Model Completeness and Quantifier Elimination

Separation Lemma

Let Σ, T be L-theories
and let $\Gamma \subseteq \mathrm{Sent}(L)$ closed under $\wedge$ and $\vee$.
Assume that, for each $\mathcal{A} \models \Sigma$, $\mathcal{B} \models T$
there is $\gamma \in \Gamma$ s.t. $\mathcal{A} \models \gamma$ and $\mathcal{B} \models \neg\gamma$.
Then there is $\gamma^* \in \Gamma$ with $\Sigma \models \gamma^*$ and $T \models \neg\gamma^*$

Proof: Fix $\mathcal{A} \models \Sigma$.
Choose for each $\mathcal{B} \models T$ some $\gamma_{\mathcal{B}} \in \Gamma$ with $\mathcal{A} \models \gamma_{\mathcal{B}}$, $\mathcal{B} \models \neg\gamma_{\mathcal{B}}$.

$\Rightarrow \quad \mathrm{Mod}(T) \subseteq \bigcup_{\mathcal{B} \models T} \mathrm{Mod}(\neg\gamma_{\mathcal{B}})$

$\Rightarrow$ by compactness, there are $\mathcal{B}_1, \dots, \mathcal{B}_m \models T$
with $\mathrm{Mod}(T) \subseteq \bigcup_{i=1}^{m} \mathrm{Mod}(\neg\gamma_{\mathcal{B}_i})$

(observe that $\mathrm{Mod}(T) \cap \bigcap_{\mathcal{B} \models T} \mathrm{Mod}(\gamma_{\mathcal{B}}) = \emptyset$)

$\bigcup_{i=1}^{m} \mathrm{Mod}(\neg\gamma_{\mathcal{B}_i}) = \mathrm{Mod}(\neg \bigwedge_{i=1}^{m} \gamma_{\mathcal{B}_i})$

$\Rightarrow \gamma_{\mathcal{A}} := \bigwedge_{i=1}^{m} \gamma_{\mathcal{B}_i}$ separates $\mathcal{A}$ from $\mathrm{Mod}(T)$:

$\mathcal{A} \models \gamma_{\mathcal{A}}$ and $\mathrm{Mod}(T) \subseteq \mathrm{Mod}(\neg\gamma_{\mathcal{A}})$

By compactness again, there are $\mathcal{A}_1, \dots, \mathcal{A}_n \models \Sigma$
s.t. $\mathrm{Mod}\,\Sigma \subseteq \bigcup_{i=1}^{n} \mathrm{Mod}(\gamma_{\mathcal{A}_i})$

(and $\mathrm{Mod}(T) \subseteq \bigcap_{i=1}^{n} \mathrm{Mod}(\neg\gamma_{\mathcal{A}_i})$)

$\Rightarrow \gamma^* := \bigvee_{i=1}^{n} \gamma_{\mathcal{A}_i}$ will do the job. ∎

<u>Corollary</u>

A class C of L-structures is <u>finitely</u> ax.^{ble} if both C and $C^c := \{\mathcal{A} \mid \mathcal{A} \notin C\}$ are ax.^{ble}

<u>Pf.</u>: „$\Rightarrow$" clear

„$\Leftarrow$" w.l.o.g. $C \neq \emptyset \neq C^c$

Let $C = \mathrm{Mod}(\Sigma)$ and $C^c = \mathrm{Mod}(T)$

Then γ^* separating Σ from T axiomatizes Σ ∎

<u>Def.</u>: An L-theory T is <u>model complete</u> if $T \cup \mathrm{Diag}(\mathcal{A})$ is complete for any $\mathcal{A} \models T$

($\Rightarrow$ $C\mathrm{Diag}(\mathcal{A})$ is the deductive closure of $T \cup \mathrm{Diag}(\mathcal{A})$)

<u>Examples</u>
- $\mathrm{Th}(\mathbb{Q}, <) = DLO$
- $\mathrm{Th}(\langle \mathbb{R}; +, \cdot, 0, 1 \rangle) = RCF$ = theory of real closed flds
- ACF = the theory of algebraically closed flds.

<u>Non-Example</u>
- the theory of abelian groups (easy)
- $\mathrm{Th}(\langle \mathbb{Z}; +, \cdot; 0, 1 \rangle)$ (hard $\rightsquigarrow$ Hilbert's 10th Problem)

<u>Def.</u>: For L-structure $\mathcal{A} \subseteq \mathcal{B}$ say $\mathcal{A}$ is <u>existentially closed</u> in $\mathcal{B}$ ("$\mathcal{A} \preccurlyeq_\exists \mathcal{B}$") if for any $\exists$-L_A sentence φ: ($\mathcal{B} \models \varphi \Rightarrow \mathcal{A} \models \varphi$)

<u>Theorem</u> For any L-theory T, TFAE:

(i) T is model complete

(ii) $\mathcal{A}, \mathcal{B} \models T$ with $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A} \preccurlyeq \mathcal{B}$

(iii) " " $\Rightarrow \mathcal{A} \preccurlyeq_\exists \mathcal{B}$

(iv) Every L-formula $\varphi = \varphi(v_1, \dots, v_n)$ is modulo T equivalent to an $\exists$-L-formula $\psi = \psi(v_1, \dots, v_n)$ i.e. $T \models \forall v_1, \dots \forall v_n \, (\varphi(v_1, \dots, v_n) \leftrightarrow \psi(v_1, \dots, v_n))$

Proof: (i) $\Rightarrow$ (ii)

Assume (i) and let $\mathcal{A}, \mathcal{B} \models T$ with $\mathcal{A} \subseteq \mathcal{B}$

$\Rightarrow$ there is an L_A-expansion $\mathcal{B}'$ of $\mathcal{B}$ wit $\mathcal{B}' \models T \cup \mathrm{Diag}\,\mathcal{A}$

$T \cup \mathrm{Diag}\,\mathcal{A}$ complete $\Rightarrow \mathcal{A}' \equiv \mathcal{B}'$

$\Rightarrow \mathcal{A} \preceq \mathcal{B}$

(ii) $\Rightarrow$ (iii) : clear

(iii) $\Rightarrow$ (iv) Assume (iii)

First show: every $\exists$-L-formula $\varphi(v_1, \ldots, v_n)$ is mod T equ. to a $\forall$-L-formula

Let $\bar{c} = (c_1, \ldots c_n)$ be new constants and let

$\Gamma = \mathrm{Sent}_\forall(L_{\bar{c}})$.

Let $\mathcal{A}', \mathcal{B}'$ be $L_{\bar{c}}$-structures with L-reducts $\mathcal{A}, \mathcal{B} \models T$

s.t. $\mathrm{Th}_\forall(\mathcal{A}') \subseteq \mathrm{Th}_\forall(\mathcal{B}')$

$\Rightarrow$ there is $\mathcal{B}'' \equiv \mathcal{B}'$ with $\mathcal{A}' \subseteq \mathcal{B}''$ (Exercise)

$\overset{(iii)}{\Longrightarrow} \mathcal{A}' \models \varphi(\bar{c})$ iff $\mathcal{B}'' \models \varphi(\bar{c})$ iff $\mathcal{B}' \models \varphi(\bar{c})$

Apply Sep. Lemma to $T \cup \{\varphi(\bar{c})\}$ and $T \cup \{\neg\varphi(\bar{c})\}$

to find $\psi(\bar{c}) \in \Gamma$ with $\mathrm{Mod}(T \cup \{\varphi(\bar{c})\}) \subseteq \mathrm{Mod}(\psi(\bar{c}))$

$\neg$ 〃 $\neg\psi(\bar{c})$

Hence $T \models \forall \bar{v}(\varphi(\bar{v}) \leftrightarrow \psi(\bar{v}))$

Rest: easy induction on φ

(iv) $\Rightarrow$ (i) Assume (iv), let $\mathcal{B}^+ \models T \cup \mathrm{Diag}(\mathcal{A})$ for $\mathcal{A} \models T$

let φ be any L_A-sentence

to show: $\mathcal{A}^+ \models \varphi$ iff $\mathcal{B}^+ \models \varphi$

(for then $\mathcal{B}^+ \equiv \mathcal{A}^+$ and so $T \cup \mathrm{Diag}\,\mathcal{A}$ is complete)

$\mathcal{A}^+, \mathcal{B}^+ \models T \overset{(iv)}{\Longrightarrow}$ w.l.o.g. φ $\exists$-L_A-sentence and $\mathcal{A} \subseteq \mathcal{B}$

so $\mathcal{A}^+ \models \varphi \Rightarrow \mathcal{B}^+ \models \varphi$

$\mathcal{A}^+ \not\models \varphi \Rightarrow \mathcal{A}^+ \models \neg\varphi \overset{(iv)}{\Longrightarrow} \neg\varphi$ equ. to $\exists\,\psi$ mod T

$\Rightarrow \mathcal{A}^+ \models \psi \Rightarrow \mathcal{B}^+ \models \psi \Rightarrow \mathcal{B}^+ \models \neg\varphi$ $\blacksquare$

Corollary 1

T model complete iff every L-formula is mod T equiv. to a $\forall$-L-formula

Corollary 2

If T is model complete then T is $\forall\exists$-axiomatizable

(Converse clearly false, e.g. T = the theory of fields: $\mathbb{R} \not\preceq_\exists \mathbb{C}$)

Example T = DLO is model complete ($L = \{<\}$)

Pf.: $\mathcal{A}, \mathcal{B} \models T$ with $\mathcal{A} \subseteq \mathcal{B}$

$\varphi \in \mathrm{Sent}_\exists(L_A)$ with $\mathcal{B}^+ \models \varphi$

to show: $\mathcal{A}^+ \models \varphi$

$\varphi = \exists v_1 \dots v_n \, \psi(\bar{a}, \bar{v})$ (ψ q.f.)

all ψ can say is the position of $v_1, \dots, v_n$ relative to $a_1 \dots a_m$

As $\mathcal{A} \models$ DLO this can be realized in $\mathcal{A}$ ∎

Def.: An L-theory T admits elimination of quantifiers (QE) if every L-formula $\varphi(v_1, \dots, v_n)$ is modulo T equivalent to a quantifier-free (q.f.) L-formula $\psi(v_1 \dots v_n)$

i.e. $T \models \forall v_1 \dots \forall v_n \, (\varphi(v_1 \dots v_n) \leftrightarrow \psi(v_1 \dots v_n))$

Note this is only possible if L contains constants or $\perp$ or $\top$.

Remark T has QE $\Rightarrow$ T is model-complete

<u>Theorem</u> Assume L contains constants.
For any L-theory T, TFAE:

(i) T admits QE

(ii) $T \cup \mathrm{Diag}\, \mathcal{C}$ is complete for each $\mathcal{C} \subseteq \mathcal{A} \models T$

(iii) For any $\mathcal{A}, \mathcal{B} \models T$ with a common L-substructure $\mathcal{C}$ and for any simple $\exists$-L_C-sentence φ (so $\varphi = \exists v\, \psi(v)$), ψ q.f.

$$\mathcal{A}^C \models \varphi \text{ iff } \mathcal{B}^C \models \varphi$$

$$\begin{tikzcd} \mathcal{A} & & \mathcal{B} \\ & \mathcal{C} \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

<u>Proof</u>: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) clear

<u>(iii) $\Rightarrow$ (i)</u>: Induction on complexity of formulas

<u>crucial step</u>: Any formula $\varphi(v_1, \dots, v_n) = \exists v\, \psi(v_1, \dots, v_n, v)$, ψ q.f.

is mod T equ. to a q.f. $\chi(v_1, \dots, v_n)$

Add new constants $\bar{c} = (c_1, \dots, c_n)$
and let $\varphi(\bar{c}) = \exists v\, \psi(c_1, \dots, c_n, v)$

Let $L_{\bar{c}} = L \cup \{c_1, \dots, c_n\}$ and let $\Gamma = \{\text{q.f. } L_{\bar{c}}\text{-sent.}\}$

want to apply Sep. Lemma to $T \cup \{\varphi(\bar{c})\}$ and $T \cup \{\neg\varphi(\bar{c})\}$

suppose $\mathcal{A}^{\bar{c}}, \mathcal{B}^{\bar{c}}$ are $L_{\bar{c}}$ structures with $\mathcal{A}, \mathcal{B} \models T$
such that $\mathrm{Th}(\mathcal{A}^{\bar{c}}) \cap \Gamma = \mathrm{Th}(\mathcal{B}^{\bar{c}}) \cap \Gamma$

$\Rightarrow$ for any two closed $L_{\bar{c}}$-terms σ, τ

$$\sigma^{\mathcal{A}^{\bar{c}}} = \tau^{\mathcal{A}^{\bar{c}}} \text{ iff } \sigma^{\mathcal{B}^{\bar{c}}} = \tau^{\mathcal{B}^{\bar{c}}}$$

$\Rightarrow$ may identify the $L_{\bar{c}}$-substructures generated by $L_{\bar{c}}$-terms in $\mathcal{A}^{\bar{c}}$ and $\mathcal{B}^{\bar{c}}$

$\overset{\text{(iii)}}{\Longrightarrow}$ $\mathcal{A}^{\bar{c}} \models \varphi(\bar{c})$ iff $\mathcal{B}^{\bar{c}} \models \varphi(\bar{c})$

Sep. Lemma $\Rightarrow$ there is $\chi(\bar{c}) \in \Gamma$ s.t.

$$\left.\begin{array}{l} \mathrm{Mod}(T \cup \varphi(\bar{c})) \subseteq \mathrm{Mod}(\chi(\bar{c})) \\ \text{--- --- } \neg \text{ --- --- } \neg\chi(\bar{c}) \end{array}\right\} \Rightarrow T \models \forall v_1 \dots \forall v_n\, \varphi(\bar{v}) \leftrightarrow \chi(\bar{v}) \ \blacksquare$$

Example

ACF admits QE in $L = \{+, \cdot ; 0, 1\}$

Pf.: Let $\mathcal{A}, \mathcal{B} \models$ ACF with common substructure $\mathcal{C}$ ($\Rightarrow \mathcal{C}$ is a subring of $\mathcal{A}$ and $\mathcal{B}$)

Let $\varphi = \exists v \, \underbrace{\psi(v)}_{\text{q.f. } L_C\text{-formula}}$

so ψ is a boolean comb. of polynomial equalities over $\mathcal{C}$

So $\bigvee_i \bigwedge_j f_{ij}(v) = 0 \wedge \bigwedge_k g_{ik}(v) \neq 0$. with $f_{ij} \in C[T]$.

$\mathcal{A}, \mathcal{B} \models$ ACF $\Rightarrow \mathcal{A}^C \models \varphi$ iff $\mathcal{B}^C \models \varphi$ ∎

e.g. $\mathcal{A}^C \models \exists v \bigwedge_j f_{ij}(v) = 0 \wedge \bigwedge_k g_{ik}(v) \neq 0$

$\Leftrightarrow$ no f_{ij} occurs [$\mathcal{A}$ is infinite so most $a \in A$ avoid the finitely many zeros of g_{ik}]

or there are f_{ij}'s and they all have a common irreducible factor h over $\mathcal{C}$ which divides none of the g_{ik}'s.

$\Rightarrow \mathcal{B}^C \models \exists v \bigwedge_j f_{ij}(v) = 0 \wedge \bigwedge_k g_{ik}(v) \neq 0$ ∎