# Noncommutative Rings

Throughout this course, $R$ is an associative but not necessarily commutative ring with an identity element 1. We will use the letter $k$ to denote a field.

## 1. Some examples of noncommutative rings

**Definition 1.1.** Let G be a group and let $R$ be a ring. The *group algebra* $RG$ consists of formal linear combinations

$$\sum_{g \in G} r_g g,$$

where $r_g \in R$ for all $g \in G$ and all but finitely many $r_g$ are zero. Addition and multiplication is given by

$$(\sum_{g \in G} r_g g) + (\sum_{g \in G} s_g g) = \sum_{g \in G} (r_g + s_g) g$$

$$(\sum_{h \in G} r_h h)(\sum_{k \in G} s_k k) = \sum_{g \in G} (\sum_{\substack{h,k \in G \\ hk = g}} r_h s_k) g.$$

Recall that a *$k$-linear representation* of $G$ is a group homomorphism

$$\varphi : G \to \mathrm{Aut}_k(V)$$

where $V$ is some vector space over $k$.

**Lemma 1.2.** There is a natural bijection between $k$-linear representations of $G$ and left $kG$-modules.

*Proof.* A group homomorphism $\varphi : G \to \mathrm{Aut}_k(V)$ extends uniquely to a $k$-algebra homomorphism $\widetilde{\varphi} : kG \to \mathrm{End}_k(V) := \{f : V \to V : f \text{ is } k\text{-linear }\}$, and $V$ may then be regarded as a left $kG$-module, via $x.v = \widetilde{\varphi}(x)(v)$ for all $x \in kG$.

Conversely, if $V$ is a left $kG$-module, there is a representation $\varphi : G \to \mathrm{Aut}_k(V)$ given by $\varphi(g)(v) = g.v$ for all $v \in V$. □

**Definition 1.3.** A *Lie algebra* over $k$ is a $k$-vector space $\mathfrak{g}$, equipped with bilinear map $[.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

(1) $[x, x] = 0$ for all $x \in \mathfrak{g}$ and hence $[y, z] = -[z, y]$ for all $y, z \in \mathfrak{g}$
(2) $[x, [y, z]] + [y, [z, x]] + [z, [y, x]] = 0$ for all $x, y, z \in \mathfrak{g}$.

Note that this bracket is not associative.

**Examples 1.4.**

(1) Any (associative) $k$-algebra $R$ becomes a Lie algebra under the commutator bracket $[x, y] = xy - yx$.
(2) $\mathfrak{gl}_n(k)$, the set of all $n \times n$ matrices over $k$ with the commutator bracket.

(3) $\mathfrak{sl}_n(k)$, the set of traceless $n \times n$ matrices over $k$ with commutator bracket.
(4) If $V$ is any vector space, we can define the trivial bracket $[x, y] = 0$ for all $x, y \in V$. This is the *abelian* Lie algebra.

A *representation of* $\mathfrak{g}$ is a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$, where

$$\mathfrak{gl}(V) := \mathrm{End}_k(V)$$

equipped with the commutator bracket.

**Question 1.5.** What is the analogue of the group algebra for representations of Lie algebras?

**Definition 1.6.** The *free associative algebra* on $n$ generators $k\langle x_1, \ldots, x_n\rangle$ is the $k$-vector space with basis given by all possible products $y_1 \cdots y_m$ where $y_1, \ldots, y_m \in \{x_1, \ldots, x_n\}$. Multiplication is given by concatenation on basis elements and is extended by $k$-linearity to the whole of $k\langle x_1, \ldots, x_n\rangle$.

Note that $k\langle x_1, \ldots, x_n\rangle$ is not finite dimensional over $k$. For example, if $n = 1$ then $k\langle x\rangle$ has $\{1, x, x^2, \ldots\}$ as a basis. In fact $k\langle x\rangle \cong k[x]$, the polynomial algebra. Similarly, $k\langle x, y\rangle$ has as a $k$-basis the set $\{1, x, y, x^2, xy, yx, y^2, x^3, x^2y, \ldots\}$. This algebra is not commutative!

**Definition 1.7.** The *universal enveloping algebra* $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ is

$$U(\mathfrak{g}) := k\langle x_1, \ldots x_n\rangle / I$$

where $\{x_1, x_2, \ldots, x_n\}$ is a basis for $\mathfrak{g}$ and $I$ is the two-sided ideal of $k\langle x_1, \ldots x_n\rangle$ generated by the set $\{x_i x_j - x_j x_i - [x_i, x_j], 1 \leqslant i, j \leqslant n\}$.

For example, if $\mathfrak{g}$ is abelian, then $U(\mathfrak{g})$ is just the polynomial algebra $k[x_1, \ldots, x_n]$.

**Lemma 1.8.** There is a natural bijection between representations of $\mathfrak{g}$ and left $U(\mathfrak{g})$-modules.

*Proof.* If $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation, we make $V$ into a left module over $k\langle x_1, \ldots, x_d\rangle$ by setting $(x_{i_1} \cdots x_{i_d}) \cdot v := \varphi(x_{i_1})\varphi(x_{i_2}) \cdots \varphi(x_{i_d})(v)$. Because $\varphi$ is a Lie algebra homomorphism, we see that $(x_i x_j - x_j x_i) \cdot v = [x_i, x_j] \cdot v$ for all $i, j$. So the ideal $I$ kills $V$ and therefore $V$ is actually a left $U(\mathfrak{g})$-module.

Conversely, if $V$ is a $U(\mathfrak{g})$-module, then there is a $k$-algebra homomorphism $U(\mathfrak{g}) \to \mathrm{End}_k(V)$ given by $r \mapsto (v \mapsto r \cdot v)$. We can view it as a Lie homomorphism $U(\mathfrak{g}) \to \mathfrak{gl}(V)$. The map $\mathfrak{g} \to U(\mathfrak{g})$ is also a Lie homomorphism, so we get a representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ by composing these. □

**Definition 1.9.** The left $R$-module $M$ is said to be *cyclic* if it can be generated by a single element: $M = Rx$ for some $x \in M$. $M$ is *finitely generated* if it can be written as a finite sum of cyclic submodules $M = Rx_1 + Rx_2 + \ldots + Rx_n$.

**Lemma 1.10.** Let $M$ be a left $R$-module. The following are equivalent:

(a) Every submodule of $M$ is finitely generated
(b) **Ascending chain condition:** There does not exist an infinite strictly ascending chain of submodules of $M$
(c) **Maximum condition:** Every non-empty subset of submodules of $M$ contains at least one maximal element. (If $\mathcal{S}$ is a set of submodules, then $N \in \mathcal{S}$ is a *maximal element* if and only if $N' \in \mathcal{S}$, $N \leqslant N'$ implies $N = N'$).

*Proof.* (a) $\Rightarrow$ (b). Suppose $M_1 \subsetneq M_2 \subsetneq \ldots$. Let $N = \cup M_n$. Then $N$ is a submodule of $M$ so $N$ is finitely generated by $m_1, \ldots, m_r$ say. If $m_i \in M_{n_i}$, then it follows that $N = M_n$ where $n = \max n_i$, a contradiction.

(b) $\Rightarrow$ (c) If $\mathcal{S}$ is a nonempty subset with no maximal element, pick $M_1 \in \mathcal{S}$. Since $\mathcal{S}$ has no maximal element, we can find $M_2 \in \mathcal{S}$ such that $M_1 \subsetneq M_2$. Continuing like this gives a strictly ascending infinite chain $M_1 \subsetneq M_2 \subsetneq \ldots$, a contradiction.

(c) $\Rightarrow$ (a) Let $N$ be a submodule of $M$ and let $\mathcal{S}$ be the set of submodules of $N$ which are finitely generated. Since $0 \in \mathcal{S}$, $\mathcal{S}$ has a maximal element $L$, say. Let $x \in N$. Since $L + Rx$ is a finitely generated submodule of $N$ and $L$ is maximal in $\mathcal{S}$, $L + Rx = L$ so $x \in L$. Hence $N = L$ is itself finitely generated. □

Dually, we have the *descending chain condition* and the *minimum condition*; these are equivalent to each other.

**Definition 1.11.** An $R$-module satisfying (a), (b), (c) of Lemma 1.10 is *Noetherian.* The ring $R$ is *left Noetherian* if it is Noetherian as a left $R$-module.

We have similar definitions "on the right hand side". Note that if the ring is commutative, there is no difference between "left" and "right". If $R$ is *both* left and right Noetherian, then we will simply say that $R$ is *Noetherian.* Artinian rings are defined similarly. Here is the main engine for proving that certain rings are left Noetherian: it is a non-commutative version of *Hilbert's Basis Theorem.*

**Theorem 1.12** (McConnell, 1968)**.** Let $S$ be a ring, $R$ a left Noetherian subring and suppose that for some $x \in S$ we have

(1) $R + xR = R + Rx$, and
(2) $S = \langle R, x \rangle$.

Then $S$ is also left Noetherian.

**Corollary 1.13.** Let $R$ be a left Noetherian subring of $S$, and $x \in S$.

(a) Suppose there is an automorphism $\varphi$ of $R$ is such that $rx = x\varphi(r)$ for all $r \in R$. If $S = \langle R, x \rangle$, then $S$ is left Noetherian.
(b) Suppose $x$ is a unit in $S$ such that $x^{-1}Rx = R$. If $S = \langle R, x, x^{-1} \rangle$, then $S$ is left Noetherian.

*Proof.* (a) If $rx = x\varphi(r)$ for all $r \in R$, then $Rx = xR$ so $R + xR = R + Rx$ and we can apply Theorem 1.12.

(b) Let $T = \langle R, x\rangle$. Then $T$ is left Noetherian by part (a). Let $I$ be a left ideal of $S$. Now, $I \cap T$ is a left ideal of $T$ and is hence finitely generated: $I \cap T = \sum_{i=1}^{n} Ts_i$, say. If $s \in I$, then $x^m s \in I \cap T$ for some $m \geq 0$, so $s = \sum_{i=1}^{n} x^{-m} a_i s_i$ for some $a_i \in T$. Hence the $s_i$'s generate $I$ as a left ideal of $S$. □

**Definition 1.14.** The group $G$ is said to be *polycyclic* if there is a chain

$$1 = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_{n-1} \triangleleft G_n = G$$

of subgroups of $G$ such that each $G_i/G_{i-1}$ is cyclic for each $i = 1, \ldots, n$.

**Examples 1.15.**

(a) Infinite cyclic $G = \langle x\rangle \cong \mathbb{Z}$.
(b) Free abelian $G = \langle x_1, \ldots, x_n\rangle \cong \mathbb{Z}^n$.
(c) $G = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$. Here we have the chain $1 \triangleleft G_1 \triangleleft G_2 \triangleleft G_3 = G$ where
$$G_1 = \begin{pmatrix} 1 & 0 & \mathbb{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } G_2 = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
(d) $\{I + N \in M_n(\mathbb{Z}) : N \text{ is strictly upper triangular}\}$ is always polycyclic.

**Proposition 1.16.** Let $R$ be a Noetherian ring and let $G$ be a polycyclic group. Then $RG$ is Noetherian.

*Proof.* Choosing a chain of subnormal subgroups with cyclic quotients

$$1 = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_{n-1} \triangleleft G_n = G$$

we see that it's sufficient to show that if $RG_{i-1}$ is left Noetherian then so is $RG_i$ for all $i = 1, \ldots, n$. Now, choose a generator $xG_{i-1}$ for the cyclic group $G_i/G_{i-1}$; then $RG_i$ is generated by $RG_{i-1}$,$x$ and $x^{-1}$. Since $G_{i-1} \triangleleft G_i$, $RG_{i-1}$ is invariant under conjugation by $x$, so $RG_i$ is left Noetherian by Corollary 1.12. □

**Question 1.17.** Suppose that $k$ is a field and $kG$ is left Noetherian. Must $G$ contain a polycyclic subgroup of finite index?

We will now introduce a new class of non-commutative rings, called the *Weyl algebras*: these are the most elementary examples of *rings of differential operators*. First, some motivation.

**Lemma 1.18.** Let $A = k[x]$ and consider the $k$-linear maps $\frac{\partial}{\partial x} : A \to A$ and $\widehat{x} : A \to A$, where $\widehat{x}$ is multiplication by $x$. Then

$$[\frac{\partial}{\partial x}, \widehat{x}] = 1.$$

*Proof.* By the product rule, $[\frac{\partial}{\partial x}, \widehat{x}](f) = (xf)' - xf' = f$ for all $f \in A$. □

Now consider the polynomial algebra $A = k[x_1, \ldots, x_n]$ and the $k$-linear maps

$$\begin{array}{rccc} \widehat{x_i}: & A & \to & A \\ & f & \mapsto & x_i f \end{array} \quad \text{and} \quad \begin{array}{rccc} \frac{\partial}{\partial x_i}: & A & \to & A \\ & f & \mapsto & \frac{\partial f}{\partial x_i}, \end{array}$$

for $1 \leqslant i \leqslant n$. These maps are examples of *differential operators* on $A$. It can be verified that all these operators commute, except for $\frac{\partial}{\partial x_i}$ and $\widehat{x_i}$, which satisfy the relation $[\frac{\partial}{\partial x_i}, \widehat{x_i}] = 1$.

**Definition 1.19.** Let $k$ be a field. The *n-th Weyl algebra* $A_n(k)$ over $k$ is

$$A_n(k) := k\langle x_1, \ldots, x_n, y_1, \ldots y_n\rangle / I$$

where $I$ is the ideal of the free algebra $k\langle x_1, \ldots, x_n, y_1, \ldots y_n\rangle$ generated by

$$\begin{array}{ll} x_i x_j - x_j x_i & 1 \leqslant i, j \leqslant n, \\ y_i y_j - y_j y_i & 1 \leqslant i, j \leqslant n, \\ y_i x_i - x_i y_i - 1 & 1 \leqslant i \leqslant n, \\ x_i y_j - y_j x_i & i \neq j. \end{array}$$

For example, if $n = 1$ then $A_1(k) = k\langle x, y\rangle / \langle yx - xy - 1\rangle$. There is a surjective $k$-algebra homomorphism from $A_n(k)$ onto the $k$-subalgebra of $\mathrm{End}_k(k[x_1, \ldots, x_n])$ generated by $\{\widehat{x_1}, \ldots, \widehat{x_n}, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\}$, mapping $x_i$ to $\widehat{x_i}$ and $y_i$ to $\frac{\partial f}{\partial x_i}$. But it is an isomorphism if and only if the characteristic of $k$ is zero.

*Proof of Theorem 1.12.* $R + Rx + \ldots + Rx^n = R + xR + \ldots + x^n R$: this follows from $R + xR = R + Rx$. To see this, use induction to show that $x^n R \subseteq R + Rx + \ldots + Rx^n$ and $Rx^n \subseteq R + xR + \ldots x^n R$ for all $n \geq 1$.

Consequences:

(a) The set of all elements of $S$ of the form

$$r_0 + xr_1 + \ldots + x^n r_n, \qquad n \geq 0 \qquad (*)$$

forms a subring of $S$. Since it contains both $R$ and $x$ and $S = \langle R, x\rangle$, we see that $S$ is the ring of all such 'polynomials'. Note that elements of $S$ need *not* be uniquely expressible in the form $(*)$.

(b) The set of polynomials of degree $\leqslant n$, namely $R + Rx + \ldots + Rx^n$, is both a left and a right $R$-submodule of $S$.

(c) For each $r \in R$ and $n \geq 0$ there exists $r' \in R$ such that $r'x^n = x^n r + s$ where $\deg s < n$.

Now, let $I$ be a left ideal in $S$. We will show that $I$ is finitely generated. Let

$$I_n := \{r_n \in R : \quad \text{there exists} \quad s \in I \quad \text{such that} \quad s = r_0 + xr_1 + \ldots + x^n r_n\}.$$

Then $I_n$ is closed under addition. Let $r \in R$. By part (c) above, we can find $r' \in R$ such that $r'x^n - x^n r$ has degree $< n$. Since $I$ is a left ideal, $r's \in I$, and

$$r's \equiv r'x^n r_n \equiv x^n(rr_n)$$

modulo terms of degree $< n$. Hence $rr_n \in I_n$ so $I_n$ is a left ideal of $R$.

Next, if $s = \sum_{i=0}^{n} x^i r_i \in I$, then $xs = \sum_{i=1}^{n+1} x^i r_{i-1} \in I$ so $r_n \in I_{n+1}$. Hence $I_n \leqslant I_{n+1}$ for all $n \geq 0$. Since $R$ is left Noetherian, the increasing chain

$$I_0 \leqslant I_1 \leqslant \ldots \leqslant I_n \leqslant \ldots$$

must terminate. Say $I_m = I_{m+1} = \ldots$. For $i = 0, \ldots, m$ let $\{r_{ij}\}$ be finitely many elements of $R$ generating $I_i$ as a left ideal of $R$. Choose $s_{ij} = x^i r_{ij} +$ lower degree terms $\in I$.

Claim: $X = \{s_{ij} : 0 \leqslant i \leqslant m, \text{ all } j\}$ generates $I$ as a left ideal.

Let $s = r_0 + xr_1 + \ldots + x^n r_n \in I$, so that $r_n \in I_n$; we'll show that $s \in RX$. Proceed by induction on $n$, the case $n = 0$ being trivial.

If $n \geq m$ then $r_n \in I_m$ so $r_n = \sum a_j r_{mj}$ for some $a_j \in R$. Choose $a'_j \in R$ such that $a'_j x^n = x^n a_j +$ lower degree terms. Then $s - \sum a'_j x^{n-m} s_{mj} \in I$ and modulo terms of degree $< n$,

$$s - \sum a'_j x^{n-m} s_{mj} \equiv x^n r_n - \sum a'_j x^n r_{mj} \equiv x^n r_n - \sum x^n a_j r_{mj} = 0.$$

So $s - \sum a'_j x^{n-m} s_{mj}$ has smaller degree than $s$ and we can apply induction.

If $n \leqslant m$ then $r_n = \sum a_j r_{nj}$ for some $a_j \in R$, so for suitable $a'_j \in R$, $s - \sum a'_j s_{nj} \in I$ also has smaller degree than $s$. By induction, these smaller degree elements of $I$ lie $RX$, as required. □

**Definition 1.20.** Let $R$ be a ring. A ($\mathbb{Z}-$)*filtration* on a $R$ is a set of additive subgroups $(R_i)_{i\in\mathbb{Z}}$ such that

- $R_i \subseteq R_{i+1}$ for all $i \in \mathbb{Z}$,
- $R_i.R_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{Z}$,
- $1 \in R_0$, and
- $\cup_{i\in\mathbb{Z}} R_i = R$.

If $R$ has a filtration, we say that $R$ is a *filtered ring*. The filtration on $R$ is *positive* if $R_i = 0$ for all $i < 0$.

Note that the axioms imply that $R_0$ is a subring of $R$ and that each $R_i$ is a left and right $R_0-$module. Note also that $\cap_{i\in\mathbb{Z}} R_i$ is always an ideal in $R$.

**Example 1.21.** Suppose $R$ is a finitely generated $k-$algebra with generating set $\{x_1, \ldots, x_n\}$. Define $R_0 = k$ and let $R_i$ be the $k-$subspace of $R$ spanned by words in the $x_j$'s of length at most $i$ for $i > 0$. Also define $R_i = 0$ whenever $i < 0$.

**Definition 1.22.** A ($\mathbb{Z}-$)*graded* ring is a ring $S$ which can be written as

$$S = \bigoplus_{i\in\mathbb{Z}} S_i$$

for some additive subgroups $S_i \subseteq S$, satisfying $S_i.S_j \subseteq S_{i+j}$ for all $i, j \in \mathbb{Z}$ and $1 \in S_0$. $S_i$ is called the $i-th$ *homogeneous component* of $S$, and an element $s \in S$ is *homogeneous* iff it lies in some $S_i$.

**Definition 1.23.** Let $R$ be a filtered ring with filtration $(R_i)_{i\in\mathbb{Z}}$. Define

$$\operatorname{gr} R = \bigoplus_{i\in\mathbb{Z}} R_i/R_{i-1}.$$

Equip $\operatorname{gr} R$ with multiplication, which is given on homogeneous components by

$$\begin{array}{ccccc} R_i/R_{i-1} & \times & R_j/R_{j-1} & \longrightarrow & R_{i+j}/R_{i+j-1} \\ r+R_{i-1} & , & s+R_{j-1} & \mapsto & rs+R_{i+j-1} \end{array}$$

and on the whole of $\operatorname{gr} R$ by bilinear extension. Then $\operatorname{gr} R$ becomes a graded ring called the *associated graded ring* of $R$.

Note that the multiplication is well-defined because $R_iR_j \subseteq R_{i+j}$, $R_{i-1}R_j \subseteq R_{i+j-1}$ and $R_iR_{j-1} \subseteq R_{i+j-1}$. One should think of $\operatorname{gr} R$ as an approximation to the ring $R$ which is often easier to understand but nonetheless contains useful information about the ring $R$ itself.

**Proposition 1.24.** Let $\mathfrak{g}$ be a Lie algebra with basis $\{x_1,\dots,x_n\}$. Equip $U(\mathfrak{g})$ with the positive filtration as in Example 1.21. Then there is a surjective homomorphism of $k-$algebras

$$\varphi : k[X_1,\dots,X_n] \twoheadrightarrow \operatorname{gr} U(\mathfrak{g})$$

given by $\varphi(X_i) = x_i + R_0$, $i = 1,\dots,n$.

*Proof.* Let $R = U(\mathfrak{g})$ and note that $x_i \in R_1$ for all $i$. Now because $x_ix_j - x_jx_i = [x_i,x_j] \in R_1$ for all $i,j$ we have

$$(x_i+R_0)(x_j+R_0) = x_ix_j + R_1 = x_jx_i + R_1 = (x_j+R_0)(x_i+R_0),$$

meaning that $\varphi(X_i)$ and $\varphi(X_j)$ commute. Hence the $k-$algebra map $\varphi$ exists. To show that $\varphi$ is surjective, it's sufficient to show that $u + R_{t-1}$ lies in $\operatorname{im}\varphi$ for any $u \in R_t\backslash R_{t-1}$. Now

$$x_{i_1}x_{i_2}\cdots x_{i_t} + R_{t-1} = \varphi(X_{i_1})\varphi(X_{i_2})\cdots\varphi(X_{i_t}) \in \operatorname{im}\varphi$$

and $u$ as a $k-$linear combination of words of length at most $t$ in the generators $\{x_1,\dots,x_n\}$. □

What about the Weyl algebra $A_n(k)$? Consider the standard monomials

$$x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}, y^\alpha = y_1^{\alpha_1}\cdots y_n^{\alpha_n} \in A_n(k) \quad \text{for all} \quad \alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n.$$

It follows from Exercise 1.3 that

$$\{x^\alpha y^\beta \in A_n(k) : \alpha,\beta \in \mathbb{N}^n\}$$

is a basis for $A_n(k)$ as a $k$-vector space. Write $|\alpha| = \sum_{i=1}^n \alpha_i$ for all $\alpha \in \mathbb{N}^n$.

**Proposition 1.25.** Let $R := A_n(k)$, set $R_0 := k[x_1,\dots,x_n]$, and define

$$R_i := \sum_{|\beta|\leqslant i} R_0 y^\beta \quad \text{for all} \quad i \in \mathbb{N}.$$

(a) $(R_i)$ is a filtration on $R$.
(b) $\operatorname{gr} R \cong k[X_1, \dots, X_n, Y_1, \dots, Y_n]$ with respect to this filtration.

*Proof.* (a) By the defining relations in the Weyl algebra we have $y_i R_0 = R_0 y_i + R_0 \subseteq R_1$ for each $i$. It follows that $y^\beta R_0 \subseteq R_{|\beta|}$ for all $\beta \in \mathbb{N}^n$. Hence $R_0 y^\beta R_0 y^\gamma \subseteq R_{|\beta|+|\gamma|}$, so that $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{N}$.

(b) There is a natural map $\varphi : R_0[Y_1, \dots, Y_n] \to \operatorname{gr} R$ of graded rings which sends $Y_i$ to $\sigma(y_i) = y_i + R_0$. Because every element in $R$ can be written as a finite sum $\sum_{\beta \in \mathbb{N}^n} r_\beta y^\beta$ for some $r_\beta \in R_0$, $\varphi$ is surjective. Because $\varphi$ respects the graded structure, to show that $\varphi$ is injective it is enough to show that $\ker \varphi$ contains no non-zero homogeneous elements. So let $\sum_{|\beta|=m} r_\beta Y^\beta \in \ker \varphi$; then $\sum_{|\beta|=m} r_\beta y^\beta \in R_{m-1}$, so we can find $r_\beta \in R$ whenever $|\beta| < m$ such that

$$\sum_{|\beta|=m} r_\beta y^\beta = \sum_{|\beta|<m} r_\beta y^\beta.$$

Because $\{x^\alpha y^\beta \in A_n(k) : \alpha, \beta \in \mathbb{N}^n\}$ is a basis for $R$, $r_\beta = 0$ for all $\beta$. □

**Definition 1.26.** The filtration on $A_n(k)$ constructed in Proposition 1.25 is called the *filtration by order of differential operator*.

**Theorem 1.27.** Suppose $R$ is a positively filtered ring such that $\operatorname{gr} R$ is left Noetherian. Then $R$ is left Noetherian.

*Proof.* Let $I$ be a left ideal in $R$, and consider the left ideal

$$\operatorname{gr} I := \bigoplus_{n \geq 0} \frac{(I \cap R_n) + R_{n-1}}{R_{n-1}}.$$

in $\operatorname{gr} R$. For each $n \in \mathbb{N}$, consider the projection operator $\pi_n : \operatorname{gr} R \to \operatorname{gr} R$ which sends $\sum x_i \in \operatorname{gr} R$ to $x_n \in \operatorname{gr} R$. Note that these operators preserve $\operatorname{gr} I$. This means that $\operatorname{gr} I$ contains the homogeneous components of each of its elements. Now because $\operatorname{gr} R$ is Noetherian, $\operatorname{gr} I$ has a finite generating set $\{X_1, \dots, X_m\}$, which we may without loss of generality assume to consist of homogeneous elements.

Choose some $x_i \in I \cap R_{n_i} \backslash R_{n_i - 1}$ such that $x_i + R_{n_i-1}$ equals $X_i$. To finish the proof, we prove that

$$I = \sum_{i=1}^m R x_i.$$

The inclusion $\supseteq$ is clear. For $\subseteq$, it is enough to prove that $I \cap R_n \subseteq \sum_{i=1}^m R x_i$ for all $n \geq -1$. Induct on $n$: $n = -1$ is clear because $R_{-1} = \{0\}$. If $x \in I \cap R_n$, then

$$x + R_{n-1} = \sum_{i=1}^m Y_i X_i$$

for some $Y_i \in \operatorname{gr} R$. We can again assume that each $Y_i$ is homogeneous of degree $n - n_i$, so choose $r_i \in R$ such that $Y_i = r_i + R_{n-n_i-1}$. Then $x \equiv \sum_{i=1}^m r_i x_i \pmod{R_{n-1}}$, so $x - \sum_{i=1}^m r_i x_i \in I \cap R_{n-1} \subseteq \sum_{i=1}^m R x_i$. So $x \in \sum_{i=1}^m R x_i$. □

**Corollary 1.28.**
(a) $U(\mathfrak{g})$ is Noetherian whenever $\dim_k \mathfrak{g} < \infty$.
(b) $A_n(k)$ is Noetherian.

*Proof.* (a) By Proposition 1.24, $\operatorname{gr} U(\mathfrak{g})$ is a quotient of a polynomial algebra $k[x_1, \ldots, x_n]$ for some $n$, which is Noetherian by Theorem 1.12. Hence $U(\mathfrak{g})$ is Noetherian by Theorem 1.27.

(b) Similar, using Proposition 1.25 instead. □

**Question 1.29.** Let $\mathfrak{g}$ be a Lie algebra over a field $k$ such that $U(\mathfrak{g})$ is Noetherian. Must $\dim_k \mathfrak{g} < \infty$?

## 2. Simple modules and Artinian rings

Throughout this chapter, $R$ denotes an arbitrary ring, unless stated otherwise.

**Definition 2.1.** An $R$-module $M$ is *simple* or *irreducible* if $M \neq 0$ and the only submodules of $M$ are 0 and $M$.

Suppose $M$ is simple. Choose $0 \neq x \in M$; then $M = Rx$ so $M \cong R/I$ where $I = \operatorname{ann}(x)$ is the point annihilator of $x$. Note that $\operatorname{ann}(x)$ need *not* be equal to $\operatorname{ann}(y)$ if $x, y$ are distinct nonzero elements of $M$, unless $R$ is commutative.

Note that $M = Rx$ is simple if and only if $\operatorname{ann}(x)$ is a maximal left ideal of $R$.

**Definition 2.2.** A *poset* is a set equipped with a binary relation $\leqslant$ which is reflexive, transitive and antisymmetric. A *chain* in a poset $\mathcal{S}$ is totally ordered subset $\mathcal{C}$ of $\mathcal{S}$: if $s, t \in \mathcal{C}$ then either $s \leqslant t$ or $t \leqslant s$. An *upper bound* for a subset $\mathcal{C}$ of $\mathcal{S}$ is an element $u \in \mathcal{S}$ such that $x \leqslant u$ for all $x \in \mathcal{C}$. We say that $x \in \mathcal{S}$ is a *maximal element* if $x \leqslant y$ with $y \in \mathcal{S}$ forces $x = y$.

**Theorem 2.3** (Zorn's Lemma)**.** Let $\mathcal{S}$ be a nonempty poset. Suppose every chain in $\mathcal{S}$ has an upper bound. Then $\mathcal{S}$ has a maximal element.

This is equivalent to the Axiom of Choice, which we will always assume.

**Lemma 2.4.** Suppose $L$ is a proper left ideal of $R$. Then $L$ is contained in a maximal ideal $I$ of $R$. Equivalently, every nonzero cyclic module has a simple quotient.

*Proof.* Since $L$ is proper, $1 \notin L$. Let $\mathcal{S} = \{K \triangleleft_l R : L \subseteq K, 1 \notin K\}$. Since $L \in \mathcal{S}$, this set is nonempty. $\mathcal{S}$ is partially ordered by inclusion. If $\mathcal{C}$ is a chain in $\mathcal{S}$, then $\cup\mathcal{C}$ also contains $L$ and doesn't contain 1, i.e. $\cup\mathcal{C} \in \mathcal{S}$. Hence every chain in $\mathcal{S}$ has an upper bound in $\mathcal{S}$. By Zorn's Lemma, $\mathcal{S}$ has a maximal element $I$. It's clear that $I$ is now a maximal left ideal of $R$ containing $L$. □

By an *ideal* of $R$ we mean a *two-sided* ideal.

**Definition 2.5.** Let $I$ be a two-sided ideal of $R$. Then $I$ is *left primitive* if $I$ is the annihilator of a simple left $R$-module $M$:

$$I = \mathrm{Ann}_R(M) = \{x \in R : xM = 0\} = \bigcap_{x \in M} \mathrm{ann}(x).$$

The ring $R$ itself is called *left primitive* if its zero ideal is left primitive, or equivalently, if $R$ has at least one faithful simple left module.

There are examples due to George Bergman of rings which are left primitive, but not right primitive! Note that the annihilator $I$ of any module $M$ is always an ideal of $R$.

**Lemma 2.6.** Let $M = Rx$ be a cyclic left $R$-module. Then $I = \mathrm{Ann}_R(M)$ is the largest two-sided ideal contained in $L = \mathrm{ann}(x)$.

*Proof.* Note that this largest two-sided ideal $K$ exists, since the sum of all two-sided ideals contained in $L$ is itself a two-sided ideal contained in $L$. Certainly $I \subseteq L$, so $I \subseteq K$. Now $KM = KRx \subseteq Kx \subseteq Lx = 0$ since $K$ is two-sided, so $K \subseteq I$. □

**Corollary 2.7.** Every maximal ideal of $R$ is left and right primitive. Moreover, if $R$ is commutative, every primitive ideal is maximal.

**Definition 2.8.** The *Jacobson radical* $J(R)$ of $R$ is defined to be the intersection of all left primitive ideals of $R$.

Note that $J(R)$ is the set of elements of $R$ which annihilate every simple left $R$-module.

**Lemma 2.9.** $J(R)$ is equal to the intersection $K$ of all maximal left ideals of $R$.

*Proof.* Let $I$ be a maximal left ideal. Then $P = \mathrm{Ann}_R(R/I)$ is primitive, so $J(R) \subseteq P \subseteq I$ by Definition 2.5. Hence $J(R) \subseteq K$.

Now let $P = \mathrm{Ann}_R(M)$ be a primitive ideal, where $M$ is a simple $R$-module. Note that $P = \cap_{0 \neq x \in M} \mathrm{ann}(x)$ is an intersection of maximal left ideals, so $K \subseteq P$. It follows that $K \subseteq J(R)$ as required. □

**Lemma 2.10** (Nakayama)**.** Let $M$ be a finitely generated nonzero left $R$-module and let $J = J(R)$. Then $JM$ is strictly contained in $M$.

*Proof.* Since $M$ is finitely generated, by choosing a minimal finite generating set for $M$ we see that $M$ has a non-zero cyclic quotient module $M/L$, which in turn has a simple quotient $M/K$ by Lemma 2.4. Then $J(M/K) = 0$ so $JM \subseteq K$ which is strictly contained in $M$. □

**Corollary 2.11.** Let $M$ be a finitely generated left $R-$module and let $J = J(R)$. If $N$ is a submodule of $M$ such that $M = N + JM$ then $M = N$.

*Proof.* Apply the Lemma to $M/N$. □

Recall that an element $x \in R$ is a *unit* if there exists $y \in R$ such that $xy = yx = 1$.

**Proposition 2.12.**

$$J(R) = \{x \in R : 1 - axb \quad \text{is a unit for all} \quad a, b \in R\} =: K.$$

*Proof.* Let $x \in K$, let $I$ be a maximal left ideal of $R$ and suppose that $x \notin I$. Since $I$ is maximal, $I + Rx = R$, so $1 - ax \in I$ for some $a \in R$. Since $x \in K$, $1 - ax$ is a unit, a contradicting the fact that $I$ is proper. Hence $x \in I$ so $K \subseteq I$ for all maximal left ideals $I$ of $R$. By Lemma 2.9, $K \subseteq J(R)$.

Now let $x \in J(R)$. Since $J(R)$ is a two-sided ideal, to show that $x \in K$ it's sufficient to show $1 - x$ is a unit. Now, if $R(1-x)$ is a proper left ideal, we can find a maximal left ideal $L$ containing it by Lemma 2.4. By Lemma 2.9, $x \in J(R) \subseteq L$ and $1 - x \in L$ so $1 \in L$, a contradiction. Hence there exists $y \in R$ such that

$$y(1 - x) = 1.$$

Now, $1 - y = -yx \in J(R)$, so by the above argument applied to $1 - y$, we can find $z \in R$ such that

$$z(1 - (1 - y)) = zy = 1.$$

Hence $zy(1 - x) = 1 - x = z$ so $zy = 1$ and $yz = 1$. Hence $z = 1 - x \in R^{\times}$. □

This result shows that $J(R)$ is the largest ideal $A$ of $R$ such that $1 - A$ consists entirely of units of $R$.

**Corollary 2.13.** The Jacobson radical is left-right symmetric. It follows that the intersection of all maximal left ideals of $R$ is equal to the intersection of all maximal right ideals.

We will now work towards understanding the structure of left primitive rings. Let $V$ be a left $R$-module and let $D = \operatorname{End}_R(V)$. Let us write $R$-module endomorphisms of $V$ on the *right*, and define composition of such endomorphisms by the rule

$$v(\alpha \cdot \beta) = (v\alpha)\beta \quad \text{for all} \quad v \in V, \alpha, \beta \in D.$$

Thus $\alpha \cdot \beta$ is the product of $\alpha$ and $\beta$ inside $D$ in this new notation. Naturally, $V$ is then a right $D$-module, and in fact, $V$ becomes an *$R$-$D$-bimodule*: this means that $V$ is simultaneously a left $R$-module and a right $D$-module via the rule $v \cdot \alpha = v\alpha$, and the two structures are compatible in the following sense:

$$r \cdot (v \cdot \alpha) = (r \cdot v) \cdot \alpha \quad \text{for all} \quad r \in R, \alpha \in D.$$

Of course, this just says that every element of $D$ is an endomorphism of the left $R$-module $V$.

**Theorem 2.14** (Schur's Lemma)**.** Let $V$ be a simple left $R$-module. Then $D := \operatorname{End}_R(V)$ is a division ring.

*Proof.* Let $\varphi : V \to V$ be a nonzero $R$-module homomorphism. Then $\ker(\varphi) < V$ and $\mathrm{im}(\varphi) > 0$. The simplicity of $V$ forces $\ker(\varphi) = 0$ and $\mathrm{im}(\varphi) = V$, so $\varphi$ is an isomorphism. Thus every nonzero element of $D$ is a unit. □

So whenever $V$ is a simple left $R$-module, $V$ becomes a right vector space over the division ring $D = \mathrm{End}_R(V)$. The following technical sounding Lemma will be key to the proof of Jacobson's Density Theorem.

**Lemma 2.15.** Let $V$ be a simple left $R$-module, let $D = \mathrm{End}_R(V)$, let $X$ be a finite $D$-linearly independent subset of $V$, and let $I := \mathrm{ann}(X)$. Suppose that $I \cdot y = 0$ for some $y \in V$. Then $y \in X \cdot D$, the $D$-linear span of $X$.

*Proof.* We proceed by induction on $n = |X|$. When $n = 0$, we have $\mathrm{ann}(\emptyset) = R$ and $\emptyset \cdot D = \{0\}$. So since $R \cdot y = 0$ by assumption, we have $y = 0 \in \emptyset \cdot D$.

Assume now that $n \geq 1$ and let $J = \mathrm{ann}(X \backslash \{x\})$ for some $x \in X$ so that $I = J \cap \mathrm{ann}(x)$. If $J \subseteq \mathrm{ann}(x)$ then $J = I$, so $J \cdot y = 0$ and we can apply the induction hypothesis. So we can assume that $J$ is not contained in $\mathrm{ann}(x)$. But then the $R$-submodule $J \cdot x$ of $V$ is non-zero, so $J \cdot x = V$ by the simplicity of $V$.

Define $d : V \to V$ by the rule $(r \cdot x)d = r \cdot y$, whenever $r \in J$. This is well-defined, because if $r \cdot x = 0$ for some $r \in J$ then $r \in \mathrm{ann}(x) \cap J = I$, so $r \cdot y = 0$ since $I \cdot y = 0$ by assumption. This function is left $R$-linear because $(s \cdot (r \cdot x))d = (sr \cdot x)d = sr \cdot y = s \cdot (r \cdot y) = s \cdot ((r \cdot x)d)$ for all $s \in R$. Thus we have found an element $d \in D$ such that $J \cdot (y - x \cdot d) = 0$. Hence $y - x \cdot d \in (X \backslash \{x\}) \cdot D$ by induction and therefore $y \in X \cdot D$. □

**Definition 2.16.** Let $M$ be a left $R$-module. We say that $M$ is *Artinian* if every descending chain of submodules terminates. The ring $R$ is *left Artinian* if it is Artinian as a left $R$-module.

**Corollary 2.17.** Let $R$ be a left Artinian ring, let $V$ be a simple left $R$-module and let $D = \mathrm{End}_R(V)$. Then $V$ is finite dimensional as a right $D$-vector space.

*Proof.* Since $R$ is left Artinian, by Exercise 2.4 the set $\{\mathrm{ann}(X) : X \subset V, |X| < \infty\}$ has a minimal element $I = \mathrm{ann}(X)$, say. Let $y \in V$; if $I \cdot y \neq 0$ then $\mathrm{ann}(X \cup \{y\}) < \mathrm{ann}(X)$, contradicting the minimality of $\mathrm{ann}(X)$. Hence $I \cdot y = 0$, so $y \in X \cdot D$ for any $y \in V$ by Lemma 2.15. Hence $V = X \cdot D$. □

**Theorem 2.18** (Jacobson's Density)**.** Let $V$ be a simple left $R$-module, and let $X \subset V$ be a finite $D$-linearly independent subset of $V$ where $D := \mathrm{End}_R(V)$. Then for every $\alpha \in \mathrm{End}(V_D)$ there exists $r \in R$ such that $\alpha(x) = r \cdot x$ for all $x \in X$.

*Proof.* Write $X = \{x_1, \ldots, x_n\}$, fix $i \in \{1, \ldots, n\}$ and write $X_i := X \backslash \{x_i\}$. Since $x_i \notin X_i \cdot D$ we see that $\mathrm{ann}(X_i) \cdot x_i \neq 0$ by Lemma 2.15. So there is some $r_i \in \mathrm{ann}(X_i)$ such that $r_i \cdot x_i \neq 0$. Since $V$ is simple, $R \cdot (r_i \cdot x_i) = V$, so we can find some $s_i \in R$ such that $s_i \cdot (r_i \cdot x_i) = \alpha(x_i)$. Now

$$\sum_{j=1}^{n} s_j r_j \cdot x_i = s_i \cdot r_i \cdot x_i = \alpha(x_i) \quad \text{for all} \quad i = 1, \ldots, n$$

because $r_j \in \operatorname{ann}(X_j) \subseteq \operatorname{ann}(x_i)$ whenever $j \neq i$. So we can take $r = \sum_{j=1}^{n} s_j r_j$. □

**Lemma 2.19.** Let $S$ be a ring, let $N$ be a right $S$-module and let $n \geq 1$ be an integer. Then the ring of right $S$-module endomorphisms of $(N_S)^n$ is isomorphic to the $n \times n$ matrix ring with coefficients in $T := \operatorname{End}(N_S)$:

$$\operatorname{End}((N_S)^n) \cong M_n(T).$$

*Proof.* This is best seen by writing elements of $N^n$ as column vectors $x = (x_j)_{j=1}^n$ and thinking of $S$-module endomorphisms acting by matrix multiplication on the *left* of these column vectors.

Formally, let $\sigma_j : N \hookrightarrow N^n$ and $\pi_j : N^n \twoheadrightarrow N$ for $j = 1, \ldots, n$ be given by

$$\sigma_j(x)_i = x\delta_{ij} \quad \text{and} \quad \pi_j(x) = x_j.$$

These are right $S$-module homomorphisms. We define $\alpha : \operatorname{End}((N_S)^n) \to M_n(T)$ by setting the $(i, j)$ element of $\alpha(f)$ to be the composition

$$N \xrightarrow{\sigma_j} N^n \xrightarrow{f} N^n \xrightarrow{\pi_i} N;$$

thus $\alpha(f)_{ij} = \pi_i f \sigma_j$. We can also define $\beta : M_n(T) \to \operatorname{End}((N_S)^n)$ by

$$\beta(A) = \sum_{i,j=1}^{n} \sigma_j A_{ji} \pi_i.$$

It is a pleasant exercise to show that $\alpha$ and $\beta$ are mutually inverse ring homomorphisms. □

**Theorem 2.20** (Artin-Wedderburn)**.** Let $R$ be a left primitive, left Artinian ring. Then $R \cong M_n(D)$ for some division ring $D$ and integer $n \geq 1$.

*Proof.* Let $V$ be a faithful simple left $R$-module, and let $D = \operatorname{End}_R(V)$. Then $D$ is a division ring by Theorem 2.14. Now $V_D \cong (D_D)^n$ for some positive integer $n$ by Corollary 2.17 and $\operatorname{End}(D_D) \cong D$ by Exercise 2.2(a). So

$$\operatorname{End}((D_D)^n) \cong M_n(\operatorname{End}(D_D)) \cong M_n(D)$$

by Lemma 2.19. Now we have a natural ring homomorphism

$$\psi : R \to \operatorname{End}(V_D)$$

given by $\psi(r)(v) = r \cdot v$. It is injective because $V$ is faithful, and it is surjective by Theorem 2.18. We conclude that $R \cong M_n(D)$. □

**Theorem 2.21** (Chinese Remainder)**.** Let $R$ be a ring, and let $P_1, \ldots, P_n$ be two-sided ideals in $R$ such that $P_i + P_j = R$ whenever $i \neq j$. Then

$$R/(P_1 \cap P_2 \cap \cdots \cap P_n) \cong (R/P_1) \oplus (R/P_2) \oplus \cdots \oplus (R/P_n).$$

*Proof.* There is a natural ring homomorphism $\varphi : R \to \bigoplus_{i=1}^n R/P_i$ given by $\varphi(r) = (r + P_i)_{i=1}^n$. Its kernel is $P_1 \cap \cdots \cap P_n$, so by the First Isomorphism Theorem for rings it will be sufficient to show that $\varphi$ is surjective. We prove this by induction on $n$, the case $n = 1$ being clear.

Since $P_i + P_n = R$ for all $i < n$, we can find $a_i \in P_i$ and $b_i \in P_n$ such that $a_i + b_i = 1$ for all $i = 1, \ldots, n-1$. Let $a := a_1 \cdots a_{n-1} \in P_1 \cap \cdots \cap P_{n-1}$ and let $b := 1 - a$. Then

$$b = 1 - a = (a_1 + b_1) \cdots (a_{n-1} + b_{n-1}) - a_1 \cdots a_{n-1} \in P_n.$$

Now, given $(r_i + P_i) \in \oplus_{i=1}^n R/P_i$, we can find some $s \in R$ such that $s - r_i \in P_i$ for all $i < n$ by induction. Let $r := sb + r_n a$; then $r \equiv r_n \mod P_n$ and $r \equiv sb \equiv r_i \mod P_i$ for each $i < n$. So $\varphi(r) = (r_i + P_i)_{i=1}^n$ and $\varphi$ is surjective. □

**Corollary 2.22.** Let $R$ be a left Artinian ring with $J(R) = 0$. Then there exist division rings $D_1, \ldots, D_n$ and integers $r_1, \ldots, r_n \geq 1$ such that

$$R \cong M_{r_1}(D_1) \oplus \cdots \oplus M_{r_n}(D_n).$$

*Proof.* Let $\mathcal{S}$ be the set of finite intersections of left primitive ideals of $R$; it is non-empty by Lemma 2.4. Since $R$ is left Artinian, this set has a minimal element $I := P_1 \cap \cdots \cap P_n$ say. If $Q$ is another left primitive ideal of $R$ then $I \cap Q = I$ by the minimality of $I$, so that $I \subseteq Q$. Hence $I \subseteq J(R) = \{0\}$ by assumption. Now $R/P_i \cong M_{r_i}(D_i)$ for some division ring $D_i$, and this ring is simple by Exercise 3.3(c). So each $P_i$ is a maximal two-sided ideal, and therefore $P_i + P_j = R$ whenever $i \neq j$. Now apply Theorem 2.21. □

**Proposition 2.23.** The Jacobson radical $J$ of a left Artinian ring $R$ is nilpotent.

*Proof.* The descending chain $J \supseteq J^2 \supseteq J^3 \supseteq \ldots$ must terminate since $R$ is left Artinian. Hence $J^n = J^{n+1} = \ldots$ for some $n \geq 0$. Let $X = \mathrm{rann}(J^n) = \{x \in R : J^n x = 0\}$, this is a two-sided ideal of $R$. Suppose for a contradiction that $X \neq R$. Then $R/X$ has a minimal nonzero left submodule $Y/X$, being left Artinian. This module is simple. Now $J \cdot (Y/X) = 0$ so $JY \subseteq X$. It follows that $J^n Y = J^{n+1} Y \subseteq J^n X = 0$, so $Y \subseteq \mathrm{rann}(J^n) = X$, contradicting $Y/X \neq 0$. Hence $X = R$ so $J^n = RJ^n = XJ^n = 0$. □

**Theorem 2.24** (Hopkins)**.** Let $R$ be a left Artinian ring. Then $R$ is also left Noetherian.

*Proof.* Let $J = J(R)$. For any $i \in \mathbb{N}$, $J^i/J^{i+1}$ is a left Artinian $R/J$-module, so it is also left Noetherian by Theorem 2.20 and Exercise 3.3(c). Since $J$ is nilpotent by

Proposition 2.23, $R$ is a finite extension of left Noetherian modules, and is therefore itself left Noetherian by Exercise 1.4(a). □

## 3. Noncommutative localisation

Let $A$ be a ring and let $S$ be a subset of $A$. We want to "invert $S$", meaning that we want to find a ring homomorphism

$$\varphi : A \to S^{-1}A \quad \text{such that} \quad \varphi(S) \subseteq (S^{-1}A)^{\times},$$

and we want $S^{-1}A$ to be "minimal" in some sense.

**Construction 3.1.** Form the free algebra on a set which is in bijection with $S$

$$A\langle i_s : s \in S\rangle$$

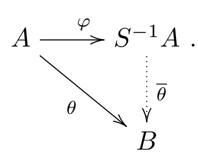
and impose the relation that $i_s$ is a two-sided inverse of $s \in S$ for each $s \in S$:

$$S^{-1}A := \frac{A\langle i_s : s \in S\rangle}{\langle s i_s - 1, i_s s - 1 : s \in S\rangle.}$$

Then define $\varphi : A \to S^{-1}A$ by letting $\varphi(s)$ be the image of $s$ in $S^{-1}A$. By definition, $\varphi(S)$ consists of units in $S^{-1}A$.

This ring $S^{-1}A$ is minimal in the following precise sense.

**Proposition 3.2** (Universal property of $S^{-1}A$)**.** Suppose that $\theta : A \to B$ is a ring homomorphism such that $\theta(S) \subseteq B^{\times}$. Then there is a unique ring homomorphism $\overline{\theta} : S^{-1}A \to B$ such that $\theta = \overline{\theta} \circ \varphi$.




**Problems.**

(1) $S^{-1}A$ could be the zero ring!
(2) *Non-examinable:* $S^{-1}A$ will not be a flat left $A$-module, in general.

**Definition 3.3.** The *left $S$-torsion* subset of $A$ is

$$t_S(A) := \{a \in A : sa = 0 \quad \text{for some} \quad s \in S\}.$$

Note that $\varphi(t_S(A)) = 0$, so that $t_S(A) \subseteq \ker\varphi$. Note also that if $\langle S\rangle$ is the sub-monoid of $A$ generated by $S$, then $\langle S\rangle^{-1}A = S^{-1}A$. For this reason, we will focus on *multiplicatively closed* subsets of $A$: by definition, these are the subsets $S$ of $A$ such that $1 \in S$ and $s, t \in S \Rightarrow st \in S$.

**Definition 3.4.** Let $S$ be a multiplicatively closed subset of $A$.

(a) $S$ is *left localisable* if
  (i) $S^{-1}A = \{\varphi(s)^{-1}\varphi(a) \mid a \in A, s \in S\}$ and

(ii) $\ker\varphi = t_S(A)$.

(b) $S$ is a *left Ore set* if

$$Sa \cap As \neq \emptyset \quad \text{for all} \quad a \in A, s \in S.$$

(c) $S$ is *left reversible* if whenever $as = 0$ for some $s \in S$ and $a \in A$, there is some $s' \in S$ such that $s'a = 0$. In other words, right $S$-torsion elements in $A$ are also left $S$-torsion.

(d) $s \in A$ is a *regular element* if $sa = 0$ or $as = 0$ imply that $a = 0$.

(e) $A$ is a *domain* if every non-zero element is regular.

Obviously if $A$ is a commutative ring, or more generally, if $S$ consists of central elements in $A$ then $S$ is a left Ore set. We will shortly see examples of multiplicatively closed sets which do not have this property.

**Proposition 3.5.** Every left localisable subset is a left reversible, left Ore set.

*Proof.* Let $a \in A$ and $s \in S$. Then by definition, the element $\varphi(a)\varphi(s)^{-1} \in S^{-1}A$ can be written as a right fraction

$$\varphi(a)\varphi(s)^{-1} = \varphi(u)^{-1}\varphi(c)$$

for some $c \in A$ and $u \in S$. Hence $ua - cs \in \ker\varphi$ so we can find $v \in S$ such that $v(ua - cs) = 0$. Hence $(vu)a = (vc)s$ so take $t = vu \in S$ and $b = vc \in A$, then

$$ta = bs$$

and hence $S$ is a left Ore set. Next, if $as = 0$ for some $s \in S$ and $a \in A$, then

$$\varphi(a) = \varphi(as)\varphi(s)^{-1} = 0$$

so $s'a = 0$ for some $s' \in A$. Hence $S$ is left reversible. □

**Examples 3.6.**

(a) Say that an element $s \in A$ is *normal* if $sA = As$. Then if every element $s \in S$ is regular and normal, then $S$ is a left Ore set. This happens, for example, whenever every element of $S$ is central in $A$.

(b) Let $A = k\langle x, y\rangle$ be a free algebra in two variables over a field. This is a domain, so $S := A\backslash\{0\}$ is multiplicatively closed. But

$$Ax \cap Ay = \{0\}$$

so $Sx \cap Ay = \emptyset$. Hence $S$ is not left localisable by Proposition 3.5.

**Theorem 3.7** (Ore, 1930)**.** Let $S$ be a left Ore set in $A$ consisting of regular elements. Then $\varphi : A \to S^{-1}A$ is injective.

*Proof (non-examinable).* Define a relation on $S \times A$ as follows:

$$(s, a) \sim (t, b) \Leftrightarrow \exists c, d \in A \quad \text{such that} \quad cs = dt \in S \quad \text{and} \quad ca = db.$$

This is an equivalence relation. Let $Q$ be the set of equivalence classes on $S \times A$ under this equivalence relation:

$$Q := (S \times A)/\sim .$$

Then $Q$ is a ring, and the map $\psi : A \to Q$ defined by $\psi(a) = [(1, a)]$ is an injective ring homomorphism which inverts $S$. So by the universal property of $S^{-1}A$, there is a map $\theta : S^{-1}A \to Q$ such that $\psi = \theta \circ \varphi$. Hence $\varphi$ is injective because $\psi$ is injective. More details can be found in the Appendix at the end of these notes. □

**Theorem 3.8** (Gabriel)**.** Let $S$ be a multiplicatively subset of $A$. Then $S$ is left localisable if and only if it is a left reversible, left Ore set.

*Proof.* We need to prove the converse of Proposition 3.5. So suppose that $S$ is a left reversible left Ore set, and consider the element

$$\varphi(s_1)^{-1}\varphi(a_1)\varphi(s_2)^{-1}\varphi(a_2)\cdots\varphi(s_n)^{-1}\varphi(a_n)$$

in $S^{-1}A$; notice that by the construction of $S^{-1}A$, since $S$ is multiplicatively closed, every element of $S^{-1}A$ is a finite sum of such elements. Using the left Ore condition, we can rewrite it in the form $\varphi(s)^{-1}\varphi(a)$ for some $a \in A$ and $s \in S$.

Next, given $a, b \in A$ and $s, t \in S$, choose $u \in A$ and $v \in S$ such that

$$ut = vs.$$

Since $s, v \in S$ and $S$ is multiplicatively closed, this element lies in $S$. So we can bring the sum of the two left fractions $\varphi(s)^{-1}\varphi(a)$ and $\varphi(t)^{-1}\varphi(b)$ to a common left denominator:

$$\varphi(s)^{-1}\varphi(a) + \varphi(t)^{-1}\varphi(b) = \varphi(vs)^{-1}\varphi(va + ub).$$

So every element of $S^{-1}A$ is of the form $\varphi(s)^{-1}\varphi(a)$ for some $a \in A$ and $s \in S$.

It remains to prove that $\ker \varphi = t_S(A)$. Now, the left $S$-torsion subset $t_S(A)$ of $A$ satisfies $t_S(A) \cdot A \subset t_S(A)$. It is also a left ideal in $A$ by Exercise 4.1. So it is a two-sided ideal. Next, let $s \in S$ and $a \in A$ and suppose that $as \in t_S(A)$. Then $tas = 0$ for some $t \in S$, but $S$ is left reversible so $s'ta = 0$ for some $s' \in S$. So $a \in t_S(A)$ because $s't \in S$. If on the other hand $sa \in t_S(A)$ then $tsa = 0$ for some $t \in S$ so $a \in t_S(A)$ because $ts \in S$.

Thus the image $\overline{S}$ of $S$ in the factor ring $\overline{A} := A/t_S(A)$ consists of non-zero divisors, and $\overline{S}$ is a left Ore set in $\overline{A}$ by Exercise 4.2. Now, the universal $\overline{S}$-inverting ring homomorphism $\overline{\varphi} : \overline{A} \to \overline{S}^{-1}\overline{A}$ is injective by Theorem 3.7. If $\pi : A \twoheadrightarrow \overline{A}$ is the natural surjection, then $\overline{\varphi}\pi : A \to \overline{S}^{-1}\overline{A}$ inverts $S$, so by the universal property of $S^{-1}A$ there is a ring homomorphism $\theta : S^{-1}A \to \overline{S}^{-1}\overline{A}$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\pi} & \overline{A} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\overline{\varphi}} \\ S^{-1}A & \xrightarrow[\theta]{} & \overline{S}^{-1}\overline{A}. \end{array}$$

Now if $a \in \ker \varphi$ then $\overline{\varphi}(\pi(a)) = \theta(\varphi(a)) = 0$, but $\overline{\varphi}$ is injective so $\pi(a) = 0$, and therefore $a \in t_S(A) = \ker \pi$. □

A similar procedure is involved in the construction of the derived category of an abelian category.

**Theorem 3.9.** [Goldie, 1957] Let $A$ be a left Noetherian domain. Then $S = A \backslash \{0\}$ is a left Ore set.

*Proof.* Let $x \in A$, $y \in S$. We want to show that $Ay \cap Sx \neq \emptyset$. Let $k \in \mathbb{N}$ and consider the left ideal

$$I_k := Ax + Axy + \cdots + Axy^k$$

of $A$. These form an ascending chain $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$ which has to terminate. Choose $k \in \mathbb{N}$ *minimal* such that $I_k = I_{k+1}$. Then $xy^{k+1} \in I_k$, so

$$xy^{k+1} = a_0 x + a_1 xy + \cdots + a_k xy^k$$

for some $a_0, \ldots, a_k \in A$. If $k = 0$ then $xy = a_0 x$; since $A$ is a domain, $xy \neq 0$ so $a_0 \neq 0$. Thus $a_0 \in S$ so that

$$xy = a_0 x \in Ay \cap Sx.$$

If $k \geq 1$ then $(xy^k - a_1 x - \cdots - a_k xy^{k-1})y = a_0 x$ and the minimality of $k$ forces $a_0 \neq 0$. So $a_0 \in S$ and $a_0 x \in Sx \cap Ay$. □

**Corollary 3.10.** Every Noetherian domain has a division ring of fractions.

*Proof.* Let $S = A \backslash \{0\}$. This is a left Ore set by Theorem 3.9, and it is consists of regular elements because $A$ is a domain. So $A$ embeds into $S^{-1}A$ by Theorem 3.7 and $S^{-1}A = \{s^{-1}a : s \in S, a \in A\}$. Now if $s^{-1}x \in S^{-1}A$ is a non-zero element then $x, s \in S$ and $x^{-1}s$ is the inverse of $s^{-1}x$. Hence every non-zero element of $S^{-1}A$ is a unit, so $S^{-1}A$ is a division ring. □

It turns out that in left Noetherian rings, we don't have to worry about the left-reversibility condition on left Ore sets.

**Proposition 3.11.** Let $A$ be a left Noetherian ring, and let $S \subset A$ be a left Ore set. Then $S$ is left reversible.

*Proof.* Suppose that $as = 0$ for some $s \in S$ and $a \in A$, and consider the ascending chain of left annihilators

$$\mathrm{lann}(s) \leqslant \mathrm{lann}(s^2) \leqslant \cdots .$$

Since $A$ is left Noetherian this chain stops, so that $\mathrm{lann}(s^{k+1}) = \mathrm{lann}(s^k)$ for some integer $k \geq 1$. Now because $S$ is a left Ore set, we can find $b \in A$ and $t \in S$ such that $ta = bs^k$. Then

$$bs^{k+1} = tas = 0$$

so $b \in \mathrm{lann}(s^{k+1}) = \mathrm{lann}(s^k)$. Hence $ta = bs^k = 0$ with $t \in S$. □

**Theorem 3.12.** [Goldie, 1958] Let $A$ be a ring, and let $S$ be the set of regular elements of $A$. The following are equivalent:

(a) (1) $S$ is a left Ore set in $A$,
  (2) $S^{-1}A$ is left Artinian,
  (3) the Jacobson radical of $S^{-1}A$ is zero.

(b) (1) $A$ has no non-trivial nilpotent two-sided ideals,
  (2) $A$ doesn't have an infinite direct sum of left ideals, and
  (3) every ascending chain of left annihilators stops.

*Proof.* Omitted. □

Rings satisfying the conditions (b2) and (b3) are called *left Goldie rings*. Clearly, every left Noetherian ring is a left Goldie ring. It follows from Corollary 2.22 that $S^{-1}A$ is the direct product of finitely many matrix rings over division rings; thus Theorem 3.12 is a generalisation of Theorem 3.9.

**Definition 3.13.** Let $S$ be a left localisable subset of $A$ and let $M$ be a left $A-$module.

(a) The *localisation of $M$ at $S$* is defined to be the set of equivalence classes

$$S^{-1}M = \{s\backslash m : m \in M, s \in S\}$$

in $S \times M$ under the equivalence relation $\sim$ given by

$$(s, m) \sim (t, n) \quad \text{if and only if} \quad ut'm = us'n \quad \text{for some} \quad u \in S,$$

where $t' \in A, s' \in S$ are such that $t's = s't \in S$.

(b) The *$S$-torsion submodule* of $M$ is defined to be

$$t_S(M) = \{m \in M : sm = 0 \quad \text{for some} \quad s \in S\}.$$

A long calculation shows that $S^{-1}M$ has the structure of an $S^{-1}A$-module. To do this, it is sufficient to check that $S^{-1}M$ is an $A$-module; then $S$ clearly acts invertibly on $S^{-1}M$ so by the universal property of $S^{-1}A$ the ring homomorphism $A \to \mathrm{End}_{\mathbb{Z}}(S^{-1}M)$ extends to $S^{-1}A$.

**Proposition 3.14.** Let $S$ be a left localisable subset of $A$ and let $N$ be a submodule of an $A$-module $M$. Then there is an $S^{-1}A$-linear isomorphism

$$S^{-1}(M/N) \cong (S^{-1}M)/(S^{-1}N).$$

*Proof.* There is a map $\alpha : S^{-1}N \to S^{-1}M$ which sends $s\backslash n \in S^{-1}N$ to $s\backslash n \in S^{-1}M$. It is left $S^{-1}A$-linear, so its image is an $S^{-1}A$-submodule. If $s\backslash n$ maps to zero, then there is $t \in S$ such that $tn = 0$. So $s\backslash n = ts\backslash tn = 0$. So $\alpha$ is injective. Now define $\beta : S^{-1}M \to S^{-1}(M/N)$ by $\beta(s\backslash m) = s\backslash(m + N)$. It is a well-defined, surjective, $S^{-1}A$-linear map. If $s\backslash m \in \ker\beta$ then $t(m + N) = 0$ for some $t \in S$ so that $tm \in N$. But then

$$s\backslash m = (ts)\backslash(tm) \in S^{-1}N$$

so that $\ker\beta = S^{-1}N$. □

**Remarks 3.15.** Here is an alternative way of constructing $S^{-1}M$:

$$S^{-1}M \cong S^{-1}A \otimes_A M.$$

So, it follows from Proposition 3.14 that $S^{-1}A$ is a flat right $A$-module, whenever $S$ is a left localisable subset of $A$.

## 4. Dimension theory for Noetherian modules

We will develop some dimension theory for finitely generated modules over Noetherian rings, with an emphasis on minimal primes.

**Definition 4.1.** Let $R$ be a ring.

(a) A proper ideal $P$ of $R$ is said to be *prime* if, whenever $I, J$ are ideals in $R$ such that $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$.
(b) The set of prime ideals in $R$ is denoted by $\operatorname{Spec}(R)$.
(c) Let $I$ be an ideal in $R$. A prime ideal $P$ of $R$ is a *minimal prime over* $I$ if $P \supseteq I$ and $I \subseteq Q \subseteq P$ with $Q$ prime forces $Q = P$.
(d) $P$ is a *minimal prime of* $R$ if it is a minimal prime over the zero ideal.
(e) $\min(I) := \{\text{minimal primes over } I\}$.

Be warned that if $R$ is not commutative and $P$ is a prime ideal, then the factor ring $R/P$ may well have zero-divisors. For example, the zero ideal in every simple ring is prime, and plenty of simple rings have zero-divisors: take, for example, any matrix algebra $M_n(k)$ over a field $k$ with $n \geq 2$.

**Proposition 4.2.** Let $R$ be a left (or right) Noetherian ring and let $I$ be a proper ideal. Then

(1) There exist primes $P_1, \dots, P_n$ containing $I$ such that $P_1 \cdots P_n \subseteq I$.
(2) The set of minimal primes over $I$ is finite and non-empty.

*Proof.* Suppose that (1) is false. Since $R$ is Noetherian, we can choose a maximal counterexample $I$. Thus $I$ contains no finite product of prime ideals containing $I$, and $I$ is maximal with respect to this property. We will show that $I$ is prime.

If $I$ is not prime, we can find $A, B \lhd R$ are such that $AB \subseteq I$ but $A \subsetneq I$ and $B \subsetneq I$. By the maximality of $I$, $I + A$ contains the product of primes $P_1, \ldots, P_n$ containing $I + A$, and similarly $Q_1 \cdots Q_m \subseteq I + B$ for some primes $Q_1, \ldots, Q_m$ containing $I + B$. Hence

$$P_1 \cdots P_n Q_1 \cdots Q_m \subseteq (I + A)(I + B) \subseteq I^2 + AI + IB + AB \subseteq I,$$

so $I$ itself contains a finite product of primes containing it. This contradicts the definition of $I$, so in fact $I$ is prime. Thus we have a contradiction, and (1) follows.

Hence we have a finite set of primes $P_1, \ldots, P_n$ containing $I$ such that $P_1 \cdots P_n \subseteq I$. After relabelling, we may assume that $\{P_1, \ldots, P_m\}$ are the distinct minimal primes of $\{P_1, \ldots, P_n\}$. Thus $I$ contains a product of primes from $\{P_1, \ldots, P_m\}$, possibly with repetition:

$$P_{i_1} \cdots P_{i_n} \subseteq I$$

for some $i_1, \ldots, i_n \in \{1, \ldots, m\}$. Now, suppose $Q$ is any prime containing $I$. Then $P_{i_1} P_{i_2} \cdots P_{i_n} \subseteq I \subseteq Q$ which forces $P_{i_j} \subseteq Q$ for some $j$. If $Q$ is a minimal prime over $I$, $Q$ must equal $P_{i_j}$.

Finally, we show that each $P_k$ is a minimal prime over $I$ for $k = 1, \ldots, m$. If $I \subseteq Q \subseteq P_k$ then $P_j \subseteq Q \subseteq P_k$ for some $j \leqslant m$ by the above. But $P_1, \ldots, P_m$ are the minimal primes in $\{P_1, \ldots, P_n\}$, so $P_j = Q = P_k$ and (2) follows. □

**Definition 4.3.** Let $I$ be an ideal in a left (or right) Noetherian ring $R$.

(a) The *prime radical* $N(R)$ of $R$ is the intersection of all prime ideals of $R$.
(b) The *prime radical* $\sqrt{I}$ of $I$ is the intersection of all prime ideals of $R$ that contain $I$.
(c) $R$ is *semiprime* if $N(R) = 0$.
(d) $I$ is said to be is *semiprime* if $I = \sqrt{I}$.

Thus $I$ is semiprime if and only if it is the intersection of some collection of prime ideals of $R$. Note that $\min(I)$ is completely determined by $\sqrt{I}$ because $\min(I) = \min(\sqrt{I})$.

**Corollary 4.4.** Let $R$ be a left (or right) Noetherian ring. Then

$$N(R) = \bigcap_{P \in \min(0)} P$$

is the largest nilpotent ideal in $R$.

*Proof.* Every nilpotent ideal is contained in every prime ideal. Thus $N(R)$ contains every nilpotent ideal. On the other hand, it follows from Proposition 4.2, that a finite product of the minimal primes of $R$ is zero. If there are $k$ terms in this product, then $N(R)^k = 0$, so $N(R)$ is nilpotent. □

**Definition 4.5.** Let $R$ be a left or right Noetherian ring. A *dimension function for $R$* is a rule which assigns to every finitely generated $R$-module $M$ a number $d(M) \in \mathbb{N}$, such that

$$d(M) = \max\{d(N), d(M/N)\}$$

whenever $N$ is a submodule of $M$.

**Theorem 4.6.** Let $R$ be a left or right Noetherian ring. Then every function

$$d : \{R/P : P \in \operatorname{Spec}(R)\} \to \mathbb{N}$$

such that $d(R/P) \geq d(R/Q)$ whenever $Q \subset P$ extends to a dimension function $d$ for $R$, given by

$$d(M) = \max\{d(R/P) : P \in \min(\operatorname{Ann}(M))\}$$

for every finitely generated $R$-module $M$.

*Proof.* Let $N$ be a submodule of a finitely generated $R$-module $M$, and write $\min(\operatorname{Ann}(M)) = \{P_\alpha\}$, $\min(\operatorname{Ann}(N)) = \{I_\beta\}$ and $\min(\operatorname{Ann}(M/N)) = \{J_\gamma\}$. Now some finite product of the $P_\alpha$'s kills $M$ by Proposition 4.2, so it kills both $N$ and $M/N$. It follows that

- every $I_\beta$ contains some $P_\alpha$, and
- every $J_\gamma$ contains some $P_\alpha$.

Now $d(N) = d(R/I_\beta)$ for some $\beta$ and $I_\beta$ contains some $P_\alpha$, so

$$d(N) = d(R/I_\beta) \leqslant d(R/P_\alpha) \leqslant d(M).$$

Similarly, $d(M/N) \leqslant d(M)$, and we have shown that $d(M) \geq \max\{d(N), d(M/N)\}$. On the other hand, some product, $A$ say, of the $I_\beta$'s kills $N$ and some product, $B$ say, of the $J_\gamma$'s kills $M/N$, again by Proposition 4.2. So $BM \subseteq N$ and $AN = 0$, whence $ABM = 0$ and $AB \subseteq \operatorname{Ann}(M)$. It follows that

- every $P_\alpha$ contains either an $I_\beta$ or a $J_\gamma$.

So if $d(M) = d(R/P_\alpha)$ for some $\alpha$ then either $P_\alpha$ contains some $I_\beta$, in which case

$$d(M) = d(R/P_\alpha) \leqslant d(R/I_\beta) \leqslant d(N),$$

or $P_\alpha$ contains some $J_\gamma$, in which case

$$d(M) = d(R/P_\alpha) \leqslant d(R/J_\gamma) \leqslant d(M/N).$$

In either case, we see that $d(M) \leqslant \max\{d(N), d(M/N)\}$. □

It can be shown that if in addition $R$ is *commutative*, then this extension is unique. More precisely, any dimension function $d'$ for $R$ such that $d(R/P) = d'(R/P)$ for all $P \in \operatorname{Spec}(R)$ must actually be equal to $d$, and is therefore completely determined by the values that it takes on modules of the form $R/P$, $P \in \operatorname{Spec}(R)$ — see Exercise 5.3.

It follows from Theorem 1.12 that every finitely generated *commutative* algebra $R$ over a field $k$ is Noetherian. Therefore $\mathfrak{m}$ is a finitely generated ideal and $\mathfrak{m}/\mathfrak{m}^2$ is a finitely generated $R/\mathfrak{m}$-module. Thus $\mathfrak{m}/\mathfrak{m}^2$ is a finite dimensional vector space over the field $R/\mathfrak{m}$.

**Definition 4.7.** Let $R$ be a finitely generated commutative $k$-algebra.

(a) If $R$ is a domain, then the *Krull dimension* of $R$ is

$$\mathrm{Kdim}(R) := \min\{\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) : \mathfrak{m} \text{ is a maximal ideal of } R\}.$$

(b) Let $M$ be a finitely generated $R$-module. Then

$$\mathrm{Kdim}(M) := \max\{\mathrm{Kdim}(R/P) : P \in \min(\mathrm{Ann}(M))\}$$

is the *Krull dimension* of $M$.

We will need to cite the following result from commutative algebra:

**Theorem 4.8.** Let $R$ be a finitely generated commutative $k$-algebra which is a domain. Then $\mathrm{Kdim}(R)$ is the length of the longest chain of prime ideals in $R$:

$$\mathrm{Kdim}(R) = \max\{n \in \mathbb{N} : \quad \text{there exist} \quad P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n, \quad P_i \in \mathrm{Spec}(R)\}.$$

*Proof.* Omitted. □

This is the classical definition of the Krull dimension of a ring. The proof uses the *Noether Normalisation Lemma* and the fact that every affine variety has a smooth, dense, open subset. Unfortunately we don't have time in this course to give all details of the proof.

**Corollary 4.9.** Let $R$ be a finitely generated commutative $k$-algebra. Then Kdim is a dimension function for $R$.

*Proof.* In order to apply Theorem 4.6, we just need to check that

$$\mathrm{Kdim}(R/P) \leqslant \mathrm{Kdim}(R/Q) \quad \text{whenever} \quad Q \subseteq P.$$

Suppose that $P = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ is the longest chain of prime ideals in $R$ starting with $P$ so that $n = \mathrm{Kdim}(R/P)$ by Theorem 4.8. Then this chain induces a chain of prime ideals of $R/Q$ of length $n$. Thus $n \leqslant \mathrm{Kdim}(R/Q)$. □

Definition 4.7 is more geometric, and more suitable to our intended applications. The vector space $(\mathfrak{m}/\mathfrak{m})^*$ is the *Zariski tangent space* to the affine algebraic variety $X := \mathrm{Spec}(R)$ at the point $\mathfrak{m}$, so $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$ is (roughly speaking) the number of linearly independent tangent vectors to $X$ at the point $x$.

We also need to borrow the following consequence of the *Weak Nullstellensatz* from C2.6 Commutative Algebra:

**Lemma 4.10.** If $k$ is an algebraically closed field, then every maximal ideal of the polynomial algebra $k[x_1, \ldots, x_n]$ is of the form $(x_1 - \alpha_1, \cdots, x_n - \alpha_n)$ for some $\alpha \in k^n$.

*Proof.* Omitted. □

**Example 4.11.** Let $R = k[x, y]/(xy)$ with $k$ algebraically closed. This is the coordinate ring of a pair of lines $X = \{(a, b) \in k^2 : ab = 0\}$ in the affine plane $k^2$. By Lemma 4.10, its maximal ideals are

$$\mathrm{MaxSpec}(R) = \{\langle x - a, y\rangle : 0 \neq a \in k\} \cup \{\langle x, y - b\rangle : 0 \neq b \in k\} \cup \{(x, y)\}.$$

If $\mathfrak{m} = \langle x - a, y\rangle$ with $a \neq 0$ then $\mathfrak{m}^2 = \langle (x - a)^2, (x - a)y, y^2\rangle = \langle (x - a)^2, y\rangle$ and $\mathfrak{m}/\mathfrak{m}^2$ is a one-dimensional vector space spanned by the image of $x - a$. Similarly, if $\mathfrak{m} = \langle x, y - b\rangle$ then $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$. However

$$\langle x, y\rangle^2 = \langle x^2, xy, y^2\rangle = \langle x^2, y^2\rangle$$

so $\dim_k \langle x, y\rangle/\langle x^2, y^2\rangle = 2$: there are two linearly independent tangent directions at the origin. Thus, as might be expected geometrically, $\mathrm{Kdim}(R) = 1$ since it is intuitively clear that $X$ is a one-dimensional space.

Now, let's return to the non-commutative setting and seek a well-behaved dimension function in the case where the ring in question doesn't necessarily have many two-sided ideals.

**Definition 4.12.** Let $R$ be a filtered ring with filtration $(R_i)_{i\in\mathbb{Z}}$ and let $M$ be a left $R-$module. A *filtration* on $M$ is a set $(M_i)_{i\in\mathbb{Z}}$ of additive subgroups of $M$ satisfying

- $M_i \subseteq M_{i+1}$ for all $i \in \mathbb{Z}$,
- $R_i \cdot M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}$,
- $\cup_{i\in\mathbb{Z}} M_i = M$.

Filtered right modules are defined similarly.

**Example 4.13.** Let $M$ be a left $R-$module with generating set $X$. Then $M_i := R_i \cdot X$ for all $i \in \mathbb{Z}$ gives a filtration of $M$, known as a *standard filtration.*

**Definition 4.14.** (a) Let $S = \oplus_{i\in\mathbb{Z}} S_i$ be a graded ring. A *graded left $S-$module* is a left $S-$module $V$ of the form

$$V = \bigoplus_{i\in\mathbb{Z}} V_i$$

such that $S_i V_j \subseteq V_{i+j}$ for all $i, j \in \mathbb{Z}$.

(b) A *graded left ideal* of $S$ is a left ideal of the form $J = \oplus_{i\in\mathbb{Z}} J_i$, where $J_i \subseteq S_i$ for each $i \in \mathbb{Z}$.

**Definition 4.15.** Let $R$ be a filtered ring and let $M$ be a filtered left $R-$module with filtration $(M_i)_{i\in\mathbb{Z}}$. Define the abelian group

$$\operatorname{gr} M = \bigoplus_{i\in\mathbb{Z}} M_i/M_{i-1}.$$

Equip $\operatorname{gr} M$ with a $\operatorname{gr} R-$action, which is given on homogeneous components by

$$\begin{array}{ccccc} R_i/R_{i-1} & \times & M_j/M_{j-1} & \longrightarrow & M_{i+j}/M_{i+j-1} \\ r+R_{i-1} & , & m+M_{j-1} & \mapsto & rm+M_{i+j-1} \end{array}$$

and on the whole of $\operatorname{gr} M$ by bilinear extension. Then $\operatorname{gr} M$ becomes a graded left $\operatorname{gr} R-$module, called the *associated graded module* of $M$.

Our next goal will be to define a well-behaved dimension function for certain filtered non-commutative rings. For this, we first need to make a digression to study Rees rings and good filtrations.

**Definition 4.16.** Let $R$ be a filtered ring with filtration $(R_i)_{i\in\mathbb{Z}}$, and let $M$ be a filtered left $R-$module with filtration $(M_i)_{i\in\mathbb{Z}}$. The *Rees ring* is the following subring $\widetilde{R}$ of the ring of Laurent polynomials $R[t,t^{-1}]$:

$$\widetilde{R} = \bigoplus_{i\in\mathbb{Z}} R_i t^i \subseteq \bigoplus_{i\in\mathbb{Z}} R t^i = R[t,t^{-1}].$$

The *Rees module* $\widetilde{M}$ of $M$ is the abelian group

$$\widetilde{M} = \bigoplus_{i\in\mathbb{Z}} M_i t^i$$

where the action of $\widetilde{R}$ is given by on homogeneous components by

$$\begin{array}{ccccc} R_i t^i & \times & M_j t^j & \to & M_{i+j} t^{i+j} \\ r_i t^i & , & m_j t^j & \mapsto & r_i m_j t^{i+j}. \end{array}$$

Note that $t \in \widetilde{R}$ is a central regular element, since $1 \in R_1$ always. There is a certain amount of interplay between the Rees ring of $R$ and the associated graded ring $\operatorname{gr} R$.

**Lemma 4.17.** Let $R$ and $M$ be as above. Then

(1) $\widetilde{R}/t\widetilde{R} \cong \operatorname{gr} R$ as rings,
(2) $\widetilde{M}/t\widetilde{M} \cong \operatorname{gr} M$ as left $\operatorname{gr} R-$modules,
(3) $\widetilde{R}/(t-1)\widetilde{R} \cong R$ as rings,
(4) $\widetilde{M}/(t-1)\widetilde{M} \cong M$ as left $R-$modules.

*Proof.* We will only prove the result for the rings, leaving the modules as an exercise.

(1). We have an isomorphism of abelian groups

$$\widetilde{R}/t\widetilde{R} = \frac{\bigoplus_{i\in\mathbb{Z}} R_i t^i}{\bigoplus_{i\in\mathbb{Z}} R_{i-1} t^i} \cong \bigoplus_{i\in\mathbb{Z}} R_i/R_{i-1} \cong \operatorname{gr} R.$$

It can be checked that this is also a ring isomorphism.

(2). Define a ring homomorphism $\pi : \widetilde{R} \to R$ by $\pi(\sum r_i t^i) = \sum r_i$. Since $\pi(R_i t^i) = R_i$ we see that $\pi$ is onto. Clearly $t - 1 \in \ker(\pi)$. Check that in fact $\ker(\pi) = (t-1)\widetilde{R}$. The result follows. □

So $\widetilde{R}$ is a ring which has both $R$ and $\operatorname{gr} R$ as epimorphic images. It follows that if $\widetilde{R}$ is right (or left) Noetherian, then so are both $R$ and $\operatorname{gr} R$.

**Definition 4.18.** Let $R$ be a filtered ring and let $M$ be a left $R-$module.

- A filtration $(M_i)$ on $M$ is said to be *good* if the Rees module $\widetilde{M}$ is finitely generated over $\widetilde{R}$.
- Two filtrations $(M_i)$ and $(M'_i)$ on $M$ are *algebraically equivalent* (or just *equivalent*) if there exist $c, d \in \mathbb{Z}$ such that

$$M'_i \subseteq M_{i+c} \quad \text{and} \quad M_j \subseteq M'_{j+d} \quad \text{for all} \quad i, j \in \mathbb{Z}.$$

Note that if $(M_i)$ is a good filtration, then $\operatorname{gr} M \cong \widetilde{M}/t\widetilde{M}$ is finitely generated over $\operatorname{gr} R$ and $M \cong \widetilde{M}/(t-1)\widetilde{M}$ is finitely generated over $R$, by Lemma 4.17.

**Proposition 4.19.** Let $R$ be a filtered ring and let $M$ be a left $R-$module.

(1) A filtration $(M_i)$ on $M$ is good if and only if there exist $k_1, k_2, \ldots, k_s \in \mathbb{Z}$ and $m_1 \in M_{k_1}, m_2 \in M_{k_2}, \ldots, m_s \in M_{k_s}$ such that

$$M_i = R_{i-k_1} m_1 + R_{i-k_2} m_2 + \cdots + R_{i-k_s} m_s \quad \text{for all} \quad i \in \mathbb{Z}.$$

(2) All good filtrations on $M$ are equivalent.

*Proof.* (1) If the graded module $\widetilde{M}$ is finitely generated, it has a finite homogeneous generating set $\{t^{k_1} m_1, \ldots, t^{k_s} m_s\}$ say, with $m_j \in M_{k_j}$. Then the $i-$th homogeneous component of $\widetilde{M}$ is

$$t^i M_i = R_{i-k_1} t^{i-k_1}(t^{k_1} m_1) + \cdots + R_{i-k_s} t^{i-k_s}(t^{k_s} m_s), \quad \text{so}$$

$$M_i = R_{i-k_1} m_1 + R_{i-k_2} m_2 + \cdots + R_{i-k_s} m_s.$$

Conversely, any filtration of this form is good, since $\{t^{k_1} m_1, \ldots, t^{k_s} m_s\}$ is then a generating set for $\widetilde{M}$.

(2) Take two good filtrations $(M_i)$ and $(M'_i)$. Then we have

$$\begin{aligned} M_i &= R_{i-k_1} m_1 + \cdots + R_{i-k_u} m_u \quad \text{for all} \quad i \\ M'_j &= R_{j-l_1} m'_1 + \cdots + R_{j-l_v} m'_v \quad \text{for all} \quad j. \end{aligned}$$

We can find $c \in \mathbb{Z}$ such that $m'_s \in M_{l_s+c}$ for all $s = 1, \ldots, v$. Then $M'_i \subseteq M_{i+c}$ for all $i$, and similarly there exists $d \in \mathbb{Z}$ such that $M_i \subseteq M'_{i+d}$ for all $i$. □

**Corollary 4.20.** Every finitely generated module over a filtered ring has at least one good filtration.

*Proof.* Let $X = \{x_1, \ldots, x_s\}$ be a finite generating set for $M$, and let

$$M_i := R_i x_1 + \ldots + R_i x_s = R_i X$$

be the standard filtration on $M$. It is good by Proposition 4.19. □

**Theorem 4.21.** Let $R$ be a filtered ring such that $\operatorname{gr} R$ is commutative and Noetherian, and let $M$ be a finitely generated $R$-module. Let $(M_i)$ and $(M_i')$ be two good filtrations on $M$, and let $\operatorname{gr} M, \operatorname{gr}' M$ be the respective associated graded modules. Then

$$\min(\operatorname{Ann}(\operatorname{gr} M)) = \min(\operatorname{Ann}(\operatorname{gr}' M)).$$

*Proof.* By Proposition 4.19, we can find an integer $c > 0$ such that

$$M_{i-c} \subseteq M_i' \subseteq M_{i+c} \quad \text{for all} \quad i \in \mathbb{Z}.$$

Let $I = \sqrt{\operatorname{Ann}(\operatorname{gr} M)}$ and $I' = \sqrt{\operatorname{Ann}(\operatorname{gr}' M)}$. Since $\min(\operatorname{Ann}(\operatorname{gr} M)) = \min(I)$, by symmetry it will be sufficient to show that $I \subseteq I'$. Because these ideals are graded and $\operatorname{gr} R$ is commutative, it will be enough to show that every homogeneous element $X \in I$ lies in $I'$. We can assume that $X = x + R_{n-1}$ for some $x \in R_n$.

Since $I/\operatorname{Ann}(\operatorname{gr} M)$ is a nilpotent ideal by Corollary 4.4, $X^m \in \operatorname{Ann}(\operatorname{gr} M)$ for some $m \in \mathbb{N}$. Thus $x^m + R_{mn-1}$ kills $\operatorname{gr} M$:

$$x^m M_i \subseteq M_{i+mn-1} \quad \text{for all} \quad i \in \mathbb{Z}.$$

Apply this relation repeatedly to deduce that

$$x^{am} M_i \subseteq M_{i+amn-a} \quad \text{for all} \quad i \in \mathbb{Z}, a \in \mathbb{N}.$$

Now, take $a = 3c$ and use $M_{i-c} \subseteq M_i' \subseteq M_{i+c}$ to obtain

$$x^{3cm} M_i' \subseteq x^{3cm} M_{i+c} \subseteq M_{i+3cmn-2c} \subseteq M'_{i+3cmn-c} \quad \text{for all} \quad i \in \mathbb{Z}.$$

Since $X = x + R_{n-1}$ and $c \geq 1$, we see that $X^{3cm}$ kills $\operatorname{gr}' M$:

$$X^{3cm} \in \operatorname{Ann}(\operatorname{gr}' M).$$

Because $\operatorname{gr} R$ is commutative, the image of $X \operatorname{gr} R$ in $\operatorname{gr} R/\operatorname{Ann}(\operatorname{gr}' M)$ is a nilpotent ideal, so $X \in \sqrt{\operatorname{Ann}(\operatorname{gr}' M)}$. □

**Definition 4.22.** Let $R$ be a filtered ring such that $\operatorname{gr} R$ is a finitely generated commutative algebra over a field $k$, and let $M$ be a finitely generated $R$-module. Choose a good filtration on $M$ using Corollary 4.20.

(a) The *set of characteristic primes* of $M$ is $\operatorname{Ch}(M) := \min(\operatorname{Ann}(\operatorname{gr} M))$.
(b) The *dimension* of $M$ is $d(M) := \operatorname{Kdim}(\operatorname{gr} M)$.

The *characteristic variety* of $M$ is the affine subvariety of $\operatorname{Spec}(\operatorname{gr} R)$ defined by $\operatorname{Ann}(\operatorname{gr} M)$. Its irreducible components are the affine varieties defined by the members of $\operatorname{Ch}(M)$. Theorem 4.21 ensures that $\operatorname{Ch}(M)$ does not depend on the choice of good filtration on $M$. Since by Definition 4.7 $\mathrm{Kdim}(\mathrm{gr}\, M)$ only depends on $\mathrm{Ch}(M)$, it also does not depend on this choice.

We can now state one of the main results in this course: the proof occupies most of Chapter 5.

**Theorem 4.23** (Bernstein's Inequality)**.** Let $k$ be an algebraically closed field of characteristic zero, and let $M$ be a finitely generated, non-zero module over the Weyl algebra $A_n(k)$. Then

$$d(M) \geq n.$$

The Weyl algebra $A_n(k)$ can be thought of as a non-commutative polynomial ring in $2n$ variables because $\mathrm{gr}\, A_n(k) \cong k[X_1, \ldots, X_{2n}]$ by Proposition 1.25. So even though $\mathrm{gr}\, A_n(k)$ has finitely generated modules of all possible dimensions between 0 and $2n$, non-zero finitely generated $A_n(k)$-modules $M$ are "large": $n \leqslant d(M) \leqslant 2n$.

To ensure that $d$ really is a dimension function for $R$ in the setting of Definition 4.22, we need to do a little more work.

**Definition 4.24.** Let $N$ be a submodule of a filtered left $R$-module $M$.

- The *subspace filtration* $(N_i)_{i\in\mathbb{Z}}$ on $N$ is given by

$$N_i := N \cap M_i.$$

- The *quotient filtration* $((M/N)_i)_{i\in\mathbb{Z}}$ on $M/N$ is given by

$$(M/N)_i := (M_i + N)/N.$$

**Proposition 4.25.** Let $R$ be a filtered ring, let $M$ be a filtered left $R-$module with filtration $(M_i)_{i\in\mathbb{Z}}$ and let $N$ be a submodule of $M$. Equip $N$ with the subspace filtration and $M/N$ with the quotient filtration. Then there exists a short exact sequence of left $\mathrm{gr}\, R-$modules

$$0 \to \mathrm{gr}\, N \xrightarrow{\alpha} \mathrm{gr}\, M \xrightarrow{\beta} \mathrm{gr}(M/N) \to 0.$$

*Proof.* The natural composition of maps $N_i \hookrightarrow M_i$ and $M_i \twoheadrightarrow M_i/M_{i-1}$ has kernel $N_i \cap M_{i-1} = N \cap M_{i-1} = N_{i-1}$. So we have an injection of abelian groups

$$\alpha_i : N_i/N_{i-1} \hookrightarrow M_i/M_{i-1}$$

for all $i \in \mathbb{Z}$. Putting these together we get an injection

$$\alpha = \oplus\alpha_i : \mathrm{gr}\, N \to \mathrm{gr}\, M.$$

Exercise: check that $\alpha$ is a left $\mathrm{gr}\, R-$module homomorphism.

Consider the composition

$$\beta_i : M_i/M_{i-1} \xrightarrow{u_i} \frac{M_i + N}{M_{i-1} + N} \xrightarrow{v_i} \frac{(M_i + N)/N}{(M_{i-1} + N)/N}$$

where $u_i(m + M_{i-1}) = m + M_{i-1} + N$ and $v_i$ is the natural isomorphism.

Note that $u_i$ is onto, whereas

$$\ker(u_i) = \frac{M_i \cap (M_{i-1} + N)}{M_{i-1}} = \frac{M_{i-1} + (M_i \cap N)}{M_{i-1}} = \frac{M_{i-1} + N_i}{M_{i-1}} = \operatorname{im}(\alpha_i)$$

by the modular law. Since $v_i$ is an isomorphism, $\beta_i$ is onto and $\ker(\beta_i) = \operatorname{im}(\alpha_i)$ for all $i \in \mathbb{Z}$. Letting

$$\beta = \oplus \beta_i : \operatorname{gr} M \to \operatorname{gr}(M/N)$$

we see that $\beta$ is onto and $\ker(\beta) = \operatorname{im}(\alpha)$.

Exercise: check that $\beta$ is a left $\operatorname{gr} R-$module homomorphism. □

**Proposition 4.26.** Let $R$ be a filtered ring such that the Rees ring $\widetilde{R}$ is left Noetherian and $\operatorname{gr} R$ is a finitely generated commutative algebra over a field $k$. Then

$$M \mapsto d(M) = \operatorname{Kdim}(\operatorname{gr} M)$$

is a dimension function for $R$.

*Proof.* We have to show that $d(M) = \max\{d(N), d(M/N)\}$ whenever $N$ is a submodule of a finitely generated $R$-module $M$. Equip $M$ with a good filtration using Corollary 4.20, and endow $N$ and $M/N$ with the subspace and quotient filtrations, respectively. Then by definition, the associated sequence of Rees modules

$$0 \to \widetilde{N} \to \widetilde{M} \to \widetilde{M/N} \to 0$$

is exact. Thus $\widetilde{N}$ and $\widetilde{M/N}$ are finitely generated over the left Noetherian ring $\widetilde{R}$, so that the filtrations on $N$ and $M$ are *good*. Now,

$$0 \to \operatorname{gr} N \to \operatorname{gr} M \to \operatorname{gr} M/N \to 0$$

is an exact sequence of finitely generated $\operatorname{gr} R$-modules by Propostion 4.25, so we can apply Proposition 4.9. □

**Theorem 4.27.** Let $R$ be a positively filtered ring such that $\operatorname{gr} R$ is left Noetherian. Then the Rees ring $\widetilde{R}$ is also left Noetherian.

□

**Remarks 4.28.** Even though Theorem 4.21 ensures that $d(M)$ does not depend on the particular choice of good filtration on $M$, the definition still depends on the choice of filtration on the ring $R$. It is quite possible that the same non-commutative ring $R$ has two "different" filtrations, in the sense that the respective associated graded rings are not isomorphic. However, using more advanced techniques from homological algebra such as the *bidualising complex*, it can be shown that in fact $d(M)$ does *not* depend on the choice of filtration on the ring $R$, and is therefore an intrinsic invariant of the $R$-module $M$.

## 5. The integrability of the characteristic variety

**Definition 5.1.** Let $R$ be a ring. A *Poisson bracket* on $R$ is a function

$$\{,\} : R \times R \to R$$

such that

(a) $\{,\}$ is bi-additive,
(b) $\{x, x\} = 0$ for all $x \in R$,
(c) $\{x, \{y, z\}\} + \{y, \{z, x\}\} + \{z, \{x, y\}\} = 0$ for all $x, y, z \in R$,
(d) $\{x, yz\} = \{x, y\}z + y\{x, z\}$ for all $x, y, z \in R$.

In other words, a Poisson bracket is a $\mathbb{Z}$-Lie bracket on $R$ which is a *bi-derivation* in the sense that the functions $\{x, -\} : R \to R$ and $\{-, y\} : R \to R$ are derivations of $R$ for all $x, y \in R$. For example, the commutator bracket on every ring is an example of a Poisson bracket. However, it can happen that a commutative ring has an interesting and non-trivial additional Poisson structure. One of the main mechanisms for constructing Poisson brackets comes from *deformation theory* as follows:

**Proposition 5.2.** Let $\mathcal{R}$ be a ring and suppose that $\tau \in \mathcal{R}$ is a central element such that $R := \mathcal{R}/\tau\mathcal{R}$ is commutative, and

$$\operatorname{ann}(\tau) = \tau\mathcal{R}.$$

Define $\{,\} : R \times R \to R$ by the rule

$$\{x + \tau R, y + \tau R\} = z + \tau\mathcal{R}$$

where $[x, y] = \tau z$. Then $\{,\}$ is a well-defined Poisson bracket on $R$.

*Proof.* We will check that $\{,\}$ is well-defined. Note that every commutator $[x, y]$ in $\mathcal{R}$ lies in $\tau\mathcal{R}$ because $\mathcal{R}/\tau\mathcal{R}$ is commutative by assumption: this ensures the existence of $z \in \mathcal{R}$ such that $[x, y] = \tau z$. Now suppose that

$$x' = x + \tau a \quad \text{and} \quad y' = y + \tau b \quad \text{for some} \quad a, b \in \mathcal{R}.$$

Then because $\tau^2 = 0$ we have

$$[x', y'] = [x, y] + \tau[x, b] + \tau[a, y].$$

But $[\mathcal{R}, \mathcal{R}] \subseteq \tau\mathcal{R}$ and $\tau^2 = 0$, so in fact $[x', y'] = [x, y]$.

Finally, if $[x, y] = \tau z = \tau z'$ for some $z' \in \mathcal{R}$ then by assumption,

$$z - z' \in \operatorname{ann}(\tau) = \tau\mathcal{R}$$

so $z + \tau\mathcal{R} = z' + \tau\mathcal{R}$. The rest is straightforward. □

The following elementary Lemma will be useful many times in what follows. It transforms questions about filtered modules into a problem in deformation theory.

**Lemma 5.3.** Let $R$ be a filtered ring and let $M$ be a filtered left $R$-module. Let $\mathcal{R} := \widetilde{R}/t^2\widetilde{R}$, $\tau := t + t^2\widetilde{R} \in \mathcal{R}$ and let $\mathcal{N} := \widetilde{M}/t^2\widetilde{M}$. Then $\tau \in \mathcal{R}$ is a central element such that $\tau^2 = 0$, and $\{m \in \mathcal{N} : \tau m = 0\} = \tau\mathcal{N}$.

*Proof.* Only the last part requires proof. Suppose that $\tau m = 0$ for some $m \in \mathcal{N}$. To show that $m \in \tau\mathcal{N}$ we may assume that $m$ is homogeneous, and thus of the form $m = at^i + t^2\widetilde{M}$ for some $a \in M_i$. Now

$$0 = \tau m = (t + t^2\widetilde{R})(at^i + t^2\widetilde{M}) = at^{i+1} + t^2\widetilde{M}$$

implies that $at^{i+1} \in t^2\widetilde{M}$. But the homogeneous component of $t^2\widetilde{M}$ of degree $i+1$ is $M_{i-1}t^{i+1}$, so $a \in M_{i-1}$. Hence $at^i = t(at^{i-1}) \in t\widetilde{M}$ and thus $m \in \tau\mathcal{N}$. □

**Corollary 5.4.** Let $R$ be a filtered ring such that $\operatorname{gr} R$ is commutative. Then there is a Poisson bracket

$$\{,\} : \operatorname{gr} R \times \operatorname{gr} R \to \operatorname{gr} R$$

such that

$$\{x + R_{i-1}, y + R_{j-1}\} = [x, y] + R_{i+j-2}$$

whenever $x \in R_i$ and $y \in R_j$.

*Proof.* We form the Rees ring $\widetilde{R}$ and set $\mathcal{R} := \widetilde{R}/t^2\widetilde{R}$. Let $\tau \in \mathcal{R}$ be the image of $t \in \widetilde{R}$ in $\mathcal{R}$; then $\tau^2 = 0$ and $\tau$ is central in $\mathcal{R}$ by Lemma 5.3. Also, $\mathcal{R}/\tau\mathcal{R} \cong \widetilde{R}/t\widetilde{R} \cong \operatorname{gr} R$ by Lemma 4.17(1), and because $R$ is itself a filtered left $R$-module,

$$\operatorname{ann}(\tau) = \tau\mathcal{R}$$

by Lemma 5.3. Proposition 5.2 now gives a Poisson bracket $\{,\}$ on $\operatorname{gr} R \cong \mathcal{R}/\tau\mathcal{R}$. If $x \in R_i$ and $y \in R_j$ then $x + R_{i-1}$ and $y + R_{j-1}$ are the images of $xt^i + t^2\widetilde{R}$ and $yt^j + t^2\widetilde{R}$ respectively under the map $\mathcal{R} \twoheadrightarrow \operatorname{gr} R$. Now since $\operatorname{gr} R$ is commutative, $[x, y] \in R_{i+j-1}$ so $[x, y]t^{i+j-1} \in R[t, t^{-1}]$ lies in $\widetilde{R}$. Hence

$$[xt^i + t^2\widetilde{R}, yt^j + t^2\widetilde{R}] = [x, y]t^{i+j} + t^2\widetilde{R} = t([x, y]t^{i+j-1} + t^2\widetilde{R})$$

so that

$$\Big\{(xt^i + t^2\widetilde{R}) + \tau\mathcal{R}, (yt^j + t^2\widetilde{R}) + \tau\mathcal{R}\Big\} = ([x, y]t^{i+j-1} + t^2\widetilde{R}) + \tau\mathcal{R}.$$

Therefore $\{x + R_{i-1}, y + R_{j-1}\} = [x, y] + R_{i+j-2}$ by the definition of $\{,\}$. □

We will next calculate the Poisson bracket induced by the Weyl algebra $A_n(k)$. Equip $A_n(k)$ with the filtration by order of differential operator, and recall that

$$\operatorname{gr} A_n(k) \cong k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$$

by Proposition 1.25 with $X_i$ in degree zero and $Y_j$ in degree one.

**Example 5.5.** The Poisson bracket on $\operatorname{gr} A_n(k)$ is given by

$$\{Y_i, X_j\} = \delta_{ij}, \quad \text{and} \quad \{X_i, X_j\} = \{Y_i, Y_j\} = 0 \quad \text{for all} \quad i, j = 1, \ldots n.$$

The goal of this Chapter is to prove the following Theorem.

**Theorem 5.6** (Gabber)**.** Let $R$ be a filtered ring such that $\operatorname{gr} R$ is a commutative Noetherian $\mathbb{Q}$-algebra and let $M$ be a finitely generated $R$-module. Then

$$\{P, P\} \subset P$$

for every $P \in \operatorname{Ch}(M) = \min(\operatorname{Ann}(\operatorname{gr} M))$.

To see how powerful this Theorem is, we will use it to prove Bernstein's Inequality (Theorem 4.23) after the next Lemma.

**Lemma 5.7.** Let $(,)$ be a non-degenerate bilinear form on a finite dimensional $k$-vector space $V$, and let $W$ be a subspace of $V$ such that $(W, W) = 0$. Then

$$\dim W \leqslant \frac{1}{2} \dim V.$$

*Proof.* Since $(,)$ is non-degenerate, the map $\Phi : V \to V^*$ given by $\Phi(v)(w) = (v, w)$ is injective. Pick a basis $\{f_1, \ldots, f_r\}$ for $\Phi(W)$, extend it to a basis $\{f_1, \ldots, f_m\}$ for $V^*$ and let $\{v_1, \ldots, v_m\}$ be the dual basis for $V$. Then $\{v_{r+1}, \ldots, v_m\}$ is a basis for $W^\perp := \{v \in V : (W, v) = 0\}$ by construction, so $\dim W + \dim W^\perp = \dim V$. But $(W, W) = 0$, so $W \leqslant W^\perp$ and hence $2 \dim W \leqslant \dim V$. □

*Proof of Theorem 4.23.* Recall from Proposition 1.25(b) that the associated graded ring of $A_n(k)$ with respect to the filtration by order of differential operators is a polynomial algebra over $k$ in $2n$-variables:

$$R := \operatorname{gr} A_n(k) \cong k[X_1, \ldots, X_{2n}].$$

Choose a good filtration on $M$ and let $P \in \operatorname{Ch}(M)$. Every maximal ideal of $R/P$ is of the form $\mathfrak{m}/P$ for some maximal ideal $\mathfrak{m}$ of $R$ containing $P$, and

$$(\mathfrak{m}/P)/(\mathfrak{m}/P)^2 \cong \mathfrak{m}/(\mathfrak{m}^2 + P)$$

as vector spaces over $F := R/\mathfrak{m}$. By Definitions 4.22 and 4.7, we need to prove that

$$\dim_F \frac{\mathfrak{m}}{\mathfrak{m}^2 + P} \geq n$$

for every maximal ideal $\mathfrak{m}$ of $R$ containing $P$. The Poisson bracket $\{,\}$ on $R$ given in Example 5.5 induces a well-defined alternating $F$-bilinear form

$$(,)_\mathfrak{m} : \mathfrak{m}/\mathfrak{m}^2 \times \mathfrak{m}/\mathfrak{m}^2 \to F$$

given by $(v + \mathfrak{m}^2, w + \mathfrak{m}^2)_\mathfrak{m} = \{v, w\} + \mathfrak{m}$. Now because $k$ is algebraically closed, we can write $\mathfrak{m} = (X_1 - \alpha_1, \ldots, X_{2n} - \alpha_{2n})$ for some $\alpha \in k^{2n}$ by Lemma 4.10. So the natural map $k \to R/\mathfrak{m}$ is an isomorphism, and if $v_i := X_i - \alpha + \mathfrak{m}^2 \in \mathfrak{m}/\mathfrak{m}^2$ then the form $(,)_\mathfrak{m}$ is given by

$$(v_i, v_j)_\mathfrak{m} = \begin{cases} 1 & \text{if} \quad j = i + n \\ -1 & \text{if} \quad i = j + n \\ 0 & \text{otherwise.} \end{cases}$$

by Example 5.5. It follows that $(,)_{\mathfrak{m}}$ is non-degenerate. Now $\{P, P\} \subseteq P \subseteq \mathfrak{m}$ by Theorem 5.6, so $(,)_{\mathfrak{m}}$ vanishes on the subspace $(P + \mathfrak{m}^2)/\mathfrak{m}^2$ of $\mathfrak{m}/\mathfrak{m}^2$. Hence

$$\dim_k \frac{\mathfrak{m}}{\mathfrak{m}^2 + P} = \dim_k \frac{\mathfrak{m}}{\mathfrak{m}^2} - \dim_k \frac{P + \mathfrak{m}^2}{\mathfrak{m}^2} \geq 2n - n = n$$

by Lemma 5.7. □

We will use the techniques of *Rees rings* and *noncommutative localisation* to prove Theorem 5.6. First, two preliminary Lemmas.

**Lemma 5.8.** Let $R$ be a commutative Noetherian ring and let $M$ be a finitely generated, non-zero, $R$-module. Let $P \in \min(\mathrm{Ann}(M))$ and let $S = R \backslash P$. Then

$$(S^{-1}P)^w \cdot S^{-1}M = 0$$

for some $w \in \mathbb{N}$.

*Proof.* Since $M \neq 0$, $I := \mathrm{Ann}(M)$ is a proper ideal, so $\min(I)$ is non-empty by Proposition 4.2(1). Write $\min(I) = \{P_1, \ldots, P_n\}$ with $P = P_1$. Then

$$P_1^{w_1} P_2^{w_2} \cdots P_n^{w_n} \subseteq I$$

for some $w_i \in \mathbb{N}$ by Proposition 4.2(2). Now pass to the localisation $S^{-1}R$:

$$(S^{-1}P_1)^{w_1}(S^{-1}P_2)^{w_2} \cdots (S^{-1}P_n)^{w_n} \subseteq S^{-1}I.$$

Now $P_i \nsubseteq P_1$ whenever $i \geq 2$, so $P_i \cap S \neq \emptyset$ and hence $S^{-1}P_i = S^{-1}R$ for all $i \geq 2$. Hence

$$(S^{-1}P)^{w_1} \cdot (S^{-1}M) = 0$$

because $S^{-1}I \cdot S^{-1}M = 0$. □

Our next result gives a very general mechanism for creating Ore sets. It can be viewed as the start of the theory of *algebraic microlocalisation*.

**Lemma 5.9.** Let $\mathcal{R}$ be a ring contining a central element $\tau \in \mathcal{R}$ such that $\tau^2 = 0$. Let $\xi : \mathcal{R} \to R := \mathcal{R}/\tau\mathcal{R}$ be the canonical surjection. If $S \subseteq R$ is a left Ore set in $R$, then $\xi^{-1}S \subseteq \mathcal{R}$ is a left Ore set in $\mathcal{R}$.

*Proof.* Let $a \in \mathcal{R}$, $s \in \xi^{-1}(S)$. Then $ta \equiv bs \pmod{\tau\mathcal{R}}$ for some $t \in \xi^{-1}(S)$, $b \in \mathcal{R}$. So $ta = bs + \tau u$, and also $t'u = b's + \tau u'$ for some $t' \in \xi^{-1}(S)$. But then,

$$(t't)a = t'bs + \tau t'u = t'bs + \tau(b's + \tau u') = (t'b + \tau b')s$$

because $\tau^2 = 0$, and $t't \in \xi^{-1}(S)$. □

*Proof of Theorem 5.6.* As in the proof of Corollary 5.4, form the Rees ring $\widetilde{R}$ and set $\mathcal{R} := \widetilde{R}/t^2\widetilde{R}$. Let $\xi : \mathcal{R} \twoheadrightarrow \mathrm{gr}\, R$ be the map defined by $\xi(xt^i + t^2\widetilde{R}) = x + R_{i-1}$ for $x \in R_i$, and let $\tau = t + t^2\widetilde{R} \in \mathcal{R}$. Then $\tau$ is central in $\mathcal{R}$, $\tau^2 = 0$ and

$$\ker \xi = \tau\mathcal{R} = \mathrm{ann}(\tau)$$

by Lemma 5.3. Let $S = \operatorname{gr} R \backslash P$ and set $\mathcal{S} := \xi^{-1}(S)$. Then $\mathcal{S}$ is a left Ore set in $\mathcal{R}$ by Lemma 5.9, and $\mathcal{R}$ is a left Noetherian ring by Exercise 1.4(a) because $\operatorname{gr} R$ is Noetherian, so $\mathcal{S}$ is left localisable by Proposition 3.11. Form the localised ring

$$\mathcal{B} := \mathcal{S}^{-1}\mathcal{R}$$

and note that $\mathcal{B}/\tau\mathcal{B} \cong \mathcal{S}^{-1}(\mathcal{R}/\tau\mathcal{R}) \cong S^{-1}(\operatorname{gr} R)$ by Proposition 3.14. Note that $\mathcal{P} := \mathcal{S}^{-1}\xi^{-1}(P)$ is a two-sided ideal in $\mathcal{B}$, and

$$\mathcal{B}/\mathcal{P} \cong \mathcal{S}^{-1}(\mathcal{R}/\xi^{-1}(P)) \cong S^{-1}(R/P)$$

is the field of fractions of $R/P$. Thus $\mathcal{P}$ is a maximal ideal in $\mathcal{B}$.

Choose a good filtration on $M$ and let $\mathcal{N} := \widetilde{M}/t^2\widetilde{M}$. This is a finitely generated $\mathcal{R}$-module, and

$$\tau\mathcal{N} \cong \mathcal{N}/\tau\mathcal{N} \cong \operatorname{gr} M$$

as $\operatorname{gr} R \cong \mathcal{R}/\tau\mathcal{R}$-modules by Lemma 5.3 and Lemma 4.17(2). The localised module

$$\mathcal{M} := \mathcal{S}^{-1}\mathcal{N}$$

is finitely generated over $\mathcal{B}$, and

$$\tau\mathcal{M} \cong \mathcal{M}/\tau\mathcal{M} \cong S^{-1}(\operatorname{gr} M)$$

by Proposition 3.14. By Lemma 5.8, $S^{-1}(\operatorname{gr} M)$ is killed by $(S^{-1}P)^w$ for some $w \in \mathbb{N}$, so $\mathcal{P}^w \cdot \mathcal{M} \subseteq \tau\mathcal{M}$. Hence $\mathcal{P}^{2w} \cdot \mathcal{M} = 0$, so $\mathcal{M}$ is a finitely generated module over

$$\mathcal{A} := \mathcal{B}/\mathcal{P}^{2w}.$$

Let $\mathcal{J}$ be the image of $\mathcal{P}$ in $\mathcal{A}$; then $\mathcal{J}$ is a maximal ideal of $\mathcal{A}$ such that $\mathcal{A}/\mathcal{J} \cong \mathcal{B}/\mathcal{P} \cong S^{-1}(R/P)$. Since $\mathcal{J}$ is finitely generated as a one-sided ideal and $\mathcal{J}^{2w} = 0$, the ring $\mathcal{A}$ is left Artinian. So by Theorem 5.10 below,

$$[\mathcal{J}, \mathcal{J}] \subseteq \tau\mathcal{J}.$$

Pulling back to $\mathcal{B}$, we deduce that $[\mathcal{P}, \mathcal{P}] \subseteq \tau\mathcal{P} + \mathcal{P}^{2w}$. But $\mathcal{B}/\tau\mathcal{B}$ is commutative by construction, so $[\mathcal{P}, \mathcal{P}] \subseteq \tau\mathcal{B}$ and

$$[\mathcal{P}, \mathcal{P}] \subseteq (\tau\mathcal{P} + \mathcal{P}^{2w}) \cap \tau\mathcal{B} \subseteq \tau\mathcal{P} + (\mathcal{P}^{2w} \cap \tau\mathcal{B})$$

by the modular law. But if $x \in \mathcal{B}$ and $\tau x \in \mathcal{P}^{2w}$ then $\tau x$ kills $\mathcal{M}$, so $x\mathcal{M} \subseteq \tau\mathcal{M}$ and $x^2\mathcal{M} = 0$. If $x \notin \mathcal{P}$ then $x$ is a unit in $\mathcal{B}$ which would force $\mathcal{M} = 0$. But then $S^{-1}(\operatorname{gr} M) = 0$, and since $\operatorname{gr} M$ is finitely generated, it is killed by some $s \in S$. But then $s \in \operatorname{Ann}(\operatorname{gr} M) \subseteq P$, which contradicts $s \in S$. Hence $x \in \mathcal{P}$, so $[\mathcal{P}, \mathcal{P}] \subseteq \tau\mathcal{P}$.

Finally, let $x, y \in P$ and choose $a, b \in \xi^{-1}(P)$ such that $x = \xi(a)$ and $y = \xi(b)$. Then $[a, b] \in \tau\mathcal{P}$ and $\mathcal{P} = \mathcal{S}^{-1}\xi^{-1}(P)$, so there is some $s \in \mathcal{S}$ such that $s[a, b] \in \tau\xi^{-1}(P)$. Hence $\xi(s)\{x, y\} \in P$ and $\xi(s) \in S$. But $S = \operatorname{gr} R \backslash P$ and $P$ is prime, so $\{x, y\} \in P$. □

Thus it remains to prove

**Theorem 5.10** (Gabber's Local Theorem)**.** Let $\mathcal{A}$ be a left Artinian $\mathbb{Q}$-algebra with unique maximal ideal $\mathcal{J}$ and a central element $\tau \in \mathcal{J}$ such that $\tau^2 = 0$ and $\mathcal{A}/\tau\mathcal{A}$ is commutative. Suppose that $\mathcal{M}$ is a finitely generated non-zero $\mathcal{A}$-module such that

$$\{m \in \mathcal{M} : \tau m = 0\} = \tau\mathcal{M}.$$

Then $[\mathcal{J}, \mathcal{J}] \subseteq \tau\mathcal{J}$.

We begin the proof with a version of Hensel's Lemma.

**Lemma 5.11.** Let $K$ be a field of characteristic zero and let $A$ be a $K$-algebra such that $A = K[x]$ for some $x \in A$. Suppose that $I$ is a maximal, nilpotent ideal in $A$. Then there exists $y \in A$ such that $y \equiv x \mod I$ and such that $K[y]$ is a field.

*Proof.* Let $f(X) \in K[X]$ be the monic minimial polynomial of $x+I \in A/I$. We will find a sequence of elements $x_1 := x, x_2, x_3, \cdots$ such that $f(x_m) \in I^m$ and $x_m \equiv x \mod I$ for all $m \geq 0$. Assume inductively that $f(x_m) \in I^m$ and consider

$$f(x_m + h) = f(x_m) + hf'(x_m) + \frac{h^2}{2!}f''(x_m) + \cdots$$

for some $h \in I^m$; this formal Taylor series makes sense because $h \in I^m$ is nilpotent by assumption and because $K$ has characteristic zero. Now if $f'(x_m) \in I$ then $f'(x) \in I$ since $x_m \equiv x \mod I$. So $f(X)$ divides $f'(X)$ in $K[X]$. This is impossible over a field of characteristic zero, so $f'(x_m) \notin I$. Hence $f'(x_m)$ is a unit in $A$. Since

$$f(x_m + h) \equiv f(x_m) + hf'(x_m) \mod I^{m+1}$$

we can take $h := -f(x_m)f'(x_m)^{-1}$ and $x_{m+1} := x_m + h$. This completes the induction. Now since $I$ is nilpotent, $I^n = 0$ for some $n \geq 1$ and hence $f(x_n) = 0$. But then $K[x_n]$ is a homomorphic image of the field $K[X]/\langle f(X)\rangle$, so $K[x_n]$ is the required subfield of $A$ with $x_n \equiv x \mod I$. ∎

**Definition 5.12.** Let $A$ be a commutative ring.

(a) $A$ is *local* if it has a unique maximal ideal.
(b) Let $A$ be a local ring with unique maximal ideal $J$. A *coefficient field* is a subfield $K$ of $A$ such that $K + J = A$.

Every coefficient field $K$ is isomorphic to $A/J$: $(K + J)/J \cong K/J \cap K \cong K$ because every proper ideal of a field is zero. Unfortunately coefficient fields do not exist in general, as the example $A = \mathbb{Z}/4\mathbb{Z}$ shows: this ring does not contain any subfield whatsoever. In fact coefficient fields exist in *any* commutative complete Noetherian local ring that contains a field: this is the key ingredient of the proof of Cohen's famous *Structure Theorem for complete commutative Noetherian local rings*. But we will not need the full strength of this result; the following will suffice.

**Theorem 5.13.** Every commutative local Artinian $\mathbb{Q}$-algebra has a coefficient field.

*Proof.* Let $\mathcal{S}$ be the set of subfields of the Artinian $\mathbb{Q}$-algebra $A$. It is not empty because $A$ contains a copy of the rational numbers $\mathbb{Q}$ by assumption. If $\mathcal{C}$ is a chain in $\mathcal{S}$ then $\cup\mathcal{C}$ is again a subfield of $A$, so $\cup\mathcal{C} \in \mathcal{S}$. Hence by Zorn's Lemma 2.3, $\mathcal{S}$ has a maximal element $K$. We will show that $K$ is the required coefficient field.

Let $J$ be the unique maximal ideal of $A$, fix $x \in A$ and consider the subring $K[x]$ of $A$ generated by $x$. Suppose for a contradiction that $x$ is transcendental over $K$. Then $g(x) \notin J$ for any non-zero $g(X) \in K[X]$, because $J$ is nilpotent by Proposition 2.23. Hence $g(x)$ is a unit in $A$ for all non-zero $g(X) \in K[X]$, which means that the $K$-algebra homomorphism $K[X] \to A$ which sends $X$ to $x$ factors through the field of fractions $K(X)$ of $K[X]$. Then the image $K(x)$ of $K(X)$ in $A$ is a subfield of $A$ which properly contains $K$, contradicting the maximality of $K$.

Hence $x$ is algebraic over $K$. Let $I := K[x] \cap J$; then $K[x]/I$ is isomorphic to a $K$-subalgebra of the field $A/J$, and it is generated by the algebraic element $x + I$. So $K[x]/I$ is itself a field and hence $I$ is a maximal ideal in $K[x]$. It is also nilpotent because $J$ is nilpotent, so by Lemma 5.11 we can find $y \in K[x]$ such that $y \equiv x \mod I$ and such that $K[y]$ is a field. The maximality of $K$ now forces $y \in K$, and we conclude that $x \in I + K \subseteq J + K$. Hence $A = J + K$. □

Until the end of this Chapter, we assume that:

- $\mathcal{A}$ is a left Artinian ring with unique maximal ideal $\mathcal{J}$,
- $\tau \in \mathcal{A}$ is a central element,
- $\tau^2 = 0$ and $A := \mathcal{A}/\tau\mathcal{A}$ is commutative.
- $\mathcal{M}$ is a finitely generated non-zero $\mathcal{A}$-module.

Write $J := \mathcal{J}/\tau\mathcal{A}$ and $M := \mathcal{M}/\tau\mathcal{M}$. Choose a coefficient field $K \subset A$ using Theorem 5.13: $K + J = A$. Since $J$ is nilpotent, $J^{t+1}M = 0$ and $J^tM \neq 0$ for some $t \geq 0$. Consider the following chain of $K$-subspaces of $M$:

$$0 < J^tM < J^{t-1}M < \cdots < JM < M.$$

Because $M$ and $J$ are finitely generated, each $J^nM/J^{n+1}M$ is finite dimensional over $K$, so we can find a $K$-basis $\{e_1, \ldots, e_s\}$ for $M$ such that the action of every element $x \in A$ on $M$ has upper triangular matrix with respect to this basis:

$$xe_j = \sum_{i=1}^{s} \chi(x)_{ij} e_i \quad \text{for all} \quad j = 1, \ldots, s.$$

In this way we obtain a $K$-algebra homomorphism $\chi : A \to M_s(K)$ such $\chi(x)$ is *strictly* upper triangular whenever $x \in J$: $\chi(J) \subseteq \mathfrak{n}_s^+(K)$.

Let $\mathcal{K}$ be the inverse image of $K$ in $\mathcal{A}$, so that $\mathcal{K}$ contains $\tau\mathcal{A}$ as an ideal and $\mathcal{K}/\tau\mathcal{A} = K$. Choose $\epsilon_1, \ldots, \epsilon_s \in \mathcal{M}$ such that $e_i = \overline{\epsilon_i} := \epsilon_i + \tau M$ for each $i$; then $\sum_{i=1}^s \mathcal{K}\epsilon_i + \tau\mathcal{M} = \mathcal{M}$ and hence

$$\mathcal{M} = \sum_{i=1}^{s} \mathcal{K}\epsilon_i + \tau\left(\sum_{i=1}^{s} \mathcal{K}\epsilon_i + \tau\mathcal{M}\right) = \sum_{i=1}^{s} \mathcal{K}\epsilon_i.$$

**Lemma 5.14.**

(a) For all $x \in \mathcal{J}$, there exist $\widetilde{\chi}(x) \in \mathfrak{n}_s^+(\mathcal{K})$ and $\mathcal{F}(x) \in M_s(\mathcal{K})$ such that if

$$\Phi(x) := \widetilde{\chi}(x) + \tau\mathcal{F}(x) \in M_s(\mathcal{K})$$

then

$$x\epsilon_j = \sum_{i=1}^{s} \Phi(x)_{ij}\epsilon_i \quad \text{for all} \quad j.$$

(b) For all $W \in \mathfrak{n}_s^+(A)$ there exists $W' \in \mathfrak{n}_s^+(K)$ such that

$$\sum_{i=1}^{s} W_{ij}e_i = \sum_{i=1}^{s} W'_{ij}e_i \quad \text{for all} \quad j.$$

(c) For all $x, y \in \mathcal{J}$ there exists $\Gamma(x,y) \in \mathfrak{n}_s^+(\mathcal{K})$ such that

$$xy\epsilon_j = \sum_{i=1}^{s} (\Phi(x)\Phi(y) + \tau\Gamma(x,y))_{ij}\epsilon_i \quad \text{for all} \quad j.$$

*Proof.* (a) Since $xe_j \in \sum_{i<j} Ke_i$, we can find $\widetilde{\chi}(x)_{ij} \in \mathcal{K}$ such that

$$x\epsilon_j - \sum_{i<j} \widetilde{\chi}(x)_{ij}\epsilon_i \in \tau\mathcal{M} = \sum_{i=1}^{s} \tau\mathcal{K}\epsilon_i.$$

So there is a matrix $\mathcal{F}(x) \in M_s(\mathcal{K})$ such that

$$x\epsilon_j = \sum_{i<j} \widetilde{\chi}(x)_{ij}\epsilon_i + \tau\sum_{i=1}^{s} \mathcal{F}(x)_{ij}\epsilon_i.$$

Now set $\widetilde{\chi}(x)_{ij} := 0$ whenever $i \leqslant j$. Note that $\chi(x) = \overline{\widetilde{\chi}(x)}$.

(b) Since $A = K + J$, we may assume that $W \in \mathfrak{n}_s^+(J)$. Now for any $i, j, b$,

$$W_{ij}e_b = \sum_{a=1}^{s} \chi(W_{ij})_{ab}e_a$$

and $\chi(W_{ij})_{ab} = 0$ whenever $i \geq j$ or $a \geq b$. Hence

$$\sum_{i=1}^{s} W_{ij}e_i = \sum_{i=1}^{s}\sum_{a=1}^{s} \chi(W_{ij})_{ai}e_a = \sum_{a=1}^{s}\left(\sum_{i=1}^{s} \chi(W_{ij})_{ai}\right)e_a = \sum_{i=1}^{s}\left(\sum_{a=1}^{s} \chi(W_{aj})_{ia}\right)e_i.$$

Set $W'_{ij} := \sum_{a=1}^{s} \chi(W_{aj})_{ia} \in K$. Now if $\chi(W_{aj})_{ia} \neq 0$ then $i < a$ and $a < j$. Hence $\chi(W_{aj})_{ia} = 0$ whenever $i \geq j$, so $W' \in \mathfrak{n}_s^+(K)$.

(c) By (a), we have $xy\epsilon_j = \sum_{i=1}^{s} x\Phi(y)_{ij}\epsilon_i = \sum_{i=1}^{s}([x, \Phi(y)_{ij}] + \Phi(y)_{ij}x)\epsilon_i$. Now

$$\begin{aligned}
\sum_{i=1}^{s} \Phi(y)_{ij}x\epsilon_i &= \sum_{i=1}^{s} \Phi(y)_{ij}\sum_{k=1}^{s} \Phi(x)_{ki}\epsilon_k = \\
&= \sum_{k=1}^{s}\left(\sum_{i=1}^{s} \Phi(y)_{ij}\Phi(x)_{ki}\right)\epsilon_k = \\
&= \sum_{i=1}^{s}\left(\sum_{k=1}^{s} \Phi(y)_{kj}\Phi(x)_{ik}\right)\epsilon_i.
\end{aligned}$$

Therefore

$$xy\epsilon_j = \sum_{i=1}^{s}(\Phi(x)\Phi(y) + \mathcal{E}(x,y))_{ij}\epsilon_i \quad \text{for all} \quad j$$

where

$$\mathcal{E}(x,y)_{ij} := [x, \Phi(y)_{ij}] + \sum_{k=1}^{s}[\Phi(y)_{kj}, \Phi(x)_{ik}] \in \mathcal{K}.$$

Since $\Phi(x) = \widetilde{\chi}(x) + \tau\mathcal{F}(x)$ and $[\tau\mathcal{A}, \mathcal{A}] \subseteq \tau[\mathcal{A}, \mathcal{A}] \subseteq \tau^2\mathcal{A} = 0$, we have

$$\mathcal{E}(x,y)_{ij} = [x, \widetilde{\chi}(y)_{ij}] + \sum_{k=1}^{s}[\widetilde{\chi}(y)_{kj}, \widetilde{\chi}(x)_{ik}].$$

Since $\widetilde{\chi}(x), \widetilde{\chi}(y) \in \mathfrak{n}_s^+(\mathcal{K})$, $\widetilde{\chi}(y)_{kj} \neq 0$ and $\widetilde{\chi}(x)_{ik} \neq 0$ imply that $k < j$ and $i < k$. But then $i < j$, so $\mathcal{E}(x,y)_{ij} = 0$ whenever $i \geq j$. Being a sum of commutators in $\mathcal{A}$, $\mathcal{E}(x,y)_{ij}$ is also an element of $\tau\mathcal{A}$.

Hence $\mathcal{E}(x,y) \in \mathfrak{n}_s^+(\mathcal{K}) \cap M_s(\tau\mathcal{A}) = \tau\mathfrak{n}_s^+(\mathcal{A})$. Choose $W(x,y) \in \mathfrak{n}_s^+(\mathcal{A})$ such that $\mathcal{E}(x,y) = \tau W(x,y)$; then by part (b) there is some $\Gamma(x,y) \in \mathfrak{n}_s^+(\mathcal{K})$ such that

$$\sum_{i=1}^{s} W(x,y)_{ij}\epsilon_i - \sum_{i=1}^{s}\Gamma(x,y)_{ij}\epsilon_i \in \tau\mathcal{M}.$$

Hence $\sum_{i=1}^{s}\mathcal{E}(x,y)_{ij}\epsilon_i = \tau\sum_{i=1}^{s}\Gamma(x,y)_{ij}\epsilon_i$ and therefore $xy\epsilon_j = \sum_{i=1}^{s}(\Phi(x)\Phi(y) + \tau\Gamma(x,y))_{ij}$ for all $j$. □

**Proposition 5.15.** For all $x, y \in \mathcal{J}$, there exist $\widetilde{\chi}(x) \in \mathfrak{n}_s^+(\mathcal{K})$, $\mathcal{F}(x) \in M_s(\mathcal{K})$ and $\mathcal{G}(x,y) \in \mathfrak{n}_s^+(\mathcal{K})$ such that

$$[x,y]\epsilon_j = \tau\sum_{i=1}^{s}\left([\widetilde{\chi}(x), \mathcal{F}(y)] - [\widetilde{\chi}(y), \mathcal{F}(x)] + \mathcal{G}(x,y)\right)_{ij}\epsilon_i$$

for all $j$.

*Proof.* By parts (a) and (c) of Lemma 5.14 we have

$$(xy - yx)\epsilon_j = \sum_{i=1}^{s}\left([\Phi(x), \Phi(y)] + \tau\Gamma(x,y) - \tau\Gamma(y,x)\right)_{ij}\epsilon_i \quad \text{for all} \quad j$$

where $\Phi(x) = \widetilde{\chi}(x) + \tau\mathcal{F}(x)$ and $\Gamma(x,y), \Gamma(y,x) \in \mathfrak{n}_s^+(\mathcal{K})$. Because $\tau^2 = 0$, we have

$$\begin{aligned}[\Phi(x), \Phi(y)] &= [\widetilde{\chi}(x) + \tau\mathcal{F}(x), \widetilde{\chi}(y) + \tau\mathcal{F}(y)] = \\ &= [\widetilde{\chi}(x), \widetilde{\chi}(y)] + \tau[\widetilde{\chi}(x), \mathcal{F}(y)] - \tau[\widetilde{\chi}(y), \mathcal{F}(x)].\end{aligned}$$

Now $[\widetilde{\chi}(x), \widetilde{\chi}(y)] \in \mathfrak{n}_s^+(\mathcal{K}) \cap M_s(\tau\mathcal{A}) = \tau\mathfrak{n}_s^+(\mathcal{A})$. So there is some $W(x,y) \in \mathfrak{n}_s^+(\mathcal{A})$ such that $[\widetilde{\chi}(x), \widetilde{\chi}(y)] = \tau W(x,y)$. By part (b) of Lemma 5.14, we can further find some $W'(x,y) \in \mathfrak{n}_s^+(\mathcal{K})$ such that

$$\sum_{i=1}^{s} W(x,y)_{ij}\epsilon_i - \sum_{i=1}^{s} W'(x,y)_{ij}\epsilon_i \in \tau\mathcal{M}.$$

Therefore

$$[x,y]\epsilon_j = \tau \sum_{i=1}^{s} \left( [\widetilde{\chi}(x), \mathcal{F}(y)] - [\widetilde{\chi}(y), \mathcal{F}(x)] + W'(x,y) + \Gamma(x,y) - \Gamma(y,x) \right)_{ij} \epsilon_i$$

for all $j$, and we may take $\mathcal{G}(x,y) := \Gamma(x,y) - \Gamma(y,x) + W'(x,y) \in \mathfrak{n}_s^+(\mathcal{K})$. □

Finally, we can prove Theorem 5.10.

**Theorem 5.16.** Let $\mathcal{A}$ be a left Artinian $\mathbb{Q}$-algebra with unique maximal ideal $\mathcal{J}$ and a central element $\tau \in \mathcal{J}$ such that $\tau^2 = 0$ and $\mathcal{A}/\tau\mathcal{A}$ is commutative. Suppose that $\mathcal{M}$ is a finitely generated non-zero $\mathcal{A}$-module such that

$$\{m \in \mathcal{M} : \tau m = 0\} = \tau\mathcal{M}.$$

Then $[\mathcal{J}, \mathcal{J}] \subseteq \tau\mathcal{J}$.

*Proof.* Let $x, y \in \mathcal{J}$ and write $[x,y] = \tau z$ for some $z \in \mathcal{A}$. Write $\overline{z} = \lambda + u$ with $\lambda \in K$ and $u \in J$; we have to show that $\lambda = 0$. Now by Proposition 5.15, we have

$$\tau z \epsilon_j = \tau \sum_{i=1}^{s} \mathcal{Z}_{ij} \epsilon_i \quad \text{for all} \quad j$$

where $\mathcal{Z} := [\widetilde{\chi}(x), \mathcal{F}(y)] - [\widetilde{\chi}(y), \mathcal{F}(x)] + \mathcal{G}(x,y)$. By the assumption on our module $\mathcal{M}$, we can deduce that

$$\overline{z} e_j = \sum_{i=1}^{s} \overline{\mathcal{Z}_{ij}} e_i.$$

Note that $\chi(x) = \overline{\widetilde{\chi}(x)}$ for all $x \in \mathcal{J}$, and write $F(x) := \overline{\mathcal{F}(x)}$, $G(x,y) := \overline{\mathcal{G}(x,y)}$. Because $\{e_1, \ldots, e_s\}$ is a $K$-basis for $M$ and $\chi(1)$ is the identity matrix,

$$\lambda I_s + \chi(u) = [\chi(x), F(y)] - [\chi(y), F(x)] + G(x,y)$$

inside $M_s(K)$. Now $\chi(u)$ and $G(x,y)$ are strictly upper triangular, so they have trace zero. The trace of every commutator is also zero. Therefore

$$s\lambda = \operatorname{tr}([\chi(x), F(y)] - [\chi(y), F(x)] + G(x,y) - \chi(u)) = 0.$$

Because $A$ is a $\mathbb{Q}$-algebra, we can cancel the positive integer $s$ and obtain $\lambda = 0$. □

## Appendix A. Ore's Theorem

Let $R$ be a ring, and let $S$ be a left Ore set in $R$ consisting of regular elements. We will define an equivalence relation on $S \times R$, and define the structure of a ring on the set of equivalence classes.

**Definition A.1.** Define a relation $\sim$ on $S \times R$ by setting

$$(s, a) \sim (t, b)$$

if and only if there exist $c, d \in R$ such that $ca = db$ and $cs = dt \in S$.

**Lemma A.2.**

(a) Suppose $s_1, s_2, \ldots, s_n \in S$. There exist $c_1, c_2, \ldots, c_n \in R$ and $s \in S$ such that

$$c_1 s_1 = c_2 s_2 = \cdots = c_n s_n = s.$$

(b) Suppose $(s, a) \sim (t, b)$ and $c', d' \in R$ are such that $c's = d't \in S$. Then

$$c'a = d'b.$$

*Proof.* (a) Proceed by induction on $n$. When $n = 1$ we can take $c_1 = 1$, so assume $n > 1$. By induction, we can find $b_1, b_2, \ldots, b_{n-1} \in R$ such that

$$b_1 s_1 = b_2 s_2 = \cdots = b_{n-1} s_{n-1} = u \in S,$$

say. By the left Ore condition, we can find $v \in S$ and $c_n \in R$ such that $vu = c_n s_n$. Since $u, v \in S$ and $S$ multiplicatively closed, $s := vu \in S$. So if $c_i := vb_i \in R$, then

$$c_1 s_1 = \cdots = c_{n-1} s_{n-1} = vu = c_n s_n = s.$$

(b) We have $ca = db$ and $cs = dt \in S$ for some $c, d \in R$. The left Ore condition gives $x' \in S$ and $x \in R$ such that $x'(c's) = x(cs)$. Hence $x(dt) = x(d't)$. Since $s, t \in S$ are regular, $xc = x'c'$ and $xd = x'd'$. Hence

$$x'c'a = xca = xdb = xd'b$$

but $x' \in S$ is regular so $c'a = d'b$. □

The first part of this Lemma shows that "any finite collection of denominators have a common left multiple which is a denominator", and consequently that any finite set of fractions of the form $s_i \backslash a_i$ "has a common denominator", that is, each one can be written in the form $s \backslash c_i a_i$ for some $c_i \in R$. It will also be useful to us in the technical verifications below.

**Lemma A.3.** $\sim$ is an equivalence relation on $S \times R$.

*Proof.* Since $1 \in S$, we can take $c = d = 1$ and obtain $(s, a) \sim (s, a)$ for any $a \in R, s \in S$. Hence $\sim$ is reflexive. Also, $\sim$ is clearly symmetric.

Suppose $(s, a) \sim (t, b) \sim (u, c)$. By Lemma A.2(a), we can find $d, e, f \in R$ such that $ds = et = fu \in S$. By Lemma A.2(b), $da = eb = fc$, so $(s, a) \sim (u, c)$. □

Denote the equivalence class of $(r, s) \in R \times S$ by $s\backslash r$ and let $Q$ be the set of equivalence classes.

**Definition A.4.** We define the *sum* of the elements $s\backslash a$ and $t\backslash b$ of $Q$ to be

$$s\backslash a \quad + \quad t\backslash b \quad := \quad z\backslash(xa + yb)$$

where $x, y \in R$ are any elements given by Lemma A.2(a) such that $xs = yt = z \in S$.

**Lemma A.5.** Addition is well-defined.

*Proof.* Suppose $a', b', x', y' \in R$ and $s', t', z' \in S$ are such that

$$s\backslash a = s'\backslash a', \quad t\backslash b = t'\backslash b' \quad \text{and} \quad x's' = y't' = z' \in S.$$

By the left Ore condition, we can find $u, u' \in R$ such that $uz = u'z' \in S$. Hence

$$uxs = u'x's' \in S \quad \text{and} \quad uyt = u'y't' \in S.$$

By Lemma A.2(b), since $s\backslash a = s'\backslash a'$, we have $uxa = u'x'a'$ and similarly $uyb = u'y'b'$. Hence

$$u(xa + yb) = u'(x'a' + y'b') \quad \text{and} \quad uz = u'z' \in S$$

and hence

$$z\backslash(xa + yb) = z'\backslash(x'a' + y'b')$$

So addition is independent of the choices of $a, b, s, t, x$ and $y$. □

Since any two fractions can be brought to a common left denominator by Lemma A.2(a), it is easy to verify that addition is commutative and associative, that $1\backslash 0$ is the zero element and that the additive inverse of $s\backslash a$ is $s\backslash(-a)$.

**Definition A.6.** The *product* of two elements $s\backslash a$ and $t\backslash b$ in $Q$ is

$$(s\backslash a) \cdot (t\backslash b) := (us)\backslash(cb)$$

for any $c \in R$ and $u \in S$ such that $ua = ct$ given by the left Ore condition.

**Lemma A.7.** Multiplication is well-defined.

*Proof.* First we show that this is independent of the choice of $c \in R$ and $u \in S$. Suppose that $c' \in R$ and $u' \in S$ are such that $u'a = c't$. By the left Ore condition, there exist $x, x' \in R$ with $xu = x'u' \in S$. Hence $xct = xua = x'u'a = x'c't$ so $xc = x'c'$ as $t \in S$ is regular. Hence $xcb = x'c'b$ and $xus = x'u's \in S$, so $us\backslash cb = u's\backslash c'b$.

Now, suppose that $s\backslash a = s'\backslash a'$ and $t\backslash b = t'\backslash b'$; we will find $u, u' \in S$ and $c, c' \in R$ such that $us\backslash cb = u's'\backslash c'b'$. First, we bring $t\backslash b$ and $t'\backslash b'$ to a common denominator: there exist $w, w' \in R$ such that $wt = w't' \in S$, whence $wb = w'b'$ by Lemma A.2(b). By the left Ore condtion, there exist $u, u' \in S$ and $d, d' \in R$ such that $ua = dwt$ and $u'a' = d'w't'$. Then

$$(s\backslash a)(t\backslash b) = (s\backslash a)(wt\backslash wb) = us\backslash dwb \quad \text{and similarly} \quad (s'\backslash a')(t'\backslash b') = u's'\backslash d'w'b'.$$

By Lemma A.2(a) there exist $x, x' \in R$ such that $xus = x'u's' \in S$. Since $s\backslash a = s'\backslash a'$, we obtain from Lemma A.2(b) that $xua = x'u'a'$. Hence

$$xdwt = xua = x'u'a' = x'd'w't',$$

but $wt = w't' \in S$ is regular so $xd = x'd'$. Finally, as $wb = w'b'$,

$$xdwb = x'd'w'b' \quad \text{and} \quad xus = x'u's' \in S, \quad \text{so} \quad us\backslash dwb = u's'\backslash d'w'b'.$$

Set $c := dw$ and $c' := d'w'$; then $us\backslash cb = u's'\backslash c'b'$. □

**Lemma A.8.** Multiplication in $Q$ is associative.

*Proof.* Let $s\backslash a, t\backslash b, u\backslash c \in Q$. Choose $d \in R$ and $v \in S$ such that $vb = ds$ using the left Ore condition; then $(t\backslash b)(s\backslash a) = vt\backslash da$. Now choose $e \in R$ and $w \in S$ such that $wc = evt$. Now

$$\begin{aligned} (u\backslash c)((t\backslash b)(s\backslash a)) &= (u\backslash c)(vt\backslash da) &&= wu\backslash eda, \quad \text{and} \\ ((u\backslash c)(t\backslash b))(s\backslash a) &= (wu\backslash evb)(s\backslash a) &&= wu\backslash eda \end{aligned}$$

because $t\backslash b = (vt)\backslash(vb)$ and $evb = eds$. □

**Theorem A.9.** $Q$ is a ring.

*Proof.* It is easy to check that $1\backslash 1$ is the identity element in $Q$, so by Lemmas A.5, A.7 and A.8, it remains to check that the distributive laws hold in $Q$. Note first that it follows from Definition A.6 that

$$(s\backslash a).(1\backslash b) = s\backslash ab \quad \text{and} \quad (s\backslash 1).(t\backslash b) = ts\backslash b \quad \text{for any} \quad s, t \in S \quad \text{and} \quad a, b \in R.$$

Given $\alpha = s\backslash a$ and $\beta = t\backslash b \in Q$, choose $x, y \in R$ so that $xs = yt = z \in S$. Let $c \in R$; then

$$(s\backslash a + t\backslash b)(1\backslash c) = (z\backslash(xa + yb))(1\backslash c) = z\backslash(xac + ybc) = s\backslash ac + t\backslash bc.$$

We have shown that for any $c \in R$ and $\alpha, \beta \in Q$ we have

$$(\alpha + \beta)(1\backslash c) = \alpha(1\backslash c) + \beta(1\backslash c).$$

Now if $u \in S$, we can apply this to obtain

$$(\alpha(u\backslash 1) + \beta(u\backslash 1))(1\backslash u) = \alpha + \beta$$

and right multiplying this equation by $u\backslash 1$ gives

$$(\alpha + \beta)(u\backslash 1) = \alpha(u\backslash 1) + \beta(u\backslash 1).$$

Hence we obtain the right distributive law

$$(\alpha + \beta)(u\backslash c) = (\alpha + \beta)(u\backslash 1)(1\backslash c) = (\alpha(u\backslash 1) + \beta(u\backslash 1))(1\backslash c) = \alpha(u\backslash c) + \beta(u\backslash c).$$

The left distributive law

$$(s\backslash a)(\beta + \gamma) = (s\backslash a)(\beta + \gamma)$$

is obtained in a similar manner, by writing $s\backslash a$ as the product $(s\backslash 1)(1\backslash a)$ first. □