C3.5 Lie Groups

Sheet 4 — MT23

Section A contains an introductory question. Section B contains material to test understanding of the course. Section C contains further questions which are optional. No solutions should be submitted for marking.

Section A

1. Check the following properties hold for a character χ_V associated to a representation V of a compact Lie group G:

   (a) $\chi_V(e) = \dim V$;

   (b) χ_V is invariant under conjugation, $\chi_V(hgh^{-1}) = \chi_V(g)$;

   (c) $\chi_V = \chi_W$ for equivalent reps $V \simeq W$;

   (d) $\chi_{V\oplus W} = \chi_V + \chi_W$;

   (e) $\chi_{V\otimes W} = \chi_V \cdot \chi_W$;

   (f) $\chi_{V^*}(g) = \chi_V(g^{-1})$ for all $g \in G$;

   (g) if V is unitary, $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ for all $g \in G$.

   Solution:

   (a) We have $\chi_V(e) = \mathrm{trace}(\mathrm{id}_V) = \dim V$.

   (b) This follows from the identity $\mathrm{trace}(PAP^{-1}) = \mathrm{trace}(AP^{-1}P) = \mathrm{trace}(A)$ for (invertible) square matrices A and P.

   (c) The fact that $\chi_V = \chi_W$ for equivalent representations V and W is immediate from the definitions.

   (d) A basis for $V \oplus W$ is given by taking a union of bases for V and W. It follows immediately that $\chi_{V\oplus W} = \chi_V + \chi_W$.

   (e) We may assume without loss of generality that the representations are unitary. Take $g \in G$. Then we may choose bases $\{v_i\}$ and $\{w_j\}$ for V and W respectively consisting of eigenvectors for the multiplication by g map, say

$$gv_i = \lambda_i v_i, \quad gw_j = \mu_j w_j.$$

Prof Jason D. Lotay: jason.lotay@maths.ox.ac.uk

Then $\{v_i \otimes w_j\}$ forms a basis for $V \otimes W$ and

$$g(v_i \otimes w_j) = \lambda_i \mu_j (v_i \otimes w_j).$$

Therefore

$$\chi_{V \otimes W}(g) = \sum_{i,j} \lambda_i \mu_j = \left(\sum_i \lambda_i \right) \left(\sum_j \mu_j \right) = \chi_V(g) \cdot \chi_W(g).$$

(f) To show that $\chi_{V^*}(g) = \chi_V(g^{-1})$, use a basis of eigenvectors v_i as in the previous part. Let $\{v_i^*\}$ be the corresponding dual basis. Then

$$(g \cdot v_i^*)(v_j) = v_i^*(g^{-1} v_j) = v_i^*(\lambda_j^{-1} v_j) = \lambda_i^{-1} \delta_{ij},$$

so $g \cdot v_i^* = \lambda_i^{-1} v_i^*$, which then gives the result.

(g) Continuing from the previous part, unitarity implies $\lambda_i^{-1} = \overline{\lambda_i}$. The equality $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ follows.

Prof Jason D. Lotay: jason.lotay@maths.ox.ac.uk

Section B

2. Recall that the irreducible representation V_n of $\mathrm{SU}(2)$ is given by the space of homogeneous polynomials of degree n in two variables (z_1, z_2) with

$$(A \cdot p)(\mathbf{z}) = p(A^{-1}\mathbf{z}), \quad A \in \mathrm{SU}(2),\ p \in V_n, \mathbf{z} = (z_1, z_2).$$

(a) Which of the irreducible representations V_n of $\mathrm{SU}(2)$ may be regarded as representations of $\mathrm{SO}(3)$?

(b) Deduce that for each natural number n we have a real $(2n+1)$-dimensional representation W_n of $\mathrm{SO}(3)$.

(c) Show further that the character of W_n is given by

$$\sum_{k=0}^{2n} e^{i(n-k)t}.$$

Solution:

(a) We have an isomorphism $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\pm 1\}$, so V_n descends to a representation of $\mathrm{SO}(3)$ if and only if $A \cdot p = (-A) \cdot p$ for all $A \in \mathrm{SU}(2)$ and all $p \in V_n$, if and only if n is even.

(b) Recall that the map $(z_1, z_2) \mapsto (z_2, -z_1)$, extended to a complex anti-linear map $J : V_{2n} \to V_{2n}$, defines a real structure on V_{2n} (recalling that all polynomials in V_{2n} are even degree homogeneous polynomials). This descends to give the $(2n+1)$-dimensional $W_n = V_{2n}^J$.

(c) We know from the lecture notes that

$$\chi_{V_{2n}}\left(\mathrm{diag}(e^{it}, e^{-it})\right) = \sum_{k=0}^{2n} e^{2it(n-k)}.$$

Recall from Sheet 2 that we have a two-to-one covering map

$$\mathcal{A} : \mathrm{SU}(2) \cong \mathrm{Sp}(1) \to \mathrm{SO}(3)$$

where $\mathcal{A}_q$ is given by conjugation by q on quaternions for $q \in \mathrm{Sp}(1)$. The matrix $R(t) = \mathrm{diag}(e^{it}, e^{-it})$ corresponds to the unit quaternion $q(t) = \cos(t) + i\sin(t) \in \mathrm{Sp}(1)$. Explicitly computing $\mathcal{A}_{q(t)}$ on the basis $\{i, j, k\}$ of $\mathrm{Im}(\mathbb{H})$ gives

$$\mathcal{A}(R(t)) = S(2t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2t) & -\sin(2t) \\ 0 & \sin(2t) & \cos(2t) \end{pmatrix}.$$

Prof Jason D. Lotay: jason.lotay@maths.ox.ac.uk

The matrices $\{S(t) : t \in \mathbb{R}\}$ form a maximal torus in $SO(3)$ (by considering dimensions), and

$$\chi_{W_n}(S(t)) = \chi_{V_{2n}}(R(t/2)) = \sum_{k=0}^{2n} e^{it(n-k)}.$$

3. Show that a maximal torus in a compact Lie group is maximal among connected abelian subgroups.

Solution: Suppose A is a connected abelian subgroup of G with $T \subset A$. We claim $\overline{A}$ is also an abelian subgroup of G. Assuming this is true for the moment, $\overline{A}$ is a closed (hence embedded) subgroup of G which is abelian and connected, so is a torus, hence $T = \overline{A} = A$ by the maximality of T.

We first show $X = \overline{A}$ is a group. Take g, $h \in H$. Let U be an open neighbourhood of gh. Multiplication is continuous so there exists open neighbourhoods V, W of g, h respectively with $VW \subset U$. As V and W both meet A then $A \cap VW \neq \emptyset$, so $gh \in X$. Similarly if U is an open neighbourhood of g^{-1} then we can find an open neighbourhood V of g with $V^{-1} \subset U$. V meets A so V^{-1} also meets A and so $g^{-1} \in X$. Therefore X is a subgroup of G.

Suppose for a contradiction there exists $g, h \in X$ with $gh \neq hg$. As G is Hausdorff we may separate gh and hg by disjoint open sets U and V. We may find open neighbourhoods P_g, Q_g of g and P_h, Q_h of h with $P_gP_h \subset U$ and $Q_gQ_h \subset V$. Let $R = P_g \cap Q_g$ and $S = P_h \cap Q_h$. R and S meet A so there exists $r \in A \cap R$ and $s \in A \cap S$. As A is abelian then $rs = sr$, which contradicts the fact U and V are disjoint. Therefore X is abelian.

4. Find the Weyl group of the unitary group $\mathrm{U}(n)$, justifying your answer.

Solution: A torus in $\mathrm{U}(n)$ is given by

$$T = \left\{ \mathrm{diag}(e^{it_1}, \ldots, e^{it_n}) : t_1, \ldots, t_n \in \mathbb{R} \right\}.$$

We claim that T is in fact a maximal torus. Suppose T' is another torus in $U(n)$ containing T. Take $g \in T'$. Then g commutes with any $t \in T$. Choosing t such that the diagonal entries of t are pairwise distinct, by direct computation we see that g itself must be diagonal, so $g \in T$ and $T = T'$ is a maximal torus.

$\mathrm{U}(n)$ acts naturally on the vector space $\mathbb{C}^n$, which we give the standard basis $e_1, \ldots, e_n$. The e_i are clearly eigenvectors of any $t \in T$. Suppose $g \in N(T)$ and $t \in T$. Then $g^{-1}tg \in T$ also has as eigenvectors all of the e_i, so $ge_i = \lambda_i e_{j_i}$ for some standard basis vector e_{j_i}.

Prof Jason D. Lotay: jason.lotay@maths.ox.ac.uk

Therefore there exists a permutation matrix P_σ (for some $\sigma \in S_n = \mathrm{Sym}(\{1,\ldots,n\})$) such that $P_\sigma^{-1}g$ is diagonal. But $P_\sigma \in U(n)$ so $P_\sigma^{-1}g \in T$, thus $g \equiv P_\sigma \bmod T$. Hence there exists a surjective homomorphism $S_n \to W$ given by $\sigma \mapsto P_\sigma$. If P_σ, P_τ are congruent mod T then by considering the actions on $e_1,\ldots,e_n$ we have $\sigma = \tau$, so the map $S_n \to W$ is an isomorphism. Consequently the Weyl group is given by S_n.

5. Let B denote the subgroup of $\mathrm{GL}(3,\mathbb{C})$ consisting of invertible matrices of the form

$$\begin{pmatrix} \alpha & a & b \\ 0 & \beta & c \\ 0 & 0 & \gamma \end{pmatrix} \qquad \text{with } a,b,c \in \mathbb{C} \text{ and } \alpha,\beta,\gamma \in \mathbb{C}^*.$$

   (a) Show that B is an embedded Lie subgroup.

   (b) Show that there is a homomorphism ϕ from B onto the *complex torus* $T_{\mathbb{C}} \cong (\mathbb{C}^*)^3$ of diagonal elements of B.

   (c) Show that $\ker\phi$ may be identified with the subgroup U consisting of elements of B with diagonal entries equal to 1.

   (d) Show further that the elements of U with $a = c = 0$ form a normal subgroup of U.

   (e) What are the maximal compact connected subgroups of $T_{\mathbb{C}}$, B and U? (You need not give detailed proofs).

Solution:

(a) From standard linear algebra B is a group, and clearly B is closed in $GL(3,\mathbb{C})$, so is an embedded Lie subgroup.

(b) Define

$$\phi : \begin{pmatrix} \alpha & a & b \\ 0 & \beta & c \\ 0 & 0 & \gamma \end{pmatrix} \mapsto (\alpha,\beta,\gamma).$$

This is easily seen to be a surjective homomorphism of Lie groups.

(c) This is immediate from the definition of ϕ.

(d) We have the identities

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+a' & b+b'+ac' \\ 0 & 1 & c+c' \\ 0 & 0 & 1 \end{pmatrix}$$

Prof Jason D. Lotay: jason.lotay@maths.ox.ac.uk

and
$$\begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}1 & -a & ac-b\\ 0 & 1 & -c\\ 0 & 0 & 1\end{pmatrix}.$$
The result then follows by inspection.

(e) First consider $T_{\mathbb{C}}$. We use the fact that $\mathbb{C}^* \cong S^1 \times (0,\infty)$, so $T_{\mathbb{C}} \cong A := (S^1)^3 \times (0,\infty)^3$. The only compact subgroup of $(0,\infty)$ is the trivial group, for if $x \neq 1$ then $\{x^n : n \in \mathbb{Z}\}$ is not bounded above. $(S^1)^3$ is a compact connected subgroup of A. Therefore $(S^1)^3$ must the unique maximal compact connected subgroup of A. Back in $T_{\mathbb{C}}$ this corresponds to the subgroup

$$M_T = \{(e^{it_1}, e^{it_2}, e^{it_3}) : t_1, t_2, t_3 \in \mathbb{R}\}.$$

Now suppose G is a compact connected subgroup of U. As G is compact then G is bounded. Therefore the off diagonal entries of any element of G must all be zero, so the only compact subgroup of U is the trivial group.

Finally we consider B. Let K be a compact subgroup of B containing M_T; we claim $K = M_T$. Projecting to $T_{\mathbb{C}}$, we see that any element of K has unit complex numbers on the diagonal. Take $k \in K$. We may write $k = mu$ with $m \in M_T$ and $u \in U$; by assumption $m \in K$ so $u \in K$ as well. The element u lies in the closure L of $\langle u \rangle$, a closed hence embedded subgroup of K. As K is compact then so is L, but then L is a compact subgroup of U so is trivial, hence $u = I$. Consequently $K = M_T$, which implies that M_T (and any conjugate of M_T) is a maximal compact connected subgroup. We claim that these are all of the maximal compact connected subgroups of B.

Let K be a compact connected subgroup of B. Consider the standard representation V of K on $\mathbb{C}^3$. As K is compact then V is a semisimple $\mathbb{C}K$-module. We may decompose V as
$$V = \mathbb{C}e_1 \oplus W$$
where $\{e_1, e_2, e_3\}$ is the standard basis for $\mathbb{C}^3$ and W is an irreducible 2-dimensional $\mathbb{C}K$-module or is the direct sum of two 1-dimensional $\mathbb{C}K$-modules. There exists $\lambda \in \mathbb{C}$ such that $w = e_2 + \lambda e_1 \in W$. Take
$$k = \begin{pmatrix}\alpha & a & b\\ 0 & \beta & c\\ 0 & 0 & \gamma\end{pmatrix} \in K.$$

Then $kw = (a + \alpha\lambda)e_1 + \beta e_2 \in W$. The vectors kw and w cannot be linearly independent as the vectors $\{e_1, w, kw\}$ do not span $V = \mathbb{C}^3$, so $kw = \beta w$ by

Prof Jason D. Lotay: jason.lotay@maths.ox.ac.uk

comparing e_2 coordinates. Then $\mathbb{C}w$ is a submodule of W, so W splits as a direct sum $W = \mathbb{C}w \oplus \mathbb{C}w'$ for some $w' \in W$; by a similar argument it can be shown that $kw' = \gamma w'$. The change of basis matrix from $\{e_1, e_2, e_3\}$ to $\{e_1, w, w'\}$ then gives an element $g \in B$ such that $K \subset gM_Tg^{-1}$; it follows that any compact subgroup of B is a conjugate of a subgroup of M_T and hence the set of all maximal compact connected subgroups of B is

$$\{gM_Tg^{-1} : g \in B\}.$$

Prof Jason D. Lotay: jason.lotay@maths.ox.ac.uk

Section C

6. Let G be a compact Lie group and $C(G)$ the space of complex-valued continuous functions on G. Define a product (the *convolution product*) by

$$(f_1 * f_2)(h) = \int_G f_1(hg^{-1}) f_2(g).$$

   (a) Show that $(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$.

   (b) Prove that convolution is commutative if the group is abelian.

   (c) Let $\rho : G \to \mathrm{Aut}(V)$ be a representation of G and $f \in C(G)$ a function. Define $\rho(f) \in \mathrm{End}V$ by

$$\rho(f) = \int_G f(g)\rho(g)$$

   Show that $\rho(f_1 * f_2) = \rho(f_1)\rho(f_2)$.

   (d) Use this to give an example of a group where the convolution product is not commutative.

7. Suppose that G is compact Lie group and that the continuous function f satisfies $f(hgh^{-1}) = f(g)$ for all h. If ρ is an irreducible representation with character χ show that $\rho(f) = \alpha \cdot \mathrm{id}_V$ where

$$\alpha = \frac{1}{\dim V}\langle f, \bar{\chi}\rangle.$$

Prof Jason D. Lotay: jason.lotay@maths.ox.ac.uk