C3.11 Riemannian Geometry: Essential formulae

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Throughout we have that

- (M, g) and (N, h) are Riemannian manifolds;
- $p \in M$;
- X, Y, Z, W are tangent vectors at $p \in M$ or vector fields on M as appropriate.

Essential formulae

Global formulae

Pullback

For $f : M \to (N, h)$ smooth, pullback f^*h:

$$(f^*h)_p(X, Y) = h_{f(p)}(df_p(X), df_p(Y)).$$

Levi-Civita connection

In this section ∇ is the Levi-Civita connection on (M, g).

$$\nabla_{aX+bY} Z = a\nabla_X Z + b\nabla_Y Z, \quad \nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z, \quad \nabla_X(aY) = a\nabla_X Y + X(a)Y$$

$$X\big(g(Y, Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \quad \nabla_X Y - \nabla_Y X = [X, Y]$$

Exponential map

In this section we consider the exponential map $\exp_p : B_\epsilon(0) \subseteq T_pM \to (M, g)$.

$$\mathrm{d}(\exp_p)_0 = \mathrm{id}$$

Gauss Lemma:

$$g_{\exp_p(X)}\left(\mathrm{d}(\exp_p)_X(X), \mathrm{d}(\exp_p)_X(Y)\right) = g_p(X, Y).$$

Curves and variations

In this section:

- $\alpha : [0, L] \to (M, g)$ is a curve;
- f is a variation of α;
- V_f is the variation field of f.

Length of α:

$$L(\alpha) = \int_0^L |\alpha'(t)|\mathrm{d}t = \int_0^L \sqrt{g\big(\alpha'(t), \alpha'(t)\big)}\mathrm{d}t.$$

Energy of α:

$$E(\alpha) = \int_0^L |\alpha'(t)|^2\mathrm{d}t.$$

Length and energy satisfy

$$L(\alpha)^2 \leq LE(\alpha)$$

with equality if and only if $|\alpha'|$ is constant
Energy of f:

$$E_f(s) = \int_0^L |\frac{\partial f}{\partial t}(s,t)|^2\mathrm{d}t.$$

For proper variations f of α:

$$\frac{1}{2}E_f'(0) = -\int_0^L g(V_f, \nabla_{\alpha'}\alpha')\mathrm{d}t.$$

If α is a geodesic and f is a proper variation:

$$\frac{1}{2}E_f''(0) = -\int_0^L g(V_f'' + R(V_f,\gamma')\gamma', V_f)\mathrm{d}t = \int_0^L g(V_f', V_f') - R(V_f,\gamma',\gamma',V_f)\mathrm{d}t.$$

Curvature

In this section:

- ∇ is the Levi-Civita connection on (M, g);
- $\{E_1, \ldots, E_n\}$ is an orthonormal frame for T_pM.

Riemann curvature operator:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Riemann curvature tensor:

$$R(X,Y,Z,W) = g(R(X,Y)Z,W)$$

Properties of R:

$$R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z) = R(Z,W,X,Y),$$

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0.$$

Sectional curvature of 2-plane $\sigma = \mathrm{Span}\{X,Y\}$:

$$K(\sigma) = K(X,Y) = \frac{R(X,Y,Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

Ricci curvature:

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^n R(E_i, X, Y, E_i)$$

Scalar curvature S:

$$S(p) = \sum_{i,j=1}^n R(E_i,E_j,E_j,E_i) = \sum_{j=1}^n \mathrm{Ric}(E_j,E_j)$$

(M, g) has constant sectional curvature K if and only if

$$R(X,Y,Z,W) = K\big(g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\big).$$

Jacobi fields

In this section:

- γ is a geodesic in (M, g) with $\gamma(0) = p$;
- J is a Jacobi field along γ;
- $\exp_p$ is the exponential map at p.

Jacobi equation:

$$J'' + R(J, \gamma')\gamma' = 0.$$

If $J(0) = 0$ then

$$J(s) = \mathrm{d}(\exp_p)_{s\gamma'(0)}(sJ'(0)).$$

Multiplicity of conjugate point $\gamma(T)$ to p along γ:

$$\mathrm{mult}(\gamma(T)) = \dim \mathrm{Ker}(\mathrm{d}(\exp_p)_{T\gamma'(0)}).$$

Riemannian submanifolds

In this section:

- (M, g) is an n-dimensional Riemannian submanifold of (N, h);
- ∇^M, ∇^N are Levi-Civita connections on (M, g), (N, h) respectively;
- ξ is a normal vector field along M;
- R^M, R^N are Riemann curvature operators on (M, g), (N, h) respectively.

Second fundamental form

Second fundamental form:

$$B(X, Y) = \nabla^N_X Y - \nabla^M_X Y = (\nabla^N_X Y)^\perp.$$

Properties of B:

$$B(Y, X) = B(X, Y) \quad \text{and} \quad B(aX + bY, Z) = aB(X, Z) + bB(Y, Z).$$

Shape operator:

$$g(S_\xi(X), Y) = g(B(X, Y), \xi) = g(X, S_\xi(Y)).$$

$$S_\xi(X) = -(\nabla^N_X \xi)^T.$$

Gauss equation:

$$g(R^N(X, Y)Z, W) = g(R^M(X, Y)Z, W) + g(B(X, Z), B(Y, W)) - g(B(X, W), B(Y, Z))$$

Minimality (for $\{E_1, \ldots, E_n\}$ orthonormal frame on (M, g)):

$$\sum_{i=1}^{n} B(E_i, E_i) = 0 \quad \Leftrightarrow \quad \mathrm{tr}(S_\xi) = 0 \text{ for all } \xi.$$

Hypersurfaces

In this section:

- (M, g) is an n-dimensional hypersurface in (N, h);
- ν is a normal vector field along M;
- $\{E_1, \dots, E_n\}$ is an orthonormal basis of T_pM consisting of eigenvectors of shape operator S_ν.

Principal curvatures λ_i and principal directions E_i:

$$S_\nu E_i = \lambda_i E_i.$$

Gaussian curvature:

$$K_M(p) = \prod_{i=1}^n \lambda_i(p)$$

Mean curvature:

$$H_M(p) = \frac{1}{n}\sum_{i=1}^n \lambda_i(p)$$

If $(N, h) = (\mathbb{R}^{n+1}, g_0)$, derivative of Gauss map satisfies:

$$\mathrm{d}\nu_p = -S_{\nu(p)}$$

Local coordinate formulae

In this section:

- $(x_1, \dots, x_n)$ are local coordinates on n-dimensional (M, g);
- $X_1, \dots, X_n$ are corresponding coordinate vector fields;
- ∇ is Levi-Civita connection and R is Riemann curvature operator of (M, g).

Metric:

$$g_{ij} = g(X_i, X_j).$$

Christoffel symbols:

$$\nabla_{X_i} X_j = \sum_{k=1}^n \Gamma_{ij}^k X_k.$$

Riemannian volume form:

$$\sqrt{\det(g_{ij})}\, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n.$$

Geodesics

Geodesic equations:

$$x_k'' + \sum_{i,j=1}^n \Gamma_{ij}^k x_i' x_j' = 0 \quad \text{for } k = 1 \dots, n.$$

Let

$$L = \frac{1}{2}\sum_{i,j} g_{ij} x_i' x_j'.$$

Then geodesic equations are equivalent to:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x_k'}\right) - \frac{\partial L}{\partial x_k} = 0 \quad \text{for } k = 1, \dots, n$$

Symmetry lemma and proposition

In this section:

- $f : A \subseteq \mathbb{R}^2 \to (M, g)$ is a smooth function of $(u, v) \in A$ for suitable A;
- X is a vector field along f.

Symmetry Lemma:

$$\nabla_{\frac{\partial f}{\partial u}} \frac{\partial f}{\partial v} = \nabla_{\frac{\partial f}{\partial v}} \frac{\partial f}{\partial u}.$$

Symmetry Proposition

$$\left(\nabla_{\frac{\partial f}{\partial u}} \nabla_{\frac{\partial f}{\partial v}} - \nabla_{\frac{\partial f}{\partial v}} \nabla_{\frac{\partial f}{\partial u}}\right) X = R(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v})X.$$

Useful but not essential formulae

Global formulae

Koszul formula for Levi-Civita connection ∇ on (M, g):

$$g(\nabla_X Y, Z) = \frac{1}{2}\Big(X\big(g(Y,Z)\big) + Y\big(g(Z,X)\big) - Z\big(g(X,Y)\big) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y])\Big).$$

If (M, g) has constant sectional curvature K then

$$\mathrm{Ric} = (n-1)K \quad \text{and} \quad S = n(n-1)K.$$

Hyperbolic n-space $(\mathcal{H}^n, g)$:

$$\mathcal{H}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \,:\, \sum_{i=1}^n x_i^2 - x_{n+1}^2 = -1,\, x_{n+1} > 0\},$$

and g is the restriction of

$$\sum_{i=1}^n \mathrm{d}x_i^2 - \mathrm{d}x_{n+1}^2.$$

Poincaré disk model (B^n, g) of hyperbolic n-space:

$$B^n = \{(y_1, \ldots, y_n) \in \mathbb{R}^n \,:\, \sum_{i=1}^n y_i^2 < 1\} \quad \text{and} \quad g = \sum_{i=1}^n \frac{4\mathrm{d}y_i^2}{(1 - \sum_{i=1}^n y_i^2)^2}.$$

Upper-half space model (H^n, g) of hyperbolic n-space:

$$H^n = \{(z_1, \ldots, z_n) \in \mathbb{R}^n \,:\, z_n > 0\} \quad \text{and} \quad g = \sum_{i=1}^n \frac{\mathrm{d}z_i^2}{z_n^2}.$$

Local formulae

In this section:

- $(x_1, \ldots, x_n)$ are local coordinates on n-dimensional (M, g);
- $X_1, \ldots, X_n$ are corresponding coordinate vector fields;
- ∇ is Levi-Civita connection of (M, g).

Christoffel symbols:

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^n g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

For curve α in (M, g) and smooth functions $a_1, \ldots, a_n$ along α:

$$\nabla_{\alpha'} \sum_{i=1}^n a_i X_i = \sum_{k=1}^n \left(a_k' + \sum_{i,j=1}^n \Gamma_{ij}^k a_i x_j'\right) X_k.$$