A BRIEF EXCURSION INTO THE MATHEMATICAL THEORY OF MIXED FINITE ELEMENT METHODS

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Synopsis: Motivating examples: Stokes equations; flow in porous media. Functional analytic prerequisites: Riesz representation theorem, the Lax–Milgram theorem, Banach's closed range theorem. Abstract mixed variational problems: the inf-sup condition and its role in existence and uniqueness of solutions. Discrete mixed formulations and the discrete inf-sup condition. Error bounds. Checking the inf-sup condition: Fortin's criterion. Examples of inf-sup unstable and inf-sup stable finite element spaces.

Lecture 1

1. Introduction. Numerous mathematical models that arise in continuum mechanics in the form of systems of partial differential equations involve several physically disparate quantities, which need to be approximated simultaneously. Finite element approximations of such problems are known as *mixed finite element methods*. These lecture notes introduce some basic concepts from the theory of mixed finite element methods. For further details the reader is referred to the monographs by Boffi, Brezzi & Fortin [1], Brenner & Scott [2], Ern & Guermond [4], Gatica [6] and Girault & Raviart [7]. For questions associated with the iterative solution of systems of linear algebraic equations arising from mixed finite element approximations, and preconditioning these, the reader may wish to consult the text by Elman, Silvester and Wathen [3].

In order to motivate the theoretical considerations that will follow we begin by presenting two typical model problems that lead to mixed finite element methods.

1.1. Example 1: the Stokes equations. The *Stokes equations* govern the flow of a steady, viscous, incompressible, isothermal, Newtonian fluid. They arise by simplifying the incompressible Navier–Stokes equations through the omission of the convective derivative. This results in the following system of linear partial differential equations:

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{1.1a}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega. \tag{1.1b}$$

Here Ω is assumed to be a bounded open set in $\mathbb{R}^d$, $d = 2, 3$, with a sufficiently smooth boundary $\partial\Omega$; in what follows it will suffice to assume that $\partial\Omega$ is Lipschitz continuous. The d-component vector function $\mathbf{u} : \overline{\Omega} \to \mathbb{R}^d$ denotes the velocity of the fluid, $p : \Omega \to \mathbb{R}$ is the pressure, $\mathbf{f} : \Omega \to \mathbb{R}^d$ is the density of body forces acting on the fluid (e.g. gravitational force), and the constant kinematic viscosity of the fluid that multiplies $\Delta\mathbf{u}$ has been set to unity as its actual value plays no role in our considerations. The equation (1.1a) is called the *momentum equation*, while equation (1.1b) is referred to as the *continuity equation*. Vector-valued functions, such as $\mathbf{u}$ and $\mathbf{f}$, and the associated function spaces to which vector-valued functions belong, will be displayed throughout in **boldface**.

For the sake of simplicity we shall supplement the system of partial differential equations (1.1a), (1.1b) with the following *homogeneous Dirichlet* boundary condition:

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \partial\Omega. \tag{1.1c}$$

By taking the dot product of the momentum equation (1.1a) with a sufficiently smooth d-component vector function $\mathbf{v}$ such that $\mathbf{v}|_{\partial\Omega} = \mathbf{0}$, integrating the resulting equality over Ω, and integrating by parts in both terms on the left-hand side, noting the assumed homogenous boundary condition on $\mathbf{v}$, yields

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \ell_f(\mathbf{v}), \tag{1.2a}$$

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where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are two bilinear functionals defined, respectively, by

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega \sum_{i=1}^d \nabla u_i \cdot \nabla v_i \,\mathrm{d}x, \tag{1.2b}$$

$$b(\mathbf{v},q) := -\int_\Omega (\nabla\cdot\mathbf{v})q\,\mathrm{d}x, \tag{1.2c}$$

and $\ell_f(\cdot)$ is the linear functional defined by

$$\ell_f(\mathbf{v}) := \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x. \tag{1.2d}$$

Similarly, multiplying the continuity equation (1.1b) with a sufficiently smooth function q and integrating over Ω yields

$$b(\mathbf{u},q) = 0. \tag{1.2e}$$

Motivated by the forms of the equations (1.2a) and (1.2e), we shall now state the weak formulation of the Stokes equations, which will represent the starting point for the construction of mixed finite element approximations for this boundary-value problem. To this end, we define the function spaces

$$\mathbf{X} := H_0^1(\Omega)^d = \underbrace{H_0^1(\Omega)\times\cdots\times H_0^1(\Omega)}_{d\text{ times}}$$

and

$$M := L_0^2(\Omega) = \left\{ q\in L^2(\Omega) \,:\, \int_\Omega q\,\mathrm{d}x = 0 \right\}.$$

The weak formulation of the Stokes equations is then as follows: find a pair of functions $(\mathbf{u},p)\in \mathbf{X}\times M$ such that

$$a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = \ell_f(\mathbf{v}) \qquad \forall\mathbf{v}\in\mathbf{X}, \tag{1.3a}$$

$$b(\mathbf{u},q) = 0 \qquad \forall q\in M. \tag{1.3b}$$

We shall show later that, as long as $\mathbf{f}\in L^2(\Omega)^d$, the problem (1.3a), (1.3b) has a unique solution $(\mathbf{u},p)\in\mathbf{X}\times M$, which we shall refer to as the *weak solution* of the Stokes equations. In fact, the regularity hypothesis $\mathbf{f}\in L^2(\Omega)^d$ on the source term can be weakened by assuming that $\ell_f\in\mathbf{X}'$, where $\mathbf{X}'$ denotes the dual space of $\mathbf{X}$, consisting of all continuous linear functionals on $\mathbf{X}$.

1.2. Example 2: flow in porous media. A simple model for fluid flow in a porous medium occupying a bounded open set $\Omega\subset\mathbb{R}^d$ has the form

$$\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial p}{\partial x_j}\right) = g(x), \qquad x\in\Omega, \tag{1.4}$$

where $p:\overline{\Omega}\to\mathbb{R}$ is the pressure, and $g\in L^2(\Omega)$ is a given source term. Again, Ω will be assumed to have sufficiently smooth boundary $\partial\Omega$; for example, it will suffice to assume for our considerations that $\partial\Omega$ is Lipschitz continuous. Let us suppose that the equation is *uniformly elliptic* on Ω; that is, there exists a positive constant c_0 such that

$$\sum_{i,j=1}^d a_{ij}(x)\,\xi_i\xi_j \ge c_0\sum_{i=1}^d \xi_i^2 \qquad \forall\xi = (\xi_1,\ldots,\xi_d)^{\mathrm{T}}\in\mathbb{R}^d, \quad \forall x\in\Omega. \tag{1.5}$$

Let us suppose further that $a_{ij} \in L^\infty(\Omega)$, $i,j = 1,\ldots,d$. According to *Darcy's law* the fluid velocity $\mathbf{u} : \overline{\Omega} \to \mathbb{R}^d$ and the pressure gradient are related by

$$-\sum_{j=1}^{d} a_{ij}(x)\,\frac{\partial p}{\partial x_j}(x) = u_i(x) \qquad \text{in } \Omega, \quad i = 1,\ldots,d. \tag{1.6}$$

Let us denote by $\mathbb{A}(x)$ the *inverse* of the matrix $(a_{ij}(x))_{i,j=1}^d \in \mathbb{R}^{d\times d}$. Thus we can rewrite (1.6) in an equivalent form as

$$\mathbb{A}\mathbf{u} = -\nabla p \qquad \text{in } \Omega. \tag{1.7}$$

Thanks to (1.4) we also have that

$$-\nabla\cdot\mathbf{u} = g \qquad \text{in } \Omega. \tag{1.8}$$

We shall supplement the system of equations (1.7), (1.8) with the following *homogeneous oblique derivative boundary condition*:

$$\sum_{i,j=1}^{d} a_{ij}(x)\,\frac{\partial p}{\partial x_j}(x)\,n_i(x) = 0 \qquad \text{on } \partial\Omega, \tag{1.9}$$

where $\mathbf{n}(x) = (n_1(x),\ldots,n_d(x))^{\mathrm{T}}$ is the unit outward normal vector to $\partial\Omega$ at the point $x \in \partial\Omega$. By noting (1.6) we can rewrite the boundary condition (1.9) as

$$\mathbf{u}\cdot\mathbf{n} = 0 \qquad \text{on } \partial\Omega. \tag{1.10}$$

The weak formulation of the system (1.7), (1.8) in conjunction with the boundary condition (1.10) is then as follows: find $(\mathbf{u}, p) \in \mathbf{X}\times M$, such that

$$a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = 0 \qquad \forall \mathbf{v}\in\mathbf{X}, \tag{1.11a}$$

$$b(\mathbf{u},q) = \ell_g(q) \qquad \forall q \in M, \tag{1.11b}$$

where now $\mathbf{X}$ and M are defined by

$$\mathbf{X} := H_0(\mathrm{div};\Omega) = \left\{\mathbf{v}\in L^2(\Omega)^d \,:\, \nabla\cdot\mathbf{v}\in L^2(\Omega),\ \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\right\}$$

and

$$M := L_0^2(\Omega),$$

and the bilinear functionals $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ and the linear functional $\ell_g(\cdot)$ are defined by

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &:= \int_\Omega \sum_{i,j=1}^{d} A_{ij}\,u_i v_j\,\mathrm{d}x, \\ b(\mathbf{v},q) &:= -\int_\Omega (\nabla\cdot\mathbf{v})\,q\,\mathrm{d}x, \\ \ell_g(q) &:= \int_\Omega g\,q\,\mathrm{d}x. \end{aligned}$$

The space $\mathbf{X} := H_0(\mathrm{div};\Omega)$ is equipped with the norm

$$\|\mathbf{v}\|_{H(\mathrm{div};\Omega)} := \left(\|\mathbf{v}\|^2_{L^2(\Omega)^d} + \|\nabla\cdot\mathbf{v}\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

After stating some standard results from functional analysis in the next section, we shall develop the elements of a mathematical theory, which, under suitable assumptions on the data, guarantees the existence a unique solution to variational problems such as (1.3) and (1.11).

Lecture 2

2. Three preliminary results. The analysis presented in the next section requires three classical theorems from linear functional analysis, which we state here without proofs; for further details we refer the reader to Yosida [10].

THEOREM 2.1 (Lax–Milgram theorem). *Suppose that H is a Hilbert space over the field of real numbers, with inner product $(\cdot,\cdot)_H$ and induced norm $\|\cdot\|_H$ defined by $\|v\|_H^2 := (v,v)_H$. Suppose further that $a(\cdot,\cdot)$ is a bilinear functional on $H \times H$, ℓ is a linear functional on H, and the following additional properties hold:*

(a) The bilinear functional a is coercive; i.e., there exists a positive real number c_a such that

$$a(v,v) \geq c_a \|v\|_H^2 \qquad \forall v \in H;$$

(b) The bilinear functional a is bounded; i.e., there exists a positive real number C_a such that

$$|a(w,v)| \leq C_a \|w\|_H \|v\|_H \qquad \forall w, v \in H;$$

(c) The linear functional ℓ is bounded; i.e., there exists a positive real number C_ℓ such that

$$|\ell(v)| \leq C_\ell \|v\|_H \qquad \forall v \in H.$$

Then, there exists a unique $u \in H$ such that $a(u,v) = \ell(v)$ for all $v \in H$.

The proof of the Lax–Milgram theorem is based on the following result, known as the Riesz representation theorem.

THEOREM 2.2 (Riesz representation theorem). *Suppose that H is a Hilbert space over the field of real numbers, with inner product $(\cdot,\cdot)_H$ and induced norm $\|\cdot\|_H$ defined by $\|v\|_H^2 := (v,v)_H$, and let $\ell : H \to \mathbb{R}$ be a bounded linear functional on H. Then, there exists a unique element $z \in H$, known as the* Riesz representer of *ℓ, such that $\ell(v) = (z,v)_H$ for all $v \in H$.*

We shall require one further result, which concerns closed linear operators in Banach spaces. Let X and Y be two Banach spaces. A linear operator $T : \mathcal{D}(T) \subset X \to Y$, with domain $\mathcal{D}(T)$, is said to be *closed* if for every sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(T)$ converging to $x \in X$ such that $Tx_n \to y \in Y$ as $n \to \infty$ one has $x \in \mathcal{D}(T)$ and $Tx = y$.

THEOREM 2.3 (Banach's closed range theorem). *Suppose that X and Y are Banach spaces, and $T : \mathcal{D}(T) \to Y$ is a closed linear operator, whose domain $\mathcal{D}(T)$ is dense in X. Let* $\mathrm{Ker}(T) := \{x \in \mathcal{D}(T) : Tx = 0\}$ *denote the kernel of T and let $T' : Y' \to X'$ be the transpose of T, defined by $\langle T'y', x\rangle = \langle y', Tx\rangle$, where X' and Y' denote the dual spaces of X and Y, respectively, and $\langle\cdot,\cdot\rangle$ is the duality pairing between Y' and Y, or X' and X, as the case may be. Then, the following properties are equivalent:*

(a) $\mathcal{R}(T)$, the range of T, is closed in Y;
(b) $\mathcal{R}(T')$, the range of T', is closed in X';
(c) $\mathcal{R}(T) = [\mathrm{Ker}(T')]^\circ := \{y \in Y : \langle y', y\rangle = 0 \quad \forall y' \in \mathrm{Ker}(T')\}$;
(d) $\mathcal{R}(T') = [\mathrm{Ker}(T)]^\circ := \{x' \in X' : \langle x', x\rangle = 0 \quad \forall x \in \mathrm{Ker}(T)\}$.

An important remark is in order regarding Banach's closed range theorem, Theorem 2.3, in the context of the discussion herein.

REMARK 1. *We shall apply this theorem to bounded linear operators $T : X \to Y$, whose domain, $\mathcal{D}(T)$, is the entire space X; any such linear operator is both closed and, trivially, densely defined in X. Therefore the hypotheses of Theorem 2.3 are automatically satisfied in such cases. In particular, for a bounded linear operator $T : X \to Y$, the properties (a) to (d) above are equivalent.*

It would have been more precise to write $\langle T'y', x\rangle_{X',X} = \langle y', Tx\rangle_{Y',Y}$ in the statement of the theorem, to highlight the fact that in the duality pairing on the left-hand side the first entry belongs to X' and the second to X, and in the duality-pairing on the right the first entry belongs to Y' and the second to Y. In the interest of simplicity of notation we have however refrained from doing so, as the actual choice of the spaces in duality pairings will always be clear from the context; the second entry in a duality pairing will always belong to a Banach or Hilbert space, and the first entry will belong to the dual space of the Banach or Hilbert space in question.

3. Abstract mixed formulation. Before embarking on the study the abstract mixed formulation in (infinite-dimensional) Hilbert spaces, we shall attempt to develop some intuition for a particular condition, usually referred to as the *inf-sup* condition, which arises in connection with mixed systems. We shall do so by first focusing on finite-dimensional problems of this type.

3.1. The inf-sup condition in finite dimensions. Let us therefore consider the following system of linear algebraic equations for a vector of unknowns $(u^{\mathrm{T}}, p^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^n \times \mathbb{R}^m$ with $n > m$:

$$Au + B^{\mathrm{T}}p = f, \tag{3.1}$$

$$Bu \qquad = 0, \tag{3.2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$ and $f \in \mathbb{R}^n$. Clearly, because of (3.2), the component u of the unknown solution vector $(u^{\mathrm{T}}, p^{\mathrm{T}})^{\mathrm{T}}$ must belong to the following (closed) linear subspace of $\mathbb{R}^n$:

$$V := \mathrm{Ker}(B) = \{v \in \mathbb{R}^n \ : \ Bv = 0\}.$$

The set V is certainly nonempty, since the zero vector of $\mathbb{R}^n$ is contained in V. If $V = \{0\}$, the component u must be equal to 0 and the equation (3.1) collapses to $B^{\mathrm{T}}p = f$, whose solvability we shall study below, under suitable assumptions on B. Let us therefore assume that V is nontrivial, in the sense that $\{0\}$ is a strict subset of V and therefore V contains a nonzero element of $\mathbb{R}^n$.

As $v^{\mathrm{T}}B^{\mathrm{T}} = (Bv)^{\mathrm{T}} = 0^{\mathrm{T}}$ for all $v \in V$, by premultiplying (3.1) with v^{T}, we deduce that, because $v^{\mathrm{T}}B^{\mathrm{T}}p = 0$ for all $v \in V$, the vector $u \in V$, if it exists, must satisfy:

$$v^{\mathrm{T}}Au = v^{\mathrm{T}}f \qquad \forall v \in V.$$

In order to ensure that such a $u \in V$ exists, we shall *assume* that A is positive definite on $V = \mathrm{Ker}(B)$ is the sense that

$$v^{\mathrm{T}}Av > 0 \qquad \forall v \in V \setminus \{0\}.$$

An equivalent assumption to this is that $v^{\mathrm{T}}Av > 0$ for all $v \in V$ with $\|v\|_{\mathbb{R}^n} = 1$, where $\|\cdot\|_{\mathbb{R}^n}$ is the Euclidean norm on $\mathbb{R}^n$; and therefore, since the unit sphere in $\mathbb{R}^n$ is compact and the mapping $v \mapsto v^{\mathrm{T}}Av$ is continuous, a further equivalent restatement of this assumption is that

$$\exists c_a > 0 \ \text{ s.t.:} \quad v^{\mathrm{T}}Av \geq c_a\|v\|^2_{\mathbb{R}^n} \quad \forall v \in V, \tag{3.3}$$

which is usually referred to as *coercivity of A on the kernel of B*. By defining $a(w, v) := v^{\mathrm{T}}Aw$, with $v, w \in V$, it directly follows from the Lax–Milgram theorem (cf. Theorem 2.1 above) that there exists a unique $u \in V$ such that $v^{\mathrm{T}}Au = v^{\mathrm{T}}f$ for all $v \in V$. To summarize, we have shown that, if A is coercive on the kernel of B, then there exists a unique $u \in V$ such that $v^{\mathrm{T}}Au = v^{\mathrm{T}}f$.

Having found u, we now return to (3.1) to find a $p \in \mathbb{R}^m$ such that

$$B^{\mathrm{T}}p = f - Au. \tag{3.4}$$

As by the Fundamental Theorem of Linear Algebra, $\mathcal{R}(B^{\mathrm{T}}) = [\mathrm{Ker}(B)]^{\perp} = V^{\perp}$, and $f - Au \in V^{\perp}$, it follows that there exists a $p \in \mathbb{R}^m$ such that $B^{\mathrm{T}}p = f - Au$. In order to show the uniqueness of p we require a condition on B, similar in spirit to the coercivity condition for A, which will ensure the injectivity of the mapping $B^{\mathrm{T}} : \mathbb{R}^m \to V^{\perp}$. It is worth noting at this point though that, unlike A, which was assumed to be a square matrix, B is a rectangular matrix, so the relevant condition for B will be slightly more complicated than the coercivity condition for A.

In order to eliminate the possibility of the existence of a $p_* \neq 0$ such that $B^{\mathrm{T}}p_* = 0$, we shall suppose that:

$$\forall q \in \mathbb{R}^m \setminus \{0\} \quad \exists v \in \mathbb{R}^n \setminus \{0\} \ \text{ s.t.: } \ v^{\mathrm{T}}B^{\mathrm{T}}q > 0. \tag{3.5}$$

Indeed, if (3.5) holds, then, for $p_* \in \mathbb{R}^m \setminus \{0\}$ there exists a $v_* \in \mathbb{R}^n \setminus \{0\}$ such that $0 < v_*^{\mathrm{T}}B^{\mathrm{T}}p_*$, which rules out the possibility that $B^{\mathrm{T}}p_* = 0$ for a nonzero vector $p_* \in \mathbb{R}^m$. Consequently, if (3.5) holds, then there exists a unique $p \in \mathbb{R}^m$ such that (3.4) holds.

In summary, we have shown that, if (3.3) and (3.5) hold, then the system of linear algebraic equations (3.1), (3.2) has a unique solution $(u^{\mathrm{T}}, p^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^n \times \mathbb{R}^m$ for any choice of $f \in \mathbb{R}^n$.

An equivalent form of writing (3.5) would be to assume that

$$\forall q \in \mathbb{R}^m \setminus \{0\} \quad \exists v \in \mathbb{R}^n \setminus \{0\} \text{ s.t.: } \frac{v^{\mathrm{T}} B^{\mathrm{T}} q}{\|v\|_{\mathbb{R}^n} \|q\|_{\mathbb{R}^m}} > 0. \tag{3.6}$$

A slightly stronger requirement, usually referred to as the *inf-sup* condition, would be to assume that

$$\exists c_b > 0 \text{ s.t.: } \inf_{q \in \mathbb{R}^m \setminus \{0\}} \sup_{v \in \mathbb{R}^n \setminus \{0\}} \frac{v^{\mathrm{T}} B^{\mathrm{T}} q}{\|v\|_{\mathbb{R}^n} \|q\|_{\mathbb{R}^m}} \geq c_b, \tag{3.7}$$

where c_b a positive constant, independent of the dimensions n and m of the finite-dimensional spaces $\mathbb{R}^n$ and $\mathbb{R}^m$. Assuming that B satisfies the inf-sup condition (3.7), it follows that

$$\forall q \in \mathbb{R}^m \quad \|B^{\mathrm{T}} q\|_{\mathbb{R}^n} \geq c_b \|q\|_{\mathbb{R}^m},$$

and therefore $B^{\mathrm{T}} : \mathbb{R}^m \to [\mathrm{Ker}(B)]^{\perp} = V^{\perp} := \{g \in \mathbb{R}^n : g^{\mathrm{T}} v = 0 \ \ \forall v \in V\}$ is an isomorphism. Consequently, by transposition, $B : V^{\perp} \to \mathbb{R}^m$ is an isomorphism.

3.2. The inf-sup condition in infinite dimensions. Let us suppose that X and M are two Hilbert spaces over the field of real numbers and consider two bilinear functionals $a(\cdot, \cdot) : X \times X \to \mathbb{R}$ and $b(\cdot, \cdot) : X \times M \to \mathbb{R}$. We shall assume that each of these bilinear functionals is bounded; i.e., there exist positive constants C_a and C_b such that

$$|a(u, v)| \leq C_a \|u\|_X \|v\|_X \qquad \forall u, v \in X, \tag{3.8a}$$

$$|b(v, q)| \leq C_b \|v\|_X \|q\|_M \qquad \forall v \in X, \ \forall q \in M, \tag{3.8b}$$

where $\|\cdot\|_X$ and $\|\cdot\|_M$ denote the norm in X and M, respectively, induced by the respective inner products, $(\cdot, \cdot)_X$ and $(\cdot, \cdot)_M$, of these two Hilbert spaces.

With these assumptions in mind, we consider the following variational problem: find the pair $(u, p) \in X \times M$ such that

$$a(u, v) + b(v, p) = \ell_f(v) \qquad \forall v \in X, \tag{3.9a}$$

$$b(u, q) = \ell_g(q) \qquad \forall q \in M, \tag{3.9b}$$

where $\ell_f \in X'$ and $\ell_g \in M'$; i.e., ℓ_f and ℓ_g are bounded linear functionals on the Hilbert spaces X and M, respectively.

We begin by studying problem (3.9) in the simplified setting when $\ell_g = 0$ (i.e., $\ell_g(q) = 0$ for all $q \in M$). We shall then show how the general case, when $\ell_g \neq 0$, can be reduced to the case when $\ell_g = 0$.

Case 1: $\boxed{\ell_g = 0.}$ The problem under consideration is then the following:

$$a(u, v) + b(v, p) = \ell_f(v) \qquad \forall v \in X, \tag{3.10a}$$

$$b(u, q) = 0 \qquad \forall q \in M. \tag{3.10b}$$

Let us consider the closed linear subspace V of the Hilbert space X, defined by

$$V := \{v \in X : b(v, q) = 0 \quad \forall q \in M\}. \tag{3.11}$$

By choosing a test function $v \in V (\subset X)$ in (3.10a) we then have that

$$a(u, v) = \ell_f(v) \qquad \forall v \in V. \tag{3.12}$$

Since V is a Hilbert space when equipped with the inner product and norm of X, our assumptions that $a(\cdot, \cdot)$ is a bounded bilinear functional on $X \times X$ and ℓ_f is a bounded linear functional on X

imply that the same is true with X replaced by V. Thus, if we now additionally *assume* that the bilinear functional $a(\cdot,\cdot)$ is coercive on $V \times V$, i.e., that

$$\exists c_a > 0 \;\text{ s.t. } \forall v \in V\colon \qquad a(v,v) \geq c_a \|v\|_X^2, \tag{3.13}$$

then by applying the Lax–Milgram theorem we deduce the existence of a unique $u \in V$ such that $a(u,v) = \ell_f(v)$ for all $v \in V$. In particular, it then follows that the element $u \in V$ thus found automatically satisfies (3.10b).

It remains to prove the existence of a unique $p \in M$ such that (3.10a) also holds; i.e., we wish to show the existence of a unique $p \in M$ such that $b(v,p) = \ell_f(v) - a(u,v)$ for all $v \in X$, with $u \in V$ as determined above (and considered fixed). Since $b(\cdot,\cdot)$ is a bounded bilinear functional on $X \times M$, and $v \mapsto \ell_f(v) - a(u,v)$ is a bounded linear functional on X (for $u \in V$ fixed), the assumptions of the Lax–Milgram theorem motivate us to seek a generalization of the coercivity assumption (a) to a wider setting when, instead of having a bilinear functional on the cartesian product of a Hilbert space with itself, we have a bilinear functional on the cartesian product of two different Hilbert spaces, X and M.

In order to identify the appropriate form of such a generalized coercivity condition, let us re-examine condition (a) of the Lax–Milgram theorem, as stated in (3.13); it clearly implies that

$$\exists c_a > 0 \;\text{ s.t. } \forall v \in X\colon \qquad c_a \|v\|_X \leq \frac{a(v,v)}{\|v\|_X} \leq \sup_{w \in X\setminus\{0\}} \frac{a(w,v)}{\|w\|_X}.$$

Motivated by the form of the right-most expression, we shall *assume* that the bilinear functional b satisfies the following generalized coercivity condition:

$$\exists c_b > 0 \;\text{ s.t. } \forall q \in M\colon \qquad c_b \|q\|_M \leq \sup_{w \in X\setminus\{0\}} \frac{b(w,q)}{\|w\|_X}. \tag{3.14}$$

Equivalently, we can rewrite (3.14) as follows:

$$\exists c_b > 0 \;\text{ s.t.:} \qquad c_b \leq \inf_{q \in M\setminus\{0\}} \sup_{w \in X\setminus\{0\}} \frac{b(w,q)}{\|w\|_X \, \|q\|_M}. \tag{3.15}$$

The condition (3.15) (or, equivalently, (3.14)) is referred to as the *inf-sup condition.* ***Lecture 3***

Assuming that the bilinear functional b satisfies the inf-sup condition (3.15) (or, equivalently (3.14)), let us return to the problem of finding a unique $p \in M$ such that

$$b(v,p) = \mathcal{L}(v) \qquad \forall v \in X, \tag{3.16}$$

where $\mathcal{L}(v) := \ell_f(v) - a(u,v)$, with $u \in V$ as identified above (i.e., $a(u,v) = \ell_f(v)$ for all $v \in V$; and hence $\mathcal{L}(v) = 0$ for all $v \in V$). As both $\ell_f(\cdot)$ and $a(u,\cdot)$ are bounded linear functionals on X the same is true of $\mathcal{L}(\cdot)$. We have the following crucial result.

Lemma 3.1. *Suppose that $b(\cdot,\cdot)$ is a bilinear functional on the cartesian product $X \times M$ of two Hilbert spaces X and M over the field of real numbers, such that b is bounded in the sense that* (3.8b) *holds, and b satisfies the inf-sup condition in the sense that* (3.14) *holds. Let V be the closed linear subspace of X defined by* (3.11) *and suppose that $\mathcal{L}$ is a bounded linear functional on X such that $\mathcal{L}(v) = 0$ for all $v \in V$. Then, there exists a unique $p \in M$ such that*

$$b(v,p) = \mathcal{L}(v) \qquad \forall v \in X. \tag{3.17}$$

We need to make some preparations before embarking on the proof of this lemma, including the statement of an auxiliary result, Lemma 3.2, which we shall prove below. Having done so, we shall be ready to prove Lemma 3.1.

Let $B : X \to M'$ be the linear operator defined by

$$\langle Bv, q\rangle = b(v,q) \qquad \forall v \in X, \quad \forall q \in M,$$

where M' denotes the dual space of M, and, on the left-hand side, $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{M',M}$. Analogously, let $B' : M \to X'$ denote the transpose of the operator B, where X' denotes the dual space of X; i.e.,

$$\langle B'q, v\rangle = b(v,q) \qquad \forall v \in X, \quad \forall q \in M,$$

where now on the left-hand side $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{X',X}$. As $b(\cdot,\cdot) : X \times M \to \mathbb{R}$ is a bounded bilinear functional, it follows that B and B' are bounded linear operators.

Remark 2. *Note that $V = \mathrm{Ker}(B)$. The coercivity assumption* (3.13) *is therefore frequently referred to as* coercivity on the kernel *(of the operator B, that is).*

We are now ready to state and prove the auxiliary result alluded to above, which we require in our proof of Lemma 3.1.

Lemma 3.2. *Let V be defined by* (3.11) *and let $V^\circ := \{g \in X' : \langle g, v\rangle = 0 \quad \forall v \in V\}$. The following three properties are equivalent:*

(a) There exists a positive constant c_b such that

$$\inf_{q\in M\setminus\{0\}} \sup_{v\in X\setminus\{0\}} \frac{b(v,q)}{\|v\|_X\|q\|_M} \geq c_b; \tag{3.18}$$

(b) The operator B' is an isomorphism from M onto V° and

$$\|B'q\|_{X'} \geq c_b\|q\|_M \qquad \forall q \in M; \tag{3.19}$$

(c) The operator B is an isomorphism from $V^\perp$ onto M' and

$$\|Bv\|_{M'} \geq c_b\|v\|_X \qquad \forall v \in V^\perp. \tag{3.20}$$

Here, $V^\perp$ denotes the orthogonal complement of the closed linear space V of the Hilbert space X, where orthogonality is understood with respect to the inner product of X.

Proof.

1) Let us show that (a) $\Leftrightarrow$ (b). Thanks to the definition of the operator $B' : M \to X'$, (a) is equivalent to demanding the existence of a positive constant c_b such that

$$\sup_{v\in X\setminus\{0\}} \frac{\langle B'q, v\rangle}{\|v\|_X} \geq c_b\|q\|_M,$$

which, in turn, is equivalent to (3.19). It remains to prove that $B' : M \to V^\circ$ is an isomorphism. It follows from (3.19) that B' is a *one-to-one* operator from M onto its range $\mathcal{R}(B')$. Moreover since B' is a bounded linear operator, which, by (3.19), has a bounded inverse $(B')^{-1} : \mathcal{R}(B') \to M$, we deduce that B' is an isomorphism from M onto $\mathcal{R}(B')$. Thus, in particular, $\mathcal{R}(B')$ is a closed subspace[1] in X'. The closed range theorem then implies that

$$\mathcal{R}(B') = [\mathrm{Ker}(B)]^\circ = V^\circ.$$

Thus we have shown that (a) $\Leftrightarrow$ (b).

[1]The proof of this is simple: suppose that $(q_n)_{n\in\mathbb{N}}$ is a sequence in M such that $B'q_n \to w$ in X' as $n \to \infty$. Then $(B'q_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in X'. By (3.19), $(q_n)_{n\in\mathbb{N}}$ is then a Cauchy sequence in M. As M is a Hilbert space, and therefore every Cauchy sequence in M converges, it follows that there exists a $q \in M$ such that $q_n \to q$ as $n \to \infty$. As B' is a bounded linear operator, it then follows that $B'q_n \to B'q$ in X' as $n \to \infty$. However, by the uniqueness of the limit $B'q$ must coincide with w; thus we have shown that the limit w of the sequence $(B'q_n)_{n\in\mathbb{N}} \subset \mathcal{R}(B')$ is also contained in $\mathcal{R}(B')$. Consequently, $\mathcal{R}(B')$ is closed in X'.

(ii) Next we show that (b) $\Leftrightarrow$ (c). To this end it suffices to prove that V° can be identified isometrically with $(V^\perp)'$. We prove this as follows. For $v \in X$, let $v^\perp$ denote the orthogonal projection of v onto $V^\perp$ (in the inner product of the Hilbert space X). Then, to each $g \in (V^\perp)'$ we associate an element $\tilde{g} \in X'$ defined by

$$\langle \tilde{g}, v \rangle = \langle g, v^\perp \rangle \qquad \forall v \in X.$$

As $v^\perp = 0$ for each $v \in V$, it follows that $\langle \tilde{g}, v \rangle = 0$ for all $v \in V$; i.e., $\tilde{g} \in V^\circ$. Furthermore, the correspondence $g \mapsto \tilde{g}$ maps isometrically $(V^\perp)'$ onto V°. Thus we have shown that $(V^\perp)'$ and V° can be identified. Hence (c) follows from (b) by transposition, and vice versa. Therefore (b) and (c) are equivalent.

That completes the proof of the auxiliary lemma. ∎

We are now ready to prove Lemma 3.1.

Proof. [of Lemma 3.1.] Thanks to the assumptions of the lemma, $\mathcal{L} \in V^\circ$. As the inf-sup condition is also assumed, we deduce from the equivalence of (a) and (b) in Lemma 3.2 that B' is an isomorphism from M onto V°. Thus, there exists a unique element $p \in M$ such that $B'p = \mathcal{L}$; equivalently, $b(v, p) = \langle B'p, v \rangle = \langle \mathcal{L}, v \rangle = \mathcal{L}(v)$ for all $v \in V$. That completes the proof. ∎

We now move on to the general case, when $\ell_g \neq 0$.

Case 2: $\boxed{\ell_g \neq 0.}$ Suppose that there is an element $u_0 \in V^\perp$ such that $b(u_0, q) = \ell_g(q)$ for all $q \in M$. Then, by replacing u in (3.9) with $u - u_0$, problem (3.9) is transformed into (3.10) and the existence of a solution to (3.9) thus follows. Uniqueness of the solution to (3.9) follows from the fact that $(0, 0)$ is the unique solution to (3.10) with $\ell_f = 0$.

It remains to show that there does indeed exist an element $u_0 \in V^\perp$ such that $b(u_0, q) = \ell_g(q)$ for all $q \in M$. In fact, we shall show that there exists a unique such element u_0. It follows from the equivalence of (a) and (c) stated in Lemma 3.2 that the inf-sup condition (3.14) is equivalent to B being an isomorphism from $V^\perp$ onto M'. Hence, the assumption of Lemma 3.1 that the inf-sup condition (3.14) holds implies that for each $\ell_g \in M'$ there is a unique element $u_0 \in V^\perp(\subset X)$ such that $Bu_0 = \ell_g$. Thus we have shown the existence of a unique $u_0 \in V^\perp(\subset X)$ such that $b(u_0, q) = \ell_g(q)$.

Having dealt with both Case 1 and Case 2, we summarize our findings in the following result.

THEOREM 3.3. *Suppose that X and M are two Hilbert spaces over the field of real numbers, with norms $\|\cdot\|_X$ and $\|\cdot\|_M$ induced by the inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_M$, respectively. Suppose further that $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ and $b(\cdot,\cdot) : X \times M \to \mathbb{R}$ are two bounded bilinear functionals (i.e., (3.8) holds) and $\ell_f : X \to \mathbb{R}$ and $\ell_g : M \to \mathbb{R}$ are bounded linear functionals on X and M, respectively. Suppose further that V is defined by (3.11), that a is coercive on V (i.e., (3.13) holds), and b satisfies the inf-sup condition (3.14). Then, there exists a unique pair $(u, p) \in X \times M$ that solves the variational problem (3.9).*

Let us illustrate the relevance of this abstract result by applying it to a specific example, the Stokes equations, to deduce the existence of a unique weak solution to these equations.

EXAMPLE 1. *Consider the bilinear functionals $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ defined by (1.2b) and (1.2c), and the linear functional ℓ_f defined by (1.2d), with $\mathbf{f} \in L^2(\Omega)^d$. We note that $\mathbf{X}$ and M, as defined in Section 1.1, are Hilbert spaces, when equipped with the Sobolev seminorm $|\cdot|_{H^1(\Omega)^d} := \|\nabla \cdot\|_{L^2(\Omega)^{d\times d}}$ and the $L^2(\Omega)$ norm $\|\cdot\|_{L^2(\Omega)}$, respectively. By the Cauchy–Schwarz inequality $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are bounded bilinear functionals, and, by Poincaré's inequality, ℓ_f is a bounded linear functional on $\mathbf{X}$; i.e., $\ell_f \in \mathbf{X}'$. Further, we have that*

$$a(\mathbf{v}, \mathbf{v}) = |\mathbf{v}|^2_{H^1(\Omega)^d} = \|\mathbf{v}\|^2_{\mathbf{X}}$$

for all $\mathbf{v} \in \mathbf{X}$, and therefore, in particular,

$$a(\mathbf{v}, \mathbf{v}) = \|\mathbf{v}\|^2_{\mathbf{X}}$$

for all $\mathbf{v} \in V = \{\mathbf{v} \in \mathbf{X} : \nabla \cdot \mathbf{v} = 0\} = \{\mathbf{v} \in \mathbf{X} : b(\mathbf{v}, q) = 0 \quad \forall q \in M\}$. *Finally, one also has the following inf-sup condition*

$$c_b \|q\|_{L^2(\Omega)} \leq \sup_{\mathbf{v} \in H_0^1(\Omega)^d \setminus \{\mathbf{0}\}} \frac{(\nabla \cdot \mathbf{v}, q)_{L^2(\Omega)}}{|\mathbf{v}|_{H^1(\Omega)^d}} \qquad \forall q \in L_0^2(\Omega),$$

with a positive constant $c_b = c_b(\Omega)$*, proved by Ladyzhenskaya [8] (see also [9]), which implies the validity of* (3.14)*. Thus we deduce the existence of a unique weak solution* $(\mathbf{u}, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ *to the Stokes equations* (1.1).

Lecture 4

4. Discrete mixed formulation. Suppose that $X_h \subset X$ and $M_h \subset M$ are (in practice, finite-dimensional) linear subspaces of the Hilbert spaces X and M, respectively, parametrized by a positive parameter $h \in (0, 1)$. Let us consider the following approximation of problem (3.10): find $u_h \in X_h$ and $p_h \in M_h$ such that

$$a(u_h, v_h) + b(v_h, p_h) = \ell_f(v_h) \qquad \forall v_h \in X_h, \tag{4.1a}$$

$$b(u_h, q_h) = 0 \qquad \forall q_h \in M_h. \tag{4.1b}$$

Let us consider the closed linear subspace V_h of the linear space X_h, defined by

$$V_h := \{v_h \in X_h : b(v_h, q_h) = 0 \quad \forall q_h \in M_h\}. \tag{4.2}$$

As $0 \in V_h$, the set V_h is nonempty.

It is important to note at this point that since M_h is a proper subspace of M the fact that $b(v_h, q_h) = 0$ for all $q_h \in M_h$ does not imply that $b(v_h, q) = 0$ for all $q \in M$; hence if $v_h \in V_h$ it does not follow that $v_h \in V$. For the same reason, if the bilinear functional $b(\cdot, \cdot)$ featuring in (3.10) satisfies the inf-sup condition (3.14) it does not automatically follow that an analogous inf-sup condition will hold with X and M replaced by X_h and M_h and $w \in X$ and $q \in M$ replaced by $w_h \in X_h$ and $q_h \in M_h$ in (3.14). This, in turn, will be a source of difficulties in the construction of finite element approximations to mixed variational problems, since the validity of a discrete inf-sup condition is not inherited from the continuous problem, but has to be independently verified for each particular choice of spaces (X_h, M_h). We shall return to this point later. First however we shall derive a bound on the error between u and u_h in terms of the best approximation errors

$$\inf_{v_h \in V_h} \|u - v_h\|_X \qquad \text{and} \qquad \inf_{q_h \in M_h} \|p - q_h\|_M,$$

which can be seen as an extension of Céa's lemma from classical finite element theory to finite element approximations of mixed variational problems.

Theorem 4.1. *Suppose, in addition to the assumptions of Theorem 3.3 that the bilinear functional* a *is coercive on* V_h*, i.e.,*

$$\exists c_a > 0 \quad s.t. \quad \forall v_h \in V_h: \qquad a(v_h, v_h) \geq c_a \|v_h\|_X^2. \tag{4.3}$$

Then, there exists a unique function $u_h \in V_h$ *that satisfies (4.1a) for all* $v_h \in V_h$*. Furthermore, for such a* $u_h \in V_h$*, we have that*

$$\|u - u_h\|_X \leq \left(1 + \frac{C_a}{c_a}\right) \inf_{v_h \in V_h} \|u - v_h\|_X + \frac{C_b}{c_a} \inf_{q_h \in M_h} \|p - q_h\|_M. \tag{4.4}$$

Proof. As $a(u_h, v_h) = \ell_f(v_h)$ for all $v_h \in V_h$, the existence of a unique $u_h \in V_h$ satisfying (4.1a) for all $v_h \in V_h$ follows from the Lax–Milgram theorem.

By taking $v = w_h \in V_h \subset X$ in (3.10a) and subtracting the resulting equation from (4.1a) with $v_h = w_h \in V_h$, we have that

$$a(u - u_h, w_h) = a(u, w_h) - a(u_h, w_h) = \ell_f(w_h) - b(w_h, p) - a(u_h, w_h) = -b(w_h, p) = -b(w_h, p - q_h)$$

for all $q_h \in M_h$. Therefore,

$$a(u - u_h, w_h) + b(w_h, p - q_h) = 0 \qquad \forall w_h \in V_h \quad \text{and} \quad \forall q_h \in M_h. \tag{4.5}$$

Let us consider any $v_h \in V_h$. Then, by noting that $u_h \in V_h$, and therefore $u_h - v_h \in V_h$, and by applying (4.3) followed by (4.5) with $w_h = u_h - v_h$, we have that

$$\begin{aligned} c_a \|u_h - v_h\|_X^2 &\le a(u_h - v_h, u_h - v_h) \\ &= a(u - v_h, u_h - v_h) + a(u_h - u, u_h - v_h) \\ &= a(u - v_h, u_h - v_h) + b(u_h - v_h, p - q_h) \qquad \forall q_h \in M_h. \end{aligned}$$

Hence, by (3.8), and dividing the resulting inequality by $c_a \|u_h - v_h\|_X$, we deduce that

$$\|u_h - v_h\|_X \le \frac{C_a}{c_a} \|u - v_h\|_X + \frac{C_b}{c_a} \|p - q_h\|_M \qquad \forall q_h \in M_h.$$

The proof is then completed by inserting this inequality into the second term on the right-hand side of the following triangle inequality

$$\|u - u_h\|_X \le \|u - v_h\|_X + \|u_h - v_h\|_X,$$

and taking the infimum over all $v_h \in V_h$ and all $q_h \in M_h$. □

The main point of Theorem 4.1 is that the error $u - u_h$ can be bounded in terms of the best approximation errors $\inf_{v_h \in V_h} \|u - v_h\|_X$ and $\inf_{q_h \in M_h} \|p - q_h\|_M$ by assuming boundedness of the bilinear functionals a and b on $X \times X$ and $X \times M$ respectively, and the coercivity of a on $V \cup V_h$. As V_h may be a rather small set (without a further assumption on b, at least, which we shall next make in Definition 4.2), the approximation properties of V_h may be quite poor. Nevertheless, Theorem 4.1 guarantees that, under its hypotheses, u_h is at least stably determined. Bounds on $p - p_h$ on the other hand require additional assumptions; in fact, under the assumptions of Theorem 4.1 alone, the function p_h may not even be stably determined, and, as a matter of fact, there is no reason why p_h should even be unique.

Definition 4.2. *We shall say that the family of spaces $\{(X_h, M_h)\}_{h>0}$ satisfies the (discrete) inf-sup condition, if there exists a constant $c_b > 0$, independent of h, such that*

$$c_b \le \inf_{q_h \in M_h \setminus \{0\}} \sup_{v_h \in X_h \setminus \{0\}} \frac{b(v_h, q_h)}{\|v_h\|_X \|q_h\|_M}. \tag{4.6}$$

A remark is in order at this point: we have used the same symbols c_a and c_b for the discrete coercivity and inf-sup constants in (4.3) and (4.6), respectively, as for their counterparts appearing in the coercivity and inf-sup conditions (3.13) and (3.14), respectively, for the continuous problem. This was done purely for the sake of notational simplicity: there is no reason of course why the constants c_a and c_b in the 'continuous' coercivity and inf-sup conditions should coincide with those in their discrete counterparts. Of course, if $a(\cdot, \cdot)$ happens to be coercive on the whole of X, then it is automatically coercive on both V and V_h, with the same coercivity constant; this will be the case with the Stokes equations (our Example 1.1), but not with the porous media equations (our Example 1.2).

We are now in a position to show that if the discrete inf-sup condition (4.6) also holds, then the function $p_h \in M_h$ is uniquely determined and the error $p - p_h$ is, much like, $u - u_h$, bounded in terms of the best approximation errors $\inf_{v_h \in V_h} \|u - v_h\|_X$ and $\inf_{q_h \in M_h} \|p - q_h\|_M$.

Theorem 4.3. *Suppose, in addition to the assumptions of Theorem 4.1 that the bilinear functional b satisfies the discrete inf-sup condition (4.6). Then, there exists a unique solution pair $(u_h, p_h) \in X_h \times M_h$ to the problem (4.1). Furthermore, in addition to the bound (4.4) on $\|u - u_h\|_X$, the following bound holds:*

$$\|p - p_h\|_M \le \frac{C_a}{c_b} \left(1 + \frac{C_a}{c_a}\right) \inf_{v_h \in V_h} \|u - v_h\|_X + \left(1 + \frac{C_b}{c_b} \left(1 + \frac{C_a}{c_a}\right)\right) \inf_{q_h \in M_h} \|p - q_h\|_M. \tag{4.7}$$

Proof. The existence of a unique solution pair (u_h, p_h) follows from Theorem (3.3), with X and M replaced by X_h and M_h. By noting the discrete inf-sup condition (4.6), the identity (4.5), and the bounds (3.8), we have that, for any $q_h \in M_h$,

$$\begin{aligned}
c_b\|q_h - p_h\|_M &\leq \sup_{w_h \in X_h\setminus\{0\}} \frac{b(w_h, q_h - p_h)}{\|w_h\|_X} \\
&= \sup_{w_h \in X_h\setminus\{0\}} \frac{b(w_h, p - p_h) + b(w_h, q_h - p)}{\|w_h\|_X} \\
&\leq \sup_{w_h \in X_h\setminus\{0\}} \frac{|b(w_h, p - p_h)| + |b(w_h, q_h - p)|}{\|w_h\|_X} \\
&= \sup_{w_h \in X_h\setminus\{0\}} \frac{|a(u - u_h, w_h)| + |b(w_h, q_h - p)|}{\|w_h\|_X} \\
&\leq C_a\|u - u_h\|_X + C_b\|p - q_h\|_M.
\end{aligned}$$

Hence, by the triangle inequality,

$$\|p - p_h\|_M \leq \frac{C_a}{c_b}\|u - u_h\|_X + \left(1 + \frac{C_b}{c_b}\right)\|p - q_h\|_M.$$

Taking the infimum over all $q_h \in M_h$ and substituting (4.4) into the resulting inequality then completes the proof. $\square$

The aim of the next result is to show that, by virtue of the discrete inf-sup condition (4.6), the term $\inf_{v_h\in V_h}\|u - v_h\|_X$ appearing in (4.4) and (4.7) can be replaced by $\inf_{v_h\in X_h}\|u - v_h\|_X$. As X_h is typically a strict superset of V_h, it is expected that $\inf_{v_h\in X_h}\|u - v_h\|_X \ll \inf_{v_h\in V_h}\|u - v_h\|_X$.

THEOREM 4.4. *Under the hypotheses of Theorem 4.3, the unique solution pair $(u_h, p_h) \in X_h \times M_h$ to the problem (4.1) satisfies the following error bound:*

$$\|u - u_h\|_X + \|p - p_h\|_M \leq C\left(\inf_{v_h\in X_h}\|u - v_h\|_X + \inf_{q_h\in M_h}\|p - q_h\|_M\right), \tag{4.8}$$

where $C = C(c_a, c_b, C_a, C_b)$ is a positive constant, independent of h.

Proof. Let $v_h \in X_h$ and choose $w_h \in V_h$ such that $v_h - w_h \in V_h^\perp$. Thanks to the discrete inf-sup condition (4.6) and part (c) of Lemma 3.2, with X, M and V replaced by X_h, M_h and V_h, respectively, we have that

$$c_b\|w_h - v_h\|_X \leq \sup_{q_h\in M_h\setminus\{0\}} \frac{b(w_h - v_h, q_h)}{\|q_h\|_M}.$$

As $b(w_h, q_h) = 0$ and $b(u, q_h) = 0$ for all $q_h \in M_h \subset M$, it follows that

$$c_b\|w_h - v_h\|_X \leq \sup_{q_h\in M_h\setminus\{0\}} \frac{b(u - v_h, q_h)}{\|q_h\|_M} \leq C_b\|u - v_h\|_X.$$

Hence, by the triangle inequality,

$$\|u - w_h\|_X \leq \|u - v_h\|_X + \|v_h - w_h\|_X \leq \left(1 + \frac{C_b}{c_b}\right)\|u - v_h\|_X.$$

Thus, in particular,

$$\inf_{w_h\in V_h}\|u - w_h\|_X \leq \left(1 + \frac{C_b}{c_b}\right)\|u - v_h\|_X \qquad \forall v_h \in X_h.$$

As the left-hand side of this inequality is independent of v_h, it follows that

$$\inf_{w_h\in V_h}\|u - w_h\|_X \leq \left(1 + \frac{C_b}{c_b}\right)\inf_{v_h\in X_h}\|u - v_h\|_X.$$

Substituting the last inequality into the right-hand sides of (4.4) and (4.7) and summing the resulting inequalities we obtain (4.8). That completes the proof. $\square$

5. Checking the discrete inf-sup condition. A helpful device for verifying the discrete inf-sup condition in instances when the (continuous) inf-sup condition is already known to hold is the following result, due to Fortin [5]. Fortin's criterion is stated here in the case of Hilbert spaces; a more general version, formulated in Banach spaces, and one which also shows that the criterion is not just sufficient but also necessary for the validity of the discrete inf-sup condition, can be found in [4]. **_Lecture 5_**

THEOREM 5.1. *Let X and M be two Hilbert spaces and suppose that $b : X \times M \to \mathbb{R}$ is a bounded bilinear functional such that the inf-sup condition* (3.14) *holds. Let $X_h \subset X$ and $M_h \subset M$. Suppose that:*

- *There exists a constant $C_f > 0$ such that for each $v \in X$ there is an element $\Pi_h(v) \in X_h$ such that $b(v, q_h) = b(\Pi_h(v), q_h)$ for all $q_h \in M_h$ and $\|\Pi_h(v)\|_X \le C_f \|v\|_X$.*

Then, the discrete inf-sup condition (4.6) *also holds.*

Proof. Let $q_h \in M_h$; then,

$$\sup_{v_h \in X_h \setminus \{0\}} \frac{b(v_h, q_h)}{\|v_h\|_X} \ge \sup_{v \in X \setminus \{0\}} \frac{b(\Pi_h(v), q_h)}{\|\Pi_h(v)\|_X} = \sup_{v \in X \setminus \{0\}} \frac{b(v, q_h)}{\|\Pi_h(v)\|_X} \ge \frac{1}{C_f} \sup_{v \in X \setminus \{0\}} \frac{b(v, q_h)}{\|v\|_X}.$$

Since the right-most expression is bounded below by $(c_b/C_f)\|q_h\|_M$ thanks to (3.14), we deduce that the discrete inf-sup condition (4.6) also holds, with the discrete inf-sup constant defined as the ratio of the continuous inf-sup constant and C_f. $\square$

6. Examples of inf-sup stable and inf-sup unstable finite element spaces for the Stokes equations. We close our exposition with examples of finite element spaces that satisfy the discrete inf-sup condition, and we also list examples of finite element spaces that violate it. We start with the latter. Our exposition here is based on Sections 4.2.3–4.2.5 of [4].

6.1. Counterexamples.

1. *The* $[\mathbb{Q}_1]^2/\mathbb{P}_0$ *pair.* The most well-known example of a pair of finite element spaces that violates the discrete inf-sup condition for the Stokes equations in two space dimensions is that of continuous piecewise bilinear finite elements for the velocity and piecewise constant finite elements for the pressure. Suppose that $\Omega := (0,1)^2$ and consider a uniform square mesh on $\overline{\Omega}$ of spacing $h := 1/N$, where N is an *even* integer ≥ 2. Denote by a_{ij} the point in the mesh whose co-ordinates are (ih, jh), and let K_{ij} denote the closed square in the mesh whose bottom left corner is a_{ij}. We then define $\mathcal{T}_h$ as a collection of mesh cells K_{ij}, $i, j = 0, \dots, N-1$.
   For a mesh cell $K_{ij} \in \mathcal{T}_h$ we denote by $T_{K_{ij}} : \widehat{K} \to K_{ij}$ the C^1-diffeomorphism that maps the canonical (or master, or reference) element $\widehat{K} := [0,1]^2$ onto K. Let

   $$\begin{aligned} \mathbf{X}_h &:= \{\mathbf{v}_h \in [C(\overline{\Omega})]^2 \,:\, \forall K_{ij} \in \mathcal{T}_h,\ \mathbf{v}_h \circ T_{K_{ij}} \in [\mathbb{Q}_1]^2,\ \mathbf{v}_h|_{\partial\Omega} = \mathbf{0}\}, \\ M_h &:= \{q_h \in L^2_0(\Omega) \,:\, \forall K_{ij} \in \mathcal{T}_h,\ p_h \circ T_{K_{ij}} \in \mathbb{P}_0\}. \end{aligned}$$

   In order to demonstrate failure of the discrete inf-sup condition it suffices to show the existence of a nonzero $p_h \in M_h$ such that $b(\mathbf{v}_h, p_h) = -\int_\Omega (\nabla \cdot \mathbf{v}_h)\, p_h \,\mathrm{d}x = 0$ for all $\mathbf{v}_h \in \mathbf{X}_h$.
   To this end, we consider any function $p_h \in M_h$ and denote its (constant) value over the interior of the mesh cell K_{ij} by $p_{i+\frac{1}{2},j+\frac{1}{2}}$. Then, by the divergence theorem and noting that the trapezium rule integrates univariate affine functions exactly, we have that

   $$\begin{aligned} \int_{K_{ij}} (\nabla \cdot \mathbf{v}_h)\, p_h \,\mathrm{d}x &= p_{i+\frac{1}{2},j+\frac{1}{2}} \int_{\partial K_{ij}} \mathbf{v}_h \cdot \mathbf{n} \,\mathrm{d}s \\ &= \tfrac{1}{2} h p_{i+\frac{1}{2},j+\frac{1}{2}} \left(u_{i+1,j} + u_{i+1,j+1} + v_{i+1,j+1} + v_{i,j+1} - u_{i,j} - u_{i,j+1} - v_{i,j} - v_{i+1,j}\right), \end{aligned}$$

   where u and v denote the two components of the vector function $\mathbf{v}_h$, and $u_{i,j} := u(ih, jh)$, $v_{i,j} := v(ih, jh)$, and so on. Integrating over the entire domain followed by summation by parts yields

$$b(\mathbf{v}_h, p_h) = -\int_\Omega (\nabla \cdot \mathbf{v}_h)\, p_h \,\mathrm{d}x = -\sum_{i,j=0}^{N-1} \int_{K_{ij}} (\nabla \cdot \mathbf{v}_h)\, p_h \,\mathrm{d}x$$
$$= h^2 \sum_{i,j=1}^{N-1} \left(u_{i,j} (\partial_1^h p)_{ij} + v_{i,j} (\partial_2^h p)_{ij} \right),$$

where

$$(\partial_1^h p)_{ij} := \tfrac{1}{2h} \left(p_{i+\frac12, j+\frac12} + p_{i+\frac12, j-\frac12} - p_{i-\frac12, j+\frac12} - p_{i-\frac12, j-\frac12} \right),$$
$$(\partial_2^h p)_{ij} := \tfrac{1}{2h} \left(p_{i+\frac12, j+\frac12} + p_{i-\frac12, j+\frac12} - p_{i+\frac12, j-\frac12} - p_{i-\frac12, j-\frac12} \right).$$

We deduce that $b(\mathbf{v}_h, p_h) = 0$ for all $\mathbf{v}_h \in \mathbf{X}_h$ if, and only if, for all $i, j = 1, \dots, N-1$, we have that

$$p_{i+\frac12, j+\frac12} = p_{i-\frac12, j-\frac12} \quad \text{and} \quad p_{i-\frac12, j+\frac12} = p_{i+\frac12, j-\frac12}.$$

The set of solutions to this system of linear algebraic equations, with N^2 unknowns, is a two-dimensional linear subspace of $\mathbb{R}^{N^2}$. One basis vector of this linear space is the N^2-component vector $(1, 1, \dots, 1)^{\mathrm{T}}$, corresponding to the constant field $p_h \equiv 1$; however our assumption that $M_h \subset L_0^2(\Omega)$ demands that $\int_\Omega p_h \,\mathrm{d}x = 0$, and therefore the possibility that $p_h \equiv 1$ (or any nonzero multiple of this function) is excluded as a solution. The second basis vector of this two-dimensional linear space is an N^2-component vector with alternating entries $+1$ and -1, which corresponds to the piecewise constant, checker-board like, field p_h such that $p_h|_{K_{ij}} = (-1)^{i+j}$, which is usually referred to as a *spurious mode*. As the integral of such a checker-board pressure over Ω is equal to zero (recall that N was assumed to be an even integer and note that there are $\frac12 N^2$ mesh cells over which $p_h = 1$ and the same number of mesh cells over which $p_h = -1$), it follows that $p_h \in M_h \setminus \{0\}$ and $b(\mathbf{v}_h, p_h) = 0$ for all $\mathbf{v}_h \in \mathbf{X}_h$. Hence,

$$\sup_{\mathbf{v}_h \in \mathbf{X}_h \setminus \{\mathbf{0}\}} \frac{b(\mathbf{v}_h, p_h)}{\|\mathbf{v}_h\|_{\mathbf{X}}} = \sup_{\mathbf{v}_h \in \mathbf{X}_h \setminus \{\mathbf{0}\}} \frac{b(\mathbf{v}_h, p_h)}{|\mathbf{v}_h|_{H^1(\Omega)^2}} = 0 \qquad \text{and} \qquad \|p_h\|_{L^2(\Omega)} = 1.$$

Thus we have shown that the inf-sup condition is violated by the pair of finite element spaces $(\mathbf{X}_h, M_h)$.

2. *The* $[\mathbb{P}_1]^2/\mathbb{P}_1$ *pair.* Once again, we consider the open unit square $\Omega = (0,1)^2$, and subdivide $\overline{\Omega}$ into a square mesh of spacing $h = 1/N$, where $N+1$ is a multiple of 3, but we now further split each mesh square into two triangles with the diagonal of positive slope. Let $\mathcal{T}_h$ denote the resulting triangulation of Ω. Let $\widehat{K}$ denote the canonical (or master, or reference) element defined as the right-angle triangle, with its right angle at the point $(0,0)$ and its other two vertices at $(1,0)$ and $(0,1)$. Let further T_K denote the C^1 diffeomorphism that maps $\widehat{K}$ onto K. We define the finite element spaces for the velocity and the pressure as follows:

$$\mathbf{X}_h := \{\mathbf{v}_h \in [C(\overline{\Omega})]^2 \,:\, \forall K \in \mathcal{T}_h,\ \mathbf{v}_h \circ T_K \in [\mathbb{P}_1]^2,\ \mathbf{v}_h|_{\partial\Omega} = \mathbf{0}\},$$
$$M_h := \{q_h \in L_0^2(\Omega) \,:\, \forall K \in \mathcal{T}_h,\ p_h \circ T_K \in \mathbb{P}_1\}.$$

Given a certain triangle $K \in \mathcal{T}_h$ let us denote its three vertices by $a_{1,K}, a_{2,K}, a_{3,K}$ and consider a continuous piecewise affine pressure p_h such that on each triangle $K \in \mathcal{T}_h$ one has $\sum_{m=1}^3 p_h(a_{m,K}) = 0$. This can be achieved, for example, by defining

$$p_h(0, jh) \quad \text{for} \quad j = 0, 1, \dots, N \quad \text{as} \quad 0, +1, -1, 0, +1, -1, \dots, 0, +1, -1;$$
$$p_h(h, jh) \quad \text{for} \quad j = 0, 1, \dots, N \quad \text{as} \quad +1, -1, 0, +1, -1, 0, \dots, +1, -1, 0;$$
$$\text{etc.},$$
$$p_h(1, jh) \quad \text{for} \quad j = 0, 1, \dots, N \quad \text{as} \quad -1, 0, +1, -1, 0, +1, \dots, -1, 0, +1.$$

We have that, for each $\mathbf{v}_h \in X_h$, the function $(\nabla \cdot \mathbf{v}_h)|_K$ is constant for each $K \in \mathcal{T}_h$, and therefore

$$
\begin{aligned}
b(\mathbf{v}_h, p_h) = -\int_\Omega (\nabla \cdot \mathbf{v}_h) p_h \,\mathrm{d}x &= -\sum_{K \in \mathcal{T}_h} (\nabla \mathbf{v}_h)|_K \int_K p_h \,\mathrm{d}x \\
&= -\sum_{K \in \mathcal{T}_h} (\nabla \cdot \mathbf{v}_h)|_K \frac{|K|}{3} \sum_{m=1}^{3} p_h(a_{m,K}) = 0.
\end{aligned}
$$

By construction $\int_\Omega p_h \,\mathrm{d}x = 0$, $p_h \not\equiv 0$, while $b(\mathbf{v}_h, p_h) = 0$ for all $\mathbf{v}_h \in \mathbf{X}_h$. Thus we have constructed an example of a 'spurious pressure mode' $p_h \in M_h \setminus \{0\}$ that leads to the violation of the inf-sup condition for this pair of finite element spaces. ***Lecture 6***

After these two counterexamples, let us now present some examples of finite element spaces that do satisfy the inf-sup condition. The proofs are omitted; the interested reader is referred to sections 4.2.4–4.2.8 of [4] for further details.

6.2. Examples.

1. *The* $[\mathbb{P}_1\text{-bubble}]^2/\mathbb{P}_1$ *pair.* The reason for the failure of the $[\mathbb{P}_1]^2/\mathbb{P}_1$ pair is that the finite element space for the velocity is not rich enough to control the spurious pressure mode. The idea behind the $[\mathbb{P}_1\text{-bubble}]^2/\mathbb{P}_1$ pair is therefore to enrich the velocity space. The simplest way of achieving this is to add just one additional degree of freedom per element, associated with the barycenter (center of mass) of the element.
Let us suppose that Ω is a bounded open polyhedron in $\mathbb{R}^d$, $d = 2, 3$, whose closure $\overline{\Omega}$ has been subdivided into simplices K that form a finite element mesh $\mathcal{T}_h$. Let $\widehat{K}$ denote the reference right-angle simplex, with barycenter $\widehat{C}$, and consider the bubble function $\widehat{b} \in H^1_0(\widehat{K})$, such that $0 \le \widehat{b} \le 1$, $\widehat{b}(\widehat{C}) = 1$. A simple choice of such a function $\widehat{b}$ is to take

$$
\widehat{b} = (d+1)^{d+1} \prod_{i=1}^{d+1} \widehat{\lambda}_i,
$$

where $\lambda_1, \ldots, \lambda_{d+1}$ are the barycentric co-ordinates on the simplex $\widehat{K}$. We then define

$$
\widehat{\mathbb{P}}_{1+b} := \mathbb{P}_1(\widehat{K}) \oplus \operatorname{span}(\widehat{b})
$$

and we introduce the finite element spaces

$$
\begin{aligned}
\mathbf{X}_h &:= \{\mathbf{v}_h \in [C(\overline{\Omega})]^d \,:\, \forall K \in \mathcal{T}_h,\ \mathbf{v}_h \circ T_K \in [\mathbb{P}_{1+b}]^d,\ \mathbf{v}_h|_{\partial\Omega} = \mathbf{0}\}, \\
M_h &:= \{q_h \in L^2_0(\Omega) \,:\, \forall K \in \mathcal{T}_h,\ q_h \circ T_K \in \mathbb{P}_1\}.
\end{aligned}
$$

This pair of spaces then satisfies the discrete inf-sup condition (4.6). The proof, based on Fortin's criterion, can be found in Lemma 4.20 in [4]. It is also known (cf. Theorem 4.21 in [4]) that, on shape-regular families of finite element meshes $\{\mathcal{T}_h\}_{h>0}$, one has the approximation properties

$$
\inf_{\mathbf{v}_h \in \mathbf{X}_h} |\mathbf{u} - \mathbf{v}_h|_{H^1(\Omega)^d} \le \text{Const.}\, h \|\mathbf{u}\|_{H^2(\Omega)^d}
$$

and

$$
\inf_{q_h \in M_h} \|p - q_h\|_{L^2(\Omega)} \le \text{Const.}\, h \|p\|_{H^1(\Omega)},
$$

and therefore, by (4.8) one arrives at the error bound

$$
|\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)^d} + \|p - p_h\|_{L^2(\Omega)} \le \text{Const.}\, h \left(\|\mathbf{u}\|_{H^2(\Omega)^d} + \|p\|_{H^1(\Omega)} \right),
$$

assuming that the exact solution $(\mathbf{u}, p) \in (H^2(\Omega)^d \cap H^1_0(\Omega)^d) \times (H^1(\Omega) \cap L^2_0(\Omega))$.

2. *Taylor–Hood element and its generalizations.* Suppose, again, that Ω is a bounded open polyhedron in $\mathbb{R}^d$, $d = 2, 3$. We shall retain the $\mathbb{P}_1$ finite element space for the pressure, but instead of enriching the space of piecewise linear functions with a bubble function on each element for the velocity, we shall replace it with the space of continuous piecewise quadratic polynomials. The resulting pair of finite element spaces

$$\begin{aligned}
\mathbf{X}_h &:= \{\mathbf{v}_h \in [C(\overline{\Omega})]^d \,:\, \forall K \in \mathcal{T}_h,\ \mathbf{v}_h \circ T_K \in [\mathbb{P}_2]^d,\ \mathbf{v}_h|_{\partial\Omega} = \mathbf{0}\},\\
M_h &:= \{q_h \in C(\overline{\Omega}) \,:\, \forall K \in \mathcal{T}_h,\ q_h \circ T_K \in \mathbb{P}_1\},
\end{aligned}$$

is called the Taylor–Hood element. The pair of spaces $(\mathbf{X}_h, M_h)$ satisfies the inf-sup condition (cf. Lemma 4.24 in [4]); furthermore, on shape-regular families of finite element meshes $\{\mathcal{T}_h\}_{h>0}$, one has the approximation properties

$$\inf_{\mathbf{v}_h \in \mathbf{X}_h} |\mathbf{u} - \mathbf{v}_h|_{H^1(\Omega)^d} \le \text{Const.}\, h^2 \|\mathbf{u}\|_{H^3(\Omega)^d}$$

and

$$\inf_{q_h \in M_h} \|p - q_h\|_{L^2(\Omega)} \le \text{Const.}\, h^2 \|p\|_{H^2(\Omega)},$$

and therefore, by (4.8) one arrives at the error bound

$$|\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)^d} + \|p - p_h\|_{L^2(\Omega)} \le \text{Const.}\, h^2 \left(\|\mathbf{u}\|_{H^3(\Omega)^d} + \|p\|_{H^2(\Omega)}\right),$$

assuming that the exact solution $(\mathbf{u}, p) \in (H^3(\Omega)^d \cap H^1_0(\Omega)^d) \times (H^2(\Omega) \cap L^2_0(\Omega))$.

Higher order generalizations of the Taylor–Hood elements also exist: it is known that the pairs of velocity/pressure finite element spaces $[\mathbb{P}_k]^d/\mathbb{P}_{k-1}$ on simplices, and $[\mathbb{Q}_k]^d/\mathbb{Q}_{k-1}$ on quadrilaterals ($d = 2$) or hexahedra ($d = 3$) satisfy the inf-sup condition for all $k \ge 2$. The associated bound on the approximation errors for the velocity and the pressure is then of the form

$$|\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)^d} + \|p - p_h\|_{L^2(\Omega)} \le \text{Const.}\, h^k \left(\|\mathbf{u}\|_{H^{k+1}(\Omega)^d} + \|p\|_{H^k(\Omega)}\right),$$

assuming that the exact solution $(\mathbf{u}, p) \in (H^{k+1}(\Omega)^d \cap H^1_0(\Omega)^d) \times (H^k(\Omega) \cap L^2_0(\Omega))$.

3. $\mathbb{Q}_2/\mathbb{P}_1$-discontinuous *finite element.* The continuity requirement on the elements of the pressure space M_h in the basic Taylor–Hood finite element method can be relaxed. The resulting finite element spaces, defined by

$$\begin{aligned}
\mathbf{X}_h &:= \{\mathbf{v}_h \in [C(\overline{\Omega})]^d \,:\, \forall K \in \mathcal{T}_h,\ \mathbf{v}_h \circ T_K \in [\mathbb{Q}_2]^d,\ \mathbf{v}_h|_{\partial\Omega} = \mathbf{0}\},\\
M_h &:= \{q_h \in L^2_0(\Omega) \,:\, \forall K \in \mathcal{T}_h,\ q_h \circ T_K \in \mathbb{P}_1\},
\end{aligned}$$

satisfy the discrete inf-sup condition and exhibit the same asymptotic error bound as the basic Taylor–Hood element (corresponding to $k = 2$ in the previous example).

It may be tempting to consider the pair of finite element spaces $\mathbb{Q}_2/\mathbb{Q}_1$-discontinuous. This pair however does not satisfy the inf-sup condition: once again, the velocity space is not rich enough to control spurious pressure modes.

There is an extensive library of inf-sup stable finite element spaces on both simplicial and quadrilateral/hexahedral meshes in both two and three space dimensions. The interested reader is referred to the references [1, 2, 4, 5, 6, 7], for example, for further details.

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