

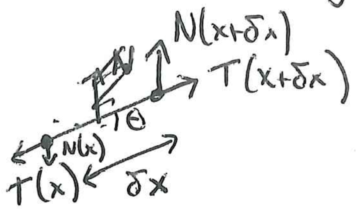
Hence $\tau_{\theta\theta}^{min} = \cancel{4}\tau_{in} - 2\tau_{out} - \Delta\tau$ (at $\theta = \pm\pi/2$)

so relative hoop compression largest when radial Tension is smallest.

2.1

raft marks

9/ · We are told to neglect gravity but incorporate inertia.



Newton's second law then gives:

$$\rho\,\delta x \begin{pmatrix} 0 \\ \frac{\partial^2 w}{\partial t^2} \end{pmatrix} = \left[\begin{pmatrix} T\cos\theta \\ T\sin\theta \end{pmatrix}\right]_x^{x+\delta x} + \left[\begin{pmatrix} 0 \\ N \end{pmatrix}\right]_x^{x+\delta x}$$

Letting $\delta x \to 0$ we find that:

$$\frac{\partial}{\partial x}\left(T\cos\theta\right) = 0$$

and:
$$\rho \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(T\sin\theta\right) + \frac{\partial N}{\partial x}.$$

Displacements are small, $|\theta| \ll 1 \Rightarrow \cos\theta \simeq 1$

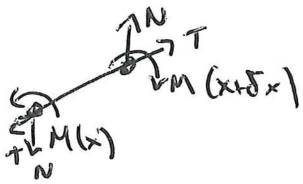
and $\sin\theta \simeq \tan\theta \simeq \frac{\partial w}{\partial x}$.

$$\therefore \frac{\partial T}{\partial x} = 0$$

and so:

$$\rho \frac{\partial^2 w}{\partial t^2} = T\frac{\partial^2 w}{\partial x^2} + \frac{\partial N}{\partial x}.$$

To determine N, use torque balance on segment:



M = bending moment exerted on each segment of beam by its neighbour.

Taking moments about x:

$$M(x+\delta x) - M(x) - N(x+\delta x)\delta x = 0$$

$$\Rightarrow \frac{\partial M}{\partial x} = N$$

But $M = -B\frac{\partial^2 w}{\partial x^2}$ given, so: $N = -B\frac{\partial^3 w}{\partial x^3}$.

and hence:
$$\rho \frac{\partial^2 w}{\partial t^2} = -B\frac{\partial^4 w}{\partial x^4} + T\frac{\partial^2 w}{\partial x^2}$$ as required.

6

Comme

Books

Explicitly covered in lectures

2.2

b) Freely supported edges $\Rightarrow$ $w(0,t) = w_{xx}(0,t) = 0$
and $w(L,t) = w_{xx}(L,t) = 0$.

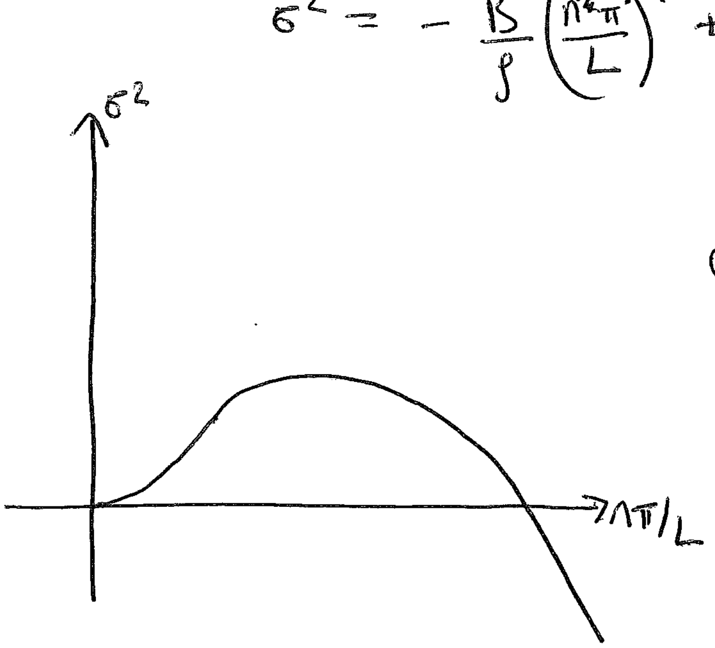
Seek a solution $w(x,t) = e^{\sigma t} f(x)$ as instructed,
and let $T = -P$ with $P > 0$ (since in compression).

Then beam equation becomes:
$$\rho \sigma^2 f = -B f'''' - P f''.$$
Choose $f(x) = \sin kx$ inspired by b.c @ $x = 0$.
$$\Rightarrow \rho \sigma^2 = -B k^4 + P k^2.$$
(sin terms cancel).

However, boundary conditions allow only certain modes.
In particular, need $k.L = n\pi$ with $n = 1, 2, \ldots$
to satisfy b.c's at $x = L$ (those at $x = 0$ are satisfied by ansatz).

Hence Have:

6? $$\sigma^2 = -\frac{B}{\rho}\left(\frac{n\pi}{L}\right)^4 + \left(\frac{n\pi}{L}\right)^2 \cdot P/\rho.$$



Only integer 'n' are allowed
Have instability if
$$P/\rho > \frac{B}{\rho}\left(\frac{n\pi}{L}\right)^2$$
ie for $n < \frac{L}{\pi}\left(\frac{P}{B}\right)^{1/2}$.

Fastest growing instability when σ largest, i.e.
when $k = \frac{n\pi}{L} \simeq \left(P/2B\right)^{1/2}$

4? ie wavelength $\lambda = \frac{2\pi}{k} = \frac{2L}{n}$ when $n = \left[\frac{L}{\pi}(P/2B)^{1/2}\right]$ or $\left[\frac{L}{\pi}(P/2B)^{1/2}\right]$

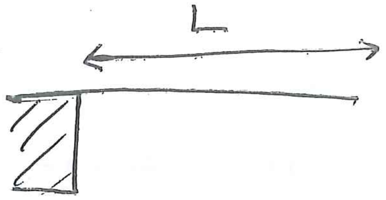
Comment
B
S
Problem of normal modes $w = e^{i(kx - \omega t)}$ covered, but no discussion of BCs there

Draft Marks

Comm

21.3

c).



BCs here are:

clamped @ $x=0 \Rightarrow w(0,t) = w_x(0,t) = 0$

free @ $x = L \Rightarrow w_{xx}(L,t) = w_{xxx}(L,t) = 0$.

B

2

Seek a solution of the beam equation of the form:

S

$$w(x,t) = e^{i\omega t} f(x)$$

Beam equation gives:

$$-\rho\omega^2 f = -B f'''' - P f''.$$

But here $P=0$ (no applied compression and end free).

so: $$f'''' = \frac{\rho\omega^2}{B} f$$

with $f(0) = f'(0) = f''(L) = f'''(L) = 0$.

Letting $\xi = x/L$ and $\Omega^2 = \frac{\rho\omega^2 L^4}{B}$ we have:

$$\frac{d^4 f}{d\xi^4} = \Omega^2 f \quad \text{with } f(0) = f'(0) = f''(1) = f'''(1) = 0.$$

N

$$\Rightarrow \quad f = A\cos\Omega^{1/2}\xi + B\sin\Omega^{1/2}\xi + C\cosh\Omega^{1/2}\xi + D\sinh\Omega^{1/2}\xi$$

$f(0) = 0 \Rightarrow A + C = 0 \Rightarrow C = -A$

$f'(0) = 0 \Rightarrow B + D = 0 \Rightarrow D = -B$

$f''(1) = 0 \Rightarrow -A\left(\cos\Omega^{1/2} + \cosh\Omega^{1/2}\right) - B\left(\sin\Omega^{1/2} + \sinh\Omega^{1/2}\right) = 0$

and $f'''(1) = 0 \Rightarrow A\left(\sin\Omega^{1/2} - \sinh\Omega^{1/2}\right) - B\left(\cos\Omega^{1/2} + \cosh\Omega^{1/2}\right) = 0$

3.

So we have:

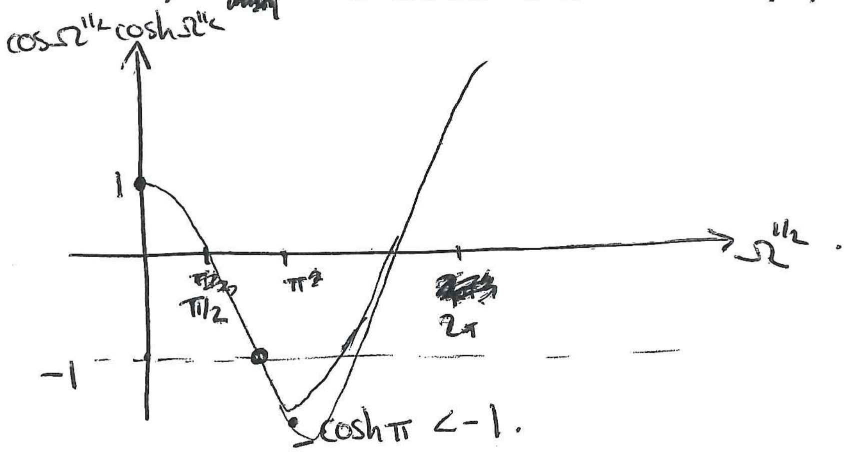
$$\begin{pmatrix} \cos\Omega^{1/2} + \cos\Omega^{1/2} & \sin\Omega^{1/2} + \sinh\Omega^{1/2} \\ \sin\Omega^{1/2} - \sinh\Omega^{1/2} & -(\cos\Omega^{1/2} + \cosh\Omega^{1/2}) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0.$$

For a non-trivial solution need $\det(M) = 0$, ie:

$$-\cos^2\Omega^{1/2} - \cosh^2\Omega^{1/2} - 2\cos\Omega^{1/2}\cdot\cosh\Omega^{1/2}$$
$$-\left(\sin^2\Omega^{1/2} - \sinh^2\Omega^{1/2}\right) = 0.$$

$$\Rightarrow \quad \cos\Omega^{1/2}\cosh\Omega^{1/2} = -1.$$

2



Clearly has solution in: $\pi/2 < \Omega^{1/2} < \pi$

ie $\pi^2/4 < \omega\sqrt{\frac{\rho L^4}{B}} < \pi^2$

So lowest frequency mode satisfies:

$$\sqrt{\frac{B}{\rho L^4}}\,\frac{\pi^2}{4} < \omega < \sqrt{\frac{B}{\rho L^4}}\,\pi^2$$

+2

$$A/B = -\frac{\sin\Omega^{1/2} + \sinh\Omega^{1/2}}{\cos\Omega^{1/2} + \cosh\Omega^{1/2}} = \frac{\cos\Omega^{1/2} + \cosh\Omega^{1/2}}{\sin\Omega^{1/2} - \sinh\Omega^{1/2}}.$$

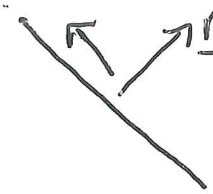
$$\Rightarrow \quad \sinh^2\Omega^{1/2} - \sin^2\Omega^{1/2} = (\cos\Omega^{1/2} + \cosh\Omega^{1/2})^2$$

Solution of normal modes problem is new, as is rough estimation of roots.

3.1

a)

$\underline{t} = (-\sin\theta, \cos\theta, 0)^T$



$\underline{n} = (\cos\theta, \sin\theta, 0)^T$

In plane strain here:

$$\tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \tau_{yy} & 0 \\ 0 & 0 & \tau_{zz} \end{pmatrix}$$

so the shear stress is:

$$\underline{t}\,\tau\,\underline{n} = (-\sin\theta, \cos\theta, 0)\begin{pmatrix} \tau_{xx}\cos\theta + \tau_{xy}\sin\theta \\ \tau_{xy}\cos\theta + \tau_{yy}\sin\theta \\ 0 \end{pmatrix}$$

$$= \frac{(\tau_{yy} - \tau_{xx})}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

To maximize this, write it as $A\sin(2\theta - \varphi)$

with $A = \sqrt{\dfrac{(\tau_{yy} - \tau_{xx})^2}{4} + \tau_{xy}^2}$

A is the max. value of shear stress so that

$$S = \sqrt{\frac{(\tau_{yy} - \tau_{xx})^2}{4} + \tau_{xy}^2}$$

as required.

6.

Tresca condition is that $S \leq \tau_Y$

Comm

B – L Discussed in lectures and covered in 2nd Prob Sheet.

Comments

3.2

b)

i) We are given the displacement field:

$$\underline{u} = [u(x,y), v(x,y), 0]^T$$

with
$$u = -Kxy$$
$$v = \frac{1}{2}Kx^2 + \frac{1}{2}\frac{\lambda}{\lambda+2\mu}Ky^2$$
with K a constant

S

Then, assuming plane strain:

$$\tau_{xx} = \lambda(u_x + v_y) + 2\mu u_x$$
$$= (\lambda+2\mu)(-Ky) + \lambda \cdot \frac{\lambda}{\lambda+2\mu} Ky$$

$$\tau_{xx} = -\frac{4\mu(\lambda+\mu)}{\lambda+2\mu} K \cdot y$$

$$\tau_{yy} = \lambda(u_x) + (\lambda+2\mu)v_y$$
$$= -\lambda Ky + \cancel{(\lambda+2\mu)}\left(\frac{Ky\lambda}{\cancel{\lambda+2\mu}}\right)$$
$$= 0.$$

and $\tau_{xy} = \mu(-Kx + Kx) = 0.$

Eqm eqns in Cartesian coordinates are:

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial}{\partial y}\tau_{xy} = 0$$

and
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = 0$$

→ Both are clearly satisfied here so the displacement field is admissible.
Further bcs of no normal tractions on $y = \pm h/2$ hold.

Example without plane strain considered in lecture notes

ii) Torque required is: $M = -\int_{-h/2}^{h/2} \tau_{xx} \cdot y \cdot dy$

$$= \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} K \cdot \left[\frac{y^3}{3}\right]_{-h/2}^{h/2} = \frac{\mu(\lambda+\mu)}{3(\lambda+2\mu)} Kh^3$$

B

Half Marks | 3.3 | Comment

c) i) The Tresca condition required that:

$$\tfrac{1}{2}|\tau_{xx}| \le \tau_y$$

$$\text{II}$$

$$2\frac{\mu(\lambda+\mu)}{\lambda+2\mu}\cdot|k||y| \le \tau_y .$$

Clearly $|\tau_{xx}|$ is largest on the two surfaces: $y = \pm h/2$

So elastic solution is valid provided ~~that~~:

$$\boxed{|k| \le k_y = \frac{(\lambda+2\mu)}{\mu(\lambda+\mu)}\cdot\frac{\tau_y}{h}.}$$

3

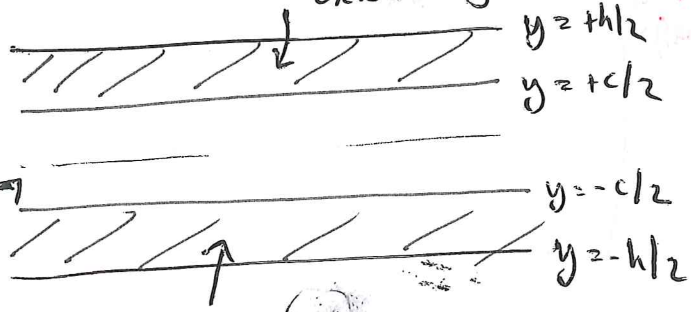
S

Standard, but not considered in lectures

ii) Beyond this point, plastic failure occurs, propagating in from edges:

Assuming $k>0$

$\tau_{xx} = -2\tau_y$



$\tau_{xx} = +2\tau_y, \quad \tau_{yy} = \tau_{xy} = 0.$

$\tau_{xx} = -\frac{\mu(\lambda+\mu)}{\lambda+2\mu}k\cdot y.$ (Clearly satisfies Tresca condition and is continuous).

N

By continuity ~~...~~ of τ_{xx} at $y=\pm c/2$ must have:

2

$$\frac{\mu(\lambda+\mu)\,k\,c/\cancel{2}}{\lambda+2\mu} = 2\tau_y \Rightarrow c = \frac{(\lambda+2\mu)}{\mu(\lambda+\mu)}\cdot\frac{\tau_y}{k}$$

(NB: $c=h$ when $k=k_y$ as expected)

Marks | Comments

3.4

So we have that the moment required to bend beam is:

iii)

$$M = \int_{-c/2}^{c/2} 4\frac{\mu(\lambda+\mu)}{\lambda+2\mu} k \cdot y^2 \, dy$$

$$+ 2\int_{c/2}^{h/2} 2 \cdot \tau_y \cdot y \, dy$$

$$= \frac{\mu(\lambda+\mu)}{\lambda+2\mu} \frac{4k}{3} \cdot \frac{c^3}{4} + 2 \cdot \tau_y \left[\frac{h^2}{4} - \frac{c^2}{4}\right].$$

But $$k = \frac{(\lambda+2\mu)}{\mu(\lambda+\mu)} \frac{\tau_y}{c}$$

so: $$M = \frac{\mu(\lambda+\mu)}{\lambda+2\mu} \cdot \frac{(\lambda+2\mu)}{\mu(\lambda+2\mu)} \frac{4}{3} \frac{\tau_y}{\cancel{c}} \cdot \frac{c^{2}}{\cancel{4}}$$

$$s^2 \frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)} \cdot \frac{\lambda+2\mu}{2\mu(\lambda+\mu)} \tau_y$$

$$-2\tau_y s^2.$$

$$-\frac{2}{3}\tau_y s^2.$$

$$+ 2\tau_y \left[\frac{h^2}{4} - \frac{c^2}{4}\right]$$

$$= \frac{1}{3}\tau_y c^2 - \frac{1}{2}\tau_y c^2 + \tau_y h^2/2$$

4 $$= \frac{1}{2}\tau_y \left[h^2 - \frac{1}{3}c^2\right]$$

Since $c \propto 1/k$, we have that $c \to 0$ as $k \to \infty$

$\Rightarrow M \to \frac{1}{2}\tau_y h^2$ a constant.

2 In reality, assumptions of small strain breaks down as $k \to \infty$.