Problem Sheet 1 (with solutions to sections A and C)

Section A

QUESTION 1. **Error Estimates for the Contraction Mapping Theorem.** Let (X, d) be a complete metric space and let $T: X \to X$ be a contractive map with constant $\kappa < 1$. Given $x_0 \in X$ consider the sequence $x_{n+1} = Tx_n$, and $x = \lim_{n\to\infty} x_n$. Show that

(1) $d(x_n, x_{n+m}) \leq \frac{\kappa^n}{1-\kappa} d(x_1, x_0)$

(2) $d(x_n, x) \leq \frac{\kappa^n}{1-\kappa} d(x_1, x_0)$

(3) $d(x_{n+1}, x) \leq \frac{\kappa}{1-\kappa} d(x_{n+1}, x_n)$

(4) $d(x_{n+1}, x) \leq \kappa d(x_n, x)$

Solution

(1) By definition of the sequence and by iterating the contraction estimate, we obtain

$$d(x_n, x_{n+1}) = d(T^n x_1, T^n x_0) \leq \kappa^n d(x_1, x_0).$$

Thus, by iterating the estimate above with the triangle inequality, and using the formula for the sum of a geometric serie we get

$$d(x_{n+m}, x_n) \leq \sum_{k=0}^{m-1} d(x_{n+k+1}, x_{n+k}) \leq \sum_{k\geq n} \kappa^n d(x_1, x_0) = \frac{\kappa^n}{1-\kappa} d(x_1, x_0).$$

(2) Enough to take the limit for $m \to \infty$ in point (1)

(3) Either repeat the argument in (1) but with

$$d(x_{n+m+1}, x_{n+m}) \leq \kappa^m d(x_{n+1}, x_n)$$

or apply (1) to the new iterates $(\tilde{x}_n)$ starting from $\tilde{x}_0 = x_n$: this gives

$$d(x_{n+1}, x_{n+1+m}) \leq \frac{\kappa}{1-\kappa} d(x_{n+1}, x_n),$$

and at this point it is enough to pass to the limit as $m \to \infty$.

(4) Recalling that $x_{n+1} = Tx_n$ and that $Tx = x$, the contraction property directly gives:

$$d(x_{n+1}, x) = d(Tx_n, Tx) \leq \kappa d(x_n, x).$$

□

QUESTION 2. **Revisions on Banach Spaces.** Which of the following spaces are Banach spaces? Please justify your answer.

(1) $C_c(\mathbb{R}) = \{u \in C(\mathbb{R})\colon \operatorname{supp}(u) \subset\subset \mathbb{R}\}$ equipped with the supremum norm $\|u\|_{sup} := \sup_{x\in\mathbb{R}}|u(x)|$.

(2) $C_V(\mathbb{R}) = \{u \in C(\mathbb{R})\colon u(x) \to 0 \text{ for } |x| \to \infty\}$ with the supremum norm $\|u\|_{sup}$.

(3) $C_b(\mathbb{R}) := \{u \in C(\mathbb{R})\colon u \text{ bounded}\}$ equipped with $\|u\| := \sup_{x\in\mathbb{R}} \frac{2+\sin(x)}{3+\cos(x)}|u(x)|$

[You may use that $(C_b(\mathbb{R}), \|\cdot\|_{sup})$ is a Banach space]

Solution

(1) $C_c(\mathbb{R})$ is not a Banach space as it is *not complete.*
E.g. Let $\varphi \in C_c(\mathbb{R})$ be a cut-off function, identically equal to 1 on $[-1/2, 1/2]$ and with $\operatorname{supp}\varphi \subset [-1, 1]$. Let $f(x) = \frac{1}{1+x^2}$ and let $f_n(x) := \varphi(x/n)\cdot f(x)$ and notice that $f_n \in C_c(\mathbb{R})$ and that $f_n \to f$ with respect to $\|\cdot\|_{\text{sup}}$. Thus, f_n is a Cauchy sequence with respect to $\|\cdot\|_{\text{sup}}$. However $f \notin C_c(\mathbb{R})$ so f_n is not a convergent sequence in $C_c(\mathbb{R})$.

(2) $C_V(\mathbb{R})$ is a Banach space. Indeed:

- It is a normed vector space as a subspace of a Banach space.
- Is is complete: Let $(f_n) \subset C_V(\mathbb{R})$ be a Cauchy sequence with respect to $\|\cdot\|_{\text{sup}}$. By completeness of $(C_b(\mathbb{R}), \|\cdot\|_{\text{sup}})$ we know that there exists $f \in C_b(\mathbb{R})$ such that $f_n \to f$ wrt $\|\cdot\|_{\text{sup}}$. It is then enough to show that $f \in C_V(\mathbb{R})$. Let's prove it. Fix $\varepsilon > 0$. From the uniform convergence, there exists $N > 0$ such that $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in \mathbb{R}$. Since $f_n \in C_V(\mathbb{R})$ then there exists $K > 0$ such that $|f_n(x)| \leq 2\varepsilon$ for all $|x| \geq K$; but then $|f(x)| \leq 2\varepsilon$ for all $|x| \geq K$.

(3) is a Banach space. Indeed $\|\cdot\|$ is a norm which is equivalent to $\|\cdot\|_{\text{sup}}$, as

$$\frac{1}{4}\|u\|_{\text{sup}} \leq \| \leq \|\frac{3}{2}\|u\|_{\text{sup}}.$$

Now, since $(C_b(\mathbb{R}), \|\cdot\|_{\text{sup}})$ is a Banach space, it follows that also $(C_b(\mathbb{R}), \|\cdot\|)$ is a Banach space (as equivalent norms give the same Cauchy sequences and the same convergent sequences).

□

QUESTION 3. **Revision on Gronwall Lemma.** Let $f\colon [t_0, t_0+c] \to [0, \infty)$ be a continuous function such that there exists two non-negative constants α and β such that

$$f(t) \leq \alpha + \beta \int_{t_0}^{t} f(s)\,ds \quad \text{for all } t \in [t_0, t_0 + c].$$

Show that

$$f(t) \leq \alpha \exp\beta(t - t_0)$$

for all $t_0 \leq t \leq t_0 + c$.

Solution You can find it the lectures notes of Differential Equations 1. Anyway, let's recall it here. Let $F(t) = \int_{t_0}^{t} f(s)\,ds$. Then

$$F'(t) \leq \alpha + \beta F(t),$$

which gives:

$$\frac{d}{dt}\left(F(t)\exp(-\beta\,t)\right) \leq \alpha\, \exp(-\beta\,t).$$

Now integrate from t_0 to t and obtain:

$$F(t) \leq \exp(\beta\,t)\,(\exp(-\beta\,t) - \exp(-\beta\,t_0))\frac{\alpha}{\beta} = \exp(\beta\,(t - t_0)(1 - \exp(-\beta\,(t - t_0)))\frac{\alpha}{\beta}.$$

So

$$f(t) \leq \alpha + \alpha\, \exp(\beta\,(t - t_0)) - \alpha = \alpha\, \exp(\beta\,(t - t_0)).$$

□

Section B

Work done in this section will be marked. Solutions will be presented in the intercollegiate classes.

QUESTION 4. **Null Lagrangian.**

(1) Give two examples of a Null-Lagrangian $L(\nabla u, u, x)$ (and explain in particular why the functions you propose are Null-Lagrangians.)

(2) Define for real $n \times n$ matrices $P \in \mathbb{R}^{n \times n}$ the map

$$L(P) = \text{tr}(P^2) - (\text{tr}(P))^2 .$$

where $\text{tr}(P)$ denotes the trace of the matrix P. Show that L is a Null-Lagrangian.

QUESTION 5. **Euler-Lagrange Equations.**

(i) Let $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^n$ be a domain. Derive the Euler-Lagrange equation for the functional

$$I(v) = \int_\Omega \frac{1}{p} |\nabla v|^p - \frac{1}{4} v^4 \, dx$$

where $v \colon \Omega \to \mathbb{R}$ and $|\nabla v| = \sqrt{(\partial_1 v)^2 + \ldots + (\partial_n v)^2}$ once by using the formula derived in the lecture and once by direct computation of $\frac{d}{dt} I(v + t\phi)$, $\phi \in C_c^\infty(\Omega)$.

(ii) Let $\Omega \subset\subset \mathbb{R}^3$ and $1 \le p \le 6$. Show that the functional

$$E(u) := \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^p}^2}$$

is well defined for all $u \in H_0^1(\Omega)$, $u \neq 0$ and satisfies $\inf\{E(u) : u \in H_0^1(\Omega)\} > 0$. Derive furthermore it's Euler-Lagrange equation.

Then consider

$$E_0(u) := \int |\nabla u|^2 dx$$

and explain what condition has to be satisfied for a function $u \in H_0^1(\Omega)$ which minimises E_0 in the set $M := \{v : \|v\|_{L^p} = 1\}$

QUESTION 6. **Counter-example to Brouwer's Fixed Point Theorem in an infinite dimensional space**. Consider the real Hilbert Space

$$l^2 = \left\{ (x_i)_{i \in \mathbb{N}} \text{ such that } \sum_{i=0}^{\infty} x_i^2 < \infty \right\} \text{ with the norm } \|x\|_{l^2} = \sqrt{\sum_{i=0}^{\infty} x_i^2}.$$

Let B be its closed unit ball.

- Consider the map

$$T \colon B \to B \text{ given by } T(x) = (\sqrt{1 - \|x\|_{l^2}^2}, x_0, x_1, x_2, \ldots).$$

Show that T is continuous and does not have a fixed point.

- Construct a continuous retraction from B to ∂B.

Section C

No work in this section will be marked. These problems are not more difficult than those in previous sections. They sit here simply because they are relevant but either slightly off or beyond the main interests of the course.

QUESTION 7. **Uniqueness of Solutions to ODEs.** Let H be a real Hilbert space endowed with the scalar product $(\cdot,\cdot)$.

(a) Show that the initial value problem for $y\colon \mathbb{R} \to H$, given by

$$y'(t) = f(t, y(t)) \text{ for } t > 0, \quad y(0) = y_0, \tag{1}$$

has at most one continuously differentiable solution on the interval $[0, T]$, provided that $f\colon \mathbb{R} \times H \to H$ is continuous and satisfies for some $L > 0$

$$(f(t,y) - f(t,z), y - z) \leq L\|y - z\|^2 \text{ for all } y, z \in H. \tag{2}$$

[Hint: Use the product rule $\frac{d}{dt}(y(t), z(t)) = (y'(t), z(t)) + (z'(t), y(t))$ for functions $y, x\colon R \to H$ and Gronwall's Lemma.]

(b) Give furthermore an example of a function f for which (2) is satisfied but for which the Lipschitz-condition of Picard's theorem does not hold.

Solution Part (a) Let y_1, y_2 be solutions of the Initial Value Problem 1. Let $g(t) := \|y_1(t) - y_2(t)\|^2$. Then

$$\begin{aligned}\frac{d}{dt}g(t) &= 2\big(y_1'(t) - y_2'(t), y_1(t) - y_2(t)\big) = 2\big(f(t, y_1(t)) - f(t, y_2(t)), y_1(t) - y_2(t)\big) \\ &\leq 2\|f(t, y_1(t)) - f(t, y_2(t))\|\,\|y_1(t) - y_2(t)\| \\ &\leq 2L\|y_1(t) - y_2(t)\|^2 \\ &= 2L\, g(t).\end{aligned}$$

So

$$\frac{d}{dt}\left(e^{-2Lt}g(t)\right) \leq 0,$$

yielding that

$$g(t) \leq e^{2Lt}g(0) = 0.$$

We conclude that $g(\cdot) \equiv 0$ and thus $y_1 \equiv y_2$.

Part (b). Let $H = \mathbb{R}$, $f(x) = x^{-\frac{1}{3}}$ (or any decreasing, non-Lipschitz function), so that $(f(x) - f(y), (x - y)) \leq 0$.

Note that we only get uniqueness for $t > 0$. With such an f the solution on $(-T, T)$ could not be unique, as the solution on $(-T, 0)$ could not be unique (e.g. look for Peano's brush on the web). □

QUESTION 8. **Equivalence between Retraction Principle and Brouwer's FPT.** Let B be the closed unit ball in $\mathbb{R}^n$. Using Brouwer's Fixed Point Theorem, show that there does not exist a retraction r from B to ∂B, i.e. a map $r : B \to \partial B$ such that r restricted to ∂B is the identity map.

Hint: by contradiction, consider the map $g(x) = -r(x)$.

Solution If $r : B \to \partial B$ is a retraction, then

$$g(x) = -r(x) \tag{3}$$

is continuous and maps B to $\partial B \subset B$. By Brouwer's Fixed point theorem there exists $x_0 \in B$ such that

$$g(x_0) = x_0. \tag{4}$$

Since $g(x_0) \in \partial B$ we infer that $x_0 \in \partial B$. Since r restricted to ∂B is the identity map, we must have

$$x_0 = r(x_0)\,. \tag{5}$$

The combination of (3), (4) and (5) gives a contradiction. □

QUESTION 9. **Application of Brouwer's FPT.** Given a map $f \in C(\mathbb{R}^n : \mathbb{R}^n)$ such that $|f(x)| \leq a + b|x|$, with $a \geq 0$ and and $0 < b < 1$, show that f has a fixed point.

Solution Choose $R > 0$ such that $|a|+bR \leq R$, i.e. $R \geq \frac{|a|}{1-b}$. Then $f : \overline{B_R(0)} \to \overline{B_R(0)}$ is continuous and has a fixed point by Brouwer's FPT. □