A12204W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.2

ELASTICITY AND PLASTICITY

TRINITY TERM 2021

Tuesday 08 June

Opening time: 09:30 (BST)

Mode of completion: Handwritten

You have 1 hour 45 minutes writing time to complete the paper and up to 30 minutes technical time to upload your answer file.

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you observe the following points:

1. Write with a black or blue pen OR with a stylus on tablet (colour set to black or blue).
2. On the first page, write
   - your candidate number
   - the paper code
   - the paper title
   - and your course title (e.g. FHS Mathematics and Statistics Part C)
   - but ***do not*** enter your name or college.
3. For each question you attempt,
   - start writing on a new sheet of paper,
   - indicate the question number clearly at the top of each sheet of paper,
   - number each page
4. Before scanning and submitting your work,
   - on the first page, in numerical order, write the question numbers attempted,
   - cross out all rough working and any working you do not want to be marked,
   - and orient all scanned pages in the same way.
5. Submit all your answers to this paper as a ***single PDF*** document

If you do not attempt any questions at all on this paper, you should still submit a single page indicating that you have opened the exam but not attempted any questions. Please make sure to write your candidate number on this single page.

1. (a) [6 marks] Seek solutions of the unsteady two-dimensional Navier equation (with no body force) of the form

$$\mathbf{u}(x,y,t) = \begin{pmatrix} u(x,y,t) \\ v(x,y,t) \end{pmatrix} = \mathbf{a}\mathrm{e}^{\mathrm{i}k(x-ct)+Ky},$$

with positive k, c, K and constant $\mathbf{a} \in \mathbb{R}^2$. Show that nontrivial solutions exist only if either $\mathbf{a} \propto (k, -\mathrm{i}K)^{\mathrm{T}}$ or $\mathbf{a} \propto (\mathrm{i}K, k)^{\mathrm{T}}$, and find the corresponding values of K, in terms of k, c, c_{s} and c_{p}, where c_{s} and c_{p} denote the S-wave and P-wave speeds, respectively.

[*The unsteady Navier equation with no body force reads*

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + \mu \nabla^2 \mathbf{u},$$

where ρ is the density and λ, μ are the Lamé constants.]

(b) [19 marks] An elastic material undergoes plane strain in the half-plane $\{(x,y) : -\infty < x < \infty,\ y < 0\}$. A thin elastic beam of surface density σ and bending stiffness B is attached to the interface at $y = 0$.

(i) Assuming that the beam performs small purely transverse displacements and that the tension in the beam is zero, derive the boundary conditions

$$u = 0, \qquad \sigma \frac{\partial^2 v}{\partial t^2} + B \frac{\partial^4 v}{\partial x^4} + \rho c_{\mathrm{p}}^2 \frac{\partial v}{\partial y} = 0 \qquad \text{at } y = 0.$$

Explain any approximations and constitutive relations that you use.

(ii) Show that the system supports waves that travel in the x-direction with speed $c > 0$ and wavenumber $k > 0$, while decaying exponentially as $y \to -\infty$, if and only if c and k satisfy

$$k\left(\sigma c^2 - Bk^2\right) H(c) = \rho c^2,$$

where

$$H(c) = \frac{c_{\mathrm{s}}}{\sqrt{c_{\mathrm{s}}^2 - c^2}} - \frac{\sqrt{c_{\mathrm{p}}^2 - c^2}}{c_{\mathrm{p}}}.$$

(iii) Deduce that such solutions can exist only if

$$k\sqrt{B/\sigma} < c < c_{\mathrm{s}} \qquad \text{and} \qquad cH(c) \geqslant \frac{3\rho\sqrt{3B}}{2\sigma^{3/2}}.$$

2. A thin beam of bending stiffness B and length $2L$ undergoes small two-dimensional deformations in the (x, z)-plane. There is no tension applied to the beam, which lies along the x-axis in its undeformed state. You may assume that, in equilibrium when subject to a downward (i.e. in the negative z-direction) body force $p(x)$ per unit length, the transverse displacement $w(x)$ satisfies the *linear beam equation*

$$Bw''''(x) + p(x) = 0.$$

The ends of the beam are fixed on the x-axis and simply supported so that $w(\pm L) = w''(\pm L) = 0$. The effects of gravity are negligible.

A smooth symmetric convex obstacle is brought into contact with the beam from above. The boundary of the obstacle is given by $z = f(x)$, where $f(-x) \equiv f(x)$, $f''(x) > 0$ and $f(0) \leqslant 0 \leqslant f(L)$. You may assume that the displacement is symmetric, i.e. $w(-x) \equiv w(x)$, and that w, w' and w'' are all continuous at points where the beam makes or loses contact with the obstacle.

(a) [4 marks] Explain why $w(x)$ satisfies the linear complementarity problem

$$w''''(x)\big(w(x) - f(x)\big) = 0, \qquad w''''(x) \leqslant 0, \qquad w(x) - f(x) \leqslant 0.$$

Show also that the upwards force exerted on the obstacle is given by $F = -2Bw'''(L)$.

(b) [5 marks] Show that

$$\int_{-L}^{L} \frac{1}{2} Bw''(x)^2 \,\mathrm{d}x \leqslant \int_{-L}^{L} \frac{1}{2} Bv''(x)^2 \,\mathrm{d}x$$

for all $v \in \mathcal{V} := \{v \in C^2[-L, L] : v(\pm L) = v''(\pm L) = 0, v \leqslant f\}$.
Interpret this result physically.

(c) [10 marks] Now focus on the case $f(x) = -\delta + \kappa x^2/2$, where $\kappa > 0$ and $0 \leqslant \delta \leqslant \kappa L^2/2$.
Show that, as δ is gradually increased from zero, the beam makes contact with the obstacle at a single point until $\delta = \kappa L^2/3$.
For $\delta > \kappa L^2/3$, show that the force F applied to the obstacle is related to the penetration distance δ by

$$F = \sqrt{\frac{2}{3}} \frac{B\kappa^{3/2}}{\sqrt{\kappa L^2/2 - \delta}}.$$

[*Hint: consider the integral* $\displaystyle\int_s^L (L - x)w''(x) \,\mathrm{d}x$.]

(d) [6 marks] The system described above is used as a catapult to launch a projectile of mass M whose bottom surface is given by $f(x)$. The projectile is pushed down to its furthest extent, with $\delta \to \kappa L^2/2$, and then released from rest. Assuming that the inertia of the beam is negligible so it remains in equilibrium throughout the motion, write down the equation of motion for the projectile and show that it is launched at a speed $\kappa\sqrt{2BL/M}$.
Explain briefly how this result is related to part (b).

3. (a) [4 marks] Consider a two-dimensional granular medium in which the granules exert a mutual adhesive stress $A > 0$, such that the normal stress N and tangential stress F on any line element inside the material satisfy the inequality $|F| \leqslant (A - N)\tan\phi$, where ϕ is the angle of friction.

Show that the stress components satisfy a yield criterion of the form $f(\tau_{xx}, \tau_{xy}, \tau_{yy}) \leqslant \tau_{\mathrm{Y}}$, where the yield function is given by

$$f(\tau_{xx}, \tau_{xy}, \tau_{yy}) = \tfrac{1}{2}\sin\phi(\tau_{xx} + \tau_{yy}) + \sqrt{\tfrac{1}{4}(\tau_{xx} - \tau_{yy})^2 + \tau_{xy}^2}$$

and the yield stress τ_{Y} is to be determined in terms of A and ϕ.

[*Properties of the* Mohr circle *may be used without proof.*]

(b) [5 marks] The granular medium described above occupies the region $r > a$ outside a circular cavity of radius a, where (r, θ) denote plane polar coordinates. The material outside the cavity undergoes a purely radial deformation with displacement $\mathbf{u} = u(r,t)\mathbf{e}_r$, where $\mathbf{e}_r$ is the unit vector in the r-direction. The stress tends to zero in the far field, while the surface of the cavity at $r = a$ is subject to a pressure $P(t)$ which is gradually increased from zero.

Assuming that the medium behaves as a linear elastic solid while $f < \tau_{\mathrm{Y}}$, show that yield first occurs at the surface of the cavity when $P = \tau_{\mathrm{Y}}$.

(c) [6 marks] For $P > \tau_{\mathrm{Y}}$, show that the material yields in a region $a < r < s$, where

$$\frac{s}{a} = \left[1 + \frac{\beta}{2}\left(\frac{P}{\tau_{\mathrm{Y}}} - 1\right)\right]^{1/\beta}, \qquad \beta = \frac{2\sin\phi}{1 + \sin\phi}.$$

(d) [10 marks] Assuming that the material obeys the *associated flow rule*, show that the displacement in the yielded region satisfies

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r} + (1 - \beta)\frac{u}{r}\right) = 0.$$

Hence evaluate the displacement everywhere in $r > a$.

[*You may use without proof the radially symmetric Navier equation:*

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0,$$

and the linear strain components

$$e_{rr} = \frac{\partial u}{\partial r}, \qquad e_{\theta\theta} = \frac{u}{r}.]$$

2D Navier eq$^{\underline{n}}$:

$$\rho \underset{\sim}{u}_{tt} = (\lambda+\mu)\,\text{grad div}\,\underline{u} + \mu \nabla^2 \underset{\sim}{u}$$

with $\underline{u} = \underset{\sim}{a}\, e^{ik(x-ct)+Ky}$ we have

$$\nabla^2 \underset{\sim}{u} = (K^2 - k^2)\underset{\sim}{u}$$

$$\text{div}\,\underset{\sim}{u} = \begin{pmatrix} ik \\ K \end{pmatrix} \cdot \underset{\sim}{a}\, e^{ik(x-ct)+Ky}$$

$$\text{grad div}\,\underline{u} = \begin{pmatrix} ik \\ K \end{pmatrix} \left[\begin{pmatrix} ik \\ K \end{pmatrix} \cdot \underset{\sim}{a} \right] e^{ik(x-ct)+Ky}$$

Denote $\underset{\sim}{k} = \begin{pmatrix} k \\ -iK \end{pmatrix}$, so Navier eq$^{\underline{n}}$ becomes

(3) $$\left(\mu(K^2-k^2) + \rho c^2 k^2\right)\underset{\sim}{a} - (\lambda+\mu)(\underset{\sim}{k}\cdot\underset{\sim}{a})\,\underset{\sim}{\mathbf{k}} = \underline{0}$$

Dot with $\underset{\sim}{k}$: $$\boxed{\left[(\lambda+2\mu)(K^2-k^2) + \rho c^2 k^2\right](\underset{\sim}{a}\cdot\underset{\sim}{k}) = 0}$$

Dot with $\underset{\sim}{k}^{\perp} = \begin{pmatrix} iK \\ k \end{pmatrix}$ $$\boxed{\left[\mu(K^2-k^2) + \rho c^2 k^2\right](\underset{\sim}{a}\cdot\underset{\sim}{k}^{\perp}) = 0}$$

Recall $$\boxed{c_p^2 = \frac{(\lambda+2\mu)}{\rho}, \quad c_s^2 = \frac{\mu}{\rho}}$$

So for non trivial $\underset{\sim}{a}$ we must have

either: (1) $\underset{\sim}{a} \cdot \underset{\sim}{k}^{\perp} = 0$, i.e. $\boxed{\underset{\sim}{a} \propto \underset{\sim}{k}}$

& then $\boxed{K = k\sqrt{1 - \frac{c^2}{c_p^2}} = K_p}$, say

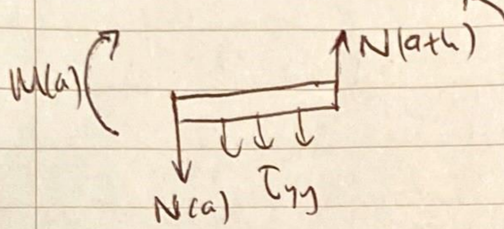
or (2) $\underset{\sim}{a} \cdot \underset{\sim}{k} = 0$, i.e. $\boxed{\underset{\sim}{a} \propto \underset{\sim}{k}^{\perp}}$

(3) & then $\boxed{K = k\sqrt{1 - \frac{c^2}{c_s^2}} = K_s}$, say

NB we need $c < c_s < c_p$ for waves of this form to occur

[6 marks — similar to problem sheet]

Consider a small section of the beam between $x=a$ and $x=a+h$ (say)



Denote transverse shear stress by $N(x,t)$. Vertical component of stress from elastic solid underneath

Vertical component of Newton's 2nd law:

$$\frac{d}{dt}\int_a^{a+h} \sigma V_t \, dx = \Big[N\Big]_a^{a+h} - \int_a^{a+h} \tau_{yy} \, dx$$

$$\therefore \int_a^{a+h} \sigma V_{tt} - \frac{\partial N}{\partial x} + \tau_{yy} \, dx = 0$$

(2) a, h arbitrary $\Rightarrow$ $\boxed{\sigma V_{tt} - \frac{\partial N}{\partial x} + \tau_{yy} = 0 \quad \text{at } y=0}$

Moment balance: Denote bending moment by $M(x,t)$ — here defined anticlockwise, i.e. around z-axis.

Take moments about (e.g.) left-hand end $x=a$:

$$hN(a+h) + M(a+h) - M(a) - \int_a^{a+h} (x-a)\,\tau_{yy} \, dx = 0$$

Note final term is of order h^2. [Also angular momentum is negligible when the beam is thin]

(2) so letting $h \to 0$ we get $\boxed{N + \frac{\partial M}{\partial x} = 0}$

Finally, constitutive relation — assume linear relation between bending moment & curvature:

$$\boxed{M = B V_{xx} \quad \text{at } y=0}$$

Put the pieces together:

(2) $$\boxed{\sigma V_{tt} + B V_{xxxx} + \tau_{yy} = 0 \quad \text{at } y=0}$$

We are also told that beam displacement is purely transverse, so

$$\boxed{u=0 \quad \text{at } y=0}$$

then $\tau_{yy} = (\lambda+2\mu) V_y + \lambda U_x$

$= \rho c_p^2 V_y$ at $y=0$ (since $u=0$)

(1) so $$\boxed{\sigma V_{tt} + B V_{xxxx} + \rho c_p^2 V_y = 0 \quad \text{at } y=0}$$

[7 marks — derivation of beam equation from lecture notes but coupling to half-space is new]

Plug in general solution from part (a):

$$\underset{\sim}{u} = C_1 \begin{pmatrix} k \\ -ik_p \end{pmatrix} e^{ik(x-ct)+k_p y} + C_2 \begin{pmatrix} ik_s \\ k \end{pmatrix} e^{ik(x-ct)+k_s y}$$

<u>BCs at $y=0$:</u>

$$\boxed{k C_1 + i k_s C_2 = 0}$$

(3)

$$\sigma\left[C_1 k^2 c^2 i k_p - C_2 k^3 c^2\right] + B\left[-ik_p C_1 k^4 + C_2 k^5\right] + \rho c_p^2\left[-ik_p^2 C_1 + k k_s C_2\right] = 0$$

$$\therefore \begin{pmatrix} k & -k_s \\ k_p\left[k^2(\sigma c^2 - Bk^2) - \rho c_p^2 k_p^2\right] & k\left[\rho c_p^2 k_s - k^2(\sigma c^2 - Bk^2)\right] \end{pmatrix} \begin{pmatrix} iC_1 \\ C_2 \end{pmatrix} = \underset{\sim}{0}$$

Nontrivial solutions $\Leftrightarrow$ determinant of this matrix is zero, i.e.

$$k^2\left[k^2(\sigma c^2 - Bk^2) - \rho c_p^2 k_s\right] = k_s k_p\left[k^2(\sigma c^2 - Bk^2) - \rho c_p^2 k_p\right]$$

<u>Rearrange:</u>

$$(\sigma c^2 - Bk^2)\left[1 - \frac{k_s k_p}{k^2}\right] = \frac{\rho c_p^2}{k}\left[\frac{k_s}{k} - \frac{k_s k_p^2}{k^3}\right]$$

$$\therefore (\sigma c^2 - Bk^2)\left[1 - \frac{\sqrt{c_s^2 - c^2}\sqrt{c_p^2 - c^2}}{c_s c_p}\right] = \frac{\rho c_p^2}{k}\frac{\sqrt{c_s^2 - c^2}}{c_s}\left[1 - 1 + \frac{c^2}{c_p^2}\right]$$

$$\therefore \quad \boxed{(\sigma c^2 - Bk^2)\, H(c) = \frac{\rho c^2}{k}}$$

(3) where $\boxed{H(c) = \dfrac{c_1}{\sqrt{c_s^2 - c^2}} - \dfrac{\sqrt{c_p^2 - c^2}}{c_p}}$

[6 marks — familiar approach but slightly awkward calculation]

Note: $H'(c) = \dfrac{c\,c_s}{(c_s^2-c^2)^{3/2}} + \dfrac{c}{c_p\sqrt{c_p^2-c^2}} > 0$ for $c\in(0,c_s)$

with $H(0)=0$ and $H(c)\to\infty$ as $c\to c_s$

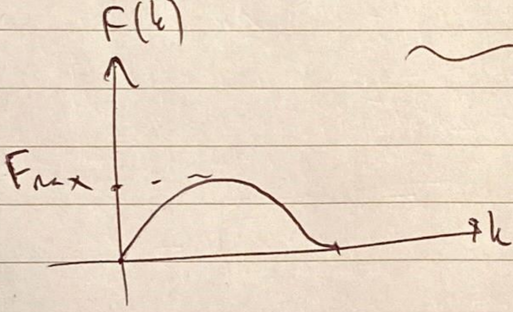
i.e. $H(c)>0$ for $c\in(0,c_s)$

So the eq^{n} has real positive solutions only if

(3) $$\boxed{c > k\sqrt{\frac{B}{\sigma}}}$$ ☒

Also consider $F(k) = k(\sigma c^2 - Bk^2)$

for $0<k<c\sqrt{\sigma/B}$



F is maximized at

$\sigma c^2 = 3Bk^2$

i.e. $k=\sqrt{\dfrac{\sigma c^2}{3B}}$,

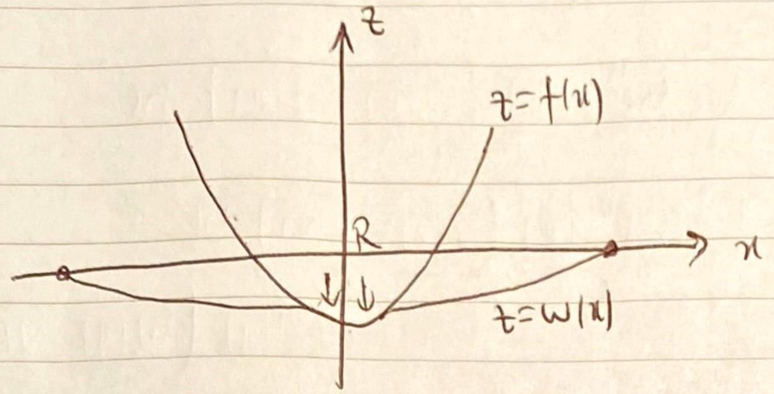
i.e. $F_{max} = \dfrac{2c^3\sigma^{3/2}}{3^{3/2}B^{1/2}}$

so $F(k)H(c) = \rho c^2 \leq \dfrac{2c^3\sigma^{3/2}}{3^{3/2}B^{1/2}}\cdot H(c)$

(3) i.e. $$\boxed{cH(c) \geq \frac{3^{3/2}\rho\sqrt{B}}{2\sigma^{3/2}}}$$ ☒

[6 marks — new]

[6 marks — new]



Either (i) beam is out of contact: then $w'''' = 0$

(given gravity is negligible) & $w < f$

Or (ii) beam is in contact: then $w = f$.

Also reaction force R in contact region must be downwards, so

$$Bw'''' = -R \leq 0$$

(2) put it together: $\boxed{\begin{array}{l} w''''(w-f) = 0 \\ w \leq f, \; w'''' \leq 0 \end{array}}$

$$\text{Force exerted on obstacle} = \int_{\substack{\text{contact}\\ \text{set}}} R\,dx = -B\int_{\substack{\text{contact}\\ \text{set}}} w''''(x)\,dx$$

$$= -B\int_{-L}^{L} w''''(x)\,dx \qquad (\text{since } w'''' = 0 \text{ in non-contact set})$$

$$= B\left(w'''(-L) - w'''(L)\right)$$

(2) $\therefore$ $\boxed{F = -2Bw'''(L)}$ (using symmetry of w)

[4 — generalising string calculations done in lectures]

When obstacle first makes contact, with $\delta = 0$, we have $w = f$, $w' = f'$ at $x=0$, but $w'' = 0 < f'' = \kappa$. There is contact at the single point $x=0$ until the curvatures match – then the contact set starts to spread.

So initially solve
$$\boxed{\begin{array}{ll} w''''(x) = 0 & x > 0 \\ w(L) = w''(L) = 0 & \\ w(0) = -\delta, \ w'(0) = 0 & \end{array}}$$

$\therefore$ $w''(x) = A(L-x)$ some constant A

$w'(x) = A\left(Lx - \frac{x^2}{2}\right)$

$w(x) = -\delta + A\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right)$

& $w(L) = 0$ gives $A = \frac{3\delta}{L^3}$, i.e.

(3) $$\boxed{w(x) = \delta\left[-1 + \frac{3x^2}{2L^2} - \frac{x^3}{2L^3}\right]}$$

$w''(0) = \frac{3\delta}{L^2}$ which reaches the curvature κ of the obstacle when

(2) $\boxed{\delta = \frac{\kappa L^2}{3}}$

For future reference, the force on the obstacle in this regime is

$$\boxed{F = -2Bw'''(0) = \frac{6B\delta}{L^3}}$$

Note $0 = \int_{-L}^{L} w''''(x)\,\big(w(x) - f(x)\big)\,dx$

$$= \int_{-L}^{L} \underbrace{w''''(x)}_{\le 0}\,\underbrace{\big(v(x) - f(x)\big)}_{\le 0}\,dx + \int_{-L}^{L} w''''(x)\,\big(w(x) - v(x)\big)\,dx$$

i.e. $0 \geq \int_{-L}^{L} w''''(x)\,\big(w(x) - v(x)\big)\,dx \quad$ for $v, w \in V$

$$= \Big[w'''(x)\,\big(w(x) - v(x)\big)\Big]_{-L}^{L} + \int_{-L}^{L} w'''(x)\,\big(v'(x) - w'(x)\big)\,dx$$

[NB w''' is piecewise smooth – contributions from points where w''' is discontinuous disappear because $w = v$ there]

(2) $$= \underbrace{\Big[w''(x)\,\big(v'(x) - w'(x)\big)\Big]_{-L}^{L}}_{=0} \cancel{+} - \int_{-L}^{L} w''(x)\,\big(v''(x) - w''(x)\big)\,dx$$

$$\therefore\ 0 \geq \int_{-L}^{L} \Big[-\tfrac{1}{2} v''(x)^2 + \tfrac{1}{2} w''(x)^2 + \tfrac{1}{2}\big(v''(x) - w''(x)\big)^2\Big]\,dx$$

(2) $$\boxed{\therefore \int_{-L}^{L} \tfrac{1}{2} B\, v''(x)^2\,dx \geq \int_{-L}^{L} \tfrac{1}{2} B\, w''(x)^2\,dx} \quad \forall\, v \in V$$

This is the elastic bending energy: w is the

(1) element of V that minimises this energy.

[5 generalising string calculations done in lectures]

For $\delta > \frac{kL^2}{3}$, introduce a contact set $-s < x < s$.

By symmetry, just focus on $s < x < L$.

So solve

$$\boxed{\begin{aligned} & w''''(x) = 0 \\ & w(L) = w''(L) = 0 \\ & w(s) = -\delta + \frac{ks^2}{2} \\ & w'(s) = ks \\ & w''(s) = k \end{aligned}}$$

(2) Integrate: $\boxed{w''(x) = \frac{k(L-x)}{L-s}}$

Use the hint:

$$\int_s^L (L-x)\, w''(x)\, dx = \Big[(L-x)\, w'(x) + w(x)\Big]_s^L$$

$$\therefore \frac{k}{(L-s)} \cdot \frac{(L-s)^3}{3} = -(L-s).ks + \delta - \frac{ks^2}{2}$$

$$= \delta + \frac{ks^2}{2} - kLs = \delta + \frac{k}{2}(L-s)^2 - \frac{kL^2}{2}$$

$$\cancel{\therefore \delta = k\left[\frac{L^2}{3} - \frac{2}{3}Ls + \frac{s^2}{3} + Ls - \frac{s^2}{2}\right]}$$

i.e. $\frac{kL^2}{2} - \delta = \frac{k}{6}(L-s)^2$

(2) $\therefore L - s = \sqrt{\frac{6}{k}} \cdot \sqrt{\frac{kL^2}{2} - \delta}$

So force $F = -2Bw'''(L) = \frac{2BK}{L-s}$

(1) ie, $\boxed{F = \sqrt{\frac{2}{3}} \frac{B K^{3/2}}{\sqrt{KL^2/2 - \delta}}}$

[10 — new but quite familiar calculations]

Newton's 2nd law for the projectile:

$$M\ddot{\delta} + F(\delta) = 0 \quad \text{with } \dot{\delta} \to 0 \text{ when } \delta \to \frac{KL^2}{2}$$

(2) $$\therefore \ \tfrac{1}{2}M\dot{\delta}^2 = \int_{\delta}^{KL^2/2} F(\delta)\,d\delta$$

Final velocity is given by $\tfrac{1}{2}MV^2 = \int_0^{KL^2/2} F(\delta)\,d\delta$

$$\tfrac{1}{2}MV^2 = \int_0^{KL^2/3} \frac{6B\delta}{L^3}\,d\delta + \int_{KL^2/3}^{KL^2/2} \sqrt{\frac{2}{3}} \cdot \frac{B \cdot K^{3/2}}{\sqrt{\frac{KL^2}{2} - \delta}}\,d\delta$$

$$= \frac{3B}{L^3} \cdot \frac{K^2L^4}{9} + \sqrt{\frac{2}{3}}BK^{3/2}\left[-2\sqrt{\frac{KL^2}{2} - \delta}\,\right]_{KL^2/3}^{KL^2/2}$$

$$= \frac{BK^2L}{3} + \sqrt{\frac{2}{3}}BK^{3/2} \cdot 2\sqrt{\frac{KL^2}{6}}$$

$$\tfrac{1}{2}MV^2 = BK^2L$$

(3) $$\text{i.e.} \quad \boxed{V = K\sqrt{\frac{2BL}{M}}}$$

[5 — new calculation]

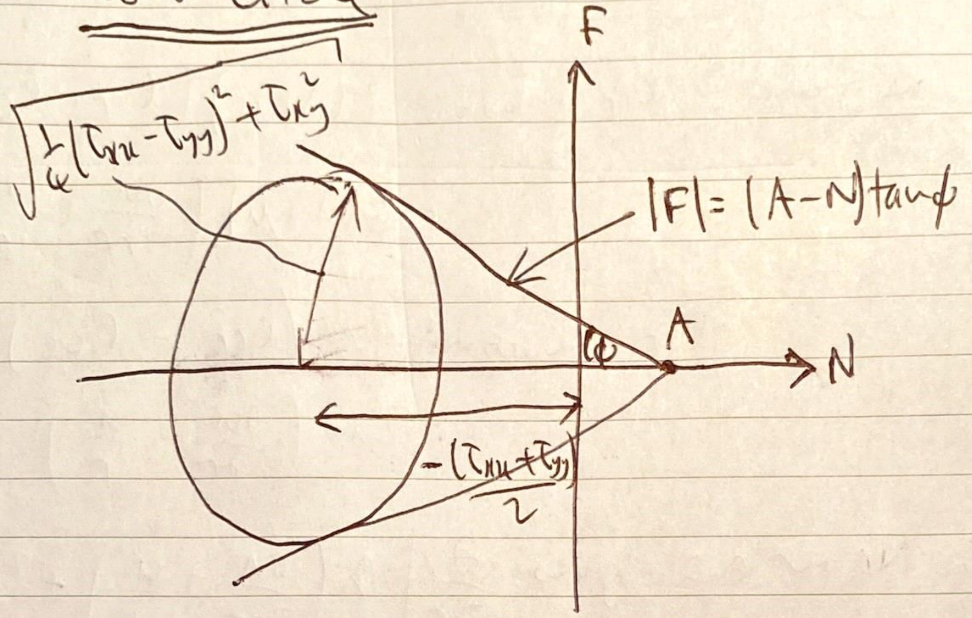
NB from (5), elastic energy in beam when its fully penetrated is

(1) $$E = \tfrac{1}{2}B\int_{-L}^{L} W''(x)^2\,dx = \tfrac{1}{2}B \cdot 2L \cdot K^2 = BLK^2$$

which is equal to the kinetic energy of the projectile when it leaves the beam.

[6 — new] [1 — new]

Mohr circle



We see from basic trig that the entire circle satisfies $|F| \le (A-N)\tan\phi$ iff

(2) $$\sin\phi \geq \frac{\sqrt{\frac{1}{4}(\tau_{xx}-\tau_{yy})^2+\tau_{xy}^2}}{A-\frac{1}{2}(\tau_{xx}+\tau_{yy})}$$

and $A - \frac{1}{2}(\tau_{xx}+\tau_{yy}) > 0$

i.e. $\boxed{f \le \tau_Y}$ where $\boxed{\tau_Y = A\sin\phi}$

(2) and $$\boxed{f = \frac{1}{2}\sin\phi\,(\tau_{xx}+\tau_{yy}) + \sqrt{\frac{1}{4}(\tau_{xx}-\tau_{yy})^2+\tau_{xy}^2}}$$

[Generalising lecture notes]

[4 — adhesion is new]

While material is elastic:

we have $\tau_{rr} + \tau_{\theta\theta} = 2(\lambda+\mu)(e_{rr} + e_{\theta\theta})$

$$= 2(\lambda+\mu)\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)$$

$$\tau_{rr} - \tau_{\theta\theta} = 2\mu(e_{rr} - e_{\theta\theta})$$

$$= 2\mu\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right)$$

& Navier eq$^=$ gives $\frac{\partial}{\partial r}\left[\lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + 2\mu\frac{\partial u}{\partial r}\right]$

$$+ \frac{2\mu}{r}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) = 0$$

$$\therefore (\lambda+2\mu)\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) = 0$$

$\therefore \boxed{\frac{\partial}{\partial r}(\tau_{rr} + \tau_{\theta\theta}) = 0}$ compatibility condition

Given stress $\to 0$ at ∞, we have

$$\boxed{\tau_{rr} + \tau_{\theta\theta} = 0}$$

$$\therefore \frac{\partial \tau_{rr}}{\partial r} + \frac{2\tau_{rr}}{r} = 0$$

$\Rightarrow \boxed{\tau_{rr} = -\frac{A}{r^2}, \quad \tau_{\theta\theta} = \frac{A}{r^2}}$ for some $A(t)$.

Given $\tau_{rr} = -P$ on $r=a$ we have

(3) $\boxed{\tau_{rr} = -\frac{Pa^2}{r^2}, \quad \tau_{\theta\theta} = \frac{Pa^2}{r^2}}$ while material is elastic

In terms of radial co-ordinates, with $\tau_{r\theta}=0$, yield function reads

$$f = \tfrac{1}{2}\sin\phi\,(\tau_{rr}+\tau_{\theta\theta}) + \tfrac{1}{2}\,|\tau_{rr}-\tau_{\theta\theta}|$$

Since $\tau_{\theta\theta} > \tau_{rr}$ here,

$$\boxed{f = \tfrac{1}{2}(1+\sin\phi)\,\tau_{\theta\theta} - \tfrac{1}{2}(1-\sin\phi)\,\tau_{rr}}$$

Here we have $f = \dfrac{Pa^2}{r^2}$ while material is elastic

This is a decreasing function of r, so

(2) $\boxed{\text{Yield first occurs at } r=a \text{ when } P=\tau_y}$

[5 — familiar]

For $p > \tau_y$, solve elastic problem in $r > s$

to get $\tau_{rr} = -\frac{A}{r^2}$, $\tau_{\theta\theta} = \frac{A}{r^2}$ again.

$f = \tau_y$ at $r = s$ (yield condition) gives

(2) $\frac{A}{s^2} = \tau_y$ $\therefore$ $\boxed{\tau_{rr} = -\frac{\tau_y s^2}{r^2}, \tau_{\theta\theta} = \frac{\tau_y s^2}{r^2}}$

in $r > s$.

In $r < s$ material satisfies yield criterion

$$2\tau_y = (1+\sin\phi)\tau_{\theta\theta} - (1-\sin\phi)\tau_{rr}$$

$$\therefore \boxed{\tau_{\theta\theta} = \frac{2\tau_y}{1+\sin\phi} + \frac{(1-\sin\phi)}{(1+\sin\phi)}\tau_{rr}}$$

So Navier eqn reads

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{\beta}{r}\tau_{rr} = \frac{2\tau_y}{r(1+\sin\phi)} = \frac{(2-\beta)\tau_y}{r}$$

with $\beta = \frac{2\sin\phi}{1+\sin\phi}$

$\therefore$ $\tau_{rr} = \frac{C_1}{r^\beta} + \frac{(2-\beta)\tau_y}{\beta}$ with $\tau_{rr} = -p$ at $r = a$

(2) $$\boxed{\tau_{rr} = \frac{(2-\beta)\tau_y}{\beta} + \left(\frac{a}{r}\right)^\beta\left[\tau_y - p - \frac{2\tau_y}{\beta}\right]}$$

Continuity (stress balance) at $r=s$:

$$\frac{(2-\beta)\tau_Y}{\beta} + \left(\frac{a}{s}\right)^{\beta}\left[\tau_Y - P - \frac{2\tau_Y}{\beta}\right] = -\tau_Y$$

$$\therefore \left(\frac{s}{a}\right)^{\beta} = \frac{\beta}{2\tau_Y}\left[\frac{2\tau_Y}{\beta} + P - \tau_Y\right]$$

(2) $$\therefore \boxed{\frac{s}{a} = \left[1 + \frac{\beta}{2}\left(\frac{P}{\tau_Y} - 1\right)\right]^{1/\beta}}$$

[6 — generalising lecture notes]

Associated flow rule:

$$\dot{e}_{rr} = \frac{\partial \dot{u}}{\partial r} = \Lambda \frac{\partial f}{\partial \tau_{rr}} = -\frac{\Lambda}{2}(1-\sin\phi)$$

$$\dot{e}_{\theta\theta} = \frac{\dot{u}}{r} = \Lambda \frac{\partial f}{\partial \tau_{\theta\theta}} = \frac{\Lambda}{2}(1+\sin\phi)$$

$$\therefore \quad \frac{\partial \dot{u}}{\partial r} + \frac{(1-\sin\phi)}{(1+\sin\phi)} \frac{\dot{u}}{r} = 0$$

(3) i.e. $\boxed{\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r} + (1-\beta)\frac{u}{r} \right) = 0}$

In elastic region $r>s$

$$\tau_{rr} = -\frac{\tau_y s^2}{r^2} = \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + 2\mu \frac{\partial u}{\partial r}$$

$$\tau_{\theta\theta} = \frac{\tau_y s^2}{r^2} = \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + 2\mu \frac{u}{r}$$

$$\therefore \quad \frac{\partial u}{\partial r} + \frac{u}{r} = 0 \quad \text{i.e.} \quad u = \frac{C_2}{r}$$

& then read off

(2) $$\boxed{u = \frac{\tau_y s^2}{2\mu r}} \quad \text{in } r>s.$$

$$\therefore \quad \frac{\partial u}{\partial r} + (1-\beta)\frac{u}{r} = -\frac{\beta \tau_y s^2}{2\mu r^2} = -\frac{\beta \tau_y}{2\mu} \quad \text{at } r=s$$

so in yielded region, $\frac{\partial u}{\partial r} + (1-\beta)\frac{u}{r}$ stays fixed at this constant value.

(3) $$\therefore \quad u = C_3 r^{-1+\beta} - \frac{\beta \tau_y r}{2\mu(2-\beta)}$$

continuity of u at $r=s$:

$$C_3 s^{-1+\beta} - \frac{\beta \tau_y s}{2\mu(2-\beta)} = \frac{\tau_y s}{2\mu}$$

$$\text{i.e.} \quad C_3 = \frac{\tau_y s^{2-\beta}}{(2-\beta)\mu}$$

(2) i.e. $$\boxed{u = \frac{\tau_Y r}{2(2-\beta)\mu}\left[2\left(\frac{s}{r}\right)^{2-\beta} - \beta\right]}$$

in $a < r < s$

[10 — new]