C3.5 Lie Groups

Sheet 3 — MT23

Section A contains an introductory question. Section B contains material to test understanding of the course. Section C contains further questions which are optional. Only answers to Section B should be submitted for marking.

Section A

1. Let $\mathfrak{sl}(2,\mathbb{R}) = \{A \in M_2(\mathbb{R}) \,:\, \operatorname{tr} A = 0\}$.

   (a) Show that $\mathfrak{sl}(2,\mathbb{R})$ is a Lie algebra with basis

   $$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

   and work out the bracket relations for e, f, h.

   (b) Show that $\mathfrak{sl}(2,\mathbb{R})$ is not isomorphic to $\mathfrak{su}(2)$.

Solution:

(a) Since $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, the trace of any commutator of square matrices is zero, so $\mathfrak{sl}(2,\mathbb{R})$ is a Lie algebra when endowed with the matrix commutator. Any traceless 2×2 real matrix is uniquely expressible in the form

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

for real numbers a, b and c, so $\{e, f, h\}$ is a basis for $\mathfrak{sl}(2,\mathbb{R})$. By direct calculation we have

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

(b) Recall $\mathfrak{su}(2) \cong (\mathbb{R}^3, \times)$ where $\times$ is the cross product. Any two linearly independent elements of $\mathbb{R}^3$ generate $(\mathbb{R}^3, \times)$ since $v \times w$ is orthogonal to both v and w and is non-zero if v and w are linearly independent. Therefore $\mathfrak{su}(2)$ has no 2-dimensional Lie subalgebras. However $\mathbb{R}\{h, e\}$ is a 2-dimensional Lie subalgebra of $\mathfrak{sl}(2,\mathbb{R})$. Thus $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$ cannot be isomorphic.

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Section B

2. (a) Let $\varphi : G_1 \to G_2$ be a Lie group homomorphism. Show that $\ker\varphi \subseteq G_1$ is a closed (hence embedded) Lie subgroup with Lie algebra $\ker(\mathrm{d}\varphi_e) \subseteq \mathfrak{g}_1$.

   (b) A vector subspace $J \subseteq (V, [\cdot,\cdot])$ of a Lie algebra is called an *ideal* if $[v, j] \in J$ for all $v \in V, j \in J$. Show that ideals are Lie subalgebras.

   (c) Let H be a Lie subgroup of G, with H, G connected. Show that H is a normal subgroup of $G \Leftrightarrow \mathfrak{h} \subseteq \mathfrak{g}$ is an ideal.

   [*You may find it helpful to show that* $ge^Y g^{-1} = e^{\mathrm{Ad}(g).Y}$ *for* $g \in G$ *and* $Y \in \mathfrak{g}$.]

   (d) The *centre* of a Lie algebra $(V, [\cdot,\cdot])$ is

   $$Z(V) = \{v \in V : [v, w] = 0 \text{ for all } w \in V\}.$$

   For G connected, prove that the centre of the group G is

   $$Z(G) = \ker(\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}))$$

   (e) Deduce that the centre of G is a closed (hence embedded) Lie subgroup of G which is abelian, normal and has Lie algebra $\mathrm{Lie}(Z(G)) = Z(\mathfrak{g})$.

   (f) Finally deduce that, for G connected, G is abelian $\Leftrightarrow \mathfrak{g}$ is abelian.

3. (a) Show that if X, Y belong to the Lie algebra of a Lie group G then

   $$[X, Y] = 0 \Rightarrow \exp(X + Y) = \exp(X)\exp(Y).$$

   (b) Prove that if G is a connected Lie group with $Z(G) = \{e\}$ then G can be identified with a Lie subgroup of $\mathrm{GL}(N, \mathbb{R})$, for some N, so $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(N, \mathbb{R})$.

   (c) If $(V, [\cdot,\cdot])$ is a Lie algebra with $Z(V) = \{0\}$, show that V is the Lie algebra of some Lie group.

4. Find all the connected Lie subgroups of $\mathrm{SO}(3)$.

5. (a) Show that Lebesgue measure is the bi-invariant Haar measure on $\mathbb{R}^n$ viewed as an additive group.

   (b) Find the bi-invariant Haar measure on $(\mathbb{R}_{>0}, \times)$, the multiplicative group of positive reals.

6. Give an example of an *irreducible* representation of S^1 on $\mathbb{R}^2$. Describe what happens to this representation when we complexify it.

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Section C

7. (a) Let $\phi : G \to \mathrm{Aut}(V)$ be a representation. If $\alpha : G \to G$ is an automorphism show that $\phi \circ \alpha$ is another representation on the same vector space.

   (b) If $\alpha(g) = hgh^{-1}$ for some $h \in G$ show that the two representations are equivalent.

   (c) Give an example of an automorphism where the two representations are not equivalent.

8. Consider the action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$ and let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth real-valued function.

   (a) For $A \in \mathrm{SO}(3)$ show that $(Af)(x) = f(A^{-1}x)$ defines an action of $\mathrm{SO}(3)$ on the space of all smooth functions.

   (b) If $r^2 = x_1^2 + x_2^2 + x_3^2$ show that $Af = f$.

   (c) Let Δ denote the Laplace operator

   $$\Delta f = \sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i^2}.$$

   Show that $A\Delta f = \Delta A f$.

   (d) Consider the vector space of functions of the form $f = p$ where $p(x_1, x_2, x_3)$ is a homogeneous polynomial of degree m. Show that this is a finite-dimensional representation V_m of $\mathrm{SO}(3)$ and calculate its dimension.

   (e) Let $H_m \subseteq V_m$ be the subspace of solutions to $\Delta f = 0$ for $f \in V_m$, the harmonic polynomials of degree m. Show that H_m is a representation space for $\mathrm{SO}(3)$ and that $V_2 = H_2 \oplus r^2 H_0$ and $V_3 = H_3 \oplus r^2 H_1$ are decompositions into inequivalent representations.

   (f) Can you generalize this?

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