Problem Sheet 2

Section A

No work in this section will be marked. The material has to be considered as preliminary/bookwork.

QUESTION 1. **Mollification.**

(1) Give an example of a function (that will play later the role of kernel for mollification) with the following properties:
- $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\mathrm{supp}(\phi) = B_1(0)$;
- $\phi(x) \geq 0$ for all $x \in \mathbb{R}^n$;
- $\int_{B_1(0)} \phi(x)dx = 1$.

(2) Given ϕ as in point 1, for every function $u \in L^1_{loc}(\mathbb{R}^n)$ define
$$u \star \phi(x) := \int_{\mathbb{R}^n} u(x-y)\phi(y)dy.$$
Show that $u \star \phi \in C^\infty(\mathbb{R}^n)$.
Hint: observe that $\int_{\mathbb{R}^n} u(x-y)\phi(y)dy = \int_{\mathbb{R}^n} \phi(x-y)u(y)dy$.

(3) Given ϕ as in point 1, for every $\epsilon \in (0,1)$, let
$$\phi_\epsilon(x) := \epsilon^{-n}\phi(x/\epsilon).$$
Show that $\mathrm{supp}(\phi_\epsilon) = B_\epsilon(0)$ and that $\int_{B_\epsilon(0)} \phi_\epsilon(x)dx = 1$.

(4) If $u \in C(\mathbb{R}^n)$, show that $u \star \phi_\epsilon$ converges to u uniformly on compact subsets of $\mathbb{R}^n$.

Solution For more on mollification see the Lecture Notes of C4.3 "Functional Analytic methods for PDEs".

(1) Define $\phi(x) := 0$ for $|x| \geq 1$ and $\phi(x) := C\exp(\frac{1}{|x|^2-1})$ for $|x| < 1$, with $C > 0$ chosen such that $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$. It is easily seen that such ϕ has all the desired properties.

(2) First of all we notice that, if $u \in L^1_{loc}(\mathbb{R}^n)$ then
$$u \star \phi(x) := \int_{\mathbb{R}^n} u(x-y)\phi(y)dy$$
is well defined for all $x \in \mathbb{R}^n$. By the change of variable $z = x - y$ we directly see that
$$u \star \phi(x) := \int_{\mathbb{R}^n} u(x-y)\phi(y)dy = \int_{\mathbb{R}^n} u(z)\phi(x-z)dz = \int_{\mathbb{R}^n} \phi(x-y)u(y)dy,$$
proving the hint. Now, since ϕ is C^1 with compact support, we can use the Differentiation Theorem (it is a corollary of Dominated Convergence Theorem) to infer that
$$\partial_{x_i}(u \star \phi)(x) = \partial_{x_i}\left(\int_{\mathbb{R}^n} \phi(x-y)u(y)dy\right) = \int_{\mathbb{R}^n} (\partial_{x_i}\phi)(x-y)u(y)dy.$$
This shows that $u \star \phi$ is C^1. By iterating the procedure, we obtain that $u \star \phi$ is C^∞.

(3) Follows directly from (1) by changing variables.

(4) If $u \in C(\mathbb{R}^n)$ then it is uniformly continuous on compact subsets. Fix a compact subset $K \Subset \mathbb{R}^n$. Using that $\phi_\epsilon \geq 0$, $\int \phi_\epsilon = 1$ and that $\mathrm{supp}(\phi_\epsilon) = B_\epsilon(0)$, for every $x \in K$ we have that
$$\begin{aligned}|u(x) - u \star \phi_\epsilon(x)| &= \left|\int_{\mathbb{R}^n} (u(x) - u(x-y))\,\phi_\epsilon(y)dy\right| \leq \int_{\mathbb{R}^n} |u(x) - u(x-y)|\,\phi_\epsilon(y)dy \\ &\leq \sup_{y\in\mathbb{R}^n, |y|\leq\epsilon} |u(x) - u(x-y)|\,.\end{aligned}$$
Denote with $K_1 := \{x \in \mathbb{R}^n : \text{there exists } y \in K \text{ such that } |x-y| \leq 1\}$, (i.e. the set of points at distance at most 1 from K) and notice that K_1 is compact as well. From the previous estimate, we obtain
$$\sup_{x\in K} |u(x) - u \star \phi_\epsilon(x)| \leq \sup_{x_1,x_2\in K_1, |x_1-x_2|\leq\epsilon} |u(x_1) - u(x_2)| \to 0 \quad \text{as } \epsilon \to 0$$
by uniform continuity of u on the compact set K_1.

$\square$

QUESTION 2. **An application of Brouwer's fixed point Theorem: zero's of continuous vector fields.**

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field. Assume that there exists $R > 0$ such that

$$g(x) \cdot x \geq 0, \qquad \text{for all } x \text{ with } |x| = R.$$

Show that there exists $x^* \in \overline{B_R(0)}$ such that $g(x^*) = 0$; in other words, show that the vector field g has a zero in $\overline{B_R(0)}$.

Hint: Argue by contradiction, consider the map $f(x) := -\frac{R}{|g(x)|} g(x)$ and apply Brouwer's fixed point Theorem.

Solution. Assume that there exists no such x^*. Then we can define

$$f(x) = -R\frac{g(x)}{|g(x)|}.$$

f is continuous and $f\colon \overline{B_R(0)} \to \overline{B_R(0)}$. Brouwer's FPT implies that there exists $x_1 \in \overline{B_R(0)}$ such that $f(x_1) = x_1$. Then $|x_1| = |f(x_1)| = R$, and thus the assumption on g implies $g(x_1) \cdot x_1 \geq 0$.

On the other hand

$$g(x_1) \cdot x_1 = -f(x_1) \cdot x_1 \frac{|g(x_1)|}{R} = -\frac{|x_1|^2 \, |g(x_1)|}{R} < 0,$$

which is a contradiction. $\square$

Section B

No work in this section will be marked. Work in this section should be done by students first without looking at the solutions. Solutions will be also discussed in intercollegiate classes.

QUESTION 3. **Leray-Schauder/Schaefer Theorem.**

(a) Prove the following result.
Let X be a Banach space and $T\colon X \to X$ be a compact map with the following property: there exists $R > 0$ such that the statement ($x = \tau Tx$ with $\tau \in [0, 1)$) implies $\|x\|_X < R$. Then T has a fixed point x^* such that $\|x^*\|_X \leq R$.
Hint: Consider the operators

$$T_n(x) := \begin{cases} Tx & \textit{if } \|Tx\|_X \leq R + \frac{1}{n}, \\ \frac{R+1/n}{\|Tx\|_X} Tx & \textit{else} \end{cases}$$

on a suitable domain and prove that they are compact.

(b) Let $T : X \to X$ be a compact map with the following property: there exists $R > 0$ such that $\|Tx - x\|_X^2 \geq \|Tx\|_X^2 - \|x\|_X^2$ for all $\|x\|_X \geq R$. Show that T admits a fixed point.

Solution. Part (a). Following the hint, consider the operators

$$T_n(x) := \begin{cases} Tx & \text{if } \|Tx\|_X \leq R + \frac{1}{n}, \\ \frac{R+1/n}{\|Tx\|_X} Tx & \text{else.} \end{cases}$$

By construction, we have that $T_n : X \to \overline{B_{R+\frac{1}{n}}(0)}$. In particular, $T_n : \overline{B_{R+\frac{1}{n}}(0)} \to \overline{B_{R+\frac{1}{n}}(0)}$.

Claim: $T_n : \overline{B_{R+\frac{1}{n}}(0)} \to X$ is a compact operator.

We first observe that $T_n : \overline{B_{R+\frac{1}{n}}(0)} \to X$ is continuous, as T is continuous, $\frac{R+1/n}{\|Tx\|_X} Tx$ is continuous for $Tx \in X \setminus \overline{B_{R+\frac{1}{n}}(0)}$, and the two maps agree for $\|Tx\|_X = R + \frac{1}{n}$.

Let $(x_j)_j \subset \overline{B_{R+\frac{1}{n}}(0)}$. Since by assumption T is a compact operator, the sequence $(Tx_j)_j$ has a converging sub-sequence $(Tx_{j_k})_k$. It easily seen that $(T_n x_{j_k})_k$ converges as well. The claim is proved.

Since $T_n : \overline{B_{R+\frac{1}{n}}(0)} \to X$ is a compact operator, from the 3rd formulation of Schauder's fixed point Theorem we infer that for every $n \in \mathbb{N}$ there exists a fixed point

$$x_n \in \overline{B_{R+\frac{1}{n}}(0)}, \quad T_n x_n = x_n.$$

Claim: $Tx_n = x_n$ for all $n \in \mathbb{N}$.

By the explicit expression of T_n, if it is not true that $x_n = T_n x_n = Tx_n$, then $x_n = T_n x_n = \tau_n T x_n$ for some $\tau_n \in (0, 1]$ and $\|T_n x_n\|_X = R + \frac{1}{n}$. Using the assumption on T, we infer that if the latter holds, then $\|x_n\|_X < R$. We thus get the contradiction:

$$R + \frac{1}{n} = \|T_n x_n\|_X = \|x_n\|_X < R,$$

proving the claim.

Recalling that T is a compact operator, we obtain that the bounded sequence $(x_n = Tx_n)_n$ has a subsequence that converges to some $\bar{x} \in \overline{B_R(0)}$ which, by continuity of T, satisfies $T\bar{x} = \bar{x}$.

Part (b). In order to apply part (a) we claim that:

Claim: if $x = \tau Tx$ with $\tau \in [0, 1)$, then $\|x\|_X < R$.

If $x = \tau Tx$ then either $\tau = 0$ and thus $\|x\|_X = 0 < R$, or $\tau \in (0, 1)$ and thus

$$\begin{aligned} \|Tx - x\|_X^2 &= \|Tx - \tau Tx\|_X^2 = (1-\tau)^2 \|Tx\|_X^2 = (1 - 2\tau + \tau^2) \|Tx\|_X^2 \\ &< (1 - \tau^2) \|Tx\|_X^2 = \|Tx\|_X^2 - \|\tau Tx\|_X^2 \\ &= \|Tx\|_X^2 - \|x\|_X^2 . \end{aligned}$$

Using the assumption on T we infer that $\|x\|_X < R$, proving the claim.

We conclude by applying part (a). □

QUESTION 4. **Integral operators on $L^2(\Omega)$ vs. $C(\bar{\Omega})$** As always, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain.

(a) Let $a : \bar{\Omega} \times \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a continuous map, and let

$$A(u)(x) = \int_\Omega a(x, y, u(y))dy.$$

show that $A : C(\bar{\Omega}) \to C(\bar{\Omega})$ is well defined and compact. *Hint: use Arzela-Ascoli Theorem.*

(b) Let $k \in L^2(\Omega \times \Omega)$ and define

$$(Ku)(x) = \int_\Omega k(x, y)u(y)dy.$$

Show that $K : L^2(\Omega) \to L^2(\Omega)$ is well defined and compact. You can use for example that $C_0^\infty(\Omega \times \Omega)$ is dense in $L^2(\Omega \times \Omega)$, and therefore there is a sequence $k_m \in C_0^\infty(\Omega \times \Omega)$ such that $k_m \to k$ in $L^2(\Omega)$.

(c) Give an example of continuous a such that A (defined as above) is not well defined as an operator from $L^2(\Omega) \to L^2(\Omega)$.

Solution of (a).

Claim. $x \mapsto A(u)(x)$ is continuous from $\bar{\Omega}$ to $\mathbb{R}$.

By assumption $a : \bar{\Omega}\times\bar{\Omega}\times\mathbb{R} \to \mathbb{R}$ is continuous. If $u : \bar{\Omega} \to \mathbb{R}$ is continuous, then also $(x, y) \mapsto a(x, y, u(y))$ is continuous from $\bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$. Since $\bar{\Omega} \times \bar{\Omega}$ is compact, we infer that $(x, y) \mapsto a(x, y, u(y))$ is uniforrmly continuous, impliying that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} |A(u)(x_1) - A(u)(x_2)| &\leq \int_{\bar{\Omega}} |a(x_1, y, u(y)) - a(x_2, y, u(y))|dy \\ &\leq |\Omega|\,\varepsilon, \quad \text{for all } x_1, x_2 \in \bar{\Omega} \text{ with } |x_1 - x_2| \leq \delta. \end{aligned} \tag{1}$$

This shows the first claim and thus the fact that $A : C(\bar{\Omega}) \to C(\bar{\Omega})$ is well defined.

Claim. For every bounded subset $\mathcal{M} \subset C(\bar{\Omega})$, the image $A(\mathcal{M})$ is precompact.

In order to show the claim, it is enough to show that $A(\mathcal{M})$ is bounded and equi-continuous, as the pre-compactness would then follow from Arzelá-Ascoli's Theorem.

We first show that $A(\mathcal{M})$ is bounded. Let $C = \sup_{u\in\mathcal{M}} \|u\|_{C^0}$. We have

$$\begin{aligned} \|A(u)\|_{C^0} &= \sup_{x\in\bar{\Omega}} |A(u)(x)| \leq \sup_{x\in\bar{\Omega}} \int_\Omega |a(x, y, u(y))|dy \\ &\leq |\Omega| \sup_{(x,y,z)\in\bar{\Omega}\times\bar{\Omega}\times[-C,C]} |a(x, y, z)| =: \bar{C} < \infty, \quad \forall u \in \mathcal{M}. \end{aligned}$$

We next show that $A(\mathcal{M})$ is equi-continuous. Since $(x, y, z) \mapsto a(x, y, z)$ is uniformly continuous on the compact set $\bar{\Omega} \times \bar{\Omega} \times [-C, C]$, the same estimates as in (1) give that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|A(u)(x_1) - A(u)(x_2)| \leq |\Omega|\,\varepsilon, \quad \text{for all } u \in \mathcal{M} \text{ and all } x_1, x_2 \in \bar{\Omega} \text{ with } |x_1 - x_2| \leq \delta.$$

Solution of (b). This can be either approached directly, working with $k \in L^2(\Omega \times \Omega)$ and arguing with Fubini-Tonelli's theorem (for the good definition and continuity of K, and using Kolmogorov-Riesz for the compactness of the map K), or arguing by approximation as suggested in the hint. We take the second approach (the advantage is that, for proving the compactness, we will only need Arzelá-Ascoli and not Kolmogorov-Riesz).

Let $k_m \in C_0^\infty(\Omega \times \Omega)$ such that $k_m \to k$ in $L^2(\Omega)$. For all $u \in L^2(\Omega)$, define

$$(K_m u)(x) = \int_\Omega k_m(x, y)u(y)dy.$$

Since $k_m(x, \cdot) \in L^2(\Omega)$, we have that $(K_m u)(x)$ is well defined for all $x \in \Omega$.

Claim 1. $K_m u \in L^2(\Omega)$ for all $u \in L^2(\Omega)$ and $K_m : L^2(\Omega) \to L^2(\Omega)$ is continuous. Using Cauchy-Schwarz and Fubini-Tonelli, we get

$$\begin{aligned}\|K_m u\|_{L^2}^2 &= \int_\Omega \left(\int_\Omega k_m(x,y)u(y)dy \right)^2 dx \le \int_\Omega \|k_m(x,\cdot)\|_{L^2}^2 \|u\|_{L^2}^2 dx \\ &= \|u\|_{L^2}^2 \int_\Omega \left(\int_\Omega k_m(x,y)^2 dy \right) dx = \|u\|_{L^2}^2 \int_{\Omega\times\Omega} k_m(x,y)^2 dxdy \\ &= \|u\|_{L^2}^2 \|k_m\|_{L^2}^2. \end{aligned} \tag{2}$$

This shows that $K_m u \in L^2$. The operator $K_m : L^2(\Omega) \to L^2(\Omega)$ is clearly linear, thus the above estimate also shows the continuity of K_m as endomorphism of L^2. The proof of the claim is complete.

From (2), it also follows that

$$\|K_m u - K_n u\|_{L^2}^2 \le \|u\|_{L^2}^2 \|k_m - k_n\|_{L^2}^2 \to 0 \quad \text{as } m, n \to \infty,$$

i.e. the sequence $(K_m u)_m$ is Cauchy in L^2 and thus converges. Define

$$Ku := \lim_{m\to\infty} K_m u.$$

We next establish compactness of K by first proving the compactness of K_m and then applying a diagonal argument.

Claim 2. Fix $m \in \mathbb{N}$. For any bounded sequence $(u_n)_n \subset L^2(\Omega)$ there exists a subsequence u_{n_j} such that $(K_m u_{n_j})_j \subset L^2$ converges.
Since $k_m \in C_0^\infty(\Omega \times \Omega)$, we get that $(K_m u_n)_n$ are uniformly bounded:

$$|K_m u_n(x)| \le \|K_m\|_{L^\infty} \|u_n\|_{L^1} \le \|K_m\|_{L^\infty} |\Omega|^{1/2} \|u_n\|_{L^2}, \qquad \text{for all } x \in \Omega$$

and equicontinuous, as k_m is uniformly continuous:

$$\begin{aligned}\sup_n \sup_{|x_1-x_2|<\delta} |K_m u_n(x_1) - K_m u_n(x_2)| \le & |\Omega|^{1/2} \sup_n \|u_n\|_{L^2} \sup_{y\in\Omega, |x_1-x_2|<\delta} |k_m(x_1,y) - k_m(x_2,y)| \\ & \to 0 \quad \text{as } \delta \to 0.\end{aligned}$$

Thus the sequence $(K_m u_n)_n$ is pre-compact in $C^0(\Omega)$ by Arzelá-Ascoli Theorem, i.e. it admits a subsequence which converges uniformly, hence in particular in L^2, showing the claim.

By a diagonal argument, from claim 2, it follows that for any bounded sequence $(u_n)_n \subset L^2(\Omega)$ there exists a subsequence u_{n_j} such that $(K_m u_{n_j})_j \subset L^2$ converges for every $m \in \mathbb{N}$.
We now show the compactness of the map K:

Claim 3. For any bounded sequence $(u_n)_n \subset L^2(\Omega)$ there exists a subsequence u_{n_j} such that $(K u_{n_j})_j \subset L^2$ converges.
Let $(u_{n_j})_j$ be the subsequence such that $(K_m u_{n_j})_j \subset L^2$ converges for every $m \in \mathbb{N}$. Then

$$\begin{aligned}\|K u_{n_{j_1}} - K u_{n_{j_2}}\|_{L^2} &\le \|K u_{n_{j_1}} - K_m u_{n_{j_1}}\|_{L^2} + \|K_m u_{n_{j_1}} - K_m u_{n_{j_2}}\|_{L^2} + \|K_m u_{n_{j_2}} - K u_{n_{j_2}}\|_{L^2} \\ &\le 2 \sup_n \|u_n\|_{L^2} \|k - k_m\|_{L^2(\Omega\times\Omega)} + \|K_m(u_{n_{j_1}} - u_{n_{j_2}})\|_{L^2}.\end{aligned}$$

For every $\varepsilon > 0$ let m be such that $\|k - k_m\|_{L^2(\Omega\times\Omega)} \le \varepsilon/(2\sup_n \|u_n\|_{L^2})$. Let also $J > 0$ be such that $\|K_m(u_{n_{j_1}} - u_{n_{j_2}})\|_{L^2} < \varepsilon/2$ for all $j_1, j_2 \ge J$. Then $\|K u_{n_{j_1}} - K u_{n_{j_2}}\|_{L^2} < \varepsilon$ for all $j_1, j_2 \ge J$, i.e. it is a Cauchy sequence. The claim follows.

Note. The argument above is very similar to an argument in the lectures, where compact operators are approximated by "finite dimensional operators".

Solution of (c). Let $a(x,y,z) = z^4$. If $u \in L^2$, it is in general not true that $\int_\Omega u^4$ exists finite and thus the corresponding Au may fail to be well defined as an L^2 function.

$\square$

QUESTION 5. **Continuous maps.** Let $g \in C(\mathbb{R} \times \mathbb{R}^n)$ be such that $g(z,p) \le a + b|z|^\alpha + c|p|$, where a,b and c are non negative constants, and $2\alpha < 2^*$, where $2^* = 2n/(n-2)$ if $n \ge 3$, and $2^* = \infty$ if $n = 1, 2$. Then the map $u \mapsto g(u, \nabla u)$ is continuous from $H_0^1(\Omega)$ to $L^2(\Omega)$ and maps bounded subsets of $H_0^1(\Omega)$ to bounded subsets of $L^2(\Omega)$.

Hint: rewrite $g(u, \nabla u) = \tilde{g}\big(u, \frac{\nabla u}{|\nabla u|^\nu}\big)$ *for a suitable function* $\tilde{g}$ *and a suitable exponent* $0 < \nu < 1$, *and apply Lemma 2.6 from the lecture notes.*

Solution. From Lemma 2.6 in the lecture notes, we know that if $f \in C(\mathbb{R})$ satisfies

$$|f(x)| \le M_1 + M_2|x|^r, \quad \forall x \in \mathbb{R}^n$$

then the map $u \mapsto f(u)$ is well defined and continuous from L^p to $L^{p/r}$, and maps bounded sets to bounded sets.

If $\alpha \le 1$ and $g \in C(\mathbb{R} \times \mathbb{R}^n)$ satisfies $g(z,p) \le a + b|z|^\alpha + c|p|$, then the claim follows immediately, as $b|z|^\alpha \le b(|z|+1)$ and as the map $u \mapsto \nabla u$ is continuous from H_0^1 to L^2.

If $\alpha > 1$ then, using that by assumption $2\alpha < 2^*$ and Sobolev embedding theorem, we get that $u \mapsto u$ is continuous as a map from H_0^1 to $L^{2\alpha}$.
At the same time, consider the map

$$v \mapsto \frac{v}{|v|^\nu}, \quad 0 < \nu < 1, \tag{3}$$

which is well defined and continuous from $L^2(\Omega, \mathbb{R}^n)$ to $L^{2/(1-\nu)}(\Omega, \mathbb{R}^n)$ and maps bounded sets to bounded sets, by Lemma 2.6. Hence, choosing ν so that $\frac{2}{1-\nu} = 2\alpha$, we get that (3) is continuous from $L^2(\Omega, \mathbb{R}^n)$ to $L^{2\alpha}(\Omega, \mathbb{R}^n)$.

Combining the above, we obtain that

$$h : \; u \mapsto \left(u, \frac{\nabla u}{|\nabla u|^\nu}\right) \tag{4}$$

is well defined and continuous from $H_0^1(\Omega)$ to $L^{2\alpha}(\Omega, \mathbb{R} \times \mathbb{R}^n)$. Choose the exponent β in

$$\tilde{g}(v, w) := g(v, |w|^\beta w)$$

so that

$$g(u, \nabla u) = \tilde{g}\left(u, \frac{\nabla u}{|\nabla u|^\nu}\right),$$

i.e. choose $\beta = \nu/(1-\nu)$. Then

$$\begin{aligned} |\tilde{g}(v,w)| &\le a + b|v|^\alpha + c|w|^{\beta+1} \le \tilde{a} + b|v|^\alpha + c|w|^{1/(1-\nu)} \\ &= \tilde{a} + + b|v|^\alpha + c|w|^\alpha. \end{aligned}$$

We conclude that

$$H_0^1(\Omega) \xrightarrow{h} L^{2\alpha}(\Omega, \mathbb{R} \times \mathbb{R}^n) \xrightarrow{\tilde{g}} L^2(\Omega)$$

is well defined, continuous, and maps bounded sets to bounded sets, as composition of maps with such properties.

$\square$

Section C

No work in this section will be marked. These problems are not more difficult than those in previous sections. They sit here simply because they are relevant but either slightly off or beyond the main interests of the course.

QUESTION 6. **Leray's eigenvalue problem**. Let $K : [a,b] \times [a,b] \to (0,\infty)$ be a continuous and positive function and consider the integral operator $T : C^0([a,b]) \to C^0([a,b])$ defined by

$$(Tu)(x) = \int_a^b K(x,t)u(t)\,dt.$$

Prove that T has at least one non-negative eigenvalue λ whose eigenvector is a continuous non-negative function u, i.e. there exist $\lambda \geq 0$ and a non-negative u so that

$$\int_a^b K(x,t)u(t)\,dt = \lambda u(x).$$

Hint: consider, on an appropriate closed convex set M, the function

$$F(u) = \frac{1}{\int_a^b Tu(t)dt} \cdot Tu.$$

and apply one of the versions of Schauder's Fixed Point Theorem with the help of Arzéla-Ascoli Theorem. To find a suitable set M think about what property all functions $F(u)$ have in common.

Solution. Since $K : [a,b] \times [a,b] \to (0,\infty)$ is continuous, there exist $c_1, c_2 \in (0,\infty)$ such that

$$c_1 \leq K(x,t) \leq c_2\,, \quad \text{for all } (x,t) \in [a,b]^2.$$

We know from First year Analysis that if $u \in C^0([a,b])$, then the function $x \mapsto \int_a^b K(x,t)u(t)\,dt := Tu(x)$ is continuous on $[a,b]$ as well. Moreover, if $u \geq 0$ then we have

$$c_1 \int_a^b u(t)\,dt \leq \int_a^b K(x,t)u(t)\,dt \leq c_2 \int_a^b u(t)\,dt, \quad \text{for all } u \geq 0.$$

Consider now

$$F(u) := \frac{1}{\int_a^b Tu(t)dt} \cdot Tu.$$

Observe that $\int_a^b (Fw)(t)\,dt = 1$ for every $w \geq 0$. Then, any fixed point of F will satisfy $u(x) = (Fu)(x)$ so in particular

$$\int_a^b u(t)\,dt = \int_a^b (Fu)(t)\,dt = 1.$$

Observe that

$$M := \left\{ u \in C^0([a,b]) \;:\; u \geq 0, \int_a^b u(t)\,dt = 1, \right\}$$

is convex, closed and non-empty. In order to apply Schauder Theorem version III, we need to prove that $F : M \to M$ is continuous and that $F(M)$ is compact.

Claim 1: $F : M \to M$ is continuous.
We know that $Tu(x) \geq c_1(b-a) > 0$. It easily follows that $Fu(x) \geq 0$ for all $x \in [a,b]$ as well. Moreover, we already observed that $\int_a^b (Fu)(t)\,dt = 1$, and thus F maps M to M.
Proof that $F : M \to M$ is continuous. since the map $u \mapsto K(u)$ is continuous on $C^0([a,b])$, so also the map $u \mapsto \int_a^b K(u)(t)\,dt$ is continuous. Moreover, this is bounded from below:

$$0 < c_1|b-a|^2 \leq \int_a^b Tu(t)\,dt \leq c_2|b-a|^2. \tag{5}$$

So claim 1 follows.

Claim 2: $F(M)$ is compact.
We first show that $F(M)$ is bounded. From (5) we obtain that

$$0 \leq \frac{c_1}{c_2(b-a)} \leq F(u)(t) \leq \frac{c_2}{c_1(b-a)}$$

for all $u \in M$ and all $t \in [a,b]$. Thus $F(M)$ is bounded.
In order to show that $F(M)$ is pre-compact it is then enough to show that it is equi-continuous (so the pre-compactness will follow from Arzelá-Ascoli's Theorem).
Let $\delta > 0$, $t_{1,2}$ be such that $|t_1 - t_2| < \delta$ and let $u \in M$. Denote $\mu(u) := \int_a^b K(u)(t)\,dt$. Then

$$|F(u)(t_1) - F(u)(t_2)| \leq \frac{1}{\mu(u)} \int |K(t_1,x) - K(t_2,x)|\, u(x)\,dx \leq \frac{1}{\mu(u)} \sup_{x,|t_1-t_2|<\delta} |K(t_1,x) - K(t_2,x)|.$$

The conclusion follows by the uniform continuity of K on the compact set $[a,b]^2$ and by lower bound $\mu(u) \geq (b-a)^2 c_1 > 0$. □