(1) (a) We want to compute $\langle \underline{R}^2 \rangle = \langle (\sum_{i=1}^{N} b\underline{t}_i)^2 \rangle$

$= \int_{(\mathbb{R}^3)^N} dV (\sum b\underline{t}_i)^2 p(\underline{r}_1, \underline{r}_2, \dots \underline{r}_N)$ where p is the prob. of the configuration [1]

Since each $\underline{t}_i$ is uniformly distributed over the unit sphere and indep. of all $\underline{t}_j$, $j \neq i$, this

$= \int d\Omega_1 \dots \int d\Omega_N (\sum b\underline{t}_i)^2 \cdot \frac{1}{(4\pi)^N}$ [1] where $d\Omega_j = \sin\theta_j \, d\theta_j \, d\varphi_j$ is solid angle.

Now we note

$$\underline{R}^2 = (\sum \underline{r}_i) \cdot (\sum \underline{r}_j) = \sum_{i=1}^{N} \underline{r}_i^2 + \sum_{i \neq j} \underline{r}_i \cdot \underline{r}_j \quad [2]$$

$$\langle \underline{r}_i \cdot \underline{r}_j \rangle \propto \underbrace{\int d\Omega_i \, \underline{t}_i}_{\parallel} \cdot \int d\Omega_j \, \underline{t}_j \quad \text{if } i \neq j$$

$$\int_0^{2\pi}\int_0^{\pi} \underline{t}_i \sin\theta_i \, d\theta_i \, d\varphi_i = \underline{0} \quad [2] \quad \text{as } \underline{t}_i = \begin{pmatrix} \sin\theta_i \cos\varphi_i \\ \sin\theta_i \sin\varphi_i \\ \cos\theta_i \end{pmatrix}$$

And $\langle \underline{r}_i^2 \rangle = \langle b^2 \rangle = b^2$ since $\langle 1 \rangle = 1$ by definition

[B]

$$\therefore \langle \underline{R}^2 \rangle = \langle \sum \underline{r}_i^2 \rangle = N b^2 \quad [1]$$

(b) We have internal energy $E = \Gamma \sum \frac{b_0}{b} - 1$.

The probability of a given chain $\{\underline{r}_1, \dots \underline{r}_N\}$

is $p(\{\underline{r}_1, \dots, \underline{r}_N\}) = \frac{1}{Z} \exp\left(-\frac{E(\{\underline{r}_1 \dots \underline{r}_N\})}{k_b T}\right)$

Here, since $\underline{r}_i = b_i \underline{t}_i$ and b_i can take any value from b to ∞, we have

[2] $$Z = \int_{(B_b)^N} e^{-\frac{E}{k_b t}} \, dV \quad \text{where } B_b = \{\underline{r} : |\underline{r}| \geq b\}$$

$$= \int d\Omega_1 \int_{b_1}^{\infty} b_1^2 \, db_1 \cdots \int d\Omega_N \int_{b_N}^{\infty} b_N^2 \, db_N \cdot \exp\left(\frac{\Gamma}{k_b T} \sum 1 - \frac{b_0}{b}\right)$$

[5] $$= \prod_{i=1}^{N} \Bigg(\underbrace{\int_0^{2\pi} \int_0^{\pi} \int_b^{\infty} b_i^2 \sin\theta_i \, e^{\frac{\Gamma}{k_b T}\left(1 - \frac{b_0}{b}\right)} db_i \, d\theta_i \, d\varphi_i}_{\substack{\| \\ 4\pi e^{\frac{\Gamma}{k_b T}} \int_b^{\infty} b_i^2 e^{-\frac{b_i}{b}} db_i}} \Bigg) \text{ [2]}$$

$$= 4\pi e^{\frac{\Gamma}{k_b T}} \cdot 5 b^3 e^{-1} \quad \text{using given formula}$$

$$\therefore Z = \left(20\pi e^{\frac{\Gamma}{k_b T} - 1} \cdot b^3\right)^N$$ [1]

(c) Now the energy consists of 2 parts :

(i) $E = -\underline{F} \cdot \underset{\sim}{R} + E_{stretch} = -F_z \sum b_i \cos\theta_i + \Gamma \sum \frac{b_i}{b} - 1$

[2]

Now $Z = \int_{(B_b)^N} dV \exp\left(-\frac{E}{k_b T}\right) = \int_{(B_b)^N} dV \, e^{\lambda_1 \sum \frac{b_i}{b} \cos\theta_i + \lambda_2 \sum 1 - \frac{b_i}{b}}$

$\lambda_1 = \frac{F_z b}{k_b T}, \lambda_2 = \frac{\Gamma}{k_b T}$

$$\rightarrow Z = \prod_{i=1}^{N} \Big(\underbrace{\int_0^{2\pi} \int_0^{\pi} \int_b^{\infty} b_i^2 \sin\theta_i \, e^{\lambda_2} \cdot e^{-\frac{b_i}{b}(\lambda_2 - \lambda_1 \cos\theta_i)} db_i \, d\theta_i \, d\varphi_i}_{} \Big)$$

$$\parallel$$

$$2\pi e^{\lambda_2} \int_b^{\infty} \frac{b}{\lambda_1 b_i} b_i^2 \, e^{-\frac{b_i}{b}(\lambda_2 - \lambda_1 \cos\theta_i)} \Big|_{\theta=0}^{\pi} db_i$$

[S/N]

$$= \frac{2\pi b e^{\lambda_2}}{\lambda_1} \int_b^{\infty} b_i e^{-\frac{b_i}{b}(\lambda_2 + \lambda_1)} - \underline{b_i e^{-\frac{b_i}{b}(\lambda_2 - \lambda_1)}} \, db_i$$

[2]

this will diverge if $\lambda_1 > \lambda_2$

ie if $F_z b > \Gamma$

ii) If $\lambda_1 < \lambda_2$, we obtain (using provided integ. formulas with $a = \lambda_2 \pm \lambda_1$, $c = b$)

$$Z = \left(\frac{2\pi b e^{\lambda_2}}{\lambda_1} \left(\frac{(1+\lambda_2+\lambda_1) b^2}{(\lambda_2+\lambda_1)^2 e^{\lambda_2-\lambda_1}} - \frac{(1+\lambda_2-\lambda_1) b^2}{(\lambda_2-\lambda_1)^2 e^{\lambda_2+\lambda_1}} \right) \right)^N$$

$$\overset{\text{call}}{=} f(\lambda_1, \lambda_2)^N$$

[2]

Now, $R_z = \sum_{i=1}^{N} b_i \cos\theta_i$, so

$$\langle R_z \rangle = \frac{1}{Z} \int_{(B_b)^N} \left(\sum b_i \cos\theta_i \right) e^{\lambda_1 \sum \frac{b_i}{b} \cos\theta_i + \lambda_2 \sum 1 - \frac{b_i}{b}} \, dV$$

[1]

Observe
$$\frac{\partial}{\partial \lambda_1} Z = \int_{(B_b)^N} \frac{1}{b} \sum b_i \cos\theta_i \, e^{(\cdot)} \, dV$$

[2]

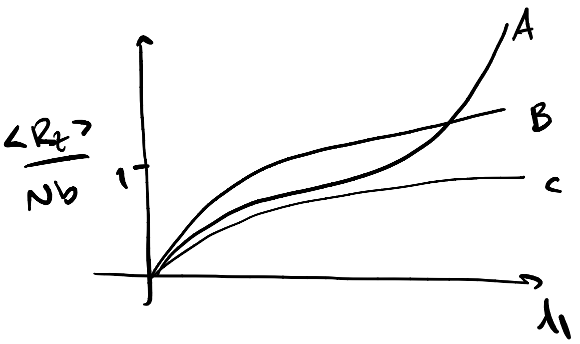
$$\therefore \langle R_z \rangle = b \cdot \frac{1}{Z} \frac{\partial}{\partial \lambda_1} Z = b \frac{\partial}{\partial \lambda_1} \log Z$$

[5]

$$= bN \frac{\partial}{\partial \lambda_1} \log f$$ with

f defined as above

(iii)



[1] A is ii) - linear energy

B is iii) - quadratic stretching energy

C is i) inextensible

· the inextensible chain asymptotes to 1 as

[1] pulling force $\lambda_1 \to \infty$ - $\langle R_z \rangle$ only reaches Nb if chain is straight

· the linear energy leads to a divergence in $\langle R_z \rangle$, as we showed, as $\lambda_1 \to \lambda_2 = 10$

[1]

(linear energy ⇒ a constant resistance force, so nothing stops the extension if $bF_z > 5$ - this is not physically realistic)

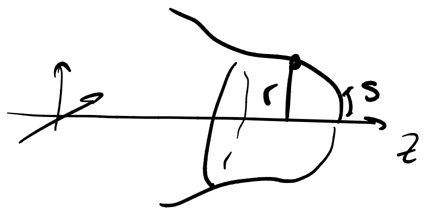
More than expected from students

· For quadratic energy, $\langle R_z \rangle$ doesn't diverge → (linear stretching force ⇒ resistance increases

[1] w/ extension), but $\langle R_z \rangle$ can exceed Nb with sufficient λ_1

[N]

(2) a)



We parameterise the surface as

$$\underline{x}(s,\varphi) = \begin{pmatrix} r(s)\cos\varphi \\ r(s)\sin\varphi \\ z(s) \end{pmatrix}$$

[2]

$$\Rightarrow \underline{x}_s = \begin{pmatrix} r'\cos\varphi \\ r'\sin\varphi \\ z' \end{pmatrix}, \quad \underline{x}_\varphi = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix}$$

$\Rightarrow$ Metric tensor has $g_{11} = \underline{x}_s \cdot \underline{x}_s = r'^2 + z'^2$

[1]

$$g_{12} = 0, \quad g_{22} = \underline{x}_\varphi \cdot \underline{x}_\varphi = r^2$$

Normal vec $\underline{n} = \dfrac{\underline{x}_s \wedge \underline{x}_\varphi}{\| \quad \|}$. $\underline{x}_s \wedge \underline{x}_\varphi = \begin{pmatrix} -rz'\cos\varphi \\ -rz'\sin\varphi \\ rr' \end{pmatrix}$

$$\Rightarrow \underline{n} = \frac{(-rz'\cos\varphi, -rz'\sin\varphi, rr')^T}{r(r'^2+z'^2)^{\frac{1}{2}}}$$

[1]

Curvature tensor: $K_{11} = -\underline{n} \cdot \dfrac{\partial^2 \underline{x}}{\partial s^2} = \begin{pmatrix} r''\cos\varphi \\ r''\sin\varphi \\ z'' \end{pmatrix}$

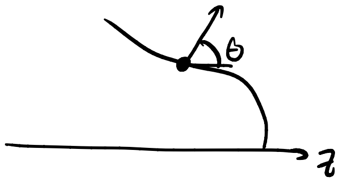
$$= \frac{-1}{\cancel{r}(r'^2+z'^2)^{\frac{1}{2}}} \cdot (\cancel{r}r'z'' - \cancel{r}r''z')$$

[2]

$$K_{22} = -\underline{n} \cdot \frac{\partial^2 \underline{x}}{\partial \varphi^2} = \frac{-r^2 z'}{\cancel{r}(r'^2+z'^2)^{\frac{1}{2}}}$$

$$\frac{\partial^2 \underline{x}}{\partial \varphi^2} = \begin{pmatrix} -r\cos\varphi \\ -r\sin\varphi \\ 0 \end{pmatrix}$$

(& $K_{12} = K_{21} = 0$ by axial symmetry – φ and s are principal directions) [1]



we note $r' = \cos\theta$, $z' = -\sin\theta$

so $r'^2 + z'^2 = 1$ ~ previous formulas.

κ_s, κ_φ are the eig vals of $L = G^{-1}K$

$$G^{-1} = \frac{1}{g_{11}g_{22}} \begin{pmatrix} g_{22} & 0 \\ 0 & g_{11} \end{pmatrix}, \quad K = \begin{pmatrix} K_{11} & 0 \\ 0 & K_{22} \end{pmatrix}$$ [1]

$\Rightarrow L$ is diag w/ $L_{11} = \dfrac{K_{11}}{g_{11}} = z'r'' - r'z''$ [2]

& $L_{22} = \dfrac{K_{22}}{g_{22}} = \dfrac{-rz'}{r^2}$

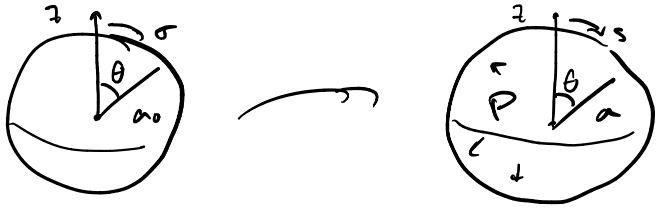
Now $r'' = -\sin\theta\,\theta'$, $z'' = -\cos\theta\,\theta'$

$\Rightarrow L_{11} = \kappa_s = \theta'(s)$, $L_{22} = \kappa_\varphi = \dfrac{\sin\theta}{r}$, [1]

as desired.

[B/S]

(b)



We have in initial state: $\begin{cases} \sigma = a_0 \theta \\ r_0 = a_0 \sin\theta \\ z_0 = a_0 \cos\theta \end{cases}$ $\xrightarrow{\text{deforms to}}$ $\begin{cases} s = a\theta \\ r = a\sin\theta \\ z = a\cos\theta \end{cases}$

[2]

The principal stretches are defined by $\lambda_s = \frac{ds}{d\sigma}$, $\lambda_\varphi = \frac{r}{r_0}$

→ here we have $s = \frac{a}{a_0}\theta \Rightarrow \lambda_s = \frac{a}{a_0}$ & $\lambda_\varphi = \frac{a\sin\theta}{a_0 \sin\theta} = \frac{a}{a_0}$.

We're given $t_s = t_\varphi = f\left(\frac{a}{a_0}\right)$

- $f(1) = 0$ so that no stress in ref. state

[2]

- $f'(1) > 0$ so that the material is in tension ($t_s = t_\varphi > 0$) when stretched ($a > a_0$)

$\frac{\partial t_s}{\partial s} = \frac{\cos\theta}{r}(t_s - t_\varphi)$ automatically satisfied, so turn to

$P = t_s \kappa_s + t_\varphi \kappa_\varphi$. We have $\kappa_s = \frac{d\theta}{ds} = \frac{1}{a}$, $\kappa_\varphi = \frac{\sin\theta}{r} = \frac{1}{a}$

(obvious for a sphere, but should verify the relations hold)

[2] Thus, $P = \frac{2}{a} f\left(\frac{a}{a_0}\right)$ defines the pressure-radius relation.

[5]

For a fluid biomembrane, we have the energy $E = \int_\Sigma dS\,(\gamma + 2\kappa H^2) - VP$

Lagrange multiplier to enforce pressure constraint.

For sphere radius a, $H = \frac{1}{a}$, $dS = 4\pi a^2$, [2]

so $E(a) = 4\pi\gamma a^2 + \frac{4\pi a^2\kappa}{a^2} - \frac{4}{3}\pi a^3 P$

We find a via $E'(a) = 0 \rightarrow 8\pi\gamma a - 4\pi a^2 P = 0$

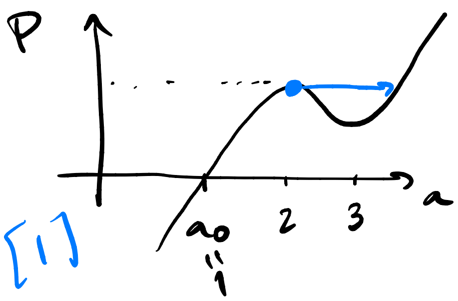
[5] $\Rightarrow$ pressure-radius relation $\boxed{P = \frac{2\gamma}{a}}$

- this agrees with (iii) if $f\left(\frac{a}{a_0}\right) = \gamma$ is constant [1]

(iv) Inflation instability: Noting that $P(a) = \frac{2f\left(\frac{a}{a_0}\right)}{a}$ (w/ $a_0 = 1$), we are given that

[S/N]

$$\frac{f'(a)}{a} - \frac{f(a)}{a^2} = \frac{d}{da}\left(\frac{f(a)}{a}\right) = \frac{d}{da}\left(\frac{P(a)}{2}\right)$$



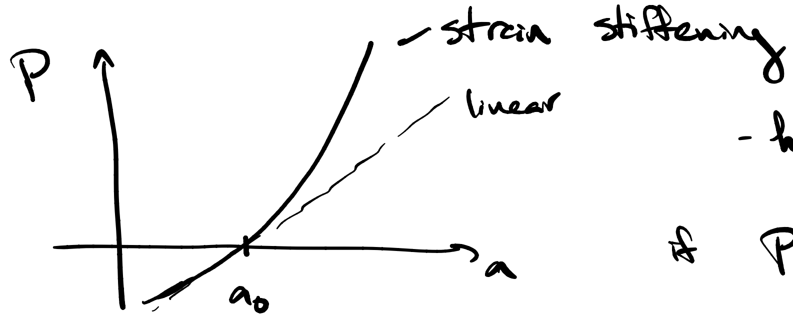
[1]

has roots at $a=2$, $a=3$ [1]

$\Rightarrow P(a)$ has shape at left

. A limit point instability will occur as P is increased past $P(2) = \frac{2f(2)}{2}$

(v) Strain-stiffening response will mean as a is increased, greater increase in P is needed to further inflate.



- holds near $a = a_0$ if $P''(a_0) > 0$. [1]

we have $P'(a) = \frac{-2}{a^2} f\left(\frac{a}{a_0}\right) + \frac{2}{a a_0} f'\left(\frac{a}{a_0}\right)$

$$\rightarrow P''(a) = \frac{-4}{a^2} f\left(\frac{a}{a_0}\right) - \frac{4}{a a_0^2} f'\left(\frac{a}{a_0}\right) + \frac{2}{a a_0^2} f''\left(\frac{a}{a_0}\right)$$

$$\Rightarrow P''(a_0) = -\frac{4}{a_0^3} f'(1) + \frac{2}{a_0^3} f''(1) \qquad (\text{as } f(1) = 0)$$

So strain stiffening observed if $\boxed{f''(1) > 2f'(1)}$

[2]

[N]

(3) $\underline{r}' = \alpha\gamma \underline{d}_3$, $\underline{d}_i' = \gamma \underline{u} \wedge \underline{d}_i$, $\underline{n}' + \underline{f} = \underline{0}$, $\underline{m}' + \underline{r}' \wedge \underline{n} = \underline{0}$

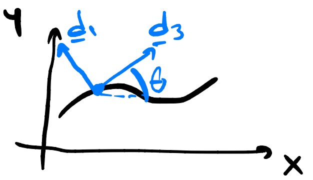
w/ $\underline{m} = K_1 u_1 \underline{d}_1 + K_2 u_2 \underline{d}_2 + K_3 u_3 \underline{d}_3$, $\underline{n} \cdot \underline{d}_3 = K_4(\alpha - 1)$

(a) $\underline{r}$. rod centreline, $\underline{u}$ curvature vector

$\{d_1, d_2, d_3\}$ material frame, gives rod's orientation

$\underline{n}$ - resultant force on rod [1]

$\underline{m}$ - resultant moment



Let $\underline{r} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$, and θ the angle between tangent $\underline{d}_3$ and x-axis

So $\underline{d}_3 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$. Choose $\underline{d}_2 = \underline{e}_z$, so $\underline{d}_1 = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$ [2]

Then $\underline{d}_3' = \theta'(s_0)\underline{d}_1 = \gamma \underline{u} \wedge \underline{d}_3 = \gamma(u_2 \underline{d}_1 - u_1 \underline{d}_2)$

[B] $\Rightarrow u_1 = 0$, $u_2 = \frac{1}{\gamma}\theta'$. Also, $\underline{d}_2' = 0 = \gamma(u_1 \underline{d}_3 - u_3 \underline{d}_1)$

$\Rightarrow u_3 = 0$. So $\underline{m} = m_2 \underline{d}_2 = K_2 \frac{1}{\gamma}\theta' \underline{e}_z$ [3]

Let $\underline{n} = \begin{pmatrix} n_x \\ n_y \\ 0 \end{pmatrix}$, $\underline{f} = \begin{pmatrix} f \\ g \\ 0 \end{pmatrix}$. $\Rightarrow$ $\begin{vmatrix} n_x' + f = 0 \\ n_y' + g = 0 \end{vmatrix}$ (1) (2)

$\underline{r}' = \alpha\gamma \underline{d}_3 \rightarrow$ $\begin{vmatrix} x' = \alpha\gamma\cos\theta \\ y' = \alpha\gamma\sin\theta \end{vmatrix}$ (3) (4)

Moment balance

$m_2' \underline{e}_z + \alpha\gamma \underline{d}_3 \wedge \underline{n} = \underline{0}$

$\Rightarrow \left| K_2\left(\frac{1}{\gamma}\theta'\right)' + \alpha\gamma(n_y\cos\theta - n_x\sin\theta) = 0 \right|$ (5) [2]

& $\underline{n} \cdot \underline{d}_3 = \left| n_x\cos\theta + n_y\sin\theta = K_4(\alpha - 1) \right|$ (6)

(b) We take $f=0$, $g=-\frac{K}{\gamma}y$.

For flat state, $y\equiv 0\equiv\theta$, so (6) $\Rightarrow$ $n_x = K_y(\alpha-1)$.

(3), (4) $\rightarrow$ $x' = \alpha\gamma$, $y'=0$. (1), (2) $\rightarrow$ n_x, n_y constant

And (5) $\rightarrow$ $n_y = 0$. [2]

Bdy cond are : At $S_0=0$, $x=0$, $y=0$, $\underline{m}=\underline{0}\Rightarrow\theta'=0$

[5] At $S_0=L_0$, $x=L_0$, $y=0$, $\theta'=0$

So $x=S_0 \Rightarrow x'=\alpha\gamma=1$ $\Rightarrow$ must have $\alpha=\frac{1}{\gamma}$ [2]

Thus, grown flat state given by $\left\{\begin{array}{ll} x^{(0)}=S_0, & \alpha^{(0)}=\frac{1}{\gamma} \\ n_x^{(0)}=K_y\left(\frac{1}{\gamma}-1\right), & \end{array}\right.$

all others $=0$.

(c) Perturb : $x\sim x^{(0)}+\varepsilon x^{(1)}$, $y\sim\varepsilon y^{(1)}$, $\theta\sim\varepsilon\theta^{(1)}$, $n_y\sim\varepsilon n_y^{(1)}$,

$n_x\sim n_x^{(0)}+\varepsilon n_x^{(1)}$, $\alpha\sim\frac{1}{\gamma}+\varepsilon\alpha^{(1)}$.

Then (2) $\rightarrow$ $n_y^{(1)\prime}=\frac{K}{\gamma}y^{(1)}$, (4) $\rightarrow$ $y^{(1)\prime}=\theta^{(1)}$

(5) $\rightarrow$ $K_2\theta'' + \gamma\left(n_y^{(1)} - n_x^{(0)}\theta^{(1)}\right) + O(\varepsilon) = 0$

$\frac{d}{dS_0}$ $\Rightarrow$ $K_2 y^{(1)''''} + \gamma\left(\frac{K}{\gamma}y^{(1)} - K_y\left(\frac{1-\gamma}{\gamma}\right)y^{(1)''}\right) \cong 0$ [2]

[5] $\Rightarrow$ $K_2 y^{(1)''''} + K_y(\gamma-1)y^{(1)''} + Ky^{(1)} = 0$

BC $y^{(1)} = y^{(1)''} = 0$ at $S_0 = 0, L_0$.

$\overset{\shortparallel}{\theta^{(1)\prime}}$

- We seek smallest $\gamma = \gamma^* > 1$ at which $y^{(1)}$ has a non-trivial sol'n. Taking $y^{(1)} = ce^{i\omega S_0}$

$$\rightarrow K_2 \omega^4 - K_4(\gamma - 1)\omega^2 + \kappa = 0$$

$$\Rightarrow \omega^2 = \frac{K_4(\gamma-1) \pm \sqrt{K_4^2(\gamma-1)^2 - 4K_2 \kappa}}{2K_2} =: \omega_\pm^2 \quad [1]$$

- for oscillatory sol'n, must have $\omega_\pm^2 > 0$

$\rightarrow (\gamma^* - 1)^2 > \frac{4K_2\kappa}{K_4^2}$. In this range, we'll have

$$y^{(1)} = A\cos\omega_+ S_0 + B\cos\omega_- S_0 + C\sin\omega_+ S_0 + D\sin\omega_- S_0$$

$$\left.\begin{aligned} y^{(1)}(0) &= A + B = 0 \\ y^{(1)''}(0) &= -A\omega_+^2 - B\omega_-^2 = 0 \end{aligned}\right\} \Rightarrow A = B = 0 \quad [1]$$

Then $y^{(1)}(L_0) = C\sin\omega_+ L_0 + D\sin\omega_- L_0 = 0$

$$y^{(1)''}(L_0) = -C\omega_+^2\sin\omega_+ L_0 - D\omega_-^2\sin\omega_- L_0 = 0$$

Non-trivial sol'ns if $\sin\omega_+ L_0 = n\pi$, $n \in \mathbb{N}$ [2]

or $\sin\omega_- L_0 = n\pi$

Critical buckling is defined by smallest $\gamma^* > 1 + \frac{2\sqrt{K_2\kappa}}{K}$ satisfying one of the above. [2]

(d) On an infinite domain (no BC),

$(\gamma^*-1)^2 = \frac{4K_2\kappa}{K_4^2}$ (smallest γ for which oscillatory sol'ns exist)

$\rightarrow \gamma^*-1 \sim \sqrt{\kappa}$ [1]

And $\omega^* = \left(\frac{K_4(\gamma^*-1)}{2K_2}\right)^{\frac{1}{2}} = \frac{\kappa^{\frac{1}{4}}}{K_2^{\frac{1}{4}}}$ [2]

[N]

Wavelength $\lambda = \frac{2\pi}{\omega} \Rightarrow \lambda^* \sim \kappa^{-\frac{1}{4}}$.

As κ increases, foundation provides increased resistance, so more growth needed to trigger instability, and observed wavelength decreases, because it costs more to have large deformation (foundation energy) relative to bending and stretching. [2]