SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.2
Honour School of Mathematical and Theoretical Physics Part C: Paper C5.2
Master of Science in Mathematical Sciences: Paper C5.2

Elasticity and Plasticity

TRINITY TERM 2022

Thursday 02 June, 14:30pm to 16:15pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.
Candidates may bring a summary sheet into this exam consisting of (both sides of) one sheet of A4 paper containing material prepared in accordance with the guidance given by the Mathematical Institute.

You should ensure that you observe the following points:

- start a new answer booklet for each question which you attempt.
- indicate on the front page of the answer booklet which question you have attempted in that booklet.
- cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.
- hand in your answers in numerical order.

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so.

1. An inextensible beam of bending stiffness B and length L in equilibrium with no body force undergoes two-dimensional deformations in the (x, z)-plane. Both ends of the beam are simply supported. One end is fixed at $(x, z) = (0, 0)$, while the other end is pushed inwards to a position $(x, z) = (\ell, 0)$, with $0 < \ell < L$. The beam is contained inside a narrow channel between fixed smooth boundaries at $z = \pm H$, with $H \ll L$.

You may assume that, in equilibrium with no body force, the angle $\theta(s)$ made between the beam and the x-axis satisfies the equation

$$B\theta''(s) + P_0 \sin\theta(s) + N_0 \cos\theta(s) = 0$$

and the boundary conditions $\theta'(0) = \theta'(L) = 0$, where s is arc-length and P_0, N_0 are constants.

(a) [6 marks] Show that, as long as $|\theta| < \pi/2$ and the beam does not make contact with the channel walls, the transverse displacement $w(x)$ satisfies the equation

$$\frac{Bw''(x)}{\left(1 + w'(x)^2\right)^{3/2}} + P_0 w(x) + N_0 x = 0,$$

and the constraint

$$L = \int_0^\ell \sqrt{1 + w'(x)^2}\,\mathrm{d}x.$$

(b) [4 marks] Now define $\epsilon = H/L \ll 1$. By using the scalings $x = LX$, $w(x) = \epsilon L W(X)$ and $\ell = L\left(1 - \epsilon^2\eta\right)$, obtain the leading-order dimensionless equations

$$W''(X) + \lambda W(X) + \nu X = 0, \qquad \int_0^1 W'(X)^2\,\mathrm{d}X = 2\eta.$$

Define the dimensionless parameters λ and ν.

(c) [5 marks] Explain why W satisfies the boundary conditions $W(X) = W''(X) = 0$ at $X = 0, 1$. Show that $\nu = 0$ while the beam remains out of contact with the channel walls, and that a possible solution for the displacement is of the form

$$W(X) = A\sin(\pi X).$$

Determine the amplitude A in terms of η and show that, as η is increased from zero, contact with one of the channel walls first occurs when $\eta = \pi^2/4$.

(d) [7 marks] For $\eta > \pi^2/4$, suppose that the beam makes contact with the upper channel wall so that $W(X) = 1$ at the single point $X = 1/2$. Assume also that the displacement is symmetric about $X = 1/2$. Show that the displacement takes the form

$$W(X) = \frac{\sin\left(X\sqrt{\lambda}\right) - X\sqrt{\lambda}\cos\left(\sqrt{\lambda}/2\right)}{\sin\left(\sqrt{\lambda}/2\right) - \left(\sqrt{\lambda}/2\right)\cos\left(\sqrt{\lambda}/2\right)}$$

in $0 < X < 1/2$, and determine η as a function of λ [*the result may be left as an unevaluated integral*].

(e) [3 marks] Explain why the assumption of point contact at $X = 1/2$ breaks down when $\lambda > 4\pi^2$, and give a brief qualitative description of what happens instead.

2. An infinite elastic medium occupies the region outside a thin stress-free crack $C \subset \mathbb{R}^3$ whose boundary is given by

$$\partial C = \left\{ (x, y, Z) : \frac{x^2}{c^2 \cosh^2 \epsilon} + \frac{y^2}{c^2 \sinh^2 \epsilon} = 1,\ -\infty < Z < \infty \right\},$$

where $c > 0$ and $0 < \epsilon \ll 1$. The entire medium undergoes torsional deformation such that the displacement field takes the form

$$\mathbf{u}(x, y, Z) = \Omega \begin{pmatrix} -yZ \\ xZ \\ \psi(x, y) \end{pmatrix}, \qquad (*)$$

where $\Omega > 0$ measures the twist about the Z-axis, and $\psi(x, y) \to 0$ as $x^2 + y^2 \to \infty$.

(a) [3 marks] Evaluate the stress corresponding to the deformation $(*)$. Hence show that ψ satisfies Laplace's equation.

(b) [6 marks] Suppose that ψ is written in the form $\psi(x, y) = \mathrm{Im}\left[f(z)\right]$, where $z = x + \mathrm{i}y$, f is holomorphic in

$$D = \left\{ (x + \mathrm{i}y) \in \mathbb{C} : \frac{x^2}{c^2 \cosh^2 \epsilon} + \frac{y^2}{c^2 \sinh^2 \epsilon} > 1 \right\},$$

and $f(z) \to 0$ as $z \to \infty$. Show that $\mathrm{Re}\left[f(z)\right] = C_1 - |z|^2/2$ on the crack boundary ∂D, where C_1 is an integration constant. Show also that the stress components satisfy

$$\tau_{yz}(x, y) + \mathrm{i}\tau_{xz}(x, y) = \mu\Omega \left(f'(z) + \overline{z} \right),$$

where μ is the shear modulus and $^{-}$ denotes complex conjugation.

(c) [5 marks] Show that D is the image of the region $\{\zeta \in \mathbb{C} : |\zeta| > \mathrm{e}^{\epsilon}\}$ under the *Joukowsky transformation*

$$z = \frac{c}{2}\left(\zeta + \frac{1}{\zeta} \right).$$

Confirm that this transformation is conformal and determine the inverse mapping from z to ζ, carefully defining the appropriate branch of any multifunction that occurs.

(d) [7 marks] Hence, or otherwise, obtain the solution

$$f(z) = \frac{\mathrm{e}^{2\epsilon}}{4} \left(c^2 - 2z^2 + 2z\sqrt{z^2 - c^2} \right).$$

(e) [4 marks] Evaluate the stress at the crack tip $(x, y) = (c \cosh \epsilon, 0)$ and show that, in the limit $\epsilon \to 0$, the stress at the crack tip diverges as

$$\tau_{yz} \sim \frac{\mu\Omega c}{2\epsilon}.$$

3. Perfectly plastic material undergoes quasi-steady radial plane strain in the annulus $a < r < b$, with displacement field given by $\mathbf{u}(r) = u(r)\mathbf{e}_r$, where (r, θ) denote plane polar coordinates. The outer boundary $r = b$ is stress-free, while the inner boundary $r = a$ is subject to a pressure $P \geqslant 0$. The material satisfies the yield condition $|\tau_{rr} - \tau_{\theta\theta}| \leqslant 2\tau_{\mathrm{Y}}$, where $\tau_{\mathrm{Y}} > 0$ denotes the yield stress.

(a) [8 marks] First supposing that the material remains elastic, evaluate the stress inside the annulus. Show that, as P increases gradually from zero, yield first occurs at $r = a$ when P reaches a critical value

$$P_{\mathrm{c1}} = \tau_{\mathrm{Y}}\left(1 - \frac{a^2}{b^2}\right).$$

(b) [6 marks] For $P > P_{\mathrm{c1}}$, suppose that the material yields in a region $a < r < s$, and obtain an implicit equation that determines s. Show that s is an increasing function of P and that the entire annulus yields at a second critical value of the applied pressure P_{c2}, that you should determine.

(c) [7 marks] Suppose P gradually increases from zero to a maximum value $P_{\mathrm{m}} \in (P_{\mathrm{c1}}, P_{\mathrm{c2}}]$, and then decreases to zero again. Assuming that the material instantaneously reverts to being elastic once the applied pressure starts to decrease, show that the material yields again at $r = a$ if $P_{\mathrm{m}} > 2P_{\mathrm{c1}}$.

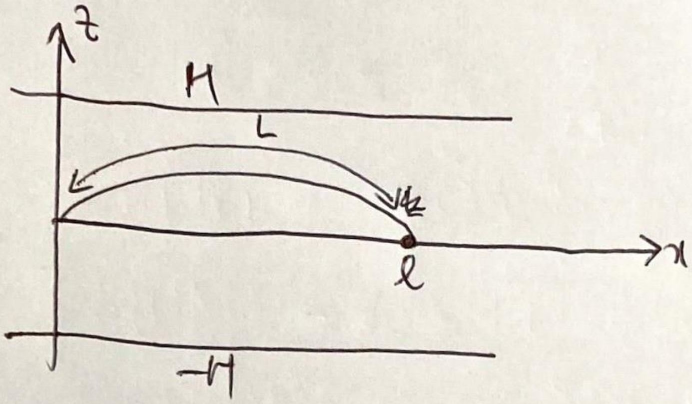
(d) [4 marks] Show that it is possible for the second onset of yielding to occur, as described in part (c), only if $b/a > \beta$, where β is a constant satisfying $\mathrm{e}^{1/2} < \beta < \mathrm{e}$.

[*You may use without proof the steady radially symmetric Navier equation and constitutive relations, namely*

$$\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0, \qquad \tau_{rr} = (\lambda + 2\mu)\frac{\mathrm{d}u}{\mathrm{d}r} + \lambda\frac{u}{r}, \qquad \tau_{\theta\theta} = \lambda\frac{\mathrm{d}u}{\mathrm{d}r} + (\lambda + 2\mu)\frac{u}{r},$$

where λ, μ are the Lamé constants.]

1(a)



Given $$B\theta'' + P_0 \sin\theta + N_0 \cos\theta = 0$$

& recall $$x' = \cos\theta, \quad z' = \sin\theta$$

so integrate w.r.t. s to get

$$B\theta' + P_0 z + N_0 x = \text{const} = 0$$

since $x = z = \theta' = 0$ at $s = 0$

Now, with $z = w(x)$ we have

$$\tan\theta = w'(x)$$

so $$\sec^2\theta \, \frac{d\theta}{ds} = w''(x) \frac{dx}{ds}$$

so $$\frac{d\theta}{ds} = w''(x)\cos^3\theta = \frac{w''(x)}{\left(1 + w'(x)^2\right)^{3/2}}$$

[that's just the curvature]

Then equilibrium equation becomes

$$\boxed{\frac{B w''(x)}{\left(1 + w'(x)^2\right)^{3/2}} + P_0 w(x) + N_0 x = 0}$$

[4] — new but straightforward calculation

The given BCs correspond to $w = w'' = 0$ at $x = 0, \ell$

Also the arc-length of the beam must be conserved (as it's inextensible) so

$$\boxed{L = \int_0^{\ell} \sqrt{1 + w'(x)^2}\, dx}$$

[2] Standard

[This is all fine until the beam contacts the channel wall, when the reaction force must also be included, i.e. as long as

$$|w(x)| < H \quad]$$

1(b)

Non-dimensionalise as suggested

$$\frac{B\varepsilon}{L} \frac{W''(X)}{[1+\varepsilon^2 W'(X)^2]^{3/2}} + P_0 \varepsilon L\, W(X) + N_0 L X = 0$$

which is of the required form, as $\varepsilon \to 0$ with

$$\boxed{\lambda = \frac{P_0 L^2}{B}, \quad V = \frac{N_0 L^2}{\varepsilon B}}$$ (both assumed $O(1)$)

[2] Standard

The constraint

$$1 = \int_0^{\ell/L} \sqrt{1 + \varepsilon^2 W'(X)^2}\, dX$$

i.e. $1 \sim \frac{\ell}{L} + \frac{\varepsilon^2}{2} \int_0^{\ell/L} W'(X)^2\, dX + O(\varepsilon^4)$

So $\ell/L \sim 1 + O(\varepsilon^2)$ and $\boxed{\int_0^1 W'(X)^2 dX = 2\eta}$

with $\boxed{\eta = \frac{L-\ell}{\varepsilon^2 L}}$

[2] New

1(c) Given simply supported BCs:

$$\boxed{W = W'' = 0 \quad \text{at} \quad X = 0, 1}$$

Note that $x = \ell$ corresponds to $X = \frac{\ell}{L} = 1 + O(\varepsilon^2)$

Evaluating the ODE at $X=1$ gives $\underline{\underline{V = 0}}$

Then general solution is

$$W(X) = A\sin(X\sqrt{\lambda}) + B\cos(X\sqrt{\lambda})$$

Apply the BCs: $B=0$ and non trivial solutions exist only if

$$\sqrt{\lambda} = n\pi, \quad n \in \mathbb{Z}_+ .$$

Given the suggestion in the question, choose $\underline{n=1}$,

i.e. $\boxed{\lambda = \pi^2}$ so $\boxed{W(X) = A\sin(\pi X)}$

[2] Standard

Apply the constraint:

$$2\eta = \int_0^1 A^2\pi^2\cos^2(\pi X)\,dX = \frac{A^2\pi^2}{2}$$

i.e. $\boxed{A = \pm\frac{2}{\pi}\sqrt{\eta}}$

Contact first occurs when $W = \pm 1$ at $X = \frac{1}{2}$, i.e.

when $A = \pm 1$, i.e. when $\boxed{\eta = \pi^2/4}$ [3] New

1(d) For $\eta > \frac{\pi^2}{4}$ we impose contact conditions at $x = \frac{1}{2}$:

$\boxed{W(\tfrac{1}{2}) = 1}$ and symmetry $\boxed{W'(\tfrac{1}{2}) = 0}$

We can no longer assume that $\nu = 0$, so general solution is

$$W(x) = -\frac{\nu x}{\lambda} + A\sin(x\sqrt{\lambda}) + B\cos(x\sqrt{\lambda})$$

$$W''(x) = -A\lambda\sin(x\sqrt{\lambda}) - B\lambda\cos(x\sqrt{\lambda})$$

So BCs at $x=0$ still give $\underline{B=0}$

At $x = \frac{1}{2}$: $-\frac{\nu}{2\lambda} + A\sin\left(\frac{\sqrt{\lambda}}{2}\right) = 1$

and $-\frac{\nu}{\lambda} + A\sqrt{\lambda}\cos\left(\frac{\sqrt{\lambda}}{2}\right) = 0$

$\therefore \frac{\nu}{\lambda} = A\sqrt{\lambda}\cos\left(\frac{\sqrt{\lambda}}{2}\right)$ & (redefine $A = C$)

$$\boxed{W(x) = C\left[\sin(x\sqrt{\lambda}) - x\sqrt{\lambda}\cos\left(\frac{\sqrt{\lambda}}{2}\right)\right]}$$ [4] New

then $W(\frac{1}{2}) = +1$ gives

$$\boxed{C = \frac{1}{\sin(\sqrt{\lambda}/2) - \frac{\sqrt{\lambda}}{2}\cos\left(\frac{\sqrt{\lambda}}{2}\right)}}$$ [1] New

Then $$2\eta = 2\int_0^{\frac{1}{2}} W'(x)^2\,dx$$

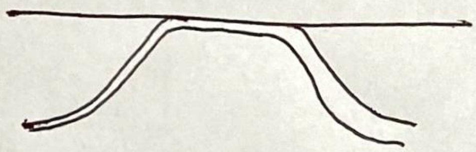
So $$\eta = c^2 \lambda \int_0^{\frac{1}{2}} \left[\cos(x\sqrt{\lambda}) - \cos(\sqrt{\lambda}/2)\right]^2 dx$$

[2] New

1(e) Note $W''(x) = -C\lambda \sin(x\sqrt{\lambda})$

So $W''(\tfrac{1}{2}) = -C\lambda \sin(\sqrt{\lambda}/2) = 0$ when $\lambda = 4\pi^2$.

So when λ increases from π^2 to $4\pi^2$, the beam contacts the wall with zero curvature, and for higher values the contact set starts to spread out



[3] New

2(a) $\underset{\sim}{u} = \Omega \begin{pmatrix} -yz \\ xz \\ \psi(x,y) \end{pmatrix}$; NB $\operatorname{div} \underset{\sim}{u} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = 0$

& $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$

So only nonzero stress components are

$$\boxed{\begin{aligned} \tau_{xz} &= \mu\Omega\left(\frac{\partial\psi}{\partial x} - y\right) \\ \tau_{yz} &= \mu\Omega\left(\frac{\partial\psi}{\partial y} + x\right) \end{aligned}}$$

So $\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \Rightarrow \boxed{\nabla^2\psi = 0}$ [3] Bookwork

2(b) If $\psi = \mathrm{Im}[f(z)]$, write

$$f(z) = \Phi(x,y) + i\psi(x,y)$$

& we have Cauchy-Riemann equations

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \Phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

so BC on crack boundary ∂D is

$$\left(-\frac{\partial \Phi}{\partial y} - y\right) y' - \left(\frac{\partial \Phi}{\partial x} + x\right) x' = 0$$

which can be integrated to give

$$\Phi(x,y) = -\frac{1}{2}(x^2+y^2) + \cancel{\text{const.}} = C \text{ wlog}$$

$$\boxed{\therefore \mathrm{Re}[f(z)] = C - \frac{1}{2}|z|^2 \text{ on } \partial D}$$

[4]

then $$f'(z) = \frac{\partial \Phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x}$$

$$\therefore \mu\Omega f'(z) = \tau_{yz} - \mu\Omega x + i(\tau_{xz} + \mu\Omega y)$$

$$\boxed{\therefore \tau_{yz} + i\tau_{xz} = \mu\Omega\left[f'(z) + \bar{z}\right]}$$

[2] Variation on problem sheet

2(c) Note $|\zeta| = e^{\varepsilon}$ gives $\zeta = e^{\varepsilon} e^{i\theta}$ so

$$z = \frac{c}{2}\left[e^{\varepsilon} e^{i\theta} + e^{-\varepsilon} e^{-i\theta} \right]$$

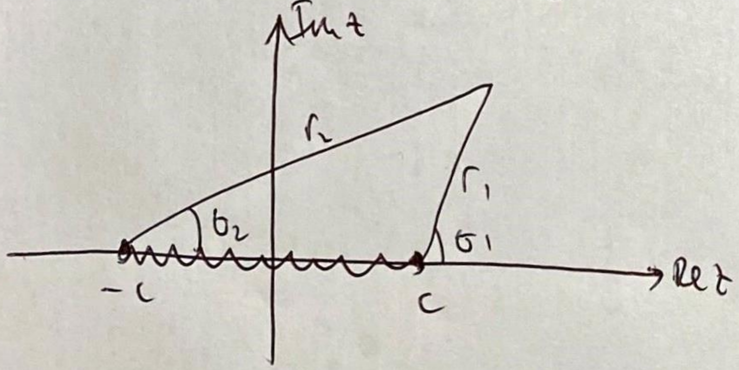
$$= c\cosh\varepsilon\cos\theta + ic\sinh\varepsilon\sin\theta$$ which parameterises ∂D

Then note e.g. that $\zeta = e^{2\varepsilon} \mapsto z = c\cosh 2\varepsilon$ which is <u>inside</u> D, so indeed $|\zeta| > e^{\varepsilon}$ is mapped to D. Note $\frac{dz}{d\zeta} = \frac{c}{2}\left(1 - \frac{1}{\zeta^2}\right) = 0$ at $\zeta = \pm 1$ so $\neq 0$ in $|\zeta| > e^{\varepsilon}$ so <u>conformal</u>.

Invert using $\zeta^2 - \frac{2z}{c}\zeta + 1 = 0$

$$\therefore \quad \zeta = \frac{z}{c} \pm \sqrt{\frac{z^2}{c^2} - 1} = \frac{1}{c}\left(z \pm \sqrt{z^2 - c^2}\right)$$

Define $\sqrt{z^2 - c^2} = \sqrt{r_1 r_2}\, e^{i(\theta_1 + \theta_2)/2}$



$r_1 = |z - c|$

$r_2 = |z + c|$

$\theta_1 = \arg(z - c) \in (-\pi, \pi]$

$\theta_2 = \arg(z + c) \in (-\pi, \pi]$

Choose the root, i.e.

$$\boxed{\zeta = \frac{1}{c}\left(z + \sqrt{z^2 - c^2}\right)}$$

Then e.g. $z = c\cosh 2\varepsilon \in D$ gets mapped to $\zeta = \cosh 2\varepsilon + \sinh 2\varepsilon = e^{2\varepsilon}$ which is in $|\zeta| > e^{\varepsilon}$

[5] Bookwork

2(d)

In the ζ-plane we have

$$\begin{cases} F(\zeta) = f(z(\zeta)) & \text{holomorphic in } |\zeta| > e^{\varepsilon} \\ F(\zeta) \to 0 & \text{as } \zeta \to \infty \\ \mathrm{Re}[F(\zeta)] = C - \frac{|z|^2}{2} = C - \frac{c^2}{8}\left(\zeta + \frac{1}{\zeta}\right)\left(\bar{\zeta} + \frac{1}{\bar{\zeta}}\right) & \text{on } |\zeta| = e^{\varepsilon} \end{cases}$$

Note on the boundary $\zeta\bar{\zeta} = e^{2\varepsilon}$ so

$$\left(\zeta + \frac{1}{\zeta}\right)\left(\bar{\zeta} + \frac{1}{\bar{\zeta}}\right) = |\zeta|^2 + \frac{e^{2\varepsilon}}{\zeta^2} + \frac{e^{2\varepsilon}}{\bar{\zeta}^2} + \frac{1}{|\zeta|^2}$$

$$= 2\cosh(2\varepsilon) + 2e^{2\varepsilon}\,\mathrm{Re}\left[\frac{1}{\zeta^2}\right]$$

So we can take

$$F(\zeta) = C - \frac{c^2}{4}\cosh(2\varepsilon) - \frac{c^2 e^{2\varepsilon}}{4\zeta^2}$$

with $F(\zeta) \to 0$ as $\zeta \to \infty$, we get $C = \frac{c^2}{4}\cosh(2\varepsilon)$

& then $\boxed{F(\zeta) = -\frac{c^2 e^{2\varepsilon}}{4\zeta^2}}$ [5] New

Now: $\frac{2z}{c} = \zeta + \frac{1}{\zeta} \Rightarrow \frac{4z^2}{c^2} = \zeta^2 + 2 + \frac{1}{\zeta^2}$

So $\frac{1}{\zeta^2} = \frac{4z^2}{c^2} - 2 - \frac{2z}{c}\zeta + 1$

$$= \frac{4z^2}{c^2} - 1 - \frac{2z}{c^2}\left(z + \sqrt{z^2 - c^2}\right) = -1 + \frac{2z}{c^2}\left(z - \sqrt{z^2 - c^2}\right)$$

i.e. $\boxed{f(z) = e^{2\varepsilon}\left[\frac{c^2}{4} - \frac{z^2}{2} + \frac{z}{2}\sqrt{z^2 - c^2}\right]}$ [2] New but standard

[Note $\sqrt{z^2-c^2} \sim z - \frac{c^2}{2z} - \frac{c^4}{8z^3} + \dots$ as $z \to \infty$

so $f(z) \sim -\frac{c^4 e^{2\varepsilon}}{16 z^2} \to 0$ as $z \to \infty$, as required.]

2(e) Use part (b)

$$\frac{\tau_{yz} + i\tau_{xz}}{\mu\Omega} = f'(z) + \bar{z} = e^{2\varepsilon}\left[-z + \frac{\sqrt{z^2-c^2}}{2} + \frac{z^2}{2\sqrt{z^2-c^2}}\right] + \bar{z}$$

At $z = c\cosh\varepsilon \in \mathbb{R}, > c$

$$\frac{\tau_{yz} + i\tau_{xz}}{\mu\Omega} = e^{2\varepsilon}\left[-c\cosh\varepsilon + \frac{1}{2}c\sinh\varepsilon + \frac{c\cosh^2\varepsilon}{2\sinh\varepsilon}\right] + c\cosh\varepsilon$$

RHS is real, so $\tau_{xz} = 0$ at crack tip

& $\frac{\tau_{yz}}{\mu\Omega} = \frac{c e^{2\varepsilon}}{2\sinh\varepsilon}\left[\cosh 2\varepsilon - \sinh 2\varepsilon\right] + c\cosh\varepsilon$

$$\boxed{\tau_{yz} = \mu\Omega c\left[\frac{1}{2}\operatorname{cosech}\varepsilon + \cosh\varepsilon\right] \text{ at crack tip}}$$

[3] New

as $\varepsilon \to 0$, $\cosh\varepsilon \sim 1 + \frac{\varepsilon^2}{2} + \dots$, $\operatorname{cosech}\varepsilon \sim \frac{1}{\varepsilon} - \frac{\varepsilon}{6} + \dots$

so $\boxed{\tau_{yz} \sim \frac{\mu\Omega c}{2\varepsilon} \text{ as } \varepsilon \to 0}$ at crack tip.

[1] New

3(a)

While material remains elastic, the displacement satisfies the Navier equation with given constitutive relation, i.e.

$$\frac{d}{dr}\left[(\lambda+2\mu)\frac{du}{dr}+\lambda\frac{u}{r}\right]+\frac{1}{r}\cdot 2\mu\left(\frac{du}{dr}-\frac{u}{r}\right)=0$$

$$\Rightarrow \quad \frac{d}{dr}\left[(\lambda+2\mu)\left(\frac{du}{dr}+\frac{u}{r}\right)\right]=0$$

& NB $\quad \tau_{rr}+\tau_{\theta\theta} = 2(\lambda+\mu)\left(\frac{du}{dr}+\frac{u}{r}\right)$

So we have elastic compatibility condition

$$\boxed{\frac{d}{dr}\left(\tau_{rr}+\tau_{\theta\theta}\right)=0}$$

so $\quad \tau_{rr}+\tau_{\theta\theta} = \text{constant} = 2A$, say.

then Navier equation becomes

$$\frac{d\tau_{rr}}{dr}+\frac{2\tau_{rr}}{r}=\frac{2A}{r}$$

$$\therefore \quad \frac{d}{dr}\left(r^2\tau_{rr}\right)=2Ar$$

$$\therefore \quad \boxed{\tau_{rr}=A-\frac{B}{r^2} \quad \& \quad \tau_{\theta\theta}=A+\frac{B}{r^2}}$$

(say) while material is elastic; A, B are integration constants.

[4] Bookwork

apply the BCs: $\tau_{rr} = \begin{cases} -P & r=a \\ 0 & r=b \end{cases}$

i.e. $\left.\begin{aligned} A - \frac{B}{a^2} &= -P \\ A - \frac{B}{b^2} &= 0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} A &= \frac{a^2 P}{b^2-a^2} \\ B &= \frac{a^2 b^2 P}{b^2-a^2} \end{aligned}\right.$

so $$\boxed{\begin{aligned} \tau_{rr} &= \frac{a^2 P}{b^2-a^2}\left(1-\frac{b^2}{r^2}\right) \\ \tau_{\theta\theta} &= \frac{a^2 P}{b^2-a^2}\left(1+\frac{b^2}{r^2}\right) \end{aligned}}$$

then $|\tau_{rr} - \tau_{\theta\theta}| = \tau_{\theta\theta} - \tau_{rr} = \frac{2a^2b^2P}{(b^2-a^2)} \cdot \frac{1}{r^2}$ [recall $P>0$]

This is a decreasing function of r, so yield first occurs at inner boundary $r=a$, when

$$\frac{2b^2 P}{(b^2-a^2)} = 2\tau_Y, \text{ i.e. } \boxed{P = P_{c1} = \tau_Y\left(1-\frac{a^2}{b^2}\right)}$$

[4] Based on problem sheet

3(b)

For $P > P_{c1}$, material must yield in a neighbourhood of $r=a$, say $a<r<s<b$.

In $r>s$ we still have elastic solution

$$\boxed{\begin{aligned} \tau_{rr} &= A\left(1-\frac{b^2}{r^2}\right) \\ \tau_{\theta\theta} &= A\left(1+\frac{b^2}{r^2}\right) \end{aligned}}$$

Yield condition at $r=s$ gives $\tau_{\theta\theta} - \tau_{rr} = \dfrac{2Ab^2}{s^2} = 2\tau_Y$

i.e. $\boxed{A = \dfrac{s^2}{b^2}\tau_Y}$

[sign determined by how yield condition was satisfied initially]

In $r<s$ apply the yield condition $\tau_{\theta\theta} - \tau_{rr} = 2\tau_Y$

so $\dfrac{d\tau_{rr}}{dr} = \dfrac{2\tau_Y}{r}$, with $\tau_{rr} = -P$ at $r=a$

$$\therefore \boxed{\tau_{rr} = -P + 2\tau_Y \log\left(\frac{r}{a}\right)}$$

stress balance at $r=s$:

$$-P + 2\tau_Y \log\left(\frac{s}{a}\right) = \frac{s^2}{b^2}\tau_Y\left(1-\frac{b^2}{s^2}\right)$$

i.e. $\boxed{P = \tau_Y\left[2\log\left(\dfrac{s}{a}\right) + 1 - \dfrac{s^2}{b^2}\right]}$ relation between s and P.

[4] New but standard

Note $P = P_{c1}$ when $s = a$

& $\dfrac{1}{\tau_Y}\dfrac{dP}{ds} = \dfrac{2}{s} - \dfrac{2s}{b^2} = \dfrac{2(b^2 - s^2)}{sb^2} > 0$ while $a < s < b$.

So s is an increasing function of P, with $s \to b$

So the whole annulus has yielded as $P \to P_{c2}$, where

$$\boxed{P_{c2} = \tau_Y \left[2\log\left(\frac{b}{a}\right) \right]}$$

& by above it's clear that $P_{c2} > P_{c1}$

[2] New

2(c)

Now impose a purely elastic stress on the plastic field obtained at the maximum pressure P_m:

$$\tau_{rr} = A\left(1 - \frac{b^2}{r^2}\right) + \begin{cases} -P_m + 2\tau_Y \log\left(\frac{r}{a}\right) & r < S_m \\ \frac{S_m^2}{b^2}\tau_Y\left(1 - \frac{b^2}{r^2}\right) & r > S_m \end{cases}$$

$$\tau_{\theta\theta} = A\left(1 + \frac{b^2}{r^2}\right) + \begin{cases} -P_m + 2\tau_Y + 2\tau_Y \log\left(\frac{r}{a}\right) & r < S_m \\ \frac{S_m^2}{b^2}\tau_Y\left(1 + \frac{b^2}{r^2}\right) & r > S_m \end{cases}$$

& BC at $r = a$ gives

$$P_m - P = A\left(1 - \frac{b^2}{a^2}\right)$$ which determines A for $P < P_m$.

when $P = 0$ we get $A = -\dfrac{P_m a^2}{b^2 - a^2}$ [4] New

for $r < S_m$:

then $\tau_{rr} - \tau_{\theta\theta} = -\dfrac{2Ab^2}{r^2} - 2\tau_Y = \dfrac{2a^2b^2 P_m}{(b^2 - a^2)r^2} - 2\tau_Y$

the material yields again as it unloads if this reaches $+2\tau_Y$ which will occur first at $r = a$ (if at all) if

$$\boxed{P_m > 2\tau_Y\left(1 - \frac{a^2}{b^2}\right) = 2P_{c1}}$$ [3] New

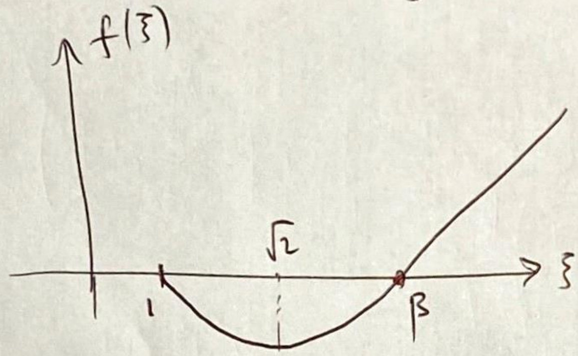
2(d)

Finally, this can only occur if $2P_{c1} < P_{c2}$, i.e. if

$$\boxed{\log\left(\frac{b}{a}\right) + \frac{a^2}{b^2} - 1 > 0}$$

$= f(b/a)$

$$f'(\xi) = \frac{1}{\xi} - \frac{2}{\xi^3} = \frac{\xi^2 - 2}{\xi^3} = 0 \text{ at } \xi = \sqrt{2}$$



$f(1) = 0$

$f(\xi) \sim \log \xi$ as $\xi \to \infty$

Possible only if $\frac{b}{a} > \beta$ where $\beta > 1$ is root of f

Note $f(\sqrt{e}) = \frac{1}{2} + \frac{1}{e} - 1 < \frac{1}{2} + \frac{1}{2} - 1 = 0$

$f(e) = 1 + \frac{1}{e^2} - 1 > 0$

So $\beta \in (\sqrt{e}, e)$.

[4] New