Mathematical Modelling

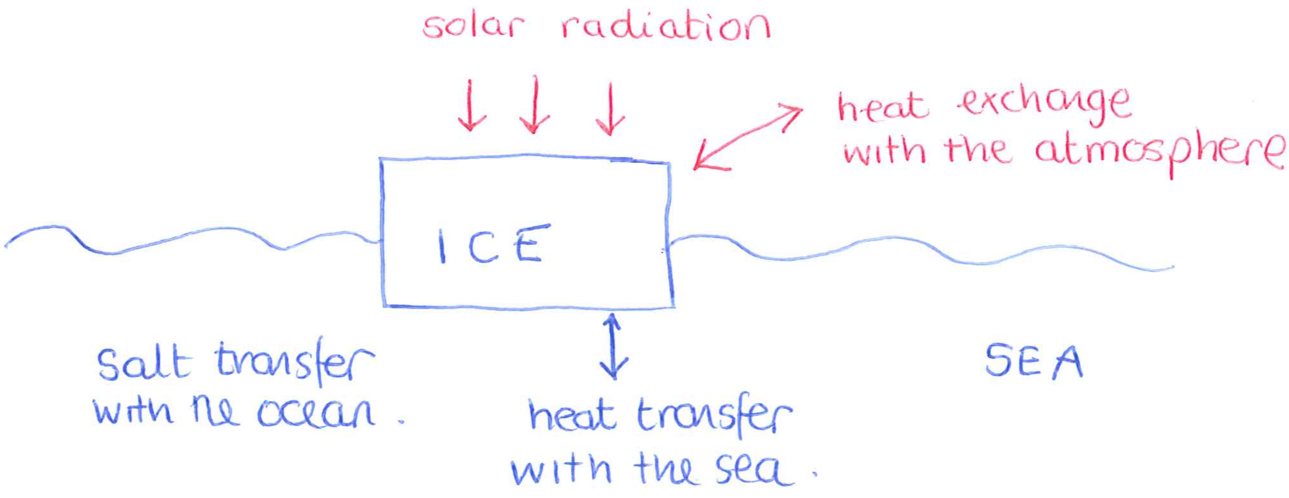
Case Study II:

Freezing and Melting Ice

Outline

1. Recap on Heat Equation and its solution
2. Introduction to
    a. Stefan problems
    b. Models for the growth and melting of ice

Modelling sea ice growth & melting



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* Neglect changes in density & salinity

$T = T_a$ (atmospheric temp)

$x = 0$

$\downarrow x$.

ICE

$$\rho c_s \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

$x = h(t)$

SEA $\quad T = T_m$

$\begin{cases} T_m = \text{melting Temp.} \\ T_a < T_m. \end{cases}$

$$\rho c_\ell \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

$T \to T_\infty$ (fixed) as $x \to \infty$.

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* In what follows, we will consider a series of models which focus on different aspects of the physical system

At the ice-sea interface

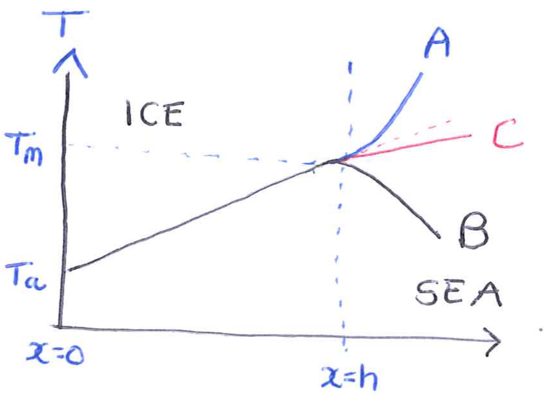
* flux out $= -k\dfrac{\partial T}{\partial x}\Big|_{x=h^+}$

* flux in $= -k\dfrac{\partial T}{\partial x}\Big|_{x=h^-}$

$$\Rightarrow \text{net flux out} = \boxed{-k\frac{\partial T}{\partial x}\Big|_{x=h^-}^{h^+} = \rho L \frac{dh}{dt}}$$: STEFAN CONDITION

where L = latent heat = energy needed to melt solid/ice per unit mass.

SANITY CHECK



$\underline{\underline{A}}$: ice should melt, so $\boxed{\dfrac{dh}{dt} < 0}$

$$\rho L \frac{dh}{dt} = -k\frac{\partial T}{\partial x}\Big|_{x=h^+} + k\frac{\partial T}{\partial x}\Big|_{x=h^-} < 0 \quad \checkmark$$

$\underline{\underline{B}}$ $\rho L \dfrac{dh}{dt} = -k\dfrac{\partial T}{\partial x}\Big|_{x=h^+} + k\dfrac{\partial T}{\partial x}\Big|_{x=h^-} > 0 \Rightarrow$ ice grows.

$\underline{\underline{C}}$: $\rho L \dfrac{dh}{dt} = -k\dfrac{\partial T}{\partial x}\Big|_{x=h^+} + k\dfrac{\partial T}{\partial x}\Big|_{x=h^-} > 0 \Rightarrow$ ice grows

Nondimensionalise:

$$\hat{x} = x/\ell \,, \quad \hat{T} = \frac{T - T_m}{T_m - T_a} \,, \quad \hat{t} = t/\tau$$

$$\Rightarrow \frac{\partial}{\partial x} = \frac{1}{\ell}\frac{\partial}{\partial \hat{x}} \,, \quad \frac{\partial}{\partial t} = \frac{1}{\tau}\frac{\partial}{\partial \hat{t}}$$

(ℓ, τ are typical length, time scales)

Assume $\boxed{c_s = c_\ell = c.}$ Then heat equation becomes

$$\rho c \frac{(T_m - T_a)}{\tau} \frac{\partial \hat{T}}{\partial \hat{t}} = \frac{k}{\ell^2}(T_m - T_a)\frac{\partial^2 \hat{T}}{\partial \hat{x}^2}$$

$$\Rightarrow \frac{\partial \hat{T}}{\partial \hat{t}} = \left(\frac{k\tau}{\rho c \ell^2}\right)\frac{\partial^2 \hat{T}}{\partial \hat{x}^2}$$

choose $\tau = \dfrac{\rho c \ell^2}{k}$ = timescale of conduction (cf $\tau = \ell^2/D$)

Exercise 2.9 Show that, with $\hat{h} = h/\ell$, the Stefan condition becomes:

$$S\frac{dh}{dt} = -\frac{\partial \hat{T}}{\partial \hat{x}}\bigg|_{\hat{x}=\hat{h}^+} + \frac{\partial \hat{T}}{\partial \hat{x}}\bigg|_{\hat{x}=\hat{h}^-}$$

$$\equiv \left[\frac{\partial \hat{T}}{\partial \hat{x}}\right]_{x=\hat{h}^+}^{x=\hat{h}^-}$$

where $\boxed{S = L/c(T_m - T_a)}$ = Stefan number.

Exercise 2.10 Suppose, instead, we take $\tau = \dfrac{\rho L \ell^2}{k(T_m - T_a)}$.

Show that the interface condition becomes:

$$\frac{d\hat{h}}{d\hat{t}} = \left.\frac{\partial \hat{T}}{\partial \hat{x}}\right|_{\hat{x}=\hat{h}^+}^{\hat{h}^-}$$

Hence, $\tau = \dfrac{\rho L \ell^2}{k(T_m - T_a)}$ = timescale of freezing

Note!

$$\frac{\text{timescale of freezing}}{\text{timescale of conduction}} = \frac{\rho L \ell^2}{k(T_m - T_a)} \cdot \frac{k}{\rho c \ell^2}$$

$$= \frac{L}{c(T_m - T_a)} \equiv S.$$

Suppose (*) $S >> 1 \Rightarrow \dfrac{dh}{dt} << 1$ ie freezes slowly.

(*) $S >> 1 \Rightarrow \dfrac{dh}{dt} >> 1$ ie freezes quickly.

Omitting $\hat{}$ etc, we have

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \quad \begin{cases} 0 < x < h(t) \\ h(t) < x < \infty . \end{cases}$$

with : $T = 0$ at $x = h(t)$

$T = -1$ at $x = 0$

$T = \dfrac{T_\infty - T_m}{T_m - T_a}$ as $x \to \infty$.

$$S \frac{dh}{dt} = \left. \frac{\partial T}{\partial x} \right]_{x=h^+}^{h^-}$$

Typically, $\begin{cases} T_m = 273K, \quad T_a - T_m = 20K, \quad L \approx 3.3 \times 10^5 \text{ J kg}^{-1} \\ c \approx 4.2 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}, \quad \rho \approx 1000 \text{ kg m}^{-3} \\ k = 0.6 \text{ Watts m}^{-1} \text{ K}^{-1} \end{cases}$

$$\Rightarrow \quad S \sim 4 .$$

We seek a similarity sol^n:

$$T(x,t) = f(\eta), \quad \eta = x/2\sqrt{t}, \quad \boxed{h = \underbrace{\lambda}_{\text{constant}} . 2\sqrt{t}}$$

Question: why is functional form for $h(t)$ "sensible"? how would you justify it?

Then

$$\frac{\partial T}{\partial t} = -\frac{1}{2t}\cdot\eta\cdot f' \quad , \quad \frac{\partial^2 T}{\partial x^2} = \frac{1}{4t}\cdot f''$$

$$\Rightarrow 0 = f'' + 2\eta f'$$

$$\Rightarrow 0 = (e^{\eta^2} . f')'$$

$$\Rightarrow f' = C_1 e^{-\eta^2}$$

$$\Rightarrow f(\eta) = C_1 \int_0^{\eta} e^{-\xi^2} d\xi + C_2 \quad , \text{ where } C_1, C_2 \text{ const.}$$

Definition : The Error function

$$\boxed{\operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\xi^2} d\xi}$$

$$\therefore f(\eta) = C^* \operatorname{erf}(\eta) + C_2 \qquad C^* = \frac{\sqrt{\pi}\, C_1}{2}$$

$f(0) = -1 \Rightarrow C_2 = -1$

$f(\lambda) = 0 \Rightarrow C^* = 1/\operatorname{erf}(\lambda)$

$$\therefore f(\eta) = -1 + \frac{\operatorname{erf}(\eta)}{\operatorname{erf}(\lambda)}$$

Assume, for simplicity, that the sea is at temp T_m (for whole of the sea!).

Then, the Stefan condition supplies

$$S\frac{dh}{dt} = k\frac{\partial T}{\partial x}\bigg|_{x=h^-} \qquad \text{since } \frac{\partial T}{\partial x}\bigg|_{x=h^+} = 0.$$

$$\Rightarrow \frac{S\lambda}{\sqrt{t}} = \frac{1}{2\sqrt{t}} \cdot f'(\lambda^-)$$

$$\Rightarrow \lambda . S = \frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \cdot \frac{e^{-\lambda^2}}{\text{erf}(\lambda)} \qquad \left(f'(\eta) = C_1 e^{-\eta^2} = \frac{2}{\sqrt{\pi}} . C^* . e^{-\eta^2} \right)$$

$$\Rightarrow \boxed{\sqrt{\pi} . \lambda . \text{erf}(\lambda) . e^{\lambda^2} = 1/S .} \quad \text{—— } (*)$$

Exercise 2.11

Show that eqn $(*)$ has a unique solution & that λ is a decreasing function of S.

Show further that, for large S, $h \sim \sqrt{\frac{2kt}{\rho c S}}$ in ~~dimensio~~ dimensional terms.

Ocean Heat flux

It is more realistic to say that there is a heat flux between the ice & water

eg $-\underline{Q}.\underline{n} = -F_0$ where : $F_0 = h_0(T_0 - T_m)$

T_0 = temp of water away from interface

$h_0 > 0$, const.

Then Stefan condition becomes:

$$\boxed{\rho L \frac{dh}{dt} = -F_0 + k \frac{\partial T}{\partial x}\bigg|_{x=h^-}}$$

Surface Boundary Condition

At the upper ice surface, heat is exchanged with the atmosphere (radiation, turbulent heat transfer, conduction). Of these, radiation is the most important

(*) Energy flow in, from sun & atmosphere : Q_{in}

(*) Energy flow out, $\boxed{Q_{out} = \sigma T^4}$ (Stefan-Boltzmann Law)

$\Rightarrow$ net heat flux $= Q_{in} - Q_{out}$ (absorbed by ice, assuming no melting)

$$\Rightarrow \boxed{-k\frac{\partial T}{\partial x} = Q_{in} - Q_{out} = Q_{in} - \sigma T^4}$$

(Robin BC)

Suppose T is close to T_m

$$\Rightarrow T^4 = (T_m + T - T_m)^4 = T_m^4 + 4T_m^3(T - T_m) + \text{HOTs}$$

$$\Rightarrow \quad -k\frac{\partial T}{\partial x} = \underbrace{Q_{in} - \sigma T_m^4}_{F_a} - k_a(T - T_m)$$

where $k_a = 4\sigma T_m^3$.

So, now we have :

ATMOSPHERE $\quad \boxed{-k_s \partial T/\partial x = F_a - k_a(T - T_m)}$ $\quad x = 0$

$\downarrow x$.

ICE $\quad \boxed{\rho c_s \frac{\partial T}{\partial t} = k_s \frac{\partial^2 T}{\partial x^2}}$

$x = h(t)$

WATER. $\quad \boxed{T = T_m}$

$$\boxed{\rho L \frac{dh}{dt} = -F_o + k_s \frac{\partial T}{\partial x}\Big|_{x=h^-}}$$

Nondimensionalise as follows :

$$\hat{T} = \frac{T - T_m}{T^*}, \quad \hat{x} = \frac{x}{X^*}, \quad \hat{h} = \frac{h}{X^*}, \quad \hat{t} = \frac{t}{\tau}.$$

Set : $\tau = \frac{\rho L . X^{*2}}{k_s}$, $\quad X^* = \frac{k_s}{k_a}$, $\quad T^* = \frac{F_a}{k_a}$, etc.

& introduce

$$\hat{F}_a = \frac{F_a . X^*}{k_s T^*}, \quad \hat{F}_o = \frac{F_o}{\rho L X^*}, \quad S = \frac{L}{c_s T^*}, \text{ etc.}$$

Eventually, we have

$$\frac{1}{S}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \qquad 0 < x < h(t)$$

with : $$T = 0, \quad \frac{dh}{dt} = \left.\frac{\partial T}{\partial x}\right|_{x=h^-} - F_0 \quad \text{on } x = h(t)$$

$$-\frac{\partial T}{\partial x} = F_a - T \qquad \text{on } x = 0.$$

Suppose $S \gg 1$. Then

$$\frac{\partial^2 T}{\partial x^2} \approx 0 \Rightarrow T = G(t)(h(t) - x) \qquad (\text{s.t. } T = 0 \text{ on } x = h(t))$$

on $x = 0$, we have

$$-\frac{\partial T}{\partial x} = G(t) = F_a - Gh \Rightarrow G(t) = \frac{F_a}{1 + h(t)}$$

$$\therefore \boxed{T(x,t) = \frac{F_a (h(t) - x)}{1 + h(t)}} \qquad 0 < x < h.$$

(We require $T \le 0 \Rightarrow F_a = Q_{in} - \sigma T_m^4 < 0$).

Stefan condition supplies:

$$\boxed{\frac{dh}{dt} = \frac{|F_a|}{1+h} - F_0}$$

$$\frac{dh}{dt} = \frac{|F_a|}{1+h} - F_o$$

Exercise 2.12 : Suppose $|F_a| > F_o$. Can you solve for $h(t)$? What is the long time behaviour?

How does the behaviour change if $|F_a| < F_o$.

Comment on the (long time) validity of the model.

Additional resources

- IJ Hewitt (2019). Lecture notes for C5.11 Mathematical Geoscience, Mathematical Institute. https://courses-archive.maths.ox.ac.uk/node/view_material/45164

- A Fowler (2011). Mathematical Geoscience. Springer. See: sections 2.1-2.5, 4.1-4.4, 5.1-5.5 and 10.1-10.4

- V Alexiades and AD Solomon (1993). Mathematical modelling of melting and freezing processes. https://www.math.utk.edu/~vasili/475/Handouts/3.PhChgbk.1+title.pdf