Final Honour School of Mathematics Part C

C3.11 Riemannian Geometry

Mock Exam: 1 hour 45 minutes

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you:

- *start a new answer booklet for each question which you attempt.*
- *indicate on the front page of the answer booklet which question you have attempted in that booklet.*
- *cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.*
- *hand in your answers in numerical order.*

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. (a) [3 marks] Define the *Ricci curvature tensor* and the *scalar curvature* of a Riemannian manifold.

Let (M_j, g_j) be complete Riemannian manifolds of dimension $n_j \geqslant 2$ for $j = 1, 2$ and let $M = M_1 \times M_2$.

Given that $T_{(p_1,p_2)}(M_1 \times M_2) \cong T_{p_1}M_1 \times T_{p_2}M_2$ for all $(p_1, p_2) \in M_1 \times M_2$, we define a section g of $S^2T^*(M_1 \times M_2)$ by

$$g_{(p_1,p_2)}((X_1, X_2), (Y_1, Y_2)) = (g_1)_{p_1}(X_1, Y_1) + (g_2)_{p_2}(X_2, Y_2)$$

for all $X_1, Y_1 \in T_{p_1}M_1$ and $X_2, Y_2 \in T_{p_2}M_2$, for all $(p_1, p_2) \in M_1 \times M_2$.

(b) [12 marks] (i) Show that g is a complete Riemannian metric on $M_1 \times M_2$.

(ii) Determine the Ricci curvature tensor and scalar curvature of (M, g) in terms of the Ricci curvature tensors and scalar curvatures of (M_1, g_1) and (M_2, g_2).

[*You may assume that if* ∇ *is the Levi-Civita connection of* g *and* ∇_j *is the Levi-Civita connection of* g_j *for* $j = 1, 2$ *then*

$$\nabla_{(X_1,X_2)}(Y_1, Y_2) = (\nabla_1)_{X_1}Y_1 + (\nabla_2)_{X_2}Y_2$$

for all vector fields X_1, Y_1 *on* M_1 *and* X_2, Y_2 *on* M_2. *You may also use any standard results about existence and uniqueness of geodesics.*]

(c) [10 marks] Suppose further that (M_j, g_j) have constant sectional curvature K_j for $j = 1, 2$.

(i) Show that if $n_1 = n_2$ then (M, g) is Einstein if and only if $K_1 = K_2$.

(ii) Show that (M, g) is scalar-flat and Einstein if and only if $K_1 = K_2 = 0$.

(iii) Show that if (M, g) is scalar-flat and M_j are simply connected for $j = 1, 2$ then M is non-compact.

[*You may use standard facts about complete Riemannian manifolds with constant sectional curvature without proof.*]

2. Let $\gamma : [0, L] \to (M, g)$ be a normalised geodesic in a Riemannian manifold (M, g) with Riemann curvature R.

   (a) [1 mark] Define what is meant by a *Jacobi field* along γ.

   (b) [12 marks] Let J be a Jacobi field along γ with $J(0) = 0$, $|J'(0)| = 1$ and $g(J'(0), \gamma'(0)) = 0$.

      (i) Show that
$$g(\nabla_{\gamma'}\big(R(J, \gamma')\gamma'\big), X)(0) = g(R(J', \gamma')\gamma', X)(0)$$
      for all vector fields X along γ.

      (ii) Show that
$$|J(t)|^2 = t^2 - \frac{1}{3}K(\gamma'(0), J'(0))t^4 + o(t^4)$$
      for t near 0, where K is the sectional curvature of (M, g).

   (c) [2 marks] Define the *conjugate locus* of $p \in (M, g)$.

   (d) [10 marks] Let (H^2, g) be given by
$$H^2 = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 > 0\} \quad \text{and} \quad g = \frac{\mathrm{d}x_1^2 + \mathrm{d}x_2^2}{x_2^2}$$
   and let $\gamma : [0, \infty) \to (H^2, g)$ be the normalised geodesic given by
$$\gamma(t) = (0, e^t).$$

      (i) Find the Jacobi fields J_k along γ such that $J_k(0) = 0$ and $J_k'(0) = \partial_k$ for $k = 1, 2$, where ∂_1, ∂_2 are the standard vector fields on H^2.

      (ii) Deduce that for all $p \in (H^2, g)$ the conjugate locus of p in (H^2, g) is empty.

   [*You may assume that the non-zero Christoffel symbols of (H^2, g) with respect to the standard vector fields ∂_1, ∂_2 are determined by*
$$\Gamma_{11}^2 = -\Gamma_{22}^2 = \frac{1}{x_2}, \quad \Gamma_{12}^1 = -\frac{1}{x_2},$$
   and that the Riemann curvature of (H^2, g) satisfies
$$R(\partial_1, \partial_2, \partial_2, \partial_1) = -\frac{1}{x_2^4}.$$
   You may assume that for any $p, q \in (H^2, g)$ and isometry $\iota : T_pH^2 \to T_qH^2$ there is a unique isometry $f \in \mathrm{Isom}(H^2, g)$ such that $f(p) = q$ and $\mathrm{d}f_p = \iota$. You may also use standard results about existence and uniqueness of geodesics.]

3. (a) [2 marks] Define the *sectional curvature* K of a Riemannian manifold.

   (b) [12 marks] Prove each of the following statements for a compact manifold M of dimension $n \geqslant 2$.

      (i) If M admits a Riemannian metric g with sectional curvature $K \leqslant 0$ then M has an infinite fundamental group.

      (ii) If M admits a Riemannian metric with sectional curvature $K > 0$ then M has a finite fundamental group.

   For the converse of each the above statements, either prove it if it is true or give a counterexample if it is false, justifying your answer in each case.

   [*You may assume the Cartan–Hadamard, Bonnet–Myers and Synge theorems, provided they are clearly stated. You may also assume the Hopf–Rinow theorem and that if M_1, M_2 are manifolds and $\pi : M_1 \to M_2$ is a covering map of finite degree, then M_1 is compact if M_2 is compact.*]

   (c) [11 marks] Let (M, g) be a complete simply connected Riemannian manifold with $K \leqslant 0$. Let $\gamma : \mathbb{R} \to M$ be a normalised geodesic and suppose that $q \in M \setminus \gamma(\mathbb{R})$.

      (i) For $s \in \mathbb{R}$ let $\delta(s) = d(q, \gamma(s))$ and let α_s be the minimizing geodesic from q to $\gamma(s)$. Show that for the variation $f(s,t) = \alpha_s(t)$ the energy E_f of f satisfies

$$\frac{1}{2}E_f'(s) = g(\gamma'(s), \alpha_s'(\delta(s))) \quad \text{and} \quad E_f''(s) > 0.$$

      (ii) Deduce that there exist a unique minimizing geodesic α from q to a point p on γ so that α is orthogonal to γ at p, i.e. $g(\alpha', \gamma') = 0$ at p, and that p is then the closest point to q on γ.

   [*You may use the First and Second Variation Formulas and their derivations without proof.*]